Counting pairs of transverse simple closed curves on a surface

Anton Zorich (with V. Delecroix, E. Goujard and P. Zograf)

Workshop on Dynamics and Moduli Spaces of Translation Surfaces

secretely related with the approaching anniversary
of eternally young Howie Masur
from whom we have already learned and continue to learn so much

Fields Institute, Toronto

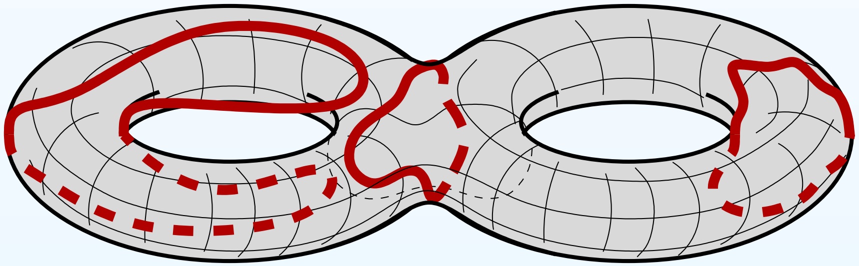
October 22, 2018

Mirzakhani's count of simple closed geodesics

Multicurves

Consider a finite collection of pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g.n}$ of genus g with n punctures.

For any hyperbolic metric X on $S_{g,n}$ and for any simple closed curve γ_i there exists a unique geodesic representative in the free homotopy class of γ_i.



Fact. *For any hyperbolic metric X and any collection $\gamma_1, \ldots, \gamma_k$ of pairwise non-intersecting simple closed curves, their geodesic representatives do not self-intersect and do not pairwise intersect either.*

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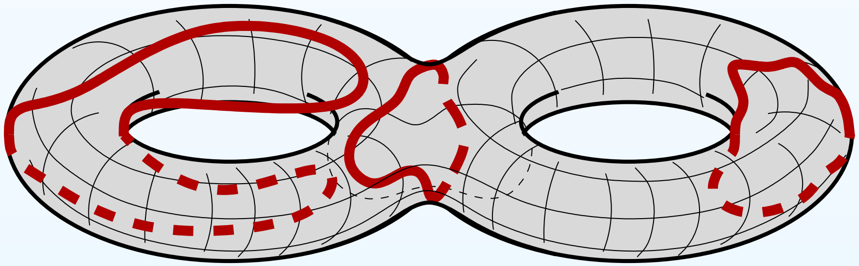

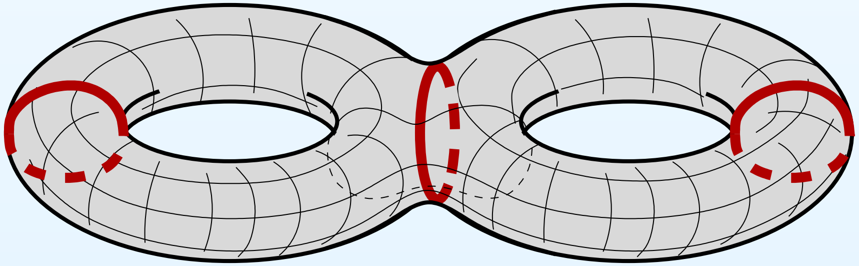

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Multicurves

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

We say that two multicurves γ, ρ have the same *topological type* $[\gamma] = [\rho]$ *if and only if they belong to the same orbit of the mapping class group:* $\rho \in \mathrm{Mod}_{g,n} \cdot \gamma$.

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Frequencies of multicurves

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ the number $s_X(L,\gamma)$ of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L has the following asymtotocs:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

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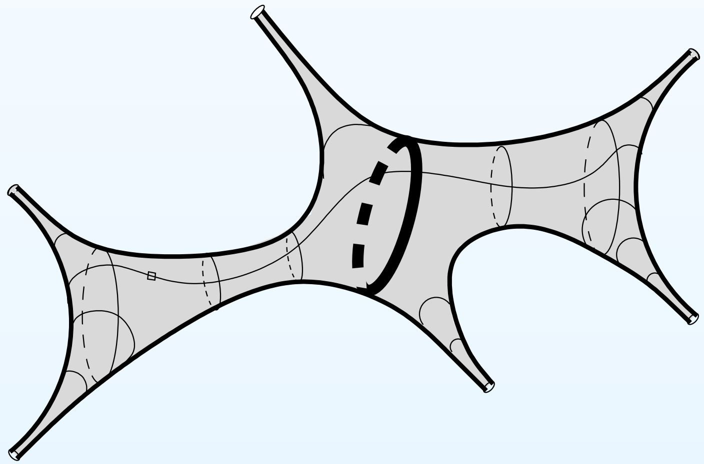
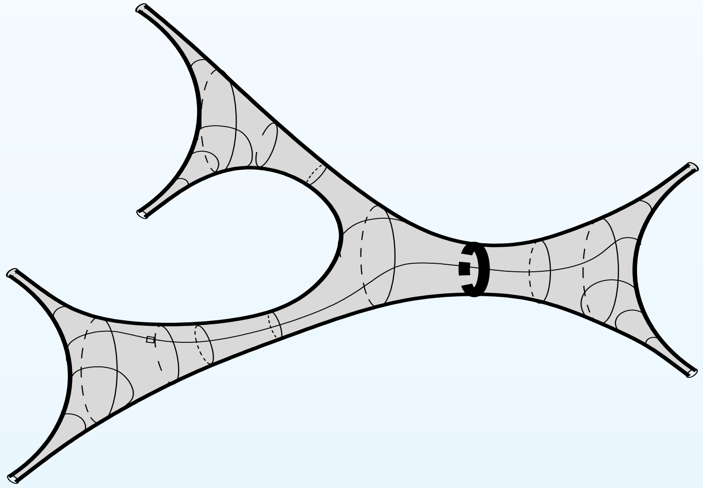
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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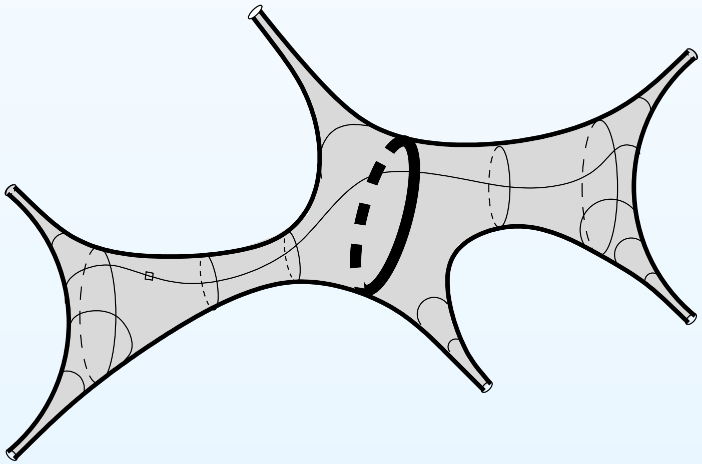

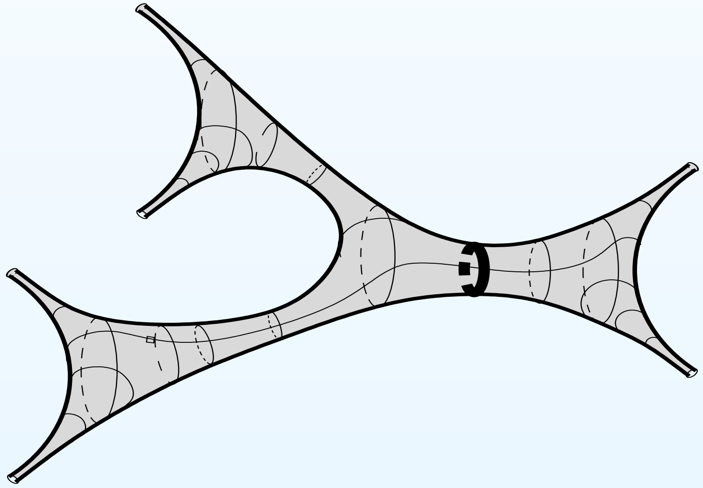

Example (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell and S. Schleimer; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

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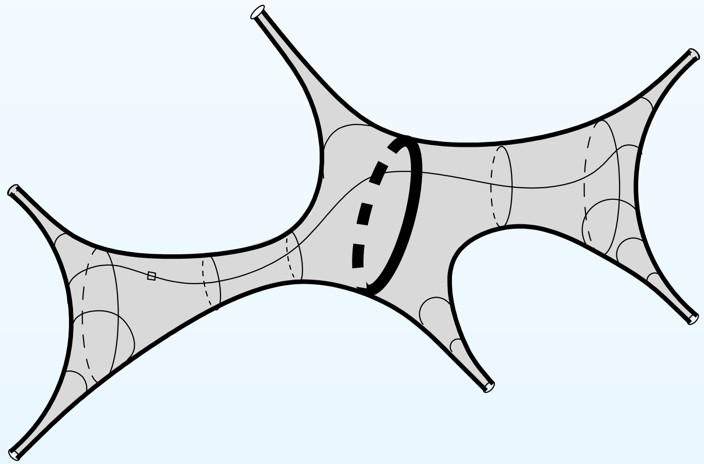

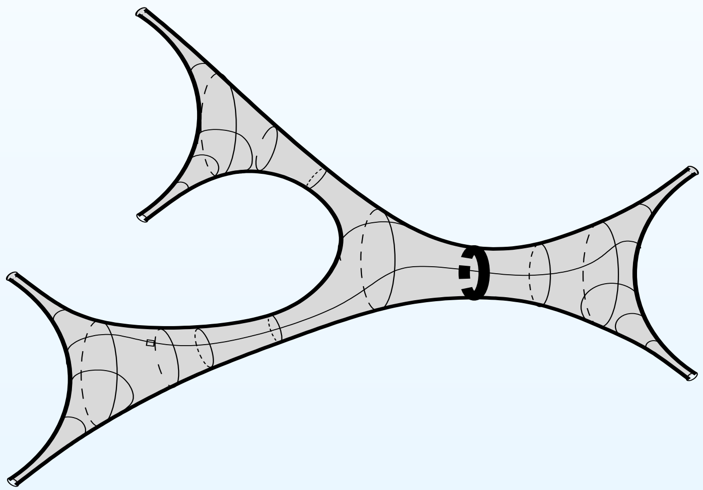

In this sense one can say that for any hyperbolic metric X on a sphere with 6 cusps, a long simple closed geodesic separates the cusps as $(3+3)$ with probability $\frac{4}{7}$ and as $(2+4)$ with probability $\frac{3}{7}$.

