Mirzakhani's count of simple closed geodesics

Anton Zorich

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Hyperbolic geometry of surfaces

Simple closed curves and simple closed geodesics

Any smooth orientable surface of genus $g \geq 2$ admits a metric of constant negative curvature (usually chosen to be -1), called *hyperbolic* metric. Allowing to metric to have several singularities (cusps), one can construct a hyperbolic metric also on a sphere and on a torus.

A smooth closed curve on a surface is called *simple* if it does not have self-intersections. Suppose that we have a simple closed curve γ on a *hyperbolic surface* (possibly with cusps). Suppose that the curve is *essential*, that is not contractible to a small curve encircling some disc or some cusp. Interpreting our curve as an elastic loop, let it slide along the surface to contract to the shortest shape in our hyperbolic metric. We get a closed geodesic, which remains to be smooth non self-intersecting curve.

Fact. *For any hyperbolic metric and any essential simple closed curve on a surface, there exists a unique geodesic representative in the free homotopy class of the curve; it is realized by a simple closed geodesic.*

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Topological types of simple closed curves

Let us say that two simple closed curves on a smooth surface have the same *topological type* if there is a diffeomorphism of the surface sending one curve to another. It immediately follows from the classification theorem of surfaces that there is a finite number of topological types of simple closed curves. For example, if the surface does not have punctures, all simple closed curves which do not separate the surface into two pieces, belong to the same class. Indeed: the classification theorem implies that cutting the surface open along such two simple closed curves we get two diffeomorphic surfaces with two boundary components. A little extra effort allows to build a diffeomorphism of the initial closed surface to itself sending the first curve to the second.

The group of all diffeomorphisms of a closed smooth orientable surface of genus g quotient over diffeomorphisms homotopic to identity is called the *mapping class group* and is denoted by Mod_g. When the surface has n marked points (punctures) we require that diffeomorphism sends marked points to marked points; the corresponding mapping class group is denoted $\mathrm{Mod}_{g,n}$.

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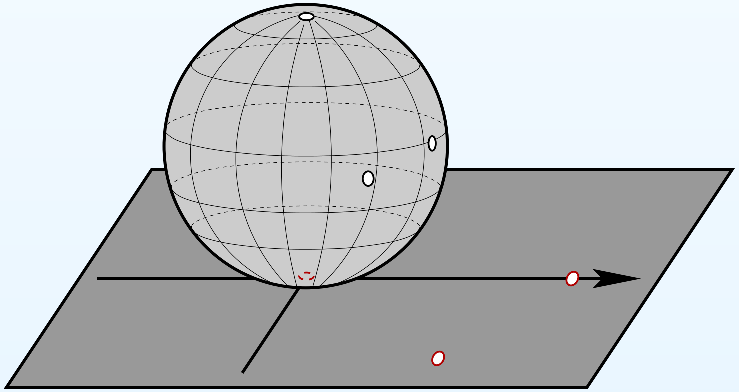
Families of hyperbolic surfaces

Consider a configuration of four distinct points on the Riemann sphere $\mathbb{CP}^1$. Using appropriate holomorphic automorphism of $\mathbb{CP}^1$ we can send three out of four points to 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of $\mathbb{CP}^1$ fixing 0, 1 and ∞ is already the identity transformation. The remaining point serves as a complex parameter in the space $\mathcal{M}_{0,4}$ of configurations of four distinct points on $\mathbb{CP}^1$ (up to a holomorphic diffeomorphism).



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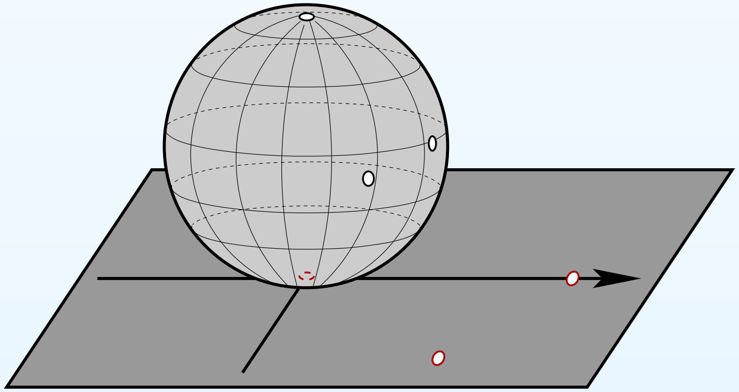

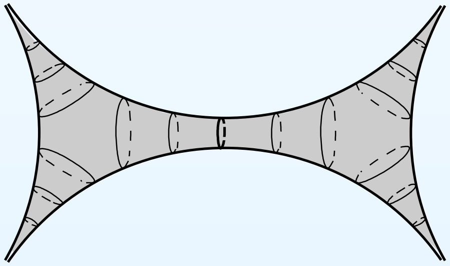

By the uniformization theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of curvature -1 with cusps at the marked points, so the *moduli space* $\mathcal{M}_{0,4}$ can be also seen as the family of hyperbolic spheres with four cusps. Deforming the configuration of points we change the shape of the corresponding hyperbolic surface.

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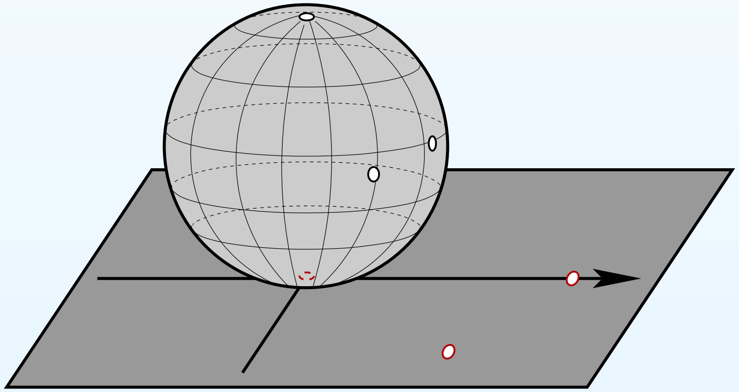

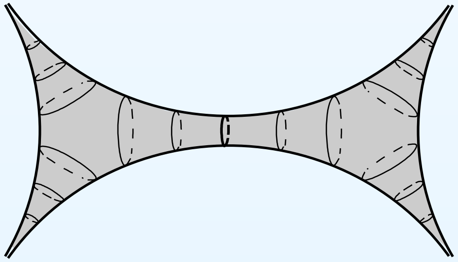

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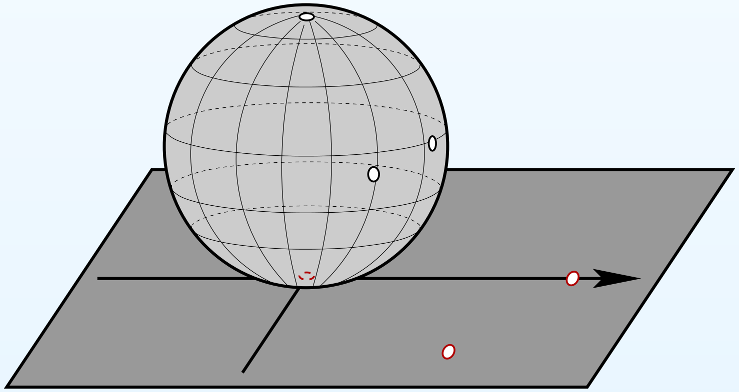

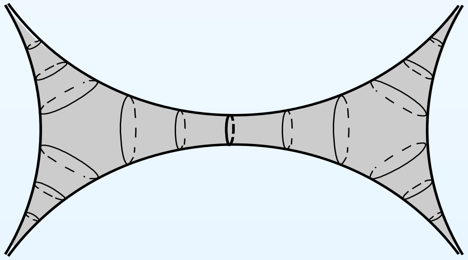

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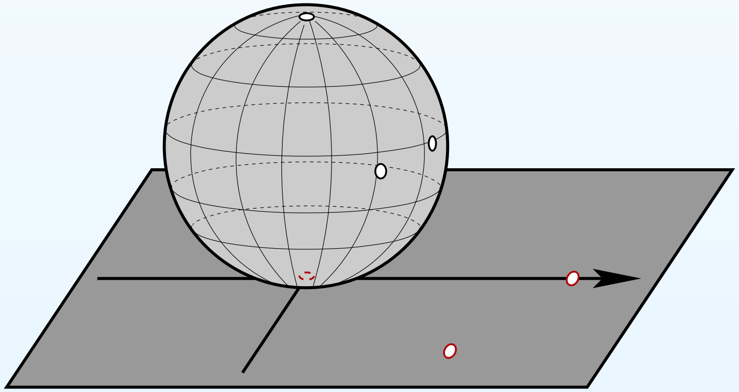

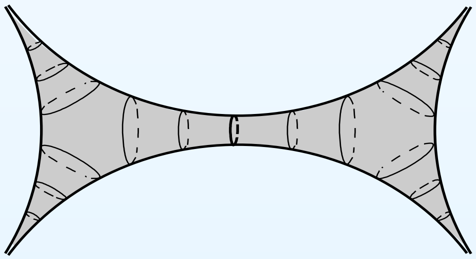

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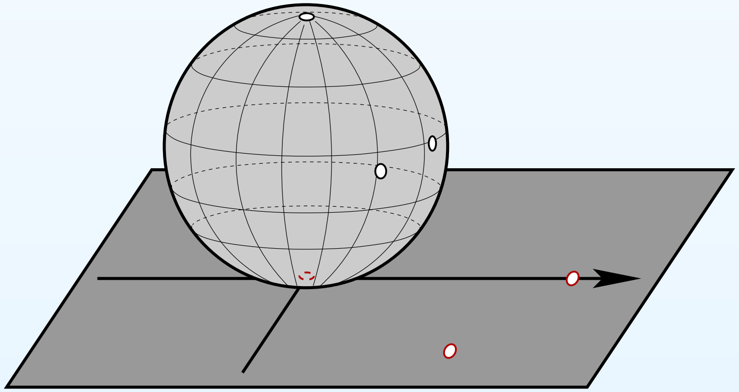

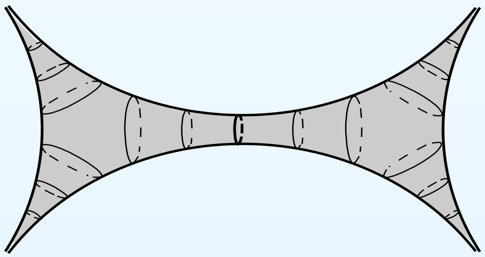

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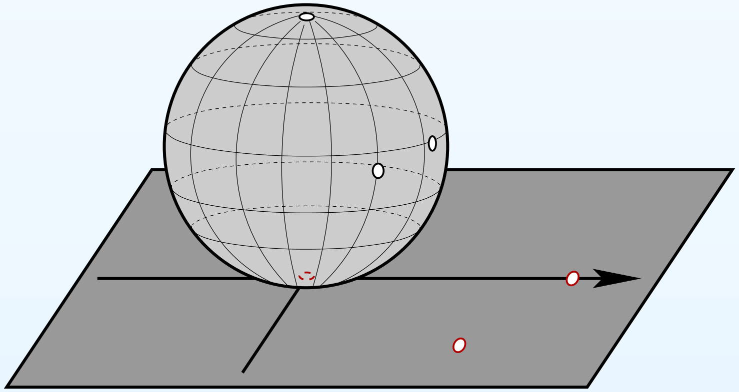

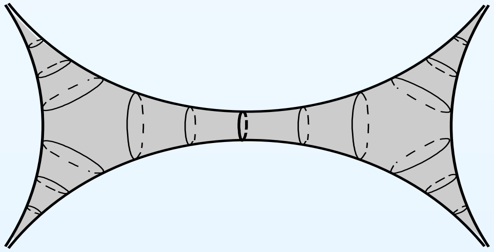

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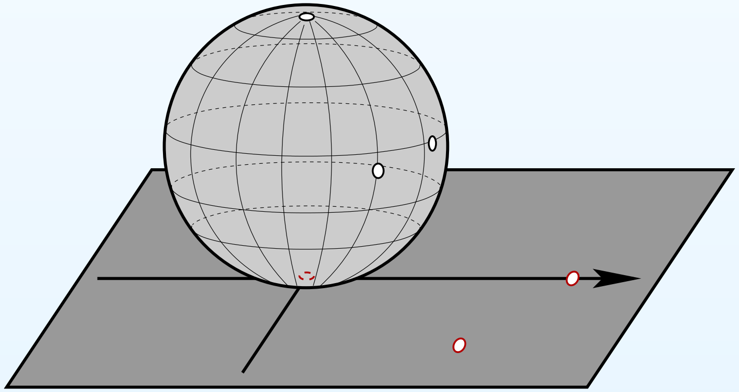

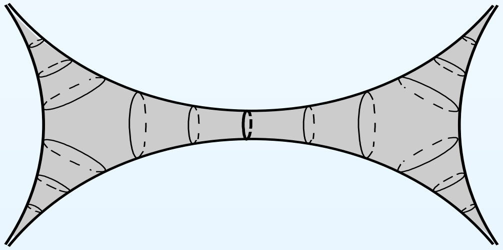

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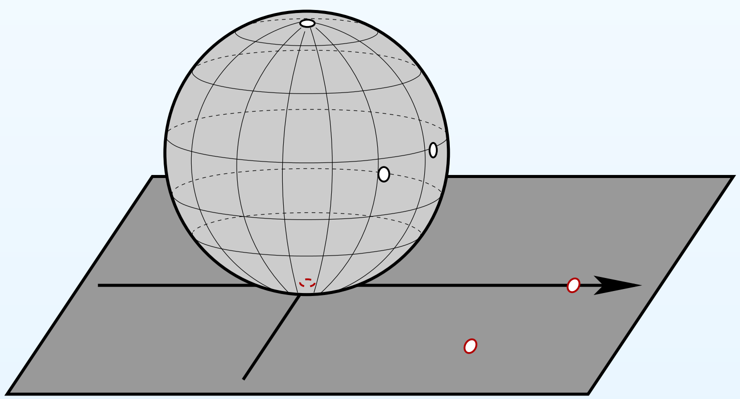