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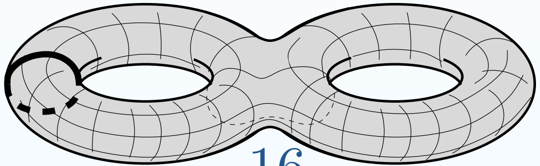
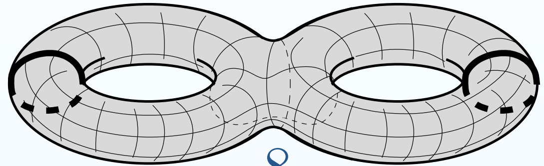
Shape of a random multicurve on a surface of large genus.

Multicurves on a surface of genus two and their frequencies

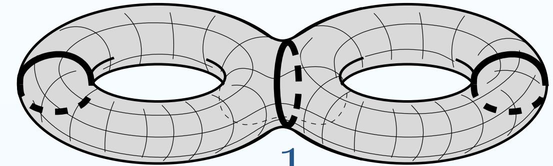
The picture below illustrates all possible types of primitive multicurves on a surface of genus two without punctures; the fractions give their frequencies.



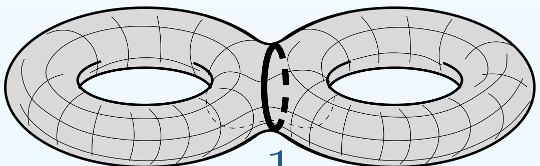
$$\frac{16}{63}$$



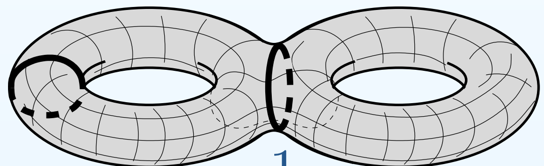
$$\frac{8}{15}$$



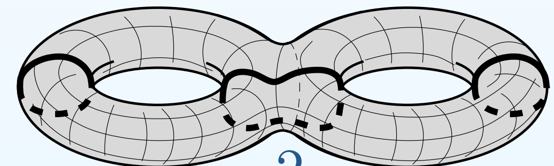
$$\frac{1}{9}$$



$$\frac{1}{189}$$



$$\frac{1}{45}$$



$$\frac{2}{27}$$

In genus 3 there are already 41 types of multicurves, and this number grows very fast when genus g grows.

To figure out a typical shape of a multicurve let us focus on the basic properties of a multicurve.

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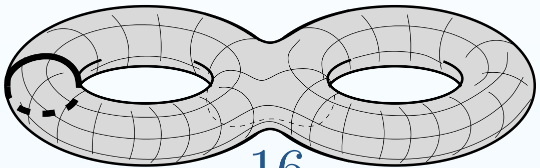

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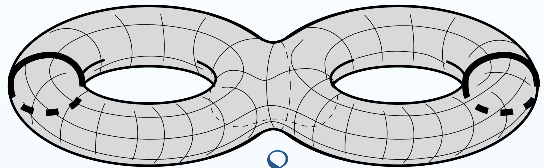

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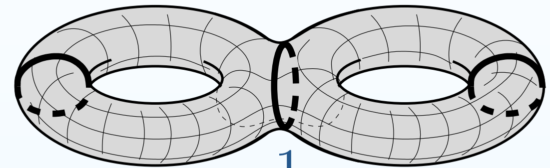

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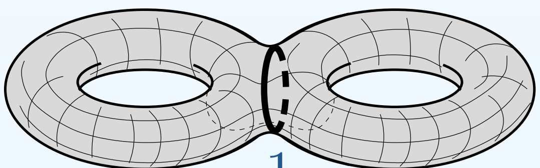

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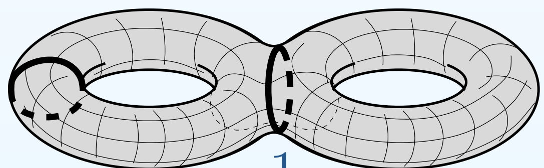

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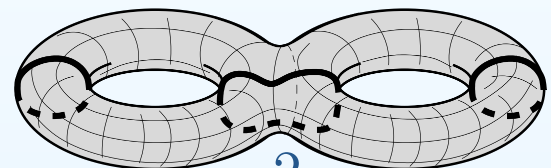

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Question 1. *With what probability a random multicurve chops a surface S_g into $j = 1, 2, \ldots, 2g-2$ connected components? Does a random simple closed geodesic separate the surface or not?*

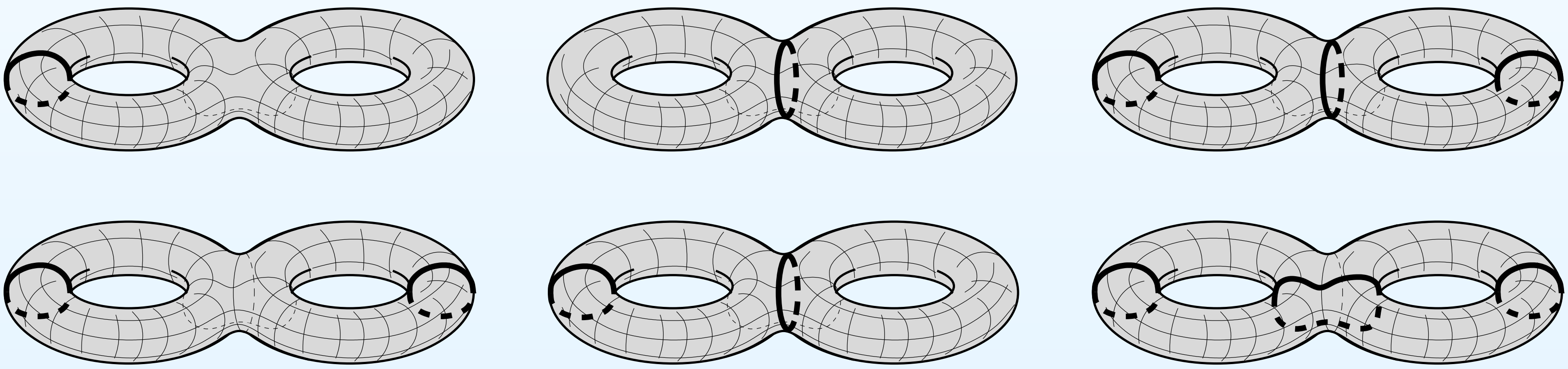
Question 2. *With what probability a random multicurve has $k = 1, 2, \ldots, 3g-3$ (primitive) connected components? How often simple closed curves show up?*

General Question. *Describe a typical multicurve on a surface of large genus.*

Typical number of subsurfaces: data

Question 1. *With what probability a random multicurve chops a surface S_g into $j = 1, 2, \ldots, 2g-2$ connected components? Does a random simple closed geodesic separate the surface or not?*

We regroup the multicurves in genus $g = 2$ so that the multicurves on the left do not separate the surface and the ones on the right separate the surface into two components; the tables give the frequencies.



$$\frac{248}{315} \approx 0.79 \qquad \Bigg| \qquad \frac{67}{315} \approx 0.21$$

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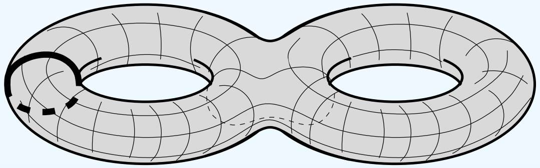

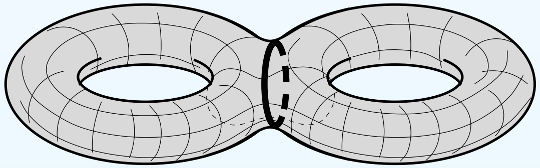

Non-separating versus separating simple closed curves on a surface of genus $g = 2$ have the following frequencies:

$\frac{48}{49} \approx 0.98$	$\frac{1}{49} \approx 0.02$

Typical number of subsurfaces: data

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The frequencies of multicurves which separate a surface of genus $g = 3$ into $1, 2, 3, 4$ components are

1 component	2 components	3 components	4 components
$\frac{4984269952}{5827696875}$	$\frac{2150659384}{17483090625}$	$\frac{7152596}{366778125}$	$\frac{115927}{52396875}$
0.855	0.123	0.0195	0.0022

and the frequencies of non-separating versus separating simple closed curves are

$\frac{1776}{1781} \approx 0.997$	$\frac{5}{1781} \approx 0.003$

Random simple closed curve (almost) never separates

Theorem. *A random simple closed curve on a surface of large genus does not separate the surface. Namely, the probability that it separates decays exponentially with the rate* 4^{-g}*: the ratio of frequencies of non-separating over separating simple closed geodesics on a closed surface of genus* g *satisfies:*

$$\lim_{g\to+\infty} \frac{1}{g} \log \frac{c(\gamma_{non-sep}, g)}{c(\gamma_{sep}, g)} = \log 4$$

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Idea of the proof. Frequencies of separating simple closed curves are expressed in terms of the intersection numbers which admit closed expression:

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g\, g!}\,.$$

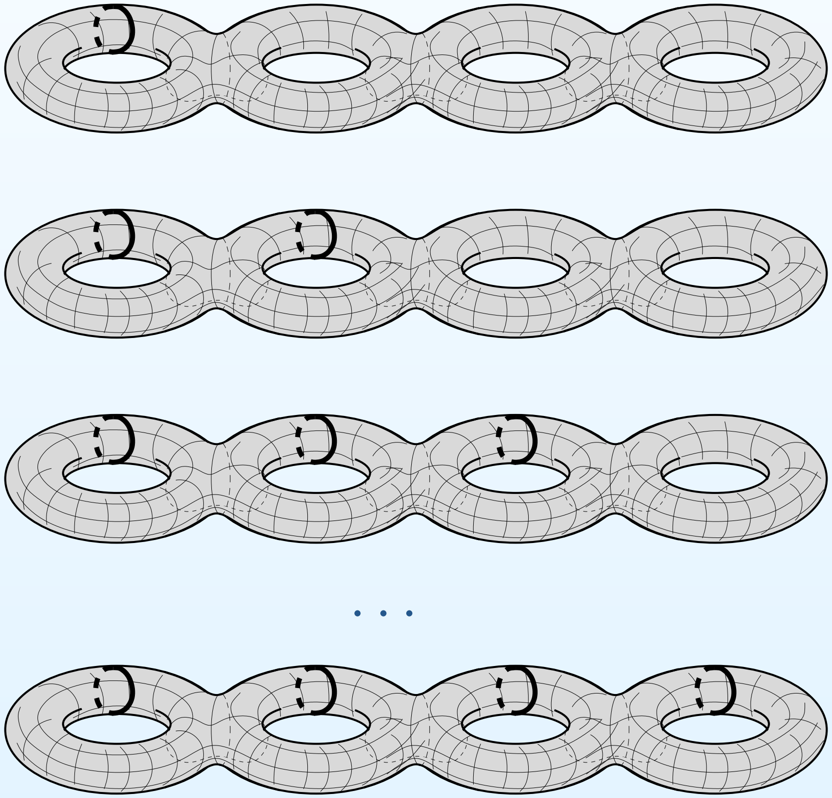
Frequencies of non-separating simple closed curves are expressed in terms of

$$\int_{\overline{\mathcal{M}}_{g,2}} \psi_1^k \psi_2^{3g-1-k}$$

for which we obtain large genus asymptotics uniform for all k in fixed genus g.

Conjecture on non-separation

Conjecture. *A randomly chosen multicurve on a surface of large genus does not separate the surface. (The sum of frequencies of all multicurves different from the following ones rapidly tends to zero as the genus grows.)*



Γ_1

Γ_2

Γ_3

$\dots \qquad \dots$

Γ_g

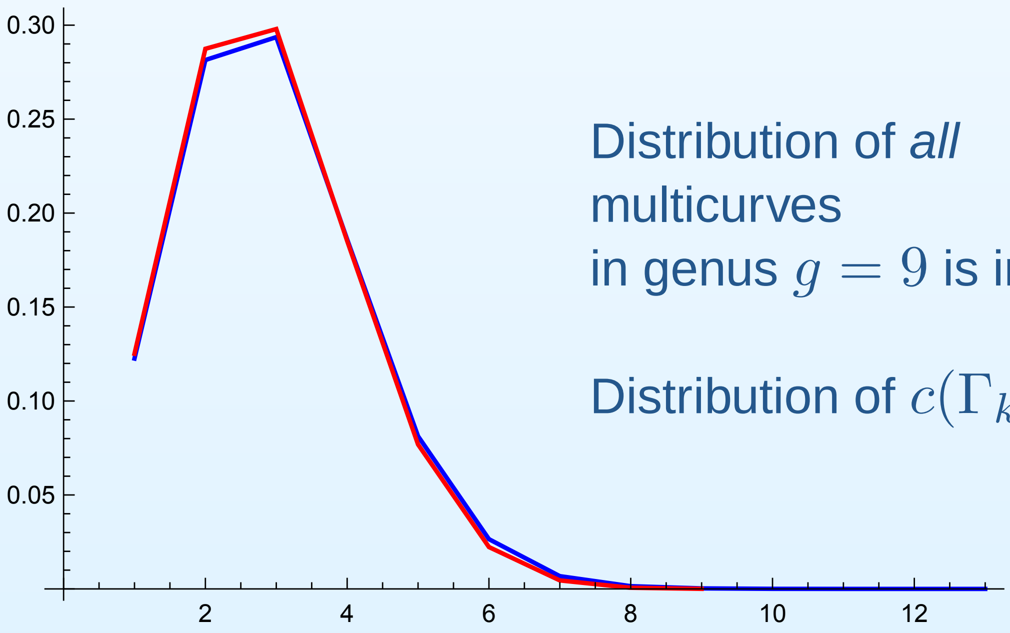
Why it is a conjecture and not just a guess?..

• Compute exact values of frequencies of the distinguished multicurves Γ_k and record the corresponding statistics (graph in red).

• Pass from *all* multicurves to *all* square-tiled surfaces and to the Masur–Veech volume of the moduli space of holomorphic quadratic differentials $\mathcal{Q}_g$.
(Élise Goujard would, probably, tell more about it in her talk on Wednesday.)

• Relate them to frequencies of interval exchange transformations having $k = 1, \dots, 3g-3$ "cylinders".

• Collect statistics of the latter frequencies using computer experiments (graph in blue). The resulting statistics approaches the first one as genus grows:



Distribution of *all* multicurves in genus $g = 9$ is in blue;

Distribution of $c(\Gamma_k, g)$ is in red.

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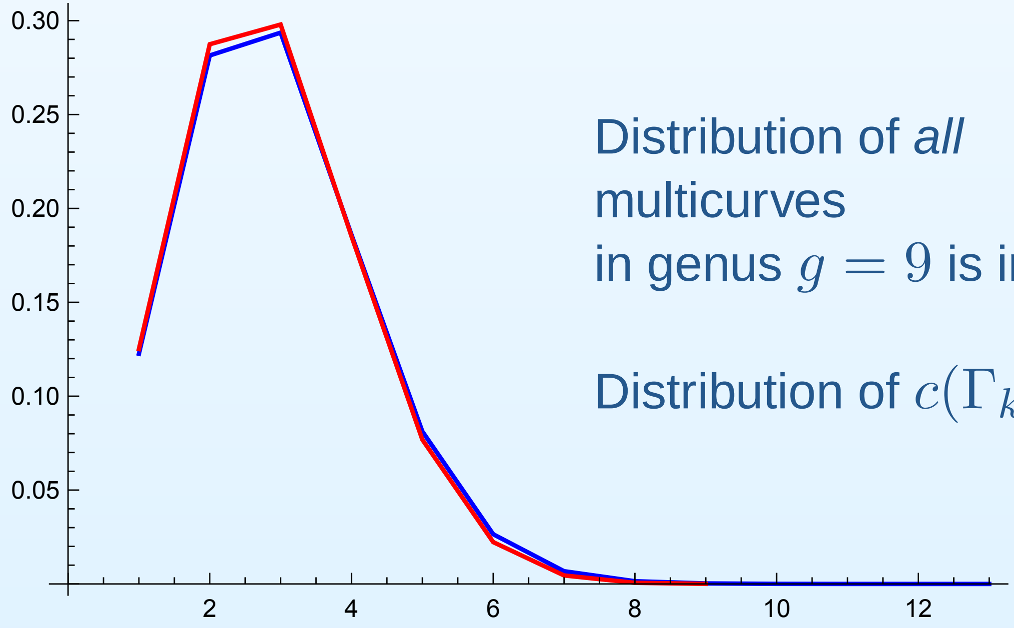

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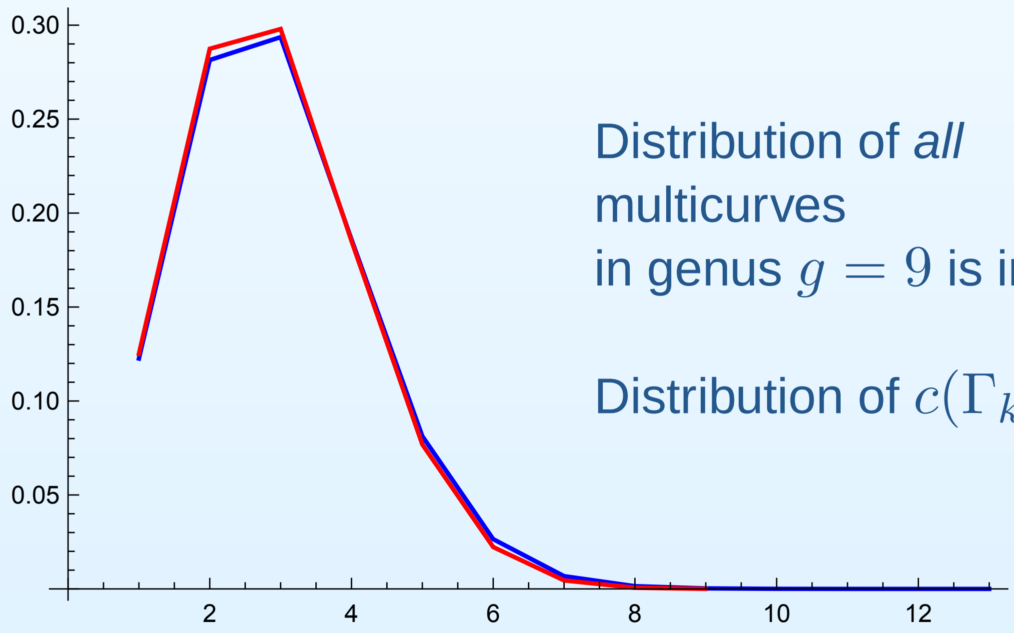

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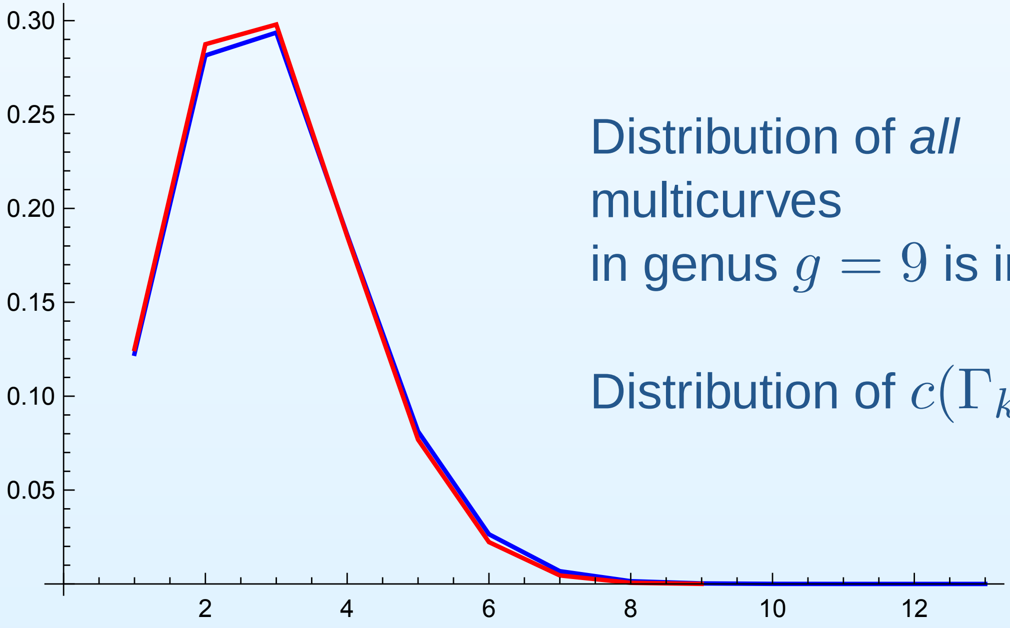