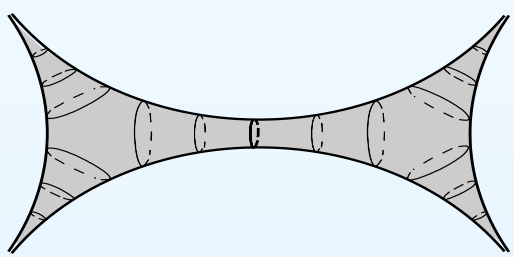

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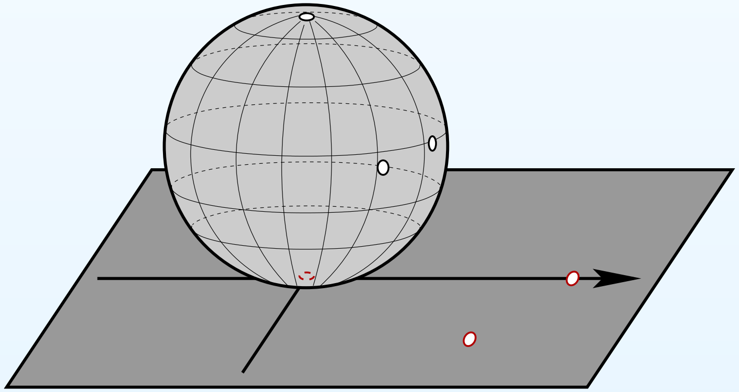

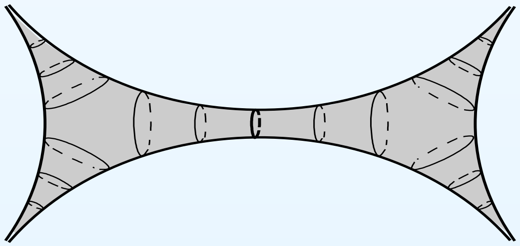

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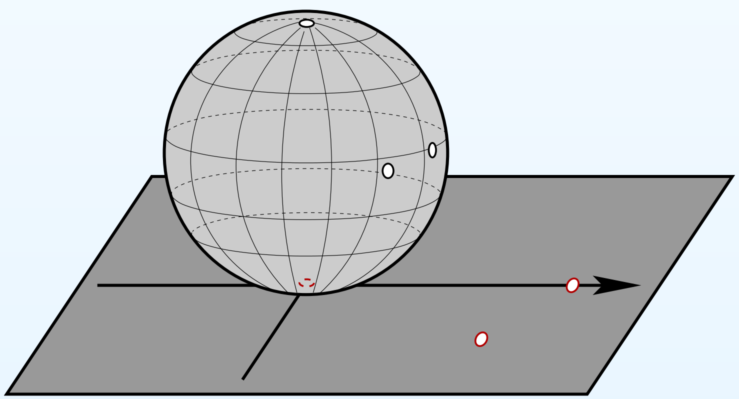

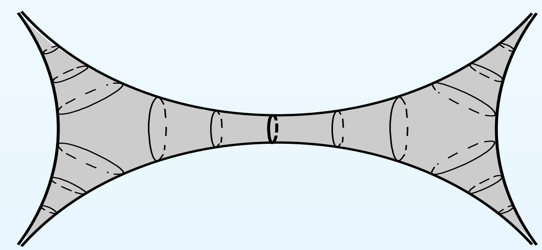

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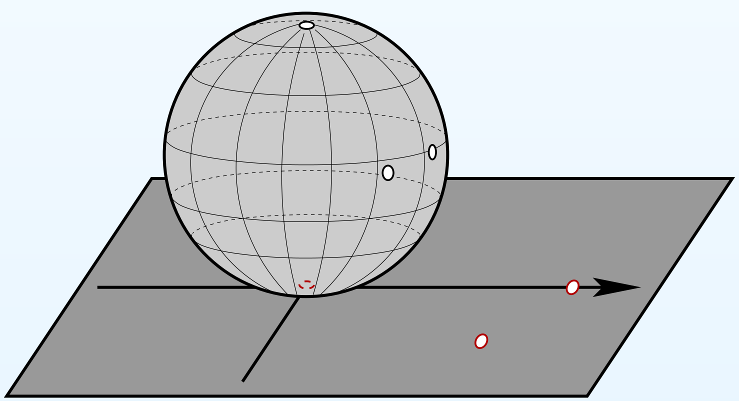

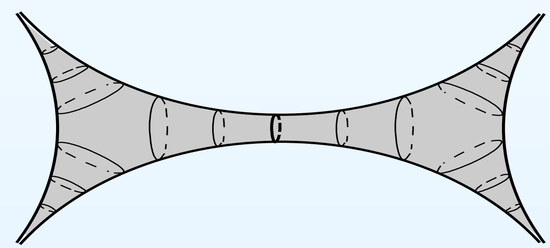

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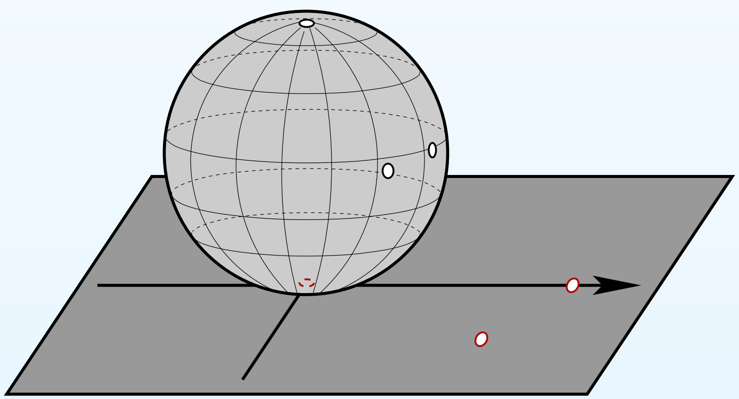

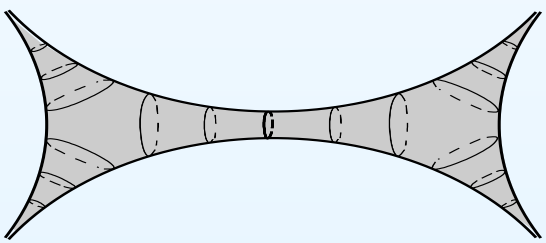

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Moduli space $\mathcal{M}_{g,n}$

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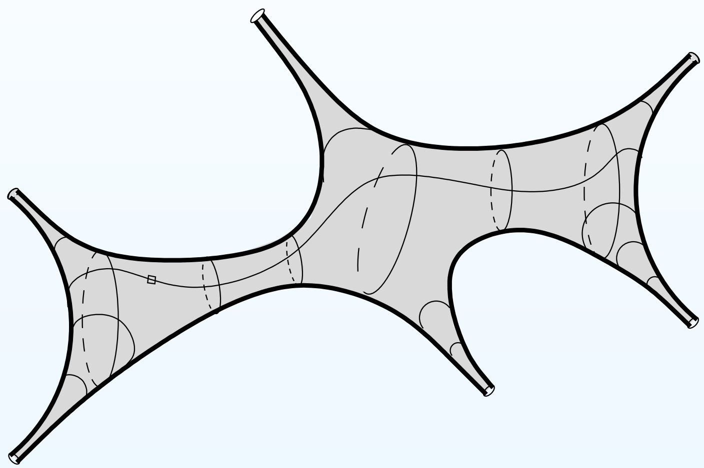

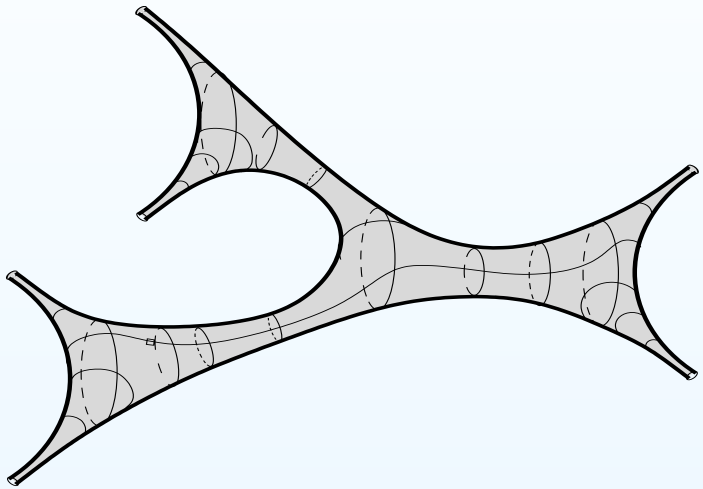

The space $\mathcal{M}_{g,n}$ of configurations of n distinct points on a smooth closed orientable Riemann surface of genus $g > 0$ is even richer. While the sphere admits only one complex structure, a surface of genus $g \geq 2$ admits complex $(3g-3)$-dimensional family of complex structures. As in the case of the Riemann sphere, complex structures on a smooth surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points. For genus $g \geq 2$ one can let $n = 0$ and consider the space $\mathcal{M}_g = \mathcal{M}_{g,0}$ of hyperbolic surfaces without cusps.

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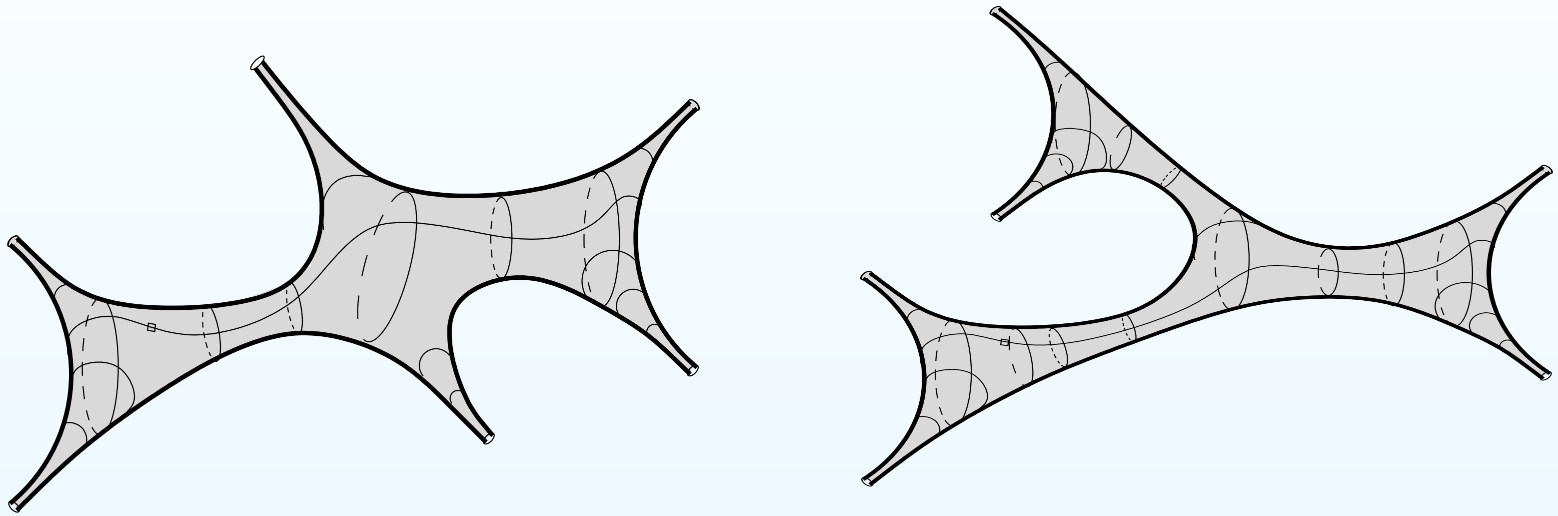

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Mirzakhani's count of simple closed geodesics: statement of results

Multicurves

Consider now several pairwise nonintersecting essential simple closed curves $\gamma_1, \dots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. We have seen that in the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves become simple closed geodesics.

Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \dots, \gamma_k$ do not have pairwise intersections.*

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

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We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients a_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

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Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *one has*

$$s_X(L,\gamma) \sim B(X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $B(X)$ depends only on the hyperbolic metric X (and would be specified later); $b_{g,n}$ is a global constant depending only on g and n (and would be specified later); $c(\gamma)$ depends only on the topological type of γ (and would be computed shortly).

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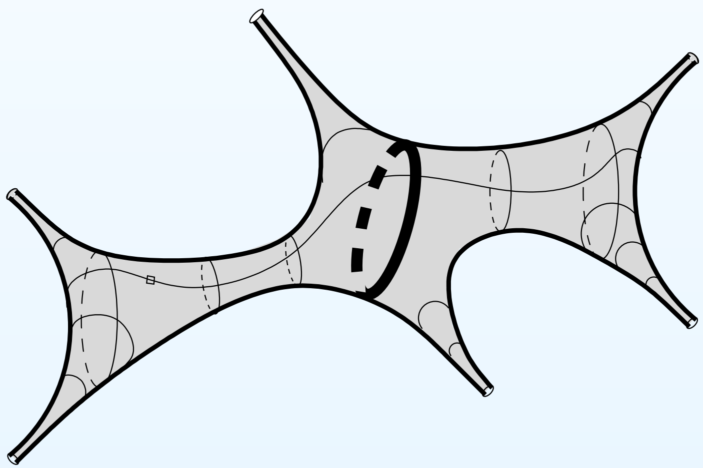
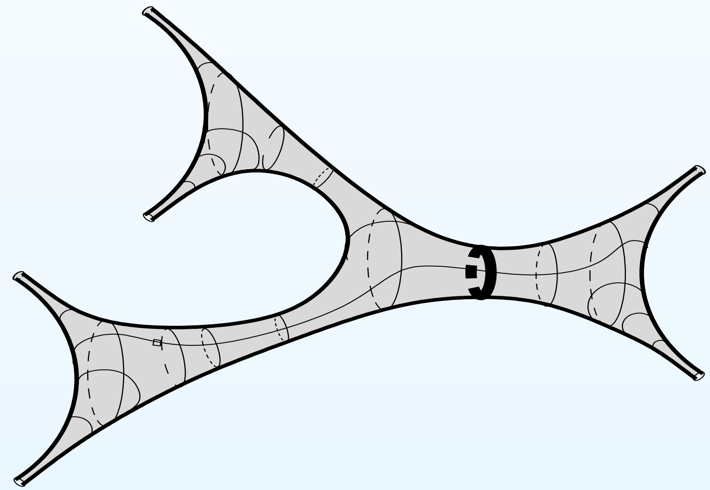
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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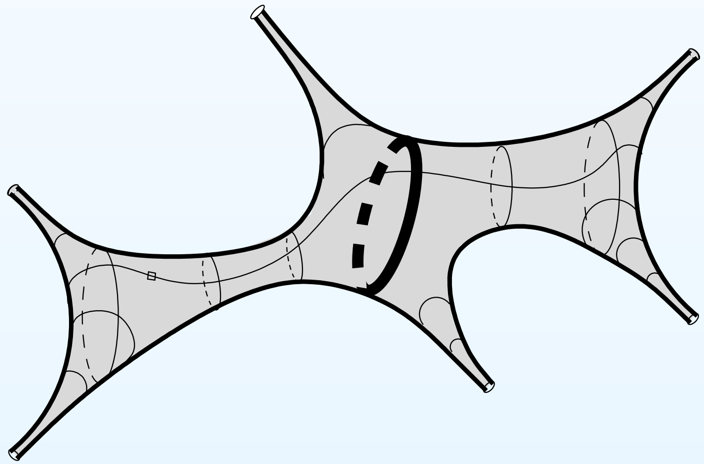

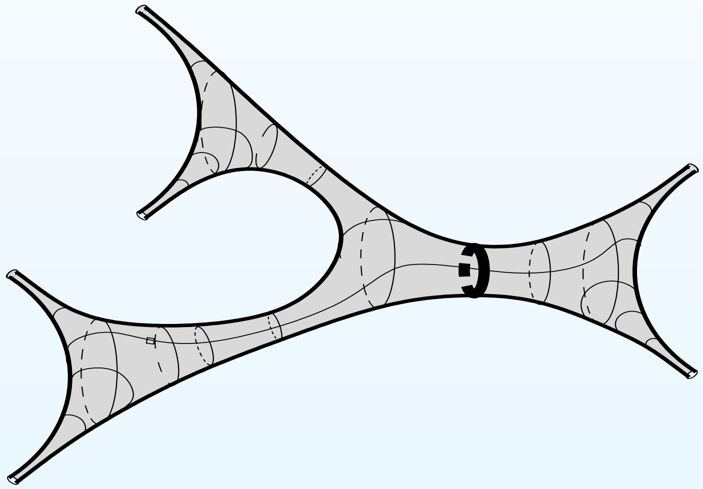

Example (M. Mirzakhani (2008); confirmed experimentally in 2017 by M. Bell and S. Schleimer); confirmed in 2017 by more implicit computer experiment of V. Delecroix and by other means.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

Average number of simple closed geodesics

Bordered hyperbolic surfaces

Cutting a hyperbolic surface by several pairwise disjoint simple closed geodesics we get one or several *bordered* hyperbolic surfaces with geodesic boundary components. Denote by $\mathcal{M}_{g,n}(b_1,\dots,b_n)$ the moduli space of hyperbolic surfaces of genus g with n geodesic boundary components of lengths $b_1,\dots,b_n$. By convention, the zero value $b_i=0$ corresponds to a cusp.

Topologically, a hyperbolic *pair of pants* $P\in\mathcal{M}_{0,3}(b_1,b_2,b_3)$ is a sphere with three holes. For any triple of nonnegative numbers $(b_1,b_2,b_3)\in\mathbb{R}^3_+$ there exists a unique hyperbolic pair of pants $P(b_1,b_2,b_3)$ with geodesic boundaries of given lengths (assuming that the boundary components of P are numbered).

Two geodesic boundary components γ_1,γ_2 of any hyperbolic pair of pants P can be joined by a single geodesic segment $\nu_{1,2}$ orthogonal to both γ_1 and γ_2. Thus, every geodesic boundary component γ of any hyperbolic pair of pants might be endowed with a canonical distinguished point.

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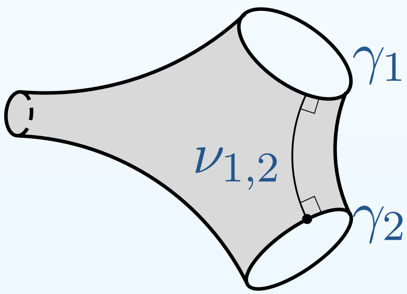

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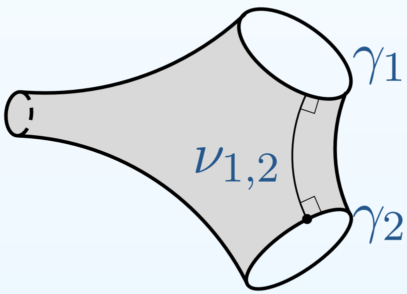

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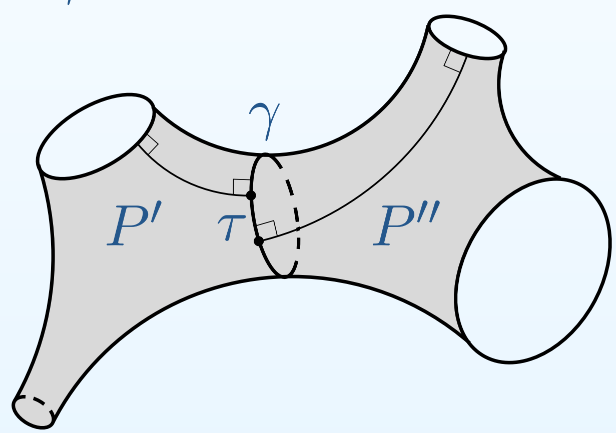
Twist parameter

Two hyperbolic pairs of pants $P'(b'_1, b'_2, \ell)$ and $P''(b''_1, b''_2, \ell)$ sharing the same length $\ell > 0$ of one of the geodesic boundary components can be glued together. The hyperbolic metric on the resulting hyperbolic surface Y is perfectly smooth and the common geodesic boundary of P' and P'' becomes a simple closed geodesic γ on Y.



Each geodesic boundary component of any pair of pants is endowed with a distinguished point. These distinguished points record how the pairs of pants P' and P'' are twisted with respect to each other. Hyperbolic surfaces $Y(\tau)$ corresponding to different values of the twist parameter τ in the range $[0, \ell[$ are generically not isometric.

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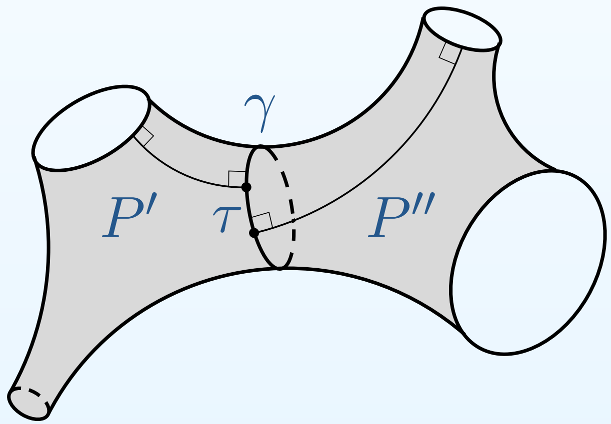

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Fenchel–Nielsen coordinates

Any hyperbolic surface X of genus g with n geodesic boundary components admits a decomposition in hyperbolic pairs of pants glued along simple closed geodesics $\gamma_1, \dots, \gamma_{3g-3+n}$. Lengths $\ell_{\gamma_i}(X)$ of the resulting simple closed geodesics γ_i involved in pants decomposition of X and twists $\tau_{\gamma_i}(X)$ along them serve as local *Fenchel–Nielsen coordinates* in $\mathcal{M}_{g,n}(b_1, \dots, b_n)$.

By the work of W. Goldman $\mathcal{M}_{g,n}(b_1, \dots, b_n)$ carries a natural closed non-degenerate 2-form ω_{WP} called the *Weil–Petersson symplectic form*.

S. Wolpert proved that ω_{WP} has particularly simple expression in Fenchel–Nielsen coordinates. No matter what pants decomposition we chose, we get

$$\omega_{WP} = \sum_{i=1}^{3g-3+n} d\ell_{\gamma_i} \wedge d\tau_{\gamma_i} \,.$$

The wedge power ω^n of a symplectic form on a manifold M^{2n} of real dimension $2n$ defines a volume form on M^{2n}. The volume $V_{g,n}(b_1, \dots, b_n)$ of the moduli space $\mathcal{M}_{g,n}(b_1, \dots, b_n)$ with respect to the volume form $\frac{1}{(3g-3+n)!}\,\omega_{WP}^{3g-3+n}$ is called the *Weil–Petersson* volume of the moduli space $\mathcal{M}_{g,n}(b_1, \dots, b_n)$; it is known to be finite.

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Averaging the counting function: statement of results

We are interested in counting the number $s_X(L,\gamma)$ of simple closed geodesic multicurves on $X \in \mathcal{M}_{g,n}$ of topological type $[\gamma]$ and of hyperbolic length at most L. Following Mirzakhani, we shall count first the *average* of the quantity $s_X(L,\gamma)$ over $\mathcal{M}_{g,n}$ with respect to the Weil–Petersson volume element:

$$P(L,\gamma) := \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX\,.$$

Theorem (M. Mirzakhani, 2008). *The average number $P(L,\gamma)$ of closed geodesic multicurves of topological type $[\gamma]$ and of hyperbolic length at most L is a polynomial in L of degree $6g-6+2n$. The leading coefficient of this polynomial*

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is expressed in terms of the Weil–Petersson volumes of the associated moduli space of bordered hyperbolic surfaces, or, more precisely, in terms of the appropriate characteristic numbers of the form

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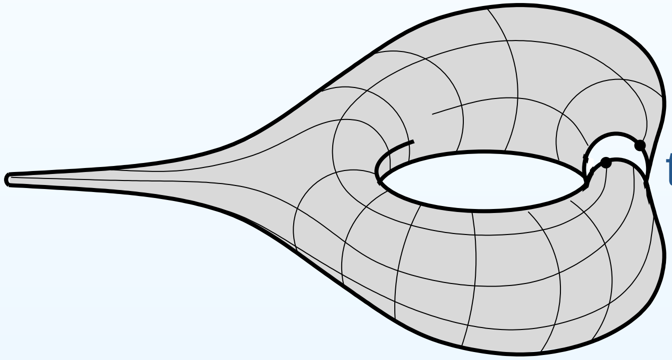
Convenient cover: model case

Consider the cover $\mathcal{M}_{1,1}^{\gamma}$ over $\mathcal{M}_{1,1}$ where the point of the cover $\mathcal{M}_{1,1}^{\gamma}$ is a hyperbolic surface X endowed with a distinguished simple closed geodesic α. The fiber of the cover can be identified with $\mathrm{Mod}_{1,1} \cdot [\gamma]$, where γ is a essential simple closed curve on a once punctured torus.



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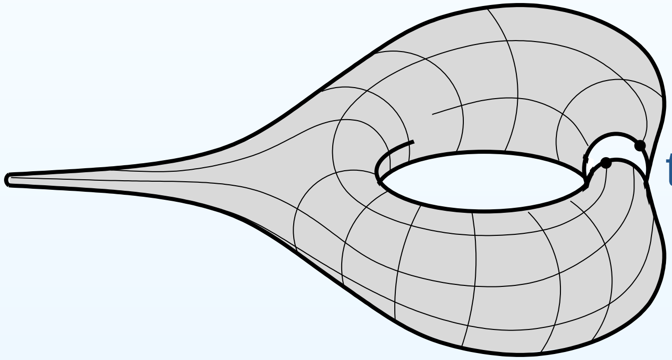

The cover $\mathcal{M}_{1,1}^{\gamma}$ admits global coordinates. Namely, given $(X, \alpha) \in \mathcal{M}_{1,1}^{\gamma}$ cut X open along the closed geodesic α. We get a hyperbolic pair of pants $P(l, l, 0)$; two geodesic boundary components of it have the same length $l = \ell_X(\alpha)$ and the third boundary component is the cusp. Reciprocally, from any hyperbolic pair of pants $P(l, l, 0)$ we can glue a hyperbolic surface X endowed with a distinguished simple closed geodesic α. Constructing X from the pair of pants $P(l, l, 0)$ we have to chose the value of the twist parameter τ in the interval $[0, l[$, where $l = \ell_X(\alpha)$ is the length of the geodesic boundary.

Integration over $\mathcal{M}_{1,1}$

Mirzakhani observed that having a continuous function $f_\gamma(X)$ on $\mathcal{M}_{1,1}$ of the form

$$f_\gamma(X) = \sum_{[\alpha]\in \mathrm{Mod}_{1,1}\cdot[\gamma]} f(\ell_X(\alpha))$$

we can integrate it over $\mathcal{M}_{1,1}$ as follows

$$\int_{\mathcal{M}_{1,1}} \sum_{[\alpha]\in \mathrm{Mod}_{1,1}\cdot[\gamma]} f(\ell_\alpha(X))\, dX = \int_{\mathcal{M}_{1,1}^\gamma} f(\ell_\alpha(X))\, dl\, d\tau =$$

$$= \int_0^\infty f(l) \int_0^l dl\, d\tau = \int_0^\infty f(l)\, l\, dl\,.$$

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$$= \int_0^\infty f(l)\int_0^l dl\,d\tau = \int_0^\infty f(l)\,l\,dl\,.$$

Note that our counting function $s_X(L,\gamma)$ is exactly of this form with $f=\chi([0,L])$. In this particular case we get

$$P(L,\gamma) := \int_{\mathcal{M}_{1,1}} s_X(L,\gamma)\,dX = \int_0^\infty \chi([0,L])\,l\,dl = \int_0^L \ell d\ell = \frac{L^2}{2}\,.$$

Integration over the moduli space $\mathcal{M}_g$

Let γ be a nonseparating simple closed curve on S_g. Consider the analogous cover $\mathcal{M}_g^{\gamma}$ over $\mathcal{M}_g$ where the point of the cover is a hyperbolic surface X endowed with a distinguished simple closed geodesic α. Cutting X open along α we get a bordered hyperbolic surface $Y(l, l)$ in $\mathcal{M}_{g-1,n+2}(l, l)$, where $l = \ell_X(\alpha)$.

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over $\mathcal{M}_g$ as before:

$$P(L,\gamma) := \int_{\mathcal{M}_g} s_X(L,\gamma)\, dX = \int_{\mathcal{M}_g^\gamma} f(\ell_X(\alpha))\, dX =$$

$$= \frac{1}{2}\int_0^L \int_0^l \int_{\mathcal{M}_{g-1,2}(l,l)} dY\, dl\, d\tau = \frac{1}{2}\int_0^L \mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{g-1,2}(l,l)\right) l\, dl\,.$$

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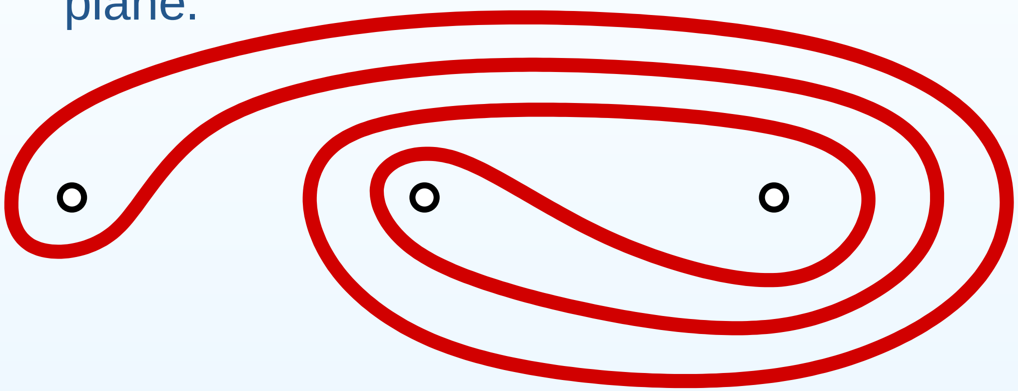
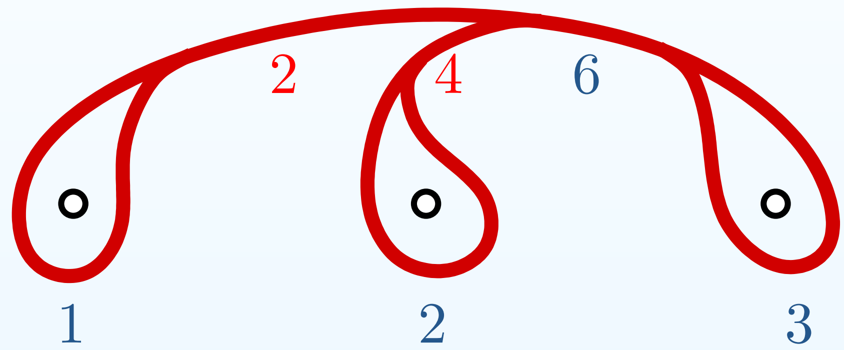
Mirzakhani proved that $\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{g-1,2}(l,l)\right)$ is an explicit polynomial in l of degree $6(g-1)-6+2\cdot 2$, so $P(L,\gamma)$ is a polynomial of degree $6g-6$.