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Number of connected components of a random multicurve

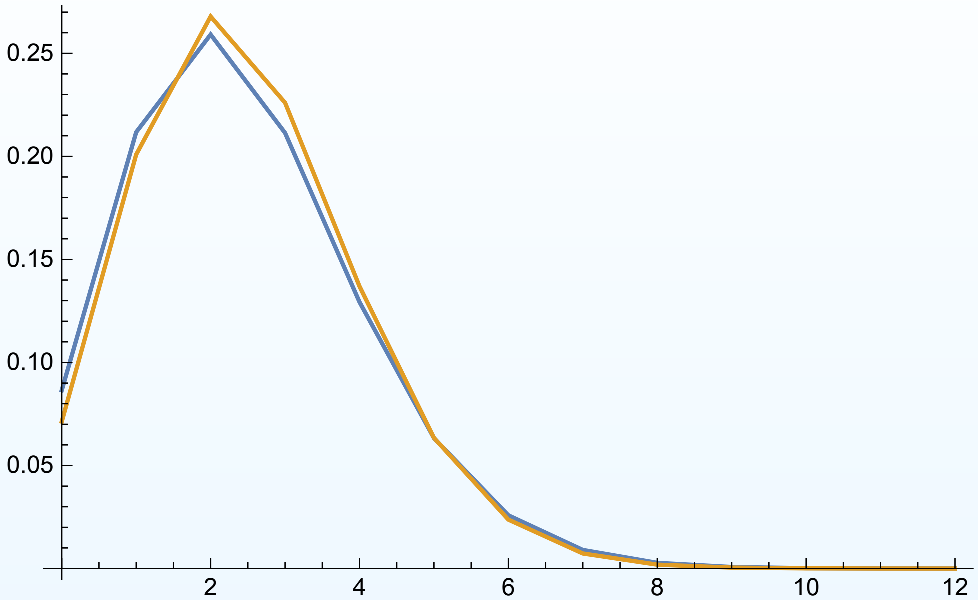
Conditional to the Connectedness Conjecture, the original question

Question 1. *With what probability a random multicurve has $k = 1, 2, \dots, 3g-3$ (primitive) connected components? How often simple closed curves show up?*

reduces in large genera to the following much more concrete problem:

Problem. *Find the distribution of $c(\Gamma_k, g)$ for $k = 1, \dots, g$. Find $c(\Gamma_1, g)$.*

Number of connected components of a random multicurve



Genus $g = 26$

Poisson distribution with λ=(log(3g−3)+γ)/2

Cylinder contribution

Conjecture. *The distribution of $c(\Gamma_k, g)$ tends to the Poisson distribution $\frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}$ with parameter $\lambda = \frac{\log(3g-3)+\gamma}{2}$, where $\gamma \approx 0.577$ is the Euler constant.*

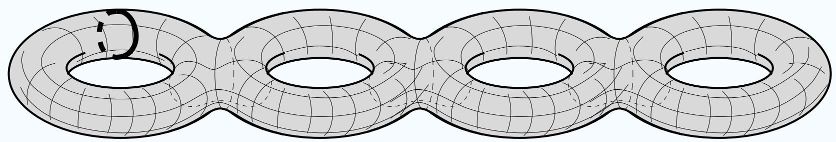
Conjectural asymptotics of the Masur–Veech volume: $\operatorname{Vol}\mathcal{Q}_g \sim \sqrt{\frac{2}{\pi}} \cdot \left(\frac{8}{3}\right)^{4g-4}$.

Conditional Theorem. Conditional to conjecture on volume asymptotics one has:

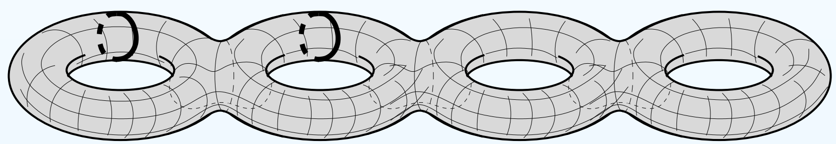
$$c(\Gamma_1, g) \sim \frac{1}{\sqrt{3g-3}} = \left(\frac{\dim \mathcal{Q}_g}{2}\right)^{-\frac{1}{2}} .$$

Summary: conjectural shape of a random multicurve

Overall Conjecture. *A randomly chosen multicurve on a surface of large genus g has one of the following shapes:*

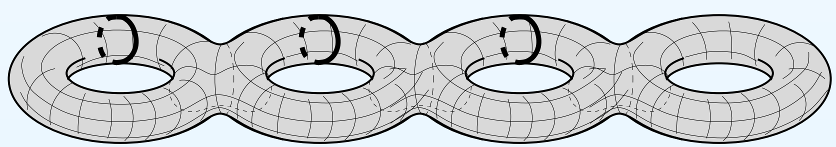


Γ_1



Γ_2

$\dots \qquad \dots$



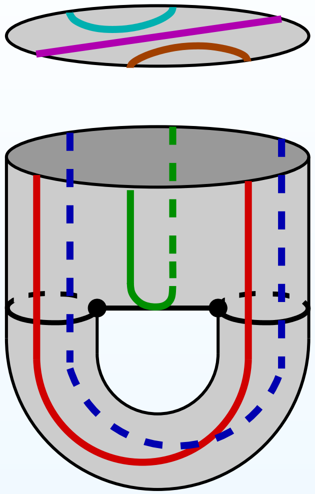
$\Gamma_{2\log g}$

● *Usually it has about $1 + \frac{\log(3g-3)+\gamma}{2}$ connected components. It has more than $2\log(g)$ components exceptionally rarely.*

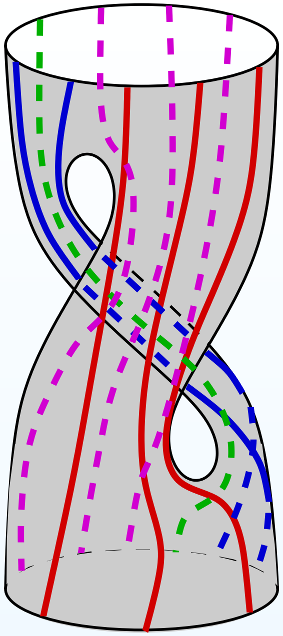
● *The distribution of frequencies of the number of components of a random multicurve very slowly tends to the normal distribution with mean $\lambda = \frac{\log(3g-3)+\gamma}{2}$ and with variance λ.*

From braids on surfaces to multicurves

Arc systems and braids on surfaces



Arc system with two bigons

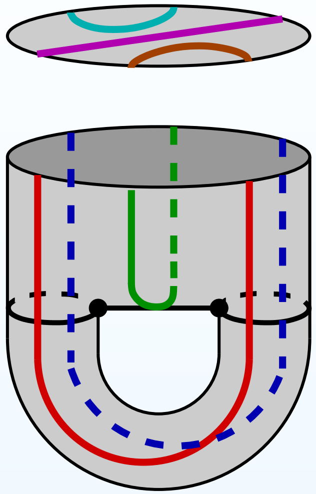


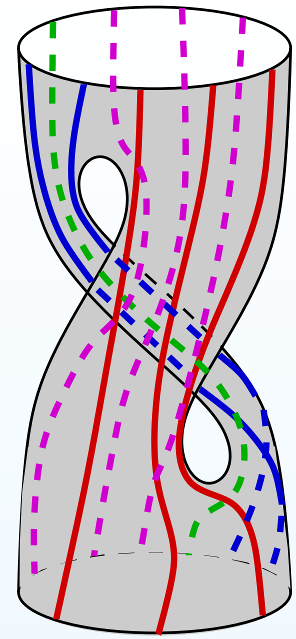
Braid

Arc system. Consider a smooth surface with two boundary components. Draw N non selfintersecting and non pairwise intersecting strands on the surface placing all endpoints at the boundary in such way that there are N endpoints at one component and N endpoints at the other.

Braid. If the surface is connected, we can let all the strands go from one component to another. These kind of braids mimic gradient flow from one regular level of a Morse function on a surface to another regular level.