Space of multicurves. Proof of the main result

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.





We can progressively deform the simple closed curve as on the left picture in transverse direction pushing it to the train track as on the right picture. Recording the number of strands projected to each segment of the train track τ we keep all homotopic information about the simple closed curve.

Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

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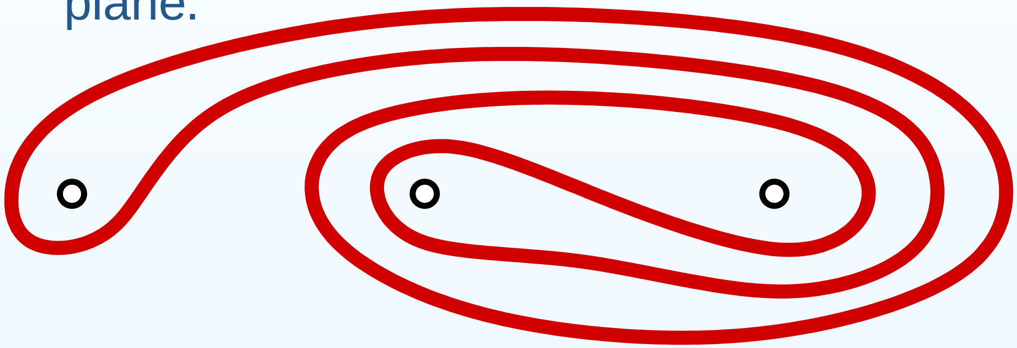

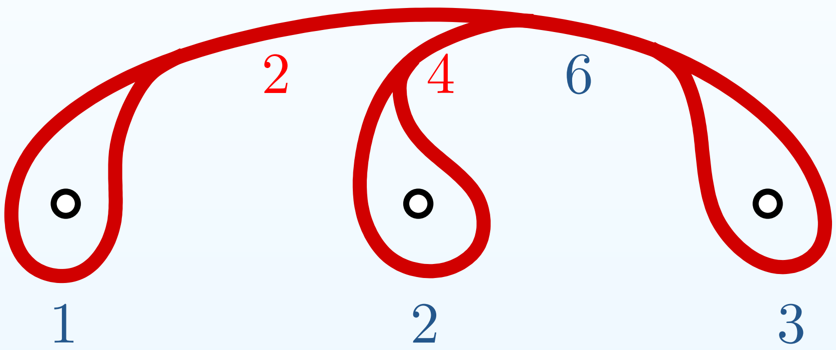

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Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

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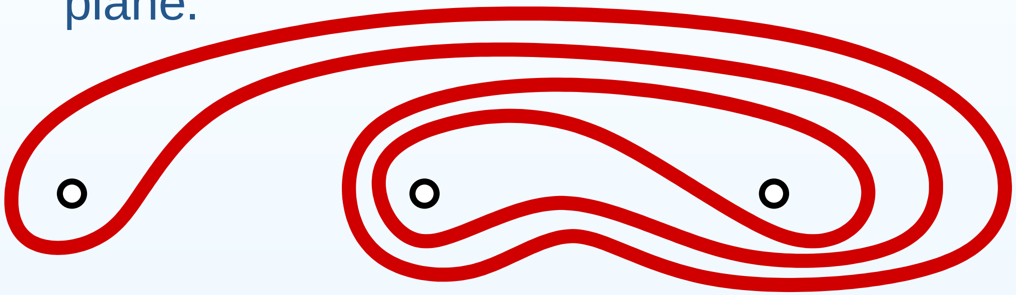

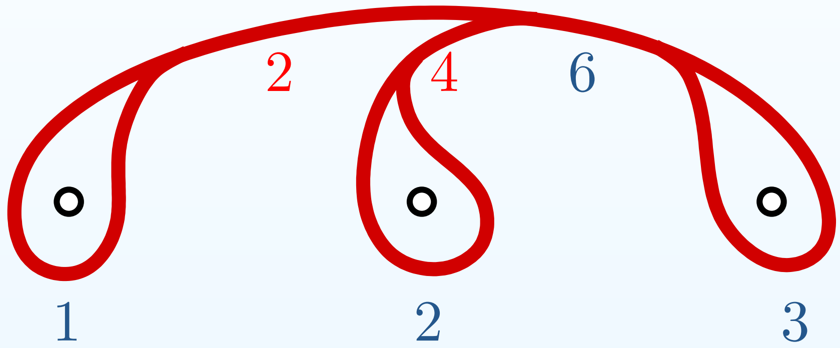

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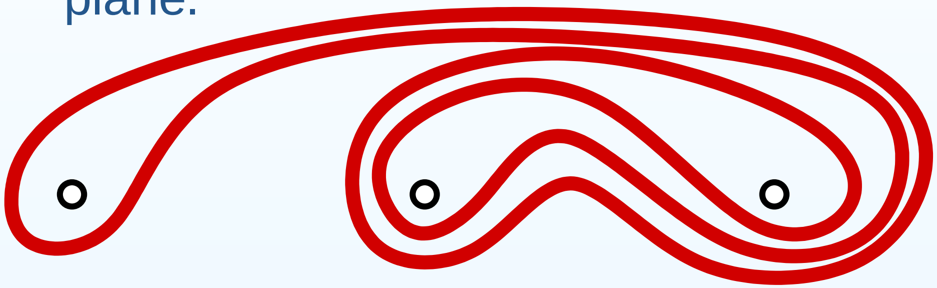

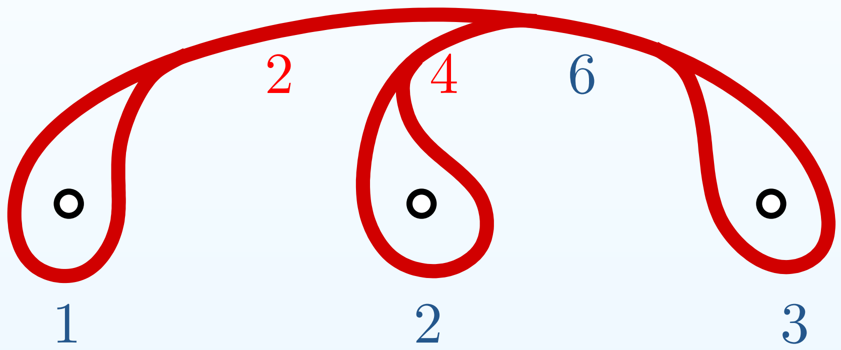

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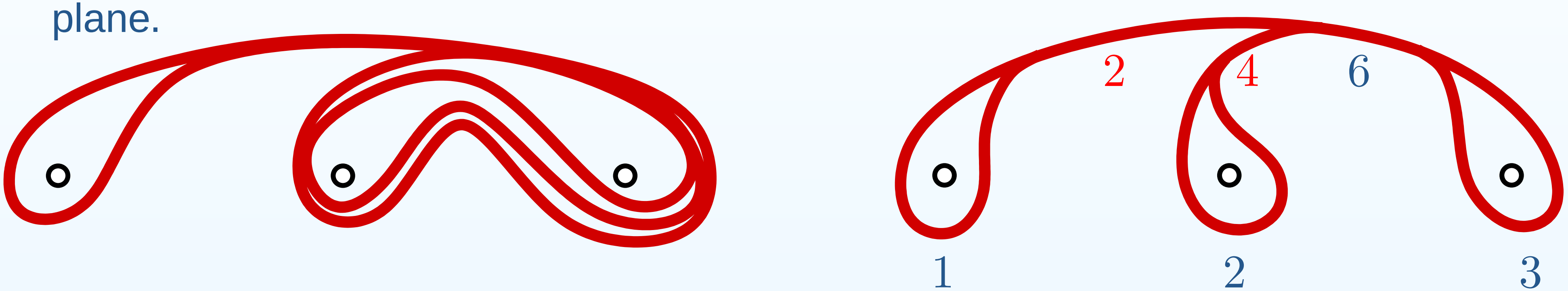

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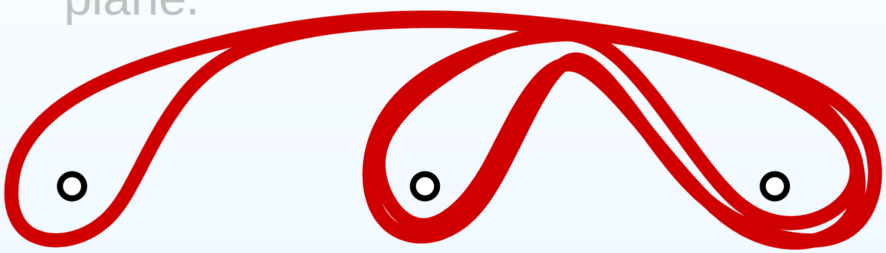

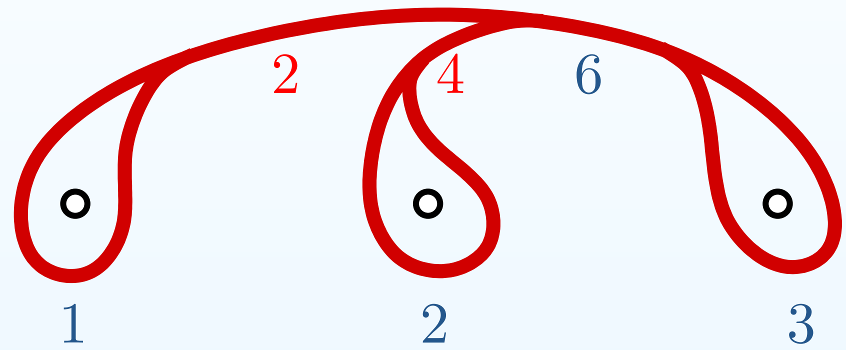

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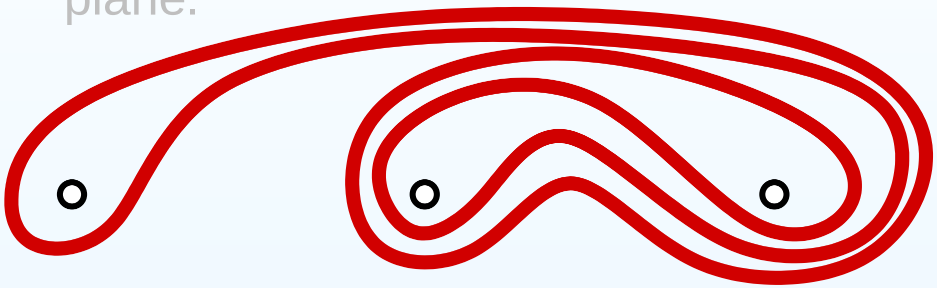

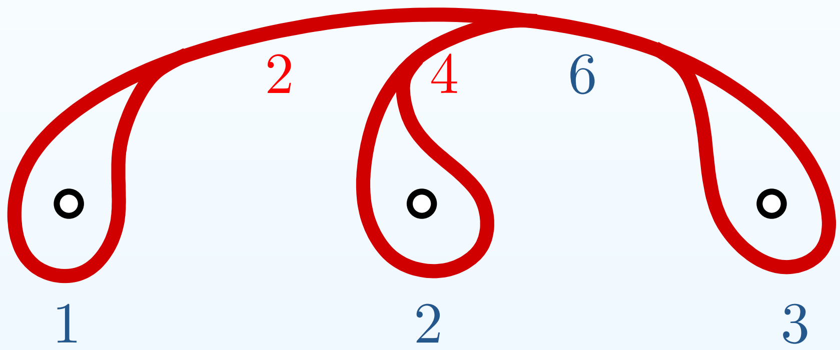

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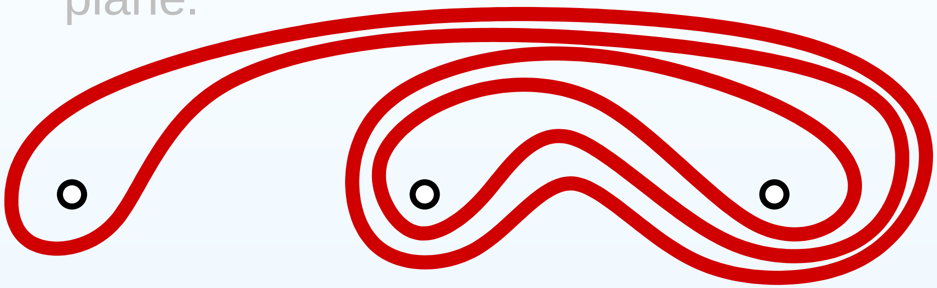

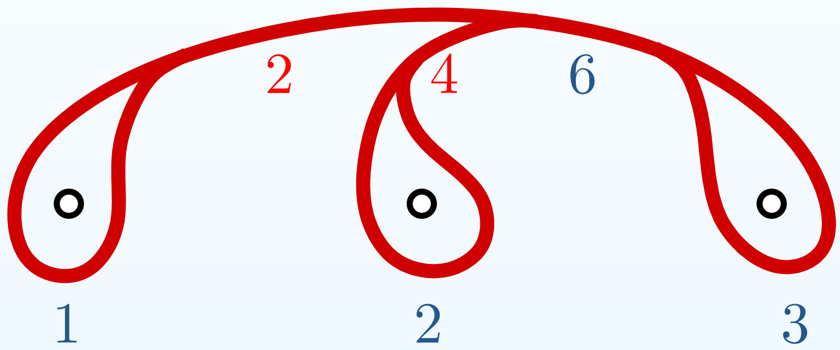

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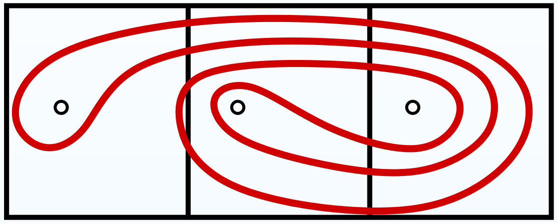
Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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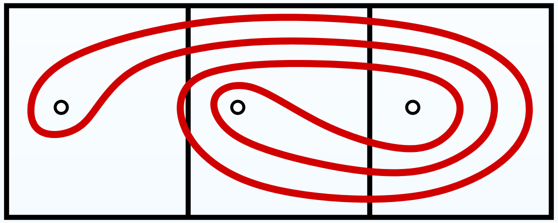