Arc systems and braids on surfaces: count





Problem. *Find the asymptotic number of arc systems/braids on a surface $S_{g,n}$ of genus g with n boundary components with (at most) $N \gg 1$ strands.*

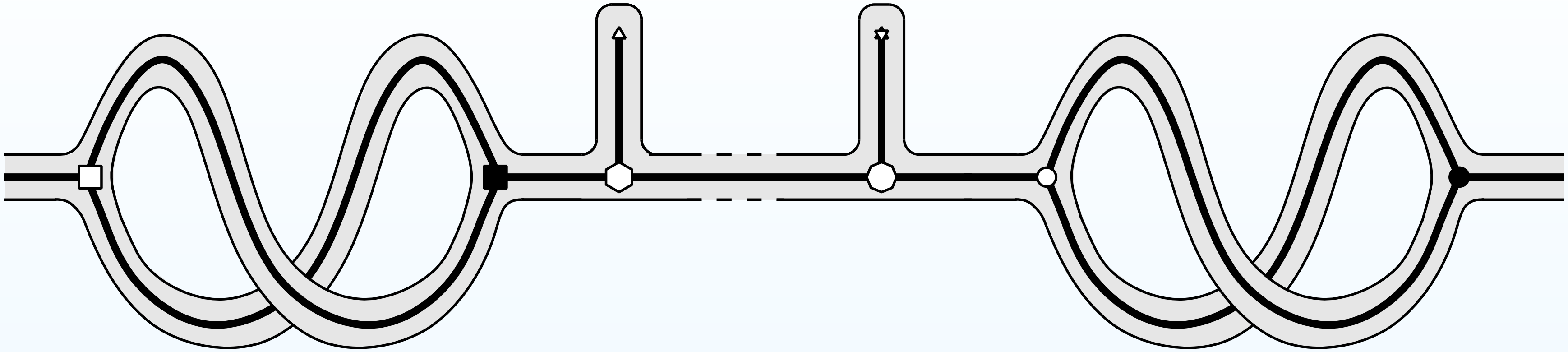
All but negligible amount of arc systems/braids chop the surface $S_{g,n}$ into polygons. Most of the polygons are 4-gons, but there are several 2-gons, 6-gons, 8-gons etc (for braids — 8-gons, 12-gons, etc). More restrictive count specifies exact numbers of these unusual polygons.

Bigons require their own convention: there are separate counts for arc systems

- without bigons;
- with exactly/(at most) p bigons;
- with no restrictions on bigons.

(Our methods do not work for the count with no restrictions on bigons.)

Ribbon graph and arc system are the same objects



Lemma. *Arc systems (resp. braids) are into one-to-one correspondence with integer metric ribbon graphs (resp. orientable metric ribbon graphs).*

Proof. A metric ribbon graph associated to an arc system is just a dual graph. Each polygon in which the strands subdivide the surface defines a vertex of the dual graph; two vertices are joined by an edge if the corresponding polygons share a common strand. The dual graph admits a natural embedding into the surface which induces the ribbon graph structure.

An arc system asssociated to an integer metric ribbon graph is the collection of segments transversally crossing the edges of the ribbon graph at all half-integer points.

Kontsevich's count of metric ribbon graphs

Let $(d_1, \ldots, d_n)$ be an ordered partition of $3g-3+n$ into a sum of nonnegative integers. Define the homogeneous polynomial $N_{g,n}$ of degree $3g-3+n$ in variables $b_1, \ldots, b_n$ in the following way:

$$N_{g,n}(b_1, \ldots, b_n) := \frac{1}{2^{5g-6+2n}} \sum_{|d|=3g-3+n} \frac{\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \rangle}{d_1! \cdots d_n!} b_1^{2d_1} \cdots b_n^{2d_n} \,,$$

where

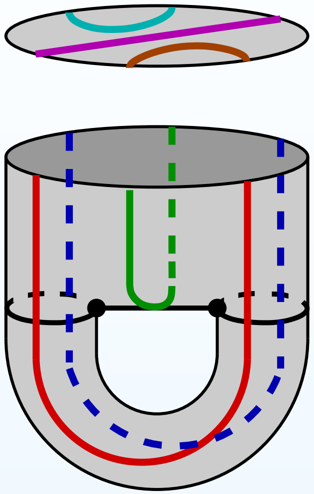
$$\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,.$$

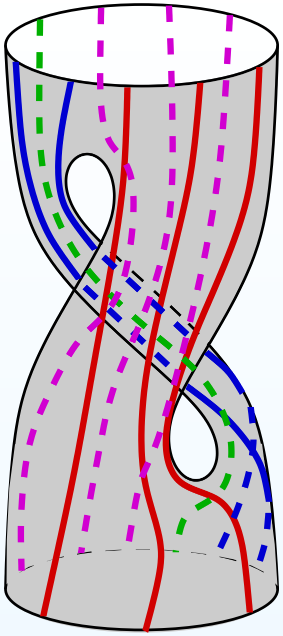
Theorem (Kontsevich). *Let $\sum_{i=1}^{n} b_i$ be even. The weighted count of genus g connected trivalent metric ribbon graphs Γ with integer edges and with n labeled boundary components of lengths $b_1, \ldots, b_n$ satisfies:*

$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n) = N_{g,n}(b_1, \ldots, b_n) + \text{lower order terms}$$

This Theorem is a part of Kontsevich's proof of Witten's conjecture.

From arc systems and braids to multicurves





Identifying pairs of boundary components in a way which matches the endpoints of the strands on each side we get a multicurve on a closed surface.

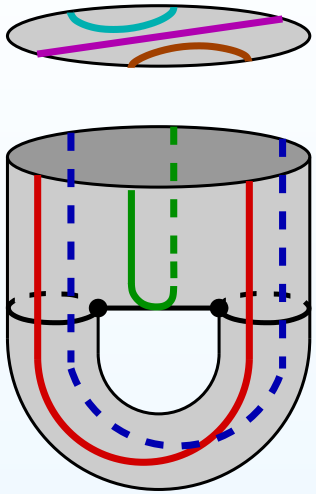
Question 1. *With what probability the resulting multicurve chops the surface into* $j = 1, 2, \ldots, 2g-2$ *connected components?*

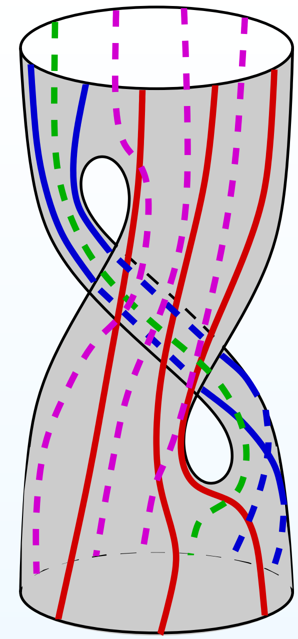
Question 2. *With what probability the resulting multicurve has* $k = 1, 2, \ldots, 3g-3$ *(primitive) connected components? How often simple closed curves show up?*

General Question. *Describe a typical multicurve obtained under this construction*

- *when genus* g *becomes very large and there are no bigons;*
- *genus* g *is fixed and the number* p *of bigons becomes very large.*

From arc systems and braids to multicurves





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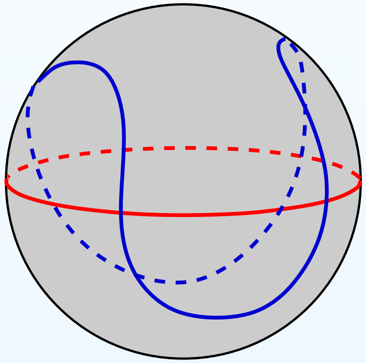
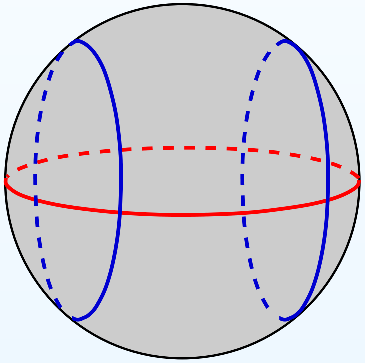
Theorem. *The corresponding probabilities are exactly the same as the frequencies of "unconditional" multicurves considered before. They depend only on the parameters* g *and* p *of the resulting surface* $S_{g,p}$.

Matching arc systems on a pair of disc

The problem is quite meaningful even for a pair of arc systems on a pair of discs. Depending on the twist chosen to identify a pair of arc systems we get a multicurve with one or several connected components.





Fix the numbers of bigons $p_1 \geq 2$ and $p_2 \geq 2$ on each of the discs.

Theorem. *The frequency $\mathrm{P}_1(p_1, p_2; N)$ of simple closed curves obtained by all possible identifications of all arc systems with at most N arcs and having p_1 bigons on one disc and with p_2 bigons on the other disc has the following limit:*

$$\lim_{N\to+\infty} \mathrm{P}_1(p_1, p_2; N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2} \qquad \text{where } p = p_1 + p_2\,.$$

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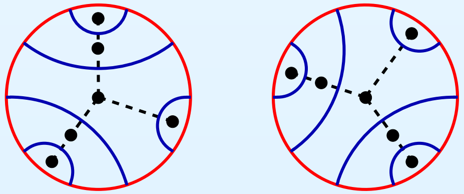
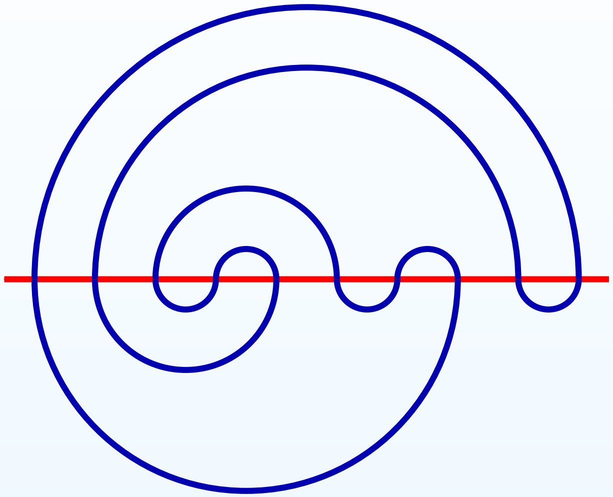
$$\lim_{N \to +\infty} \mathrm{P}_1(2,4;N) = \mathrm{P}_1(3,3;N) = \frac{280}{\pi^6} \approx 0.291245$$



Application: count of meanders



A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line. The notion “meander” was suggested by Arnold though meanders were studied already by Poincaré. They appear in various contexts, in particular in physics (P. Di Francesco, O. Golinelli, E. Guitter).

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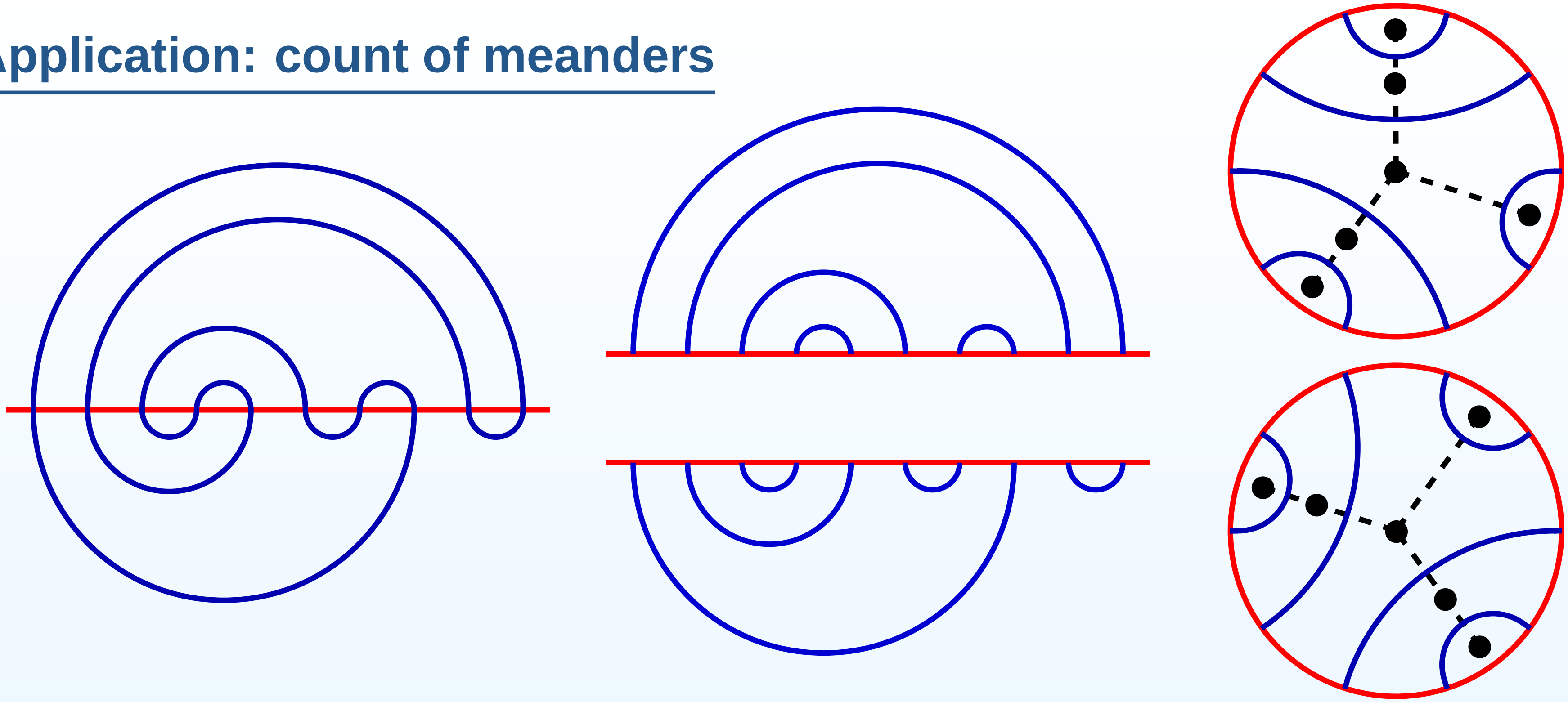

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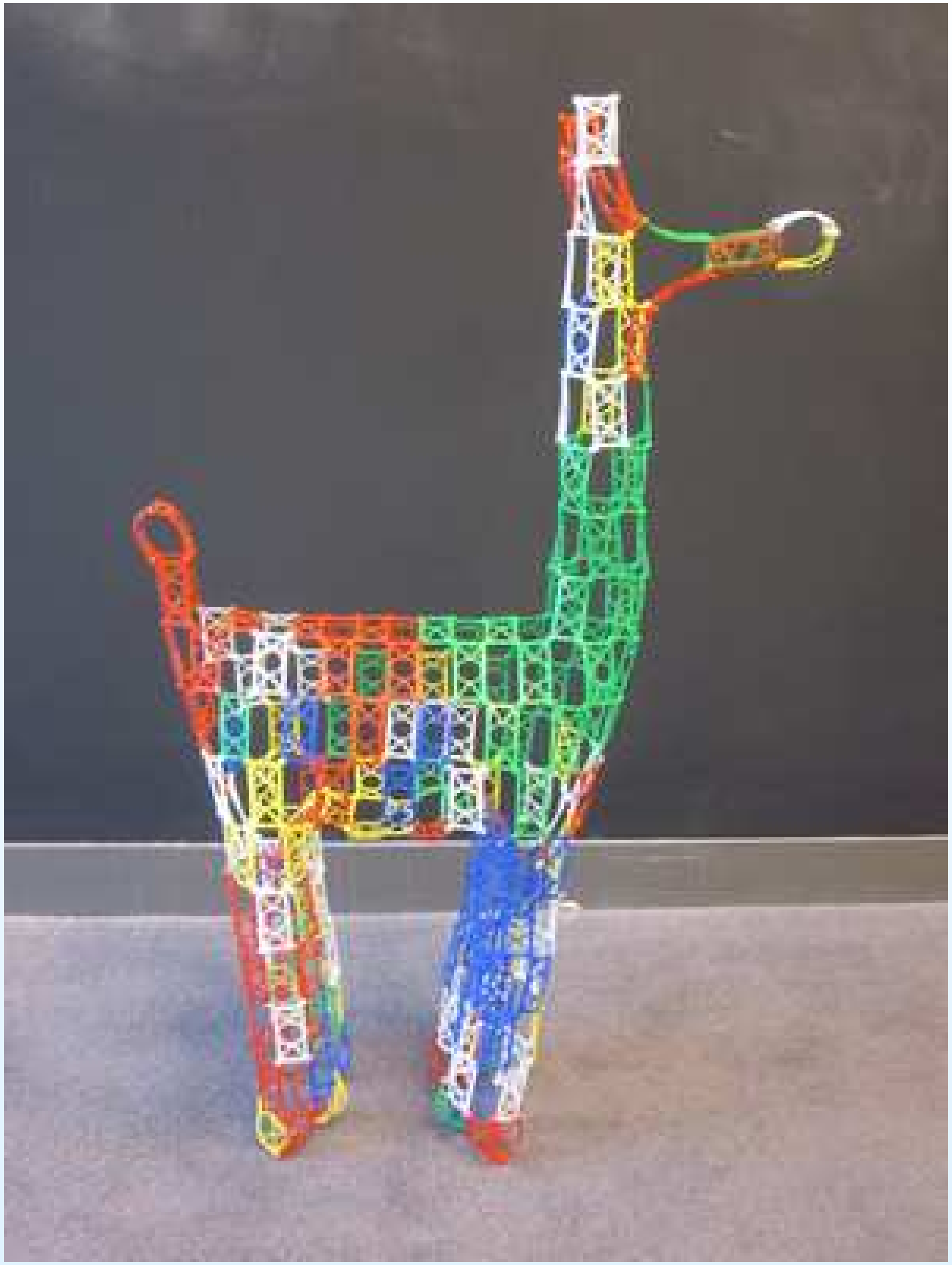
Each meander defines a pair of arc systems on a disc. It is easy to count arc systems on a disc. Thus, the frequency of connected curves (versus multicurves) obtained after gluing a pair of arc systems provides us with the asymptotic count of meanders with fixed number p of bigons.

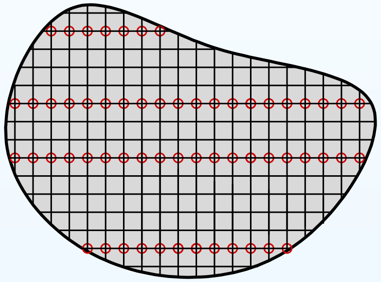
Non-correlation of vertical and horizontal foliations on square-tiled surfaces

Cylinder decomposition of a square-tiled surface



Equidistribution and Non-correlation Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \cdots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L})/\operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula! Proof: Moore ergodicity theorem.

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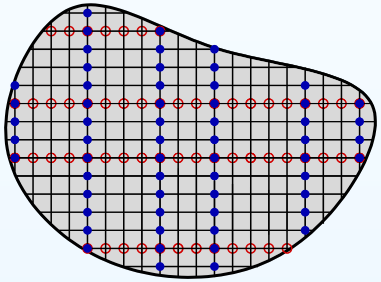

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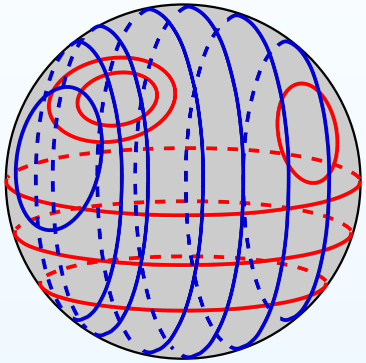
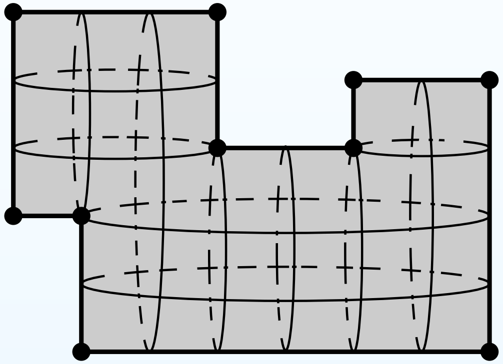
How to count meanders

Step 1. There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





Step 2. Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height 1. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

Step 3. Fixing the number of minimal arcs (“pimples”) we fix the number of simple poles p of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

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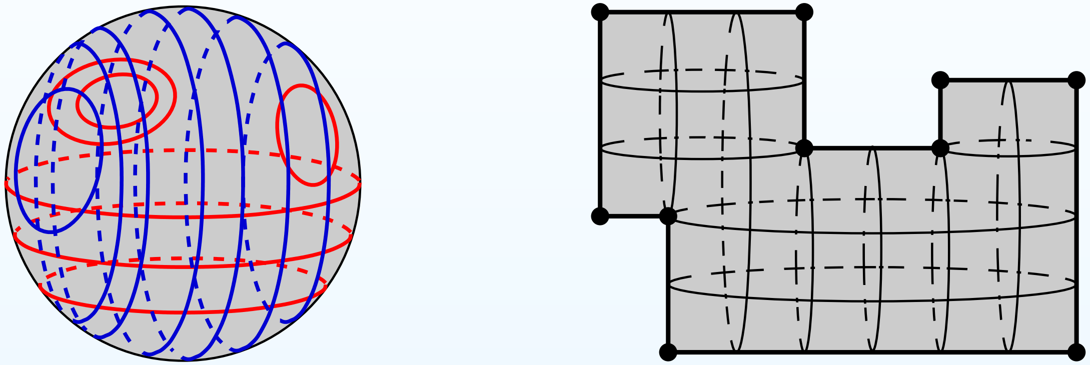