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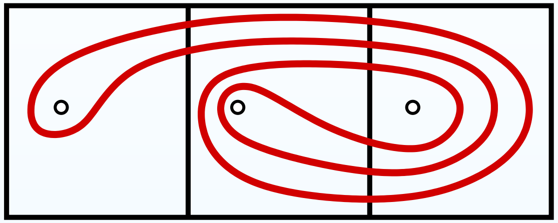

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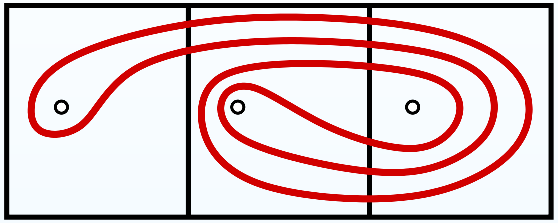

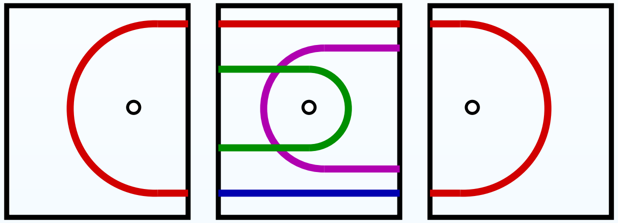

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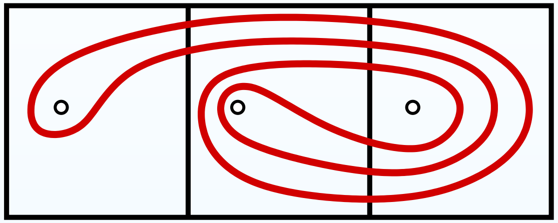

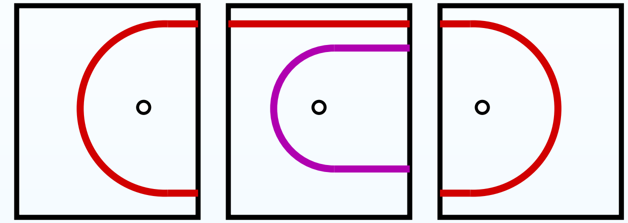

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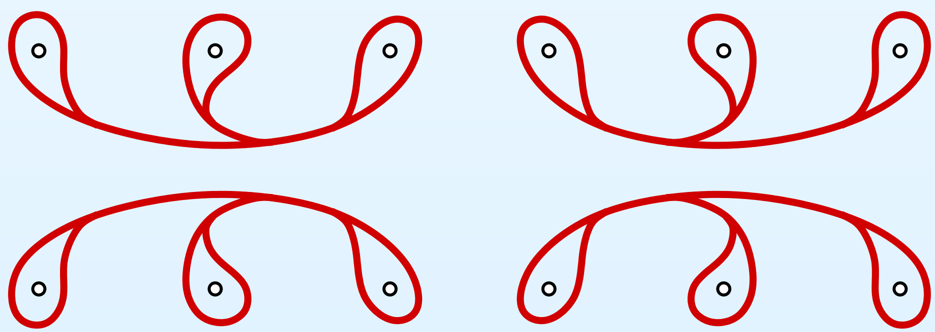
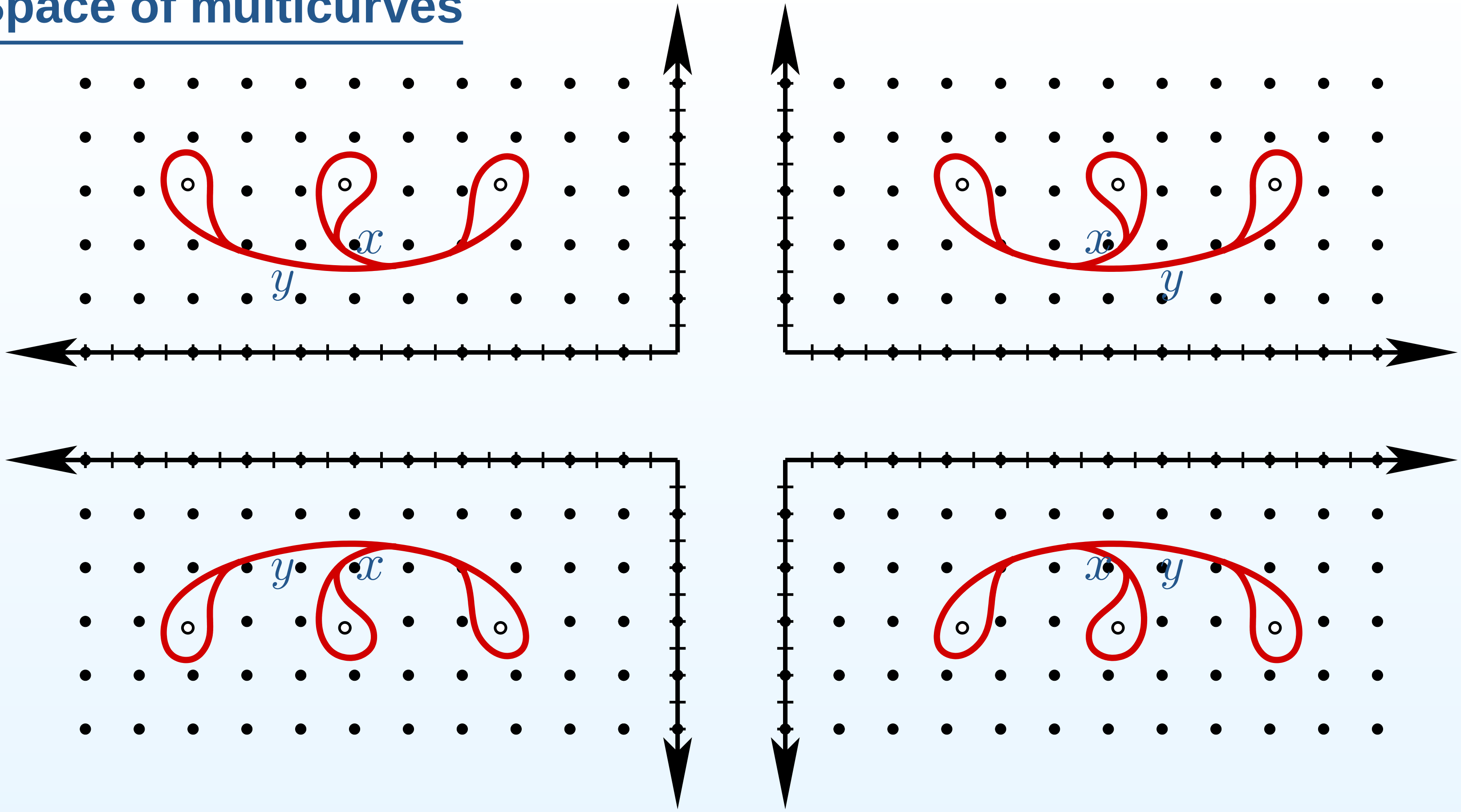
Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve in is carried by one of the following four train tracks:



Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

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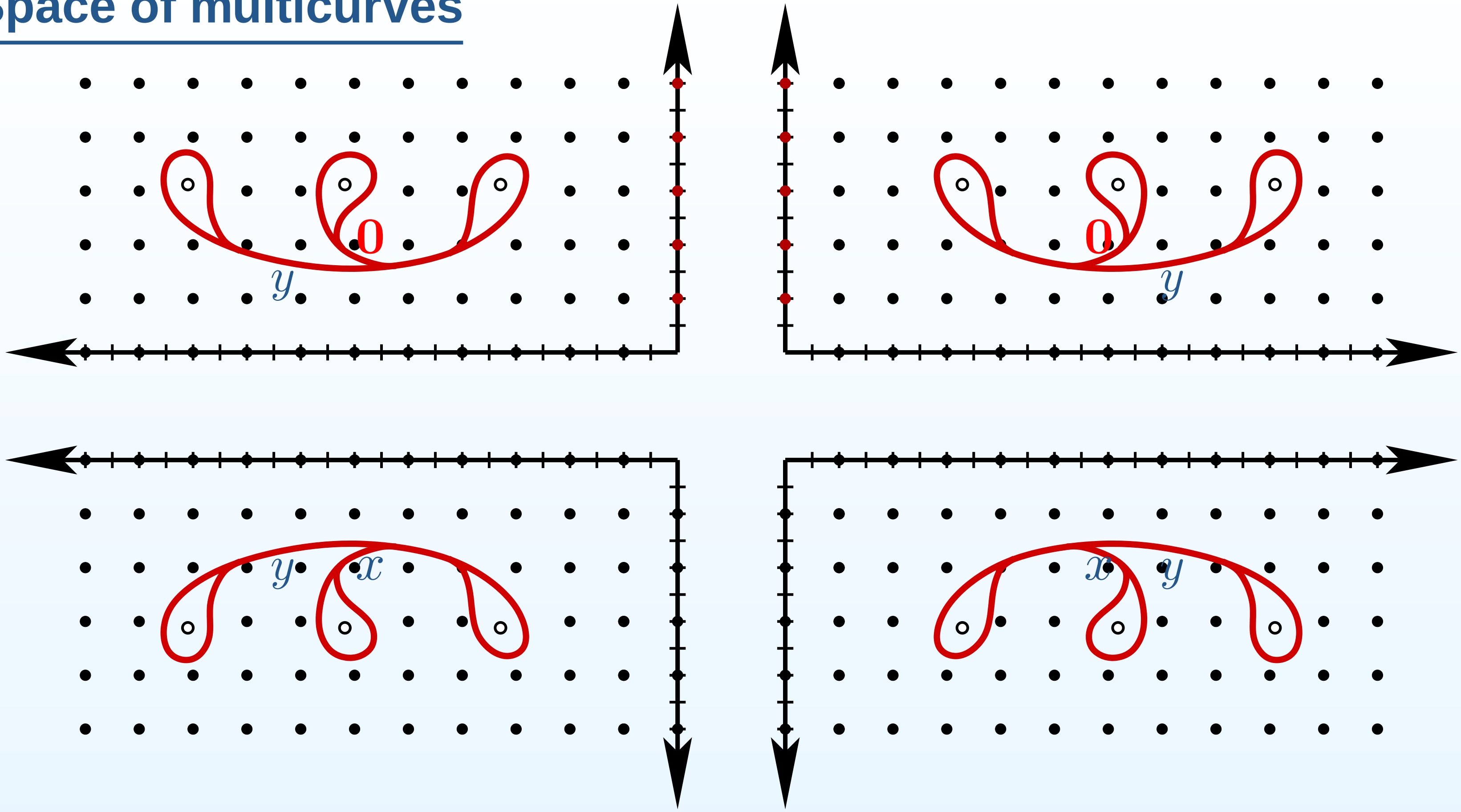

Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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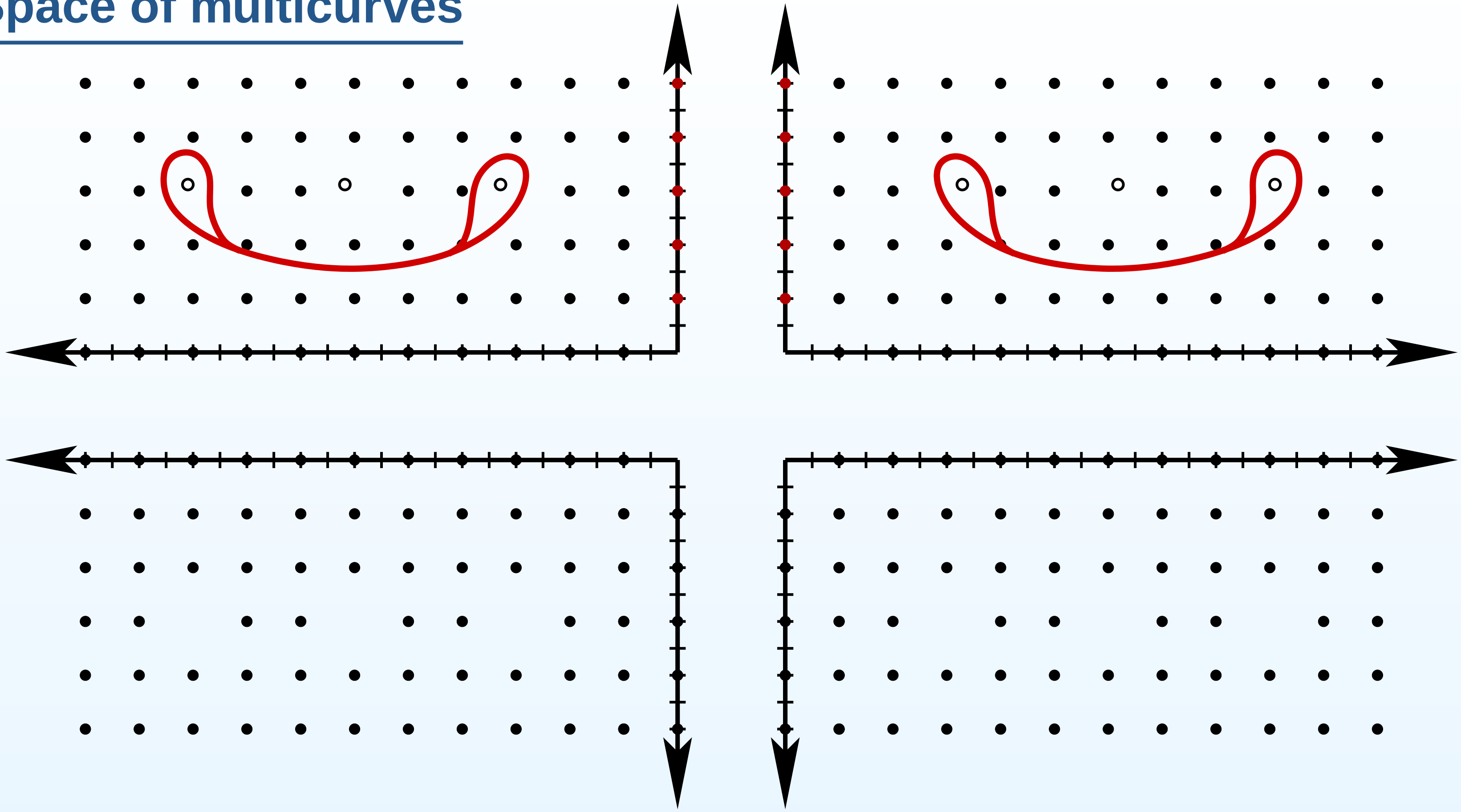

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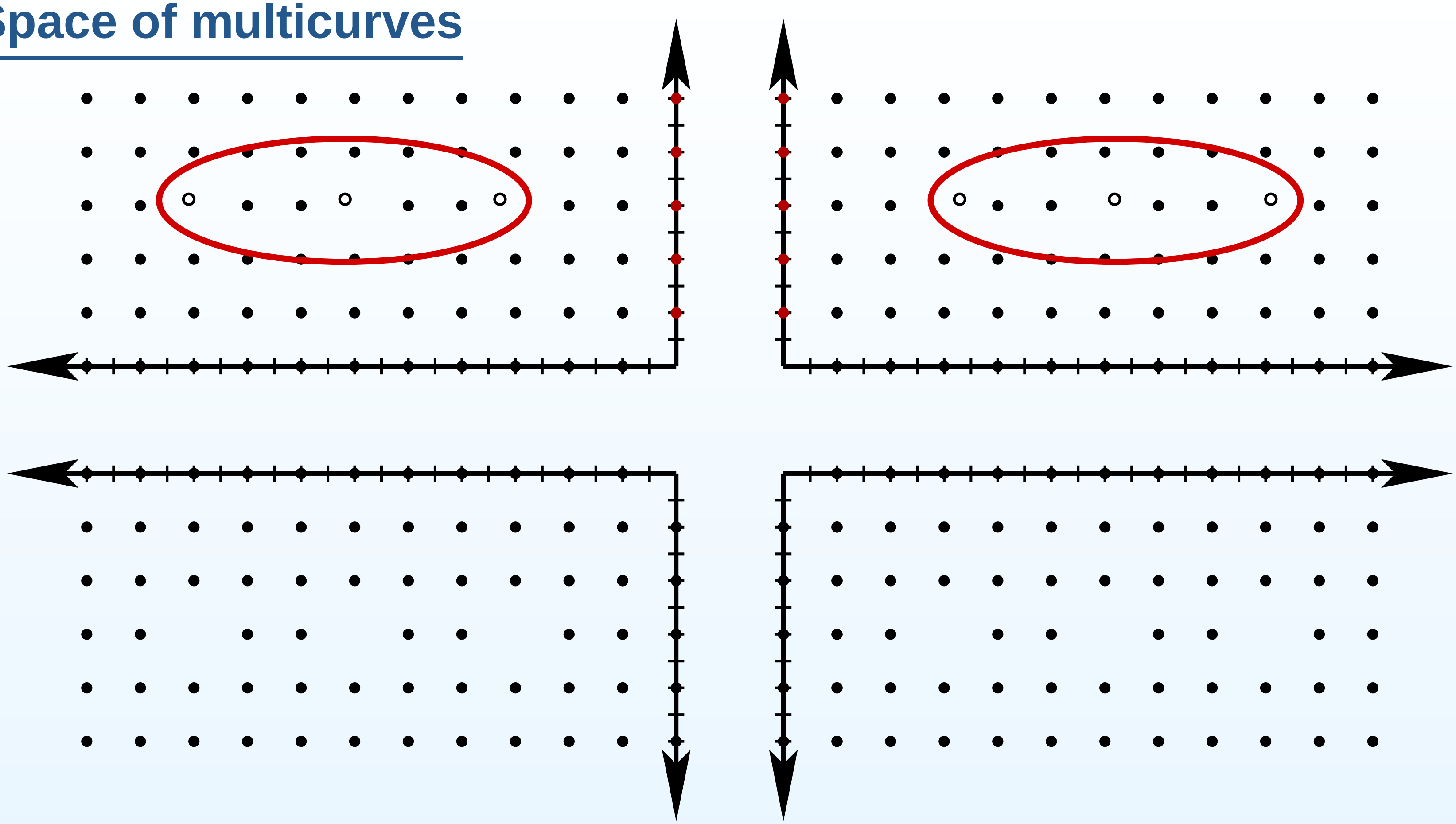

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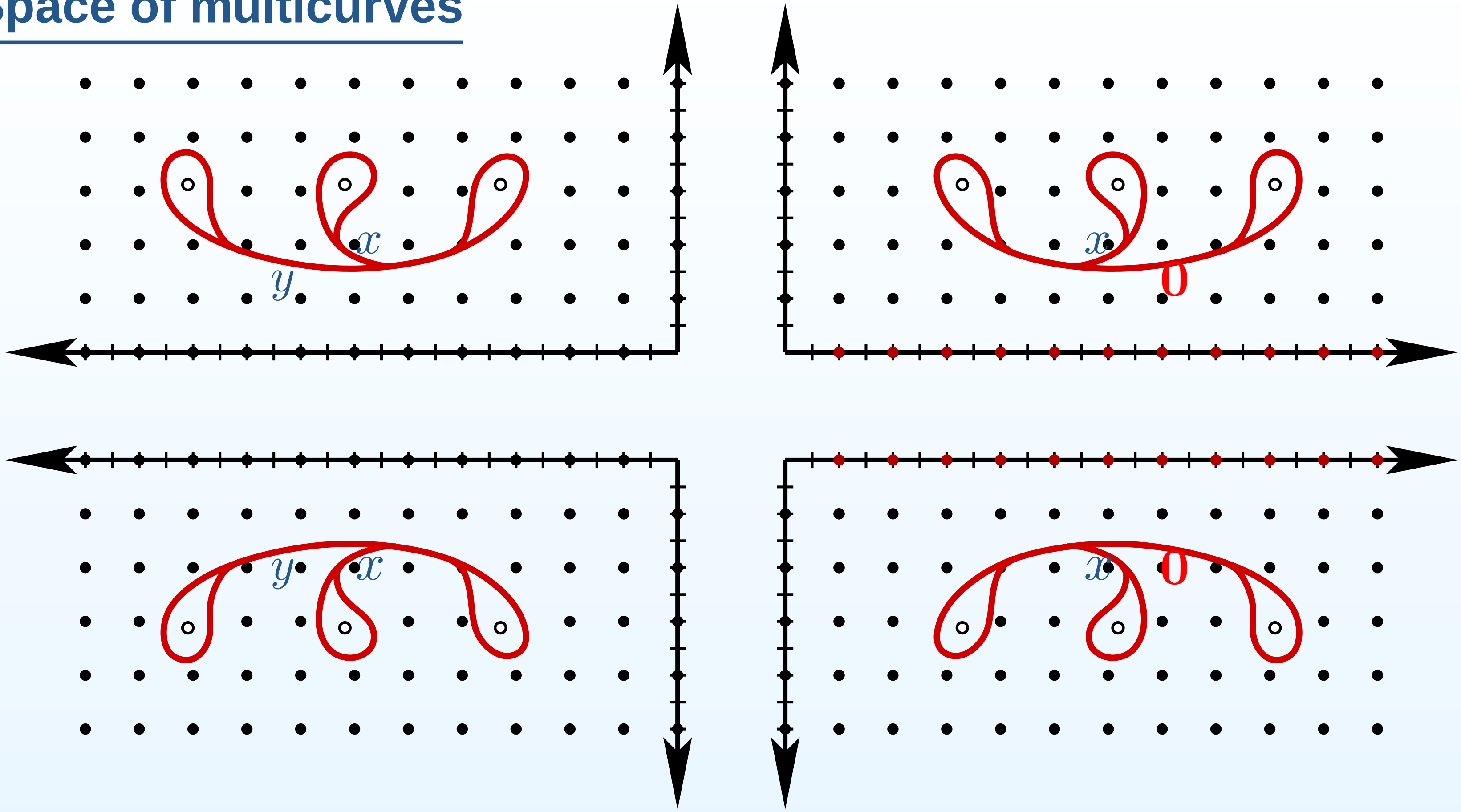

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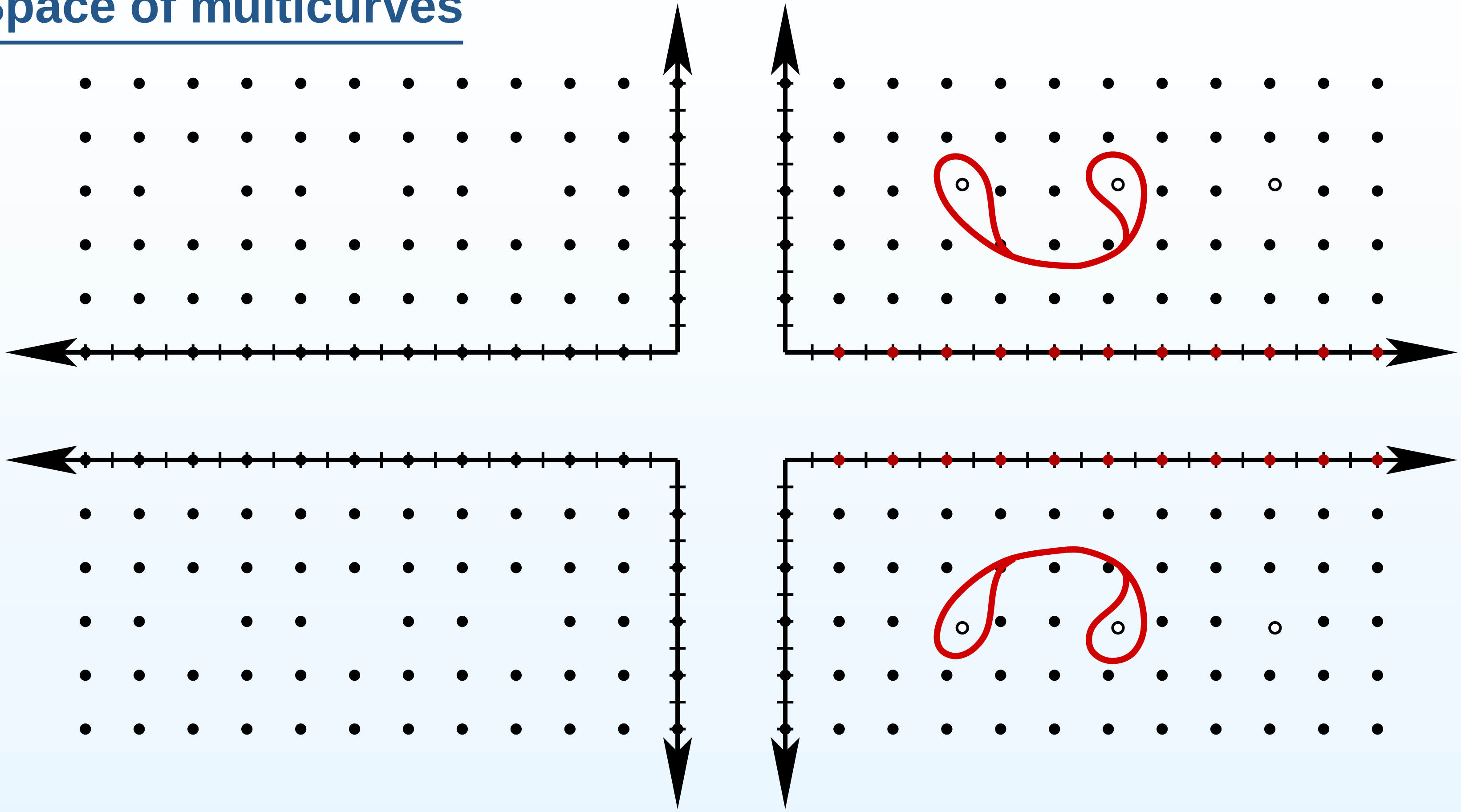

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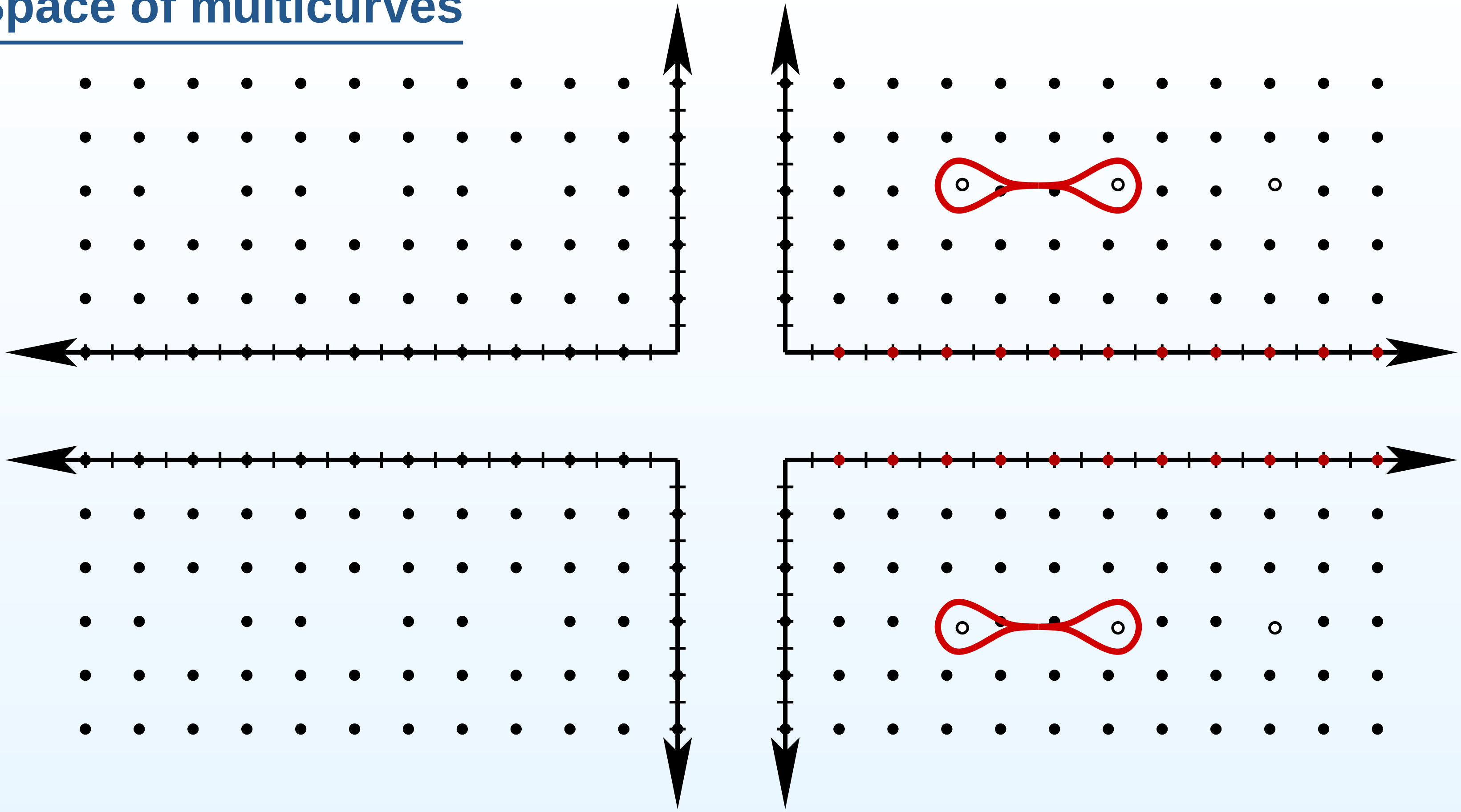