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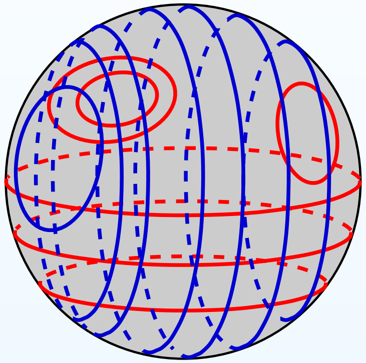

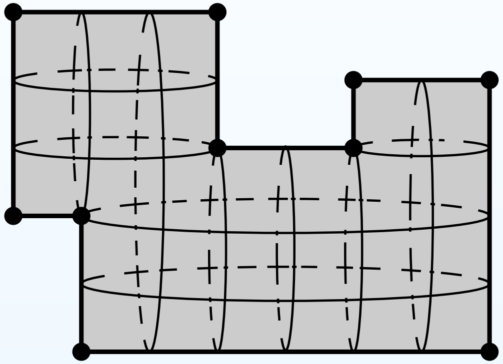

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General philosophy

● Pairs of transverse multicurves correspond to square-tiled surfaces. Thus, count of all pairs of transverse multicurves is equivalent to count of Masur–Veech volumes.

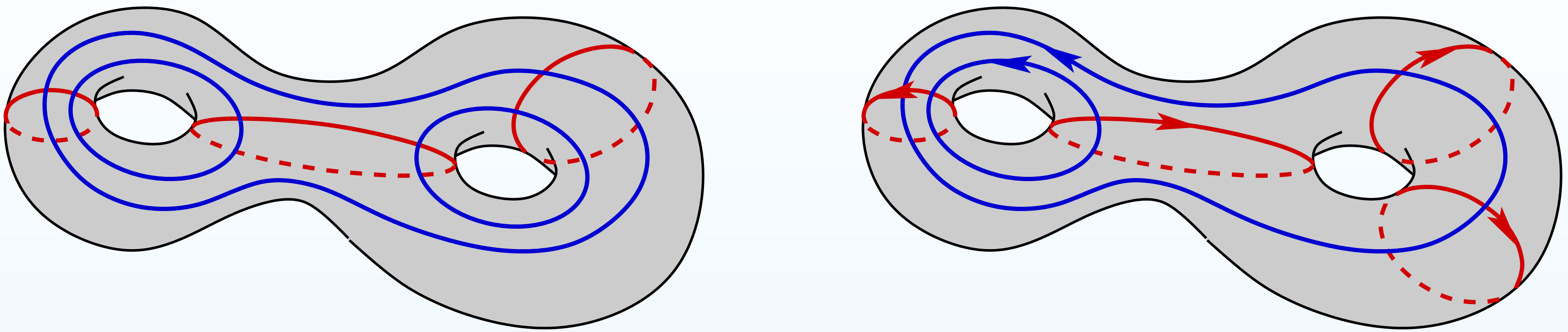
● Count of arc systems, braids, ribbon graphs, pairs: simple closed curve plus transverse multicurve, one-cylinder square-tiled surfaces is another group of (somehow equivalent) problems, which usually admits a more efficient solution.

● Consider the following three counting problems:

1. count of all square-tiled surfaces (i.e. Masur–Veech volume Vol);
2. count of horizontally one-cylinder square-tiled surfaces (i.e. c_1);
3. count of horizontally and vertically square-tiled surfaces (i.e. $c_{1,1}$).

By non-correlation, $c_{1,1} = \dfrac{c_1^2}{\mathrm{Vol}}$. Count of c_1 usually admits a relatively efficient solution. Hence, as soon as we know the appropriate Masur–Veech volume, we know $c_{1,1}$, and hence we can count meanders, pairs of transverse simple closed curves etc.

Pairs of transverse multicurves

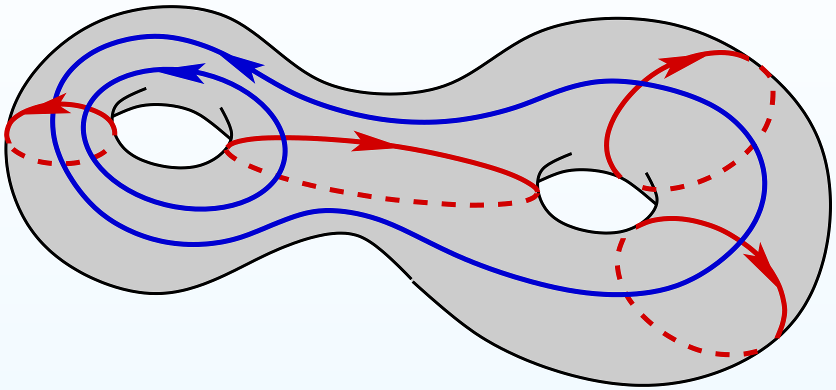


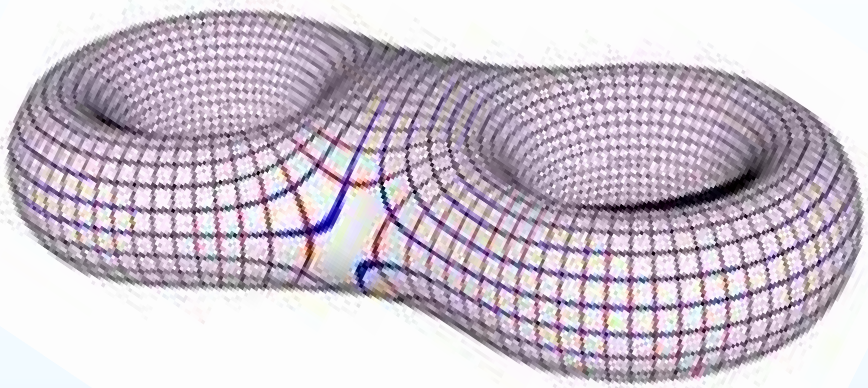
Exercise. For each of the two pairs of transverse multicurves answer to the following questions:

- Does this pair of multicurves chop the surface into polygons?
- If so, is the associated quare-tiled surface a translation surface (Abelian differential) or a half-translation surface (quadratic differential)?
- In which stratum lives the corresponding square-tiled surface?
- Describe its horizontal and vertical cylinder decomposition.

Braids on surfaces; positively intersecting multicurves; Masur–Veech volumes of strata of Abelian differentials

Pairs of positively intersecting multicurves





Picture created by Jian Jiang

Questions.

- *With what probability a random square-tiled translation surface has* $k = 1, 2, \ldots, 3g-3$ *maximal horizontal cylinders? How often it has a single horizontal cylinder?*
- *How strongly these quantities depend on the ambient stratum? On the genus?*
- *Do we get the same distribution as for random square-tiled quadratic differentials (random multicurves)?*
- *What is the shape of a random square-tiled surface of large genus?*

Contribution of k-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$0.19 \approx \mathrm{P}_1(\mathcal{H}(3,1)) = \frac{3\,\zeta(7)}{16\,\zeta(6)} \leftarrow \text{ the only quantity which is easy to compute}$$

$$0.47 \approx \mathrm{P}_2(\mathcal{H}(3,1)) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx \mathrm{P}_3(\mathcal{H}(3,1)) = \frac{1}{32\,\zeta(6)}\Big(& 12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3) \\ & + 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2) \\ & - 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4) \\ & + 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4) \\ & - 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3) \\ & - 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx \mathrm{P}_4(\mathcal{H}(3,1)) = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big)\,.$$

State of the art

We know the similar distribution of frequencies of square-tiled surfaces with respect to the number of horizontal cylinders only for several low-dimensional strata. However, we have a conjecture on this distribution for large genera based on extensive numerical experiments.

Conjecture. *The distribution of square-tiled surfaces by the number k of maximal horizontal cylinders tends to the Poisson distribution $\frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}$ with parameter $\lambda = \log(d) + \gamma$, where $\gamma \approx 0.577$ is the Euler constant. Here $d = \dim_{\mathbb{C}} \mathcal{H}$ is the dimension of the ambient stratum.*

According to this conjecture *a random translation square-tiled surface has about $\log d + 1 + \gamma$ cylinders and very rarely more that $3 \log d$ cylinders.*

Remark. Conjecturally, the distribution is the same for all strata (except hyperelliptic components) in different genera but of the same dimension.

Remark. The parameter λ in the analogous conjectural distribution for $\mathcal{Q}(1,\dots,1)$ is different. It is $\dfrac{\log\frac{d}{2}+\gamma}{2}$, where $d = \dim \mathcal{Q}(1,\dots,1)$.