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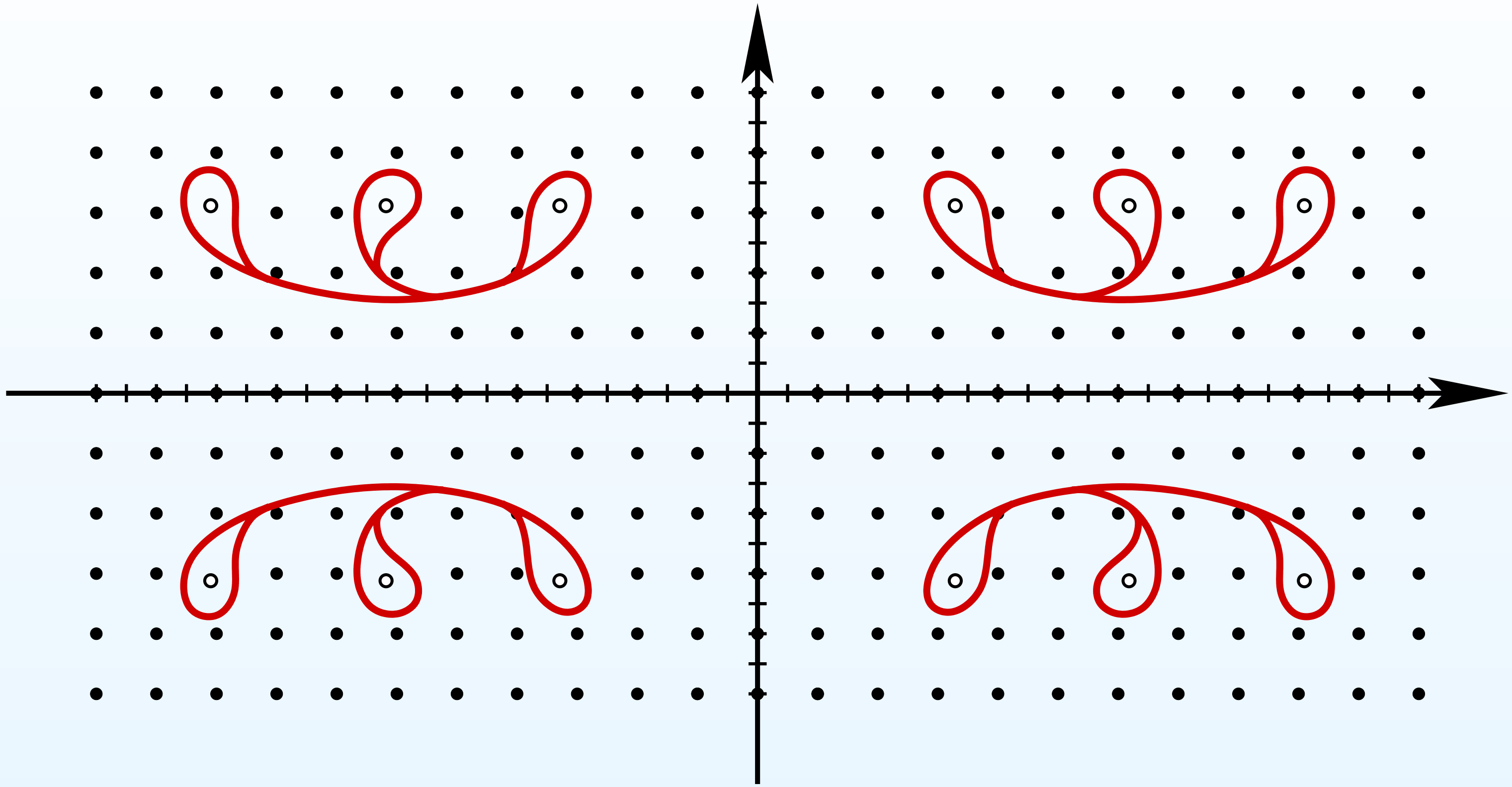

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Thurston measure on $\mathcal{ML}_{g,n}$

Similar considerations applied to a smooth surface $S_{g,n}$ lead to analogous space $\mathcal{ML}_{g,n}$. Up to now we did not use hyperbolic metric on $S_{g,n}$. In the presence of a hyperbolic metric, integral points of $\mathcal{ML}_{g,n}$ can be interpreted as simple closed geodesic multicurves. Moreover: all other points also get geometric realization as *measured geodesic laminations* — disjoint unions of non self-intersecting infinite geodesics.

Train track charts define piecewise linear structure on $\mathcal{ML}_{g,n}$. "Integral lattice" $\mathcal{ML}_{g,n}(\mathbb{Z})$ provides canonical normalization of the corresponding volume form μ_{Th} in which the fundamental domain of the lattice has unit volume. Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves, so the piecewise-linear action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the "integral lattice" $\mathcal{ML}_{g,n}(\mathbb{Z})$, and, hence, preserves the measure μ_{Th}.

Theorem (H. Masur, 1985). *The action of* $\mathrm{Mod}_{g,n}$ *on* $\mathcal{ML}_{g,n}$ *is ergodic with respect to the Lebesgue measure class (i.e. any measurable subset of* $\mathcal{ML}_{g,n}$ *invariant under* $\mathrm{Mod}_{g,n}$ *has measure zero or its complement has measure zero). Any* $\mathrm{Mod}_{g,n}$*-invariant measure in the Lebesgue measure class is just Thurston measure rescaled by some constant factor.*

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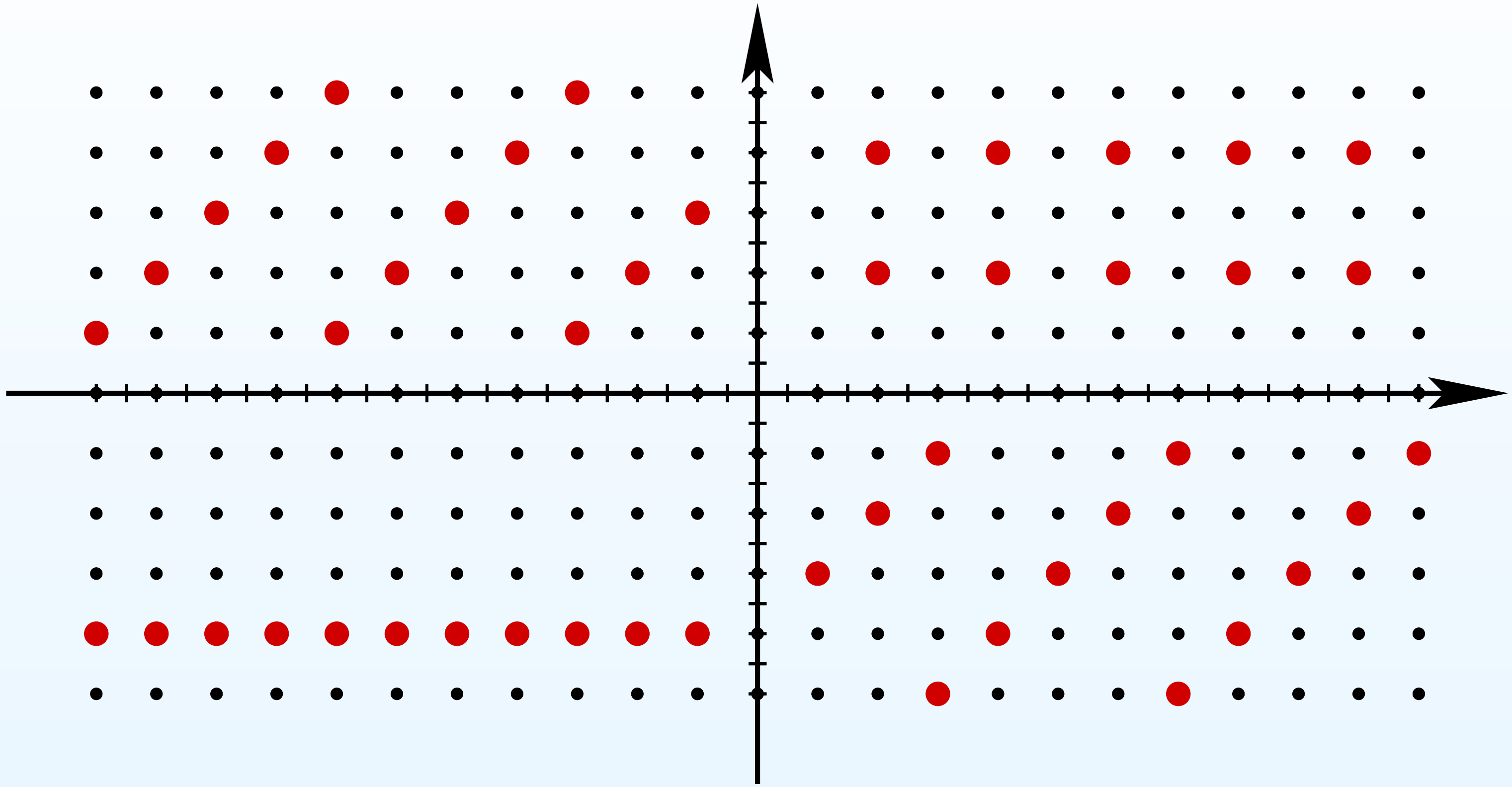

Choose some integral multicurve γ, say, a simple closed curve on $S_{g,n}$. The subset $\mathcal{O}_\gamma := \mathrm{Mod}_{g,n} \cdot \gamma$ can be seen as an analog of a "sublattice" in $\mathcal{ML}_{g,n}(\mathbb{Z})$. The insight of Mirzakhani was to realize that replacing the discrete set $\mathcal{ML}_{g,n}(\mathbb{Z})$ with the subset $\mathcal{O}_\gamma$ we get a new measure on $\mathcal{ML}_{g,n}$ which is proportional to the Thurston measure μ_{Th} with coefficient depending only on the homotopy type of γ.

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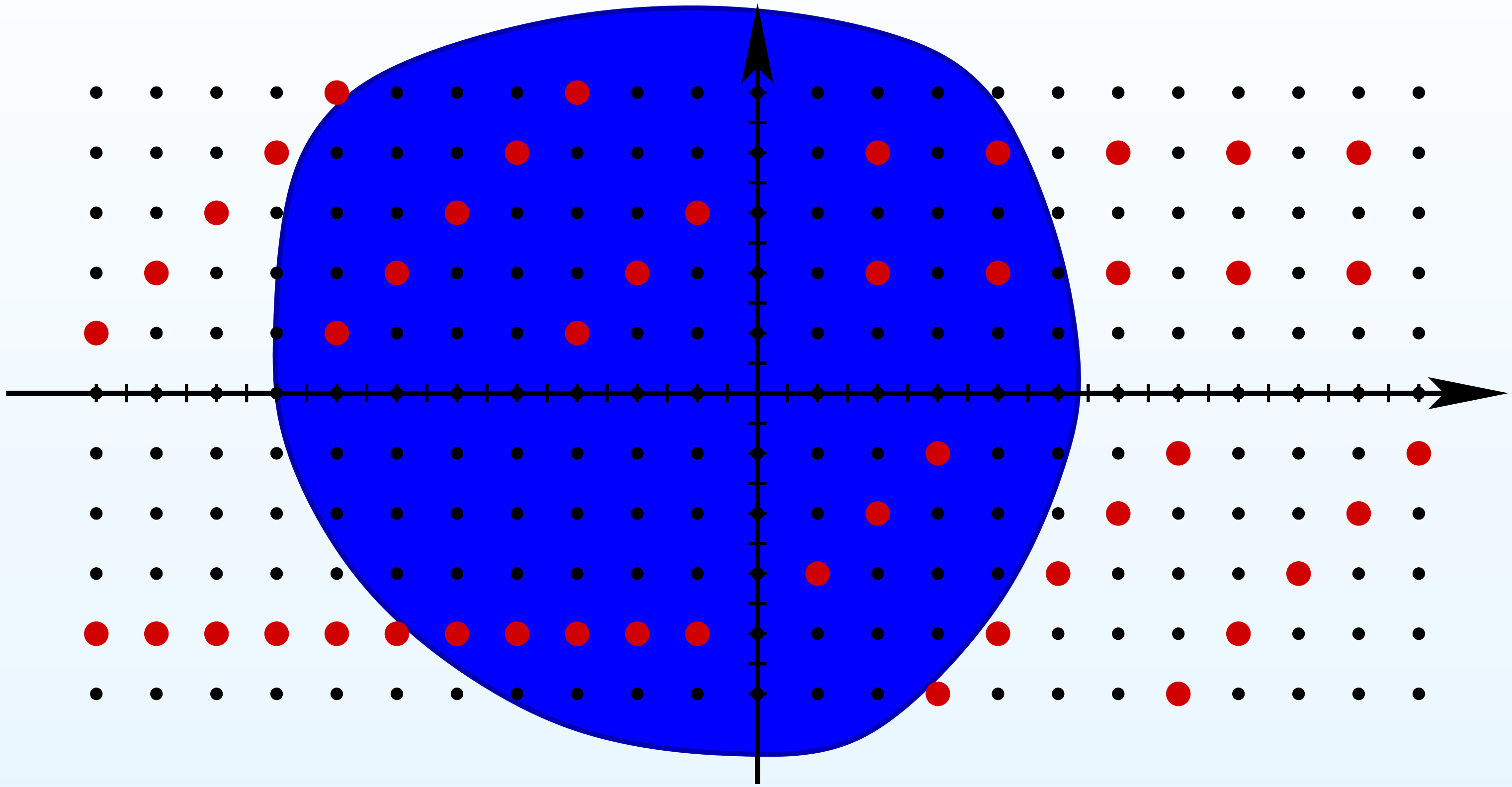