Count of 1-cylinder square-tiled surfaces

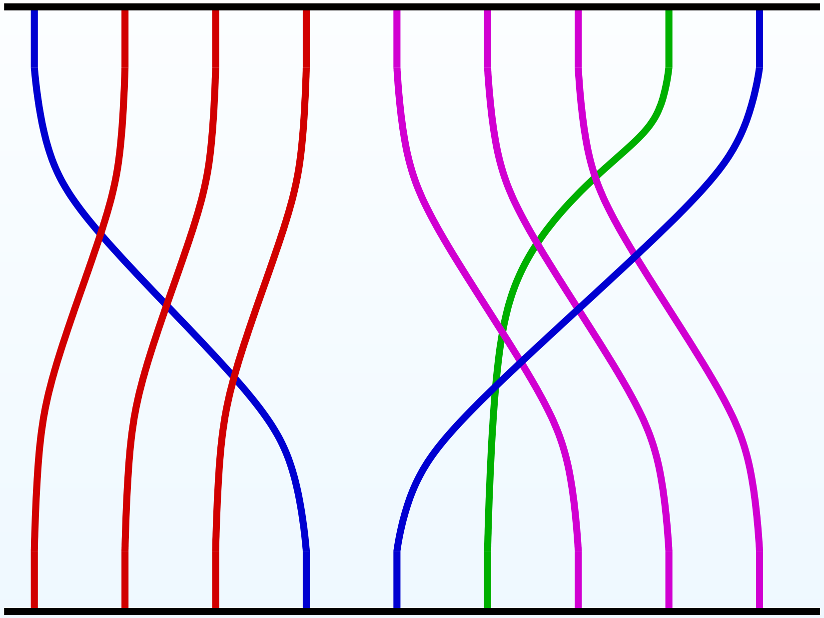
We have rather comprehensive information about Masur–Veech volumes of strata of Abelian differentials. Namely, in all low genera we know them explicitly. In higher genera the volume of the principal stratum $\mathcal{H}(1,\dots,1)$ can be computed exactly up to very high genus. When $g \to +\infty$ it can be computed approximately with arbitrary precision by the work of Chen–Möller–Zagier. Less precise but universal formula for all strata proving our conjecture with Eskin

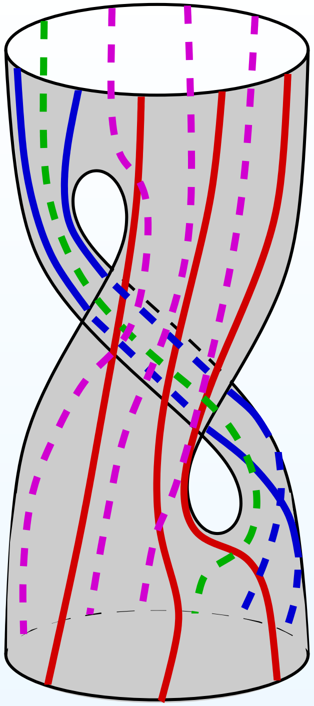
$$\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n) \sim \frac{4}{(m_1+1)\dots(m_n+1)}\,.$$

was recently proved by Aggarwal.

By the general philosophy, to compute braids, frequencies of simple closed curves, numbers of pairs of positively intersecting simple closed curves, etc in this context, one has to compute contribution of 1-cylinder square-tiled surfaces. As in the previous cases, this problem admits a solution.

Counting braids and positive meanders





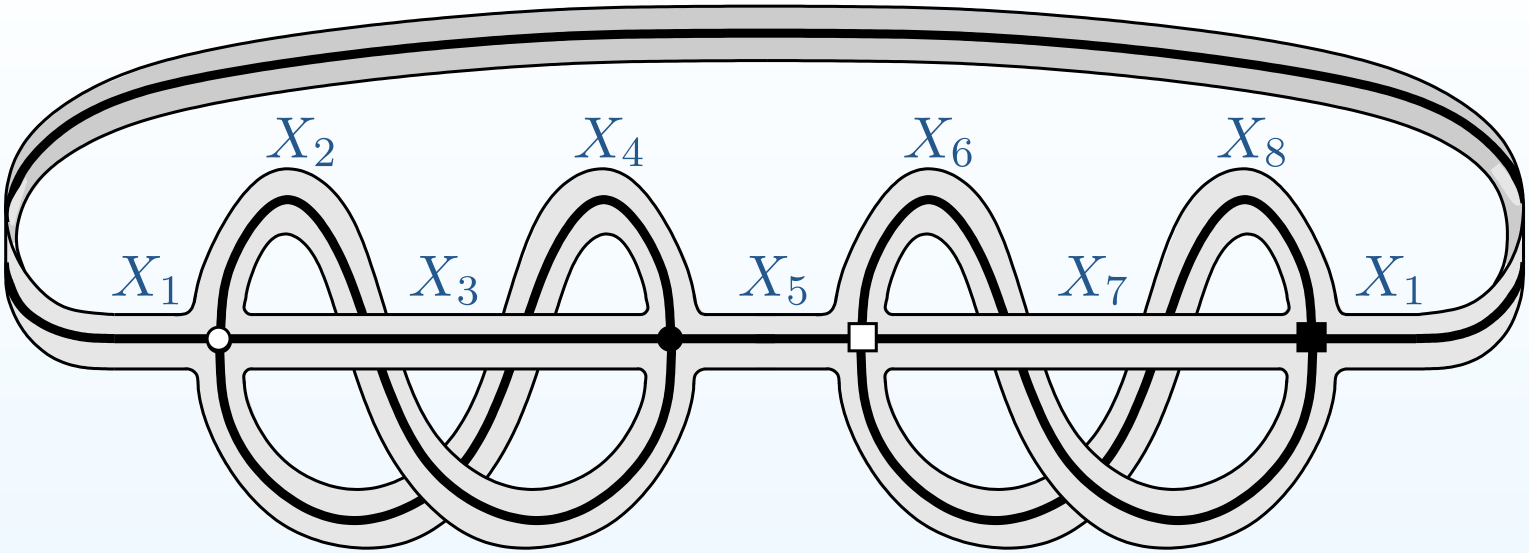
Theorem. *The relative contribution $p_1(\mathcal{H}(m_1, \ldots, m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1, \ldots, m_n)$ is of the order $\frac{1}{d}$, where $d = \dim \mathcal{H}(m_1, \ldots, m_n)$.*

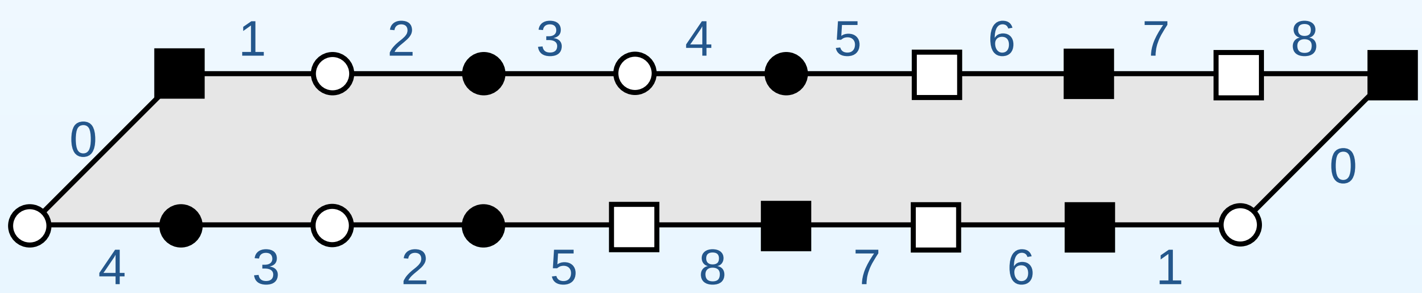
In particular, the probability to get a connected curve from a braid on a surface of genus g is of the order $\frac{1}{4g}$.

Pairs of positively intersecting transverse simple closed curves have frequency $\frac{1}{16g^2}$ among all positively intersecting pairs of multicurves.

1-cylinder square-tiled surfaces and permutations

1-cylinder surface as a pair of permutations





$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8$$

$$4 \to 3 \to 2 \to 5 \to 8 \to 7 \to 6 \to 1$$

$$c_{bot} \cdot c_{top}^{-1} = (1,3)(2,4)(5,7)(6,8)$$

Frobenius formula

The count of 1-cylinder N-square-tiled surfaces in the stratum $\mathcal{H}(m_1, \dots, m_n)$ is reduced to the count of solutions of the following equation for permutations:

$$(N - \text{cycle}) \cdot (N - \text{cycle}) = \text{product of cycles of lengths } d_1 + 1, \dots, d_n + 1$$

Frobenius formula expresses this number in terms of characters of the exterior powers of the standard representation $\mathbf{St}_n$ of the symmetric group S_n:

$$\chi_j(g) := \operatorname{tr}(g, \pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0 \le j \le n-1)\,.$$

Theorem. *The absolute contribution $c_1(\mathcal{H}(m_1, \dots, m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1, \dots, m_n)$ equals*

$$c_1 = \frac{2}{(d-1)!} \cdot \prod_k \frac{1}{(k+1)^{\mu_k}} \cdot \sum_{j=0}^{d-2} j!\,(n-1-j)!\,\chi_j(\nu)\,.$$

Here $d = \dim \mathcal{H}(m_1, \dots, m_n)$; $\nu \in S_n$ is any permutation with decomposition into cycles of lengths $(m_1+1), \dots, (m_n+1)$; μ_i is the number of zeroes of order i, i.e. the multiplicity of the entry i in the multiset $\{m_1, \dots, m_n\}$.

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Theorem. *The absolute contribution $c_1(\mathcal{H}(m_1,\dots,m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ satisfies the following bounds:*

$$\frac{1}{d+1}\cdot\frac{4}{(m_1+1)\dots(m_n+1)} \le c_1(\mathcal{H}) \le \frac{1}{d-\frac{10}{29}}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}$$

We were able to replace the formula in characters by this much more efficient estimate using the results of Zagier.

Count of one-cylinder square-tiled surfaces: answers

For permutations ν representing the principal and the minimal strata the characters $\chi_j(\nu)$ admit easier computation which leads to the following formulae:

$$c_1(\mathcal{H}(1^{2g-2})) = \frac{1}{4g-2} \cdot \frac{4}{2^{2g-2}}$$

$$c_1(\mathcal{H}(2g-2)) = \frac{1}{2g} \cdot \frac{4}{2g-1}$$

Count of one-cylinder square-tiled surfaces: answers

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$$c_1(\mathcal{H}(1^{2g-2})) = \frac{\zeta(4g-3)}{4g-2} \cdot \frac{4}{2^{2g-2}} \approx \frac{1}{d} \cdot \operatorname{Vol} \mathcal{H}(1^{2g-2})$$

$$c_1(\mathcal{H}(2g-2)) = \frac{\zeta(2g)}{2g} \cdot \frac{4}{2g-1} \approx \frac{1}{d} \cdot \operatorname{Vol} \mathcal{H}(2g-2)$$

We get an extra factor $\zeta(d)$ if we count the contribution of 1-cylinder square-tiled surfaces having arbitrary number of bands of squares (i.e. any height of the cylinder). However, $\zeta(d)$ is very close to 1 for $d \gg 1$, so this gives almost the same count.

We use results of Chen–Möller–Zagier and independent results of Aggarwal and Sauvaget for volume asymptotics.