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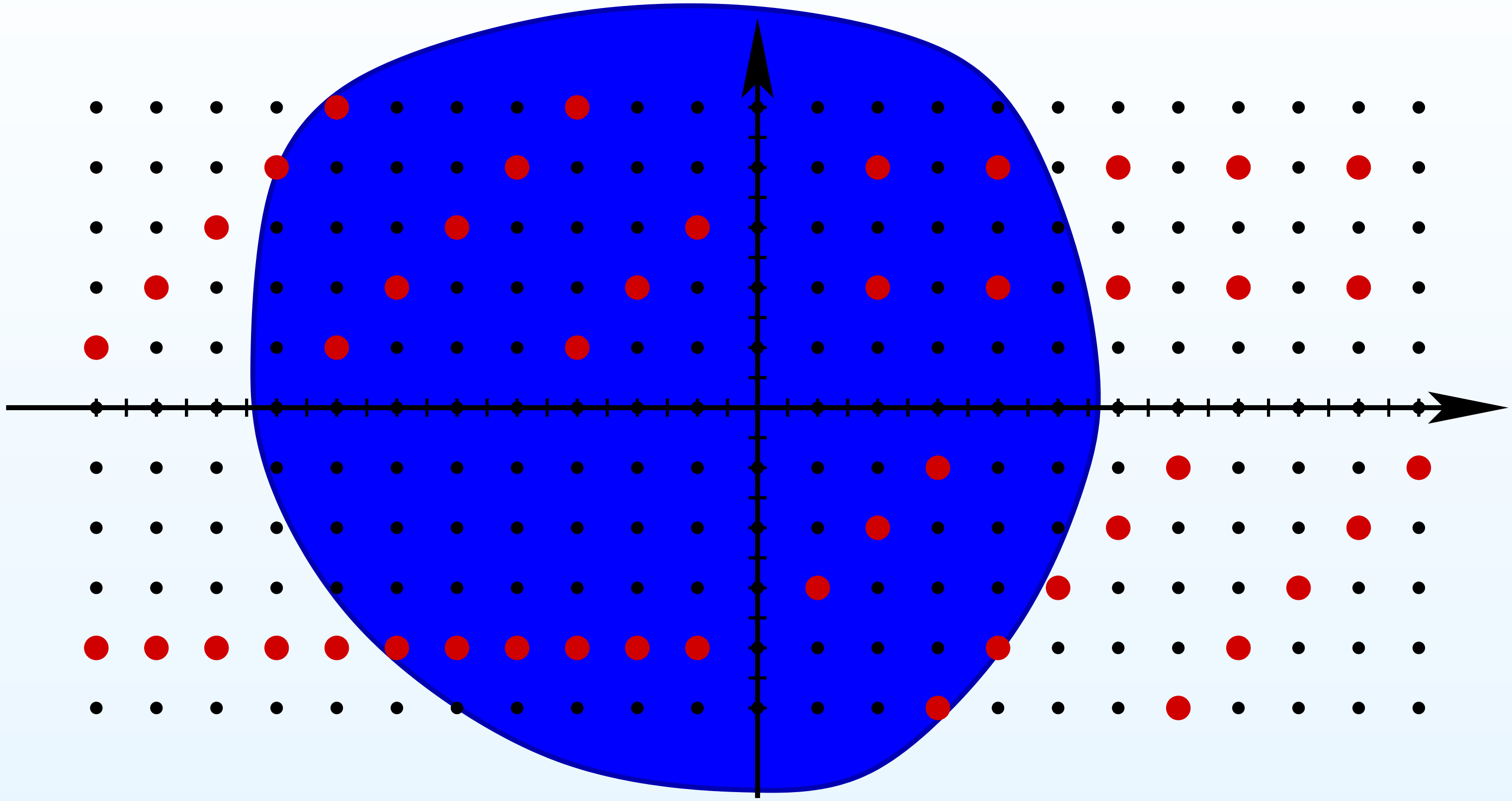

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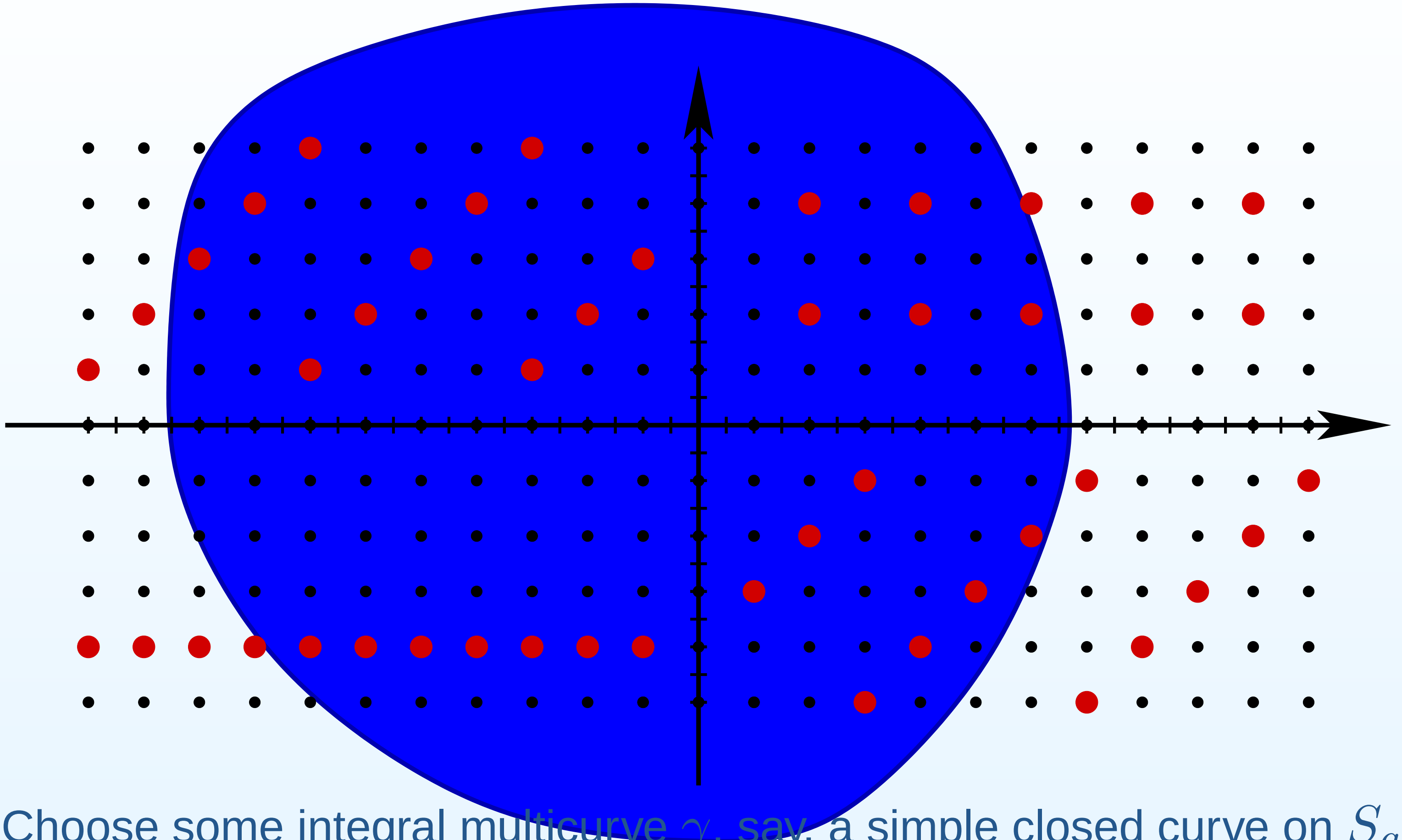

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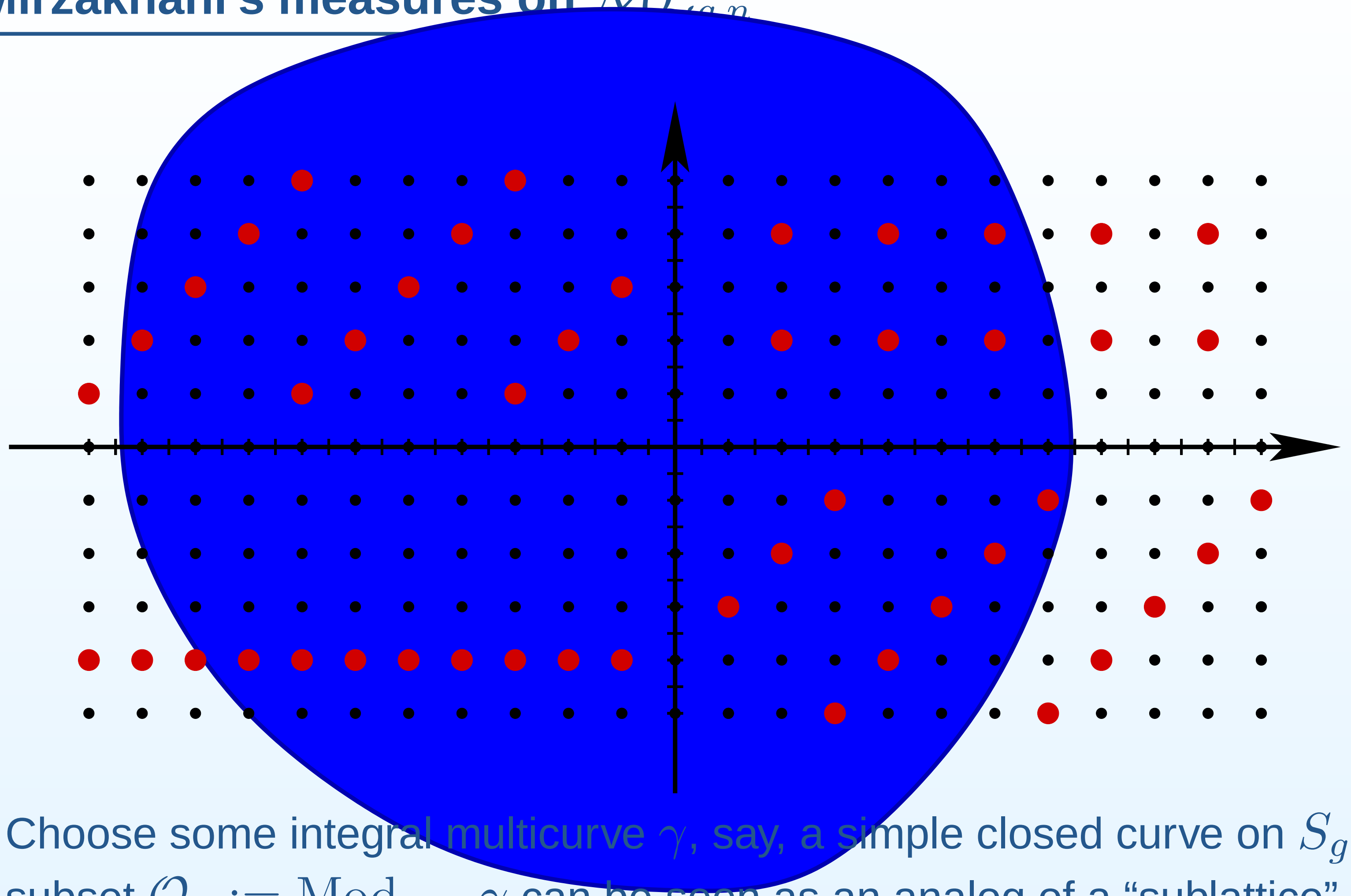

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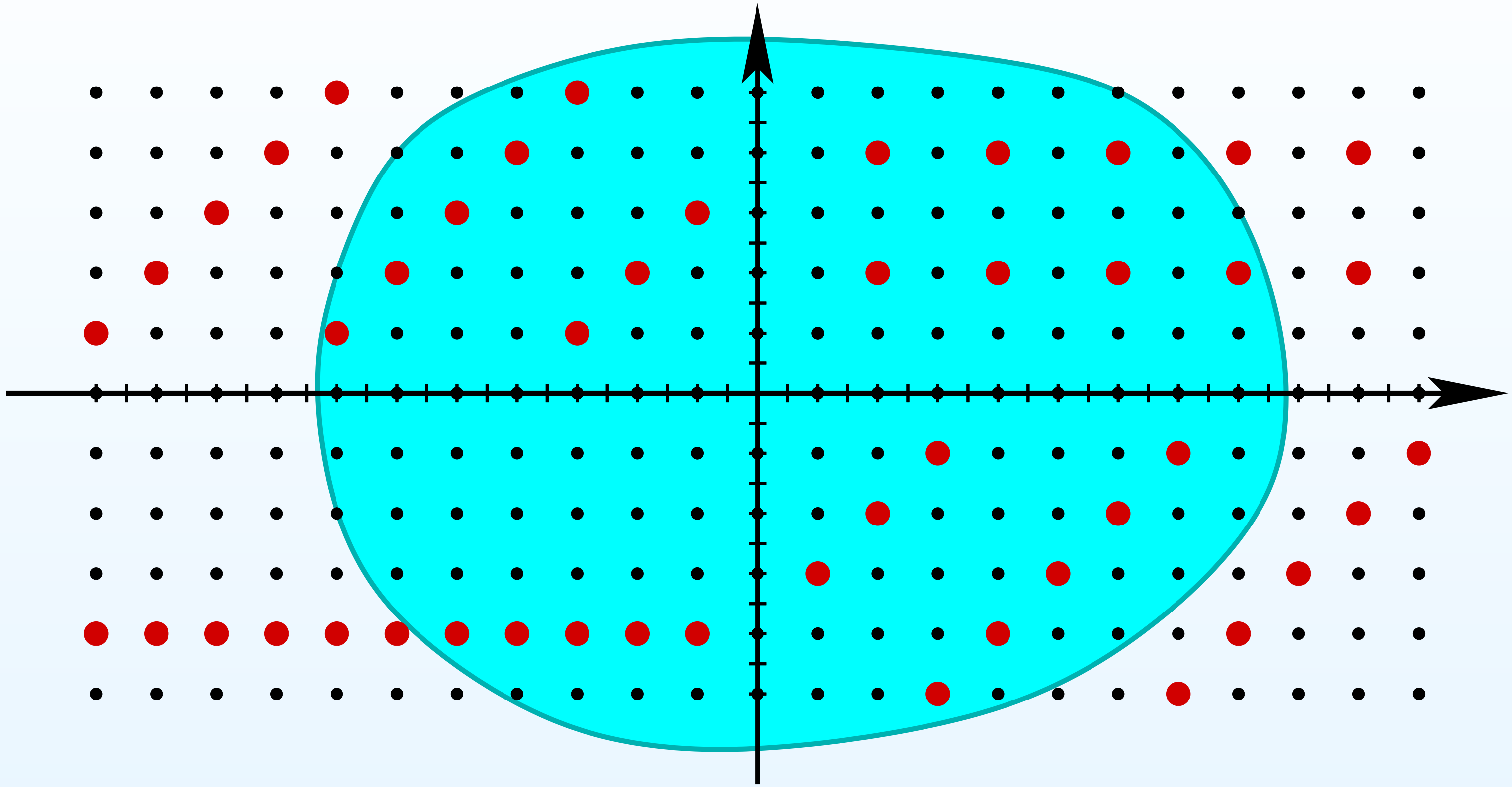

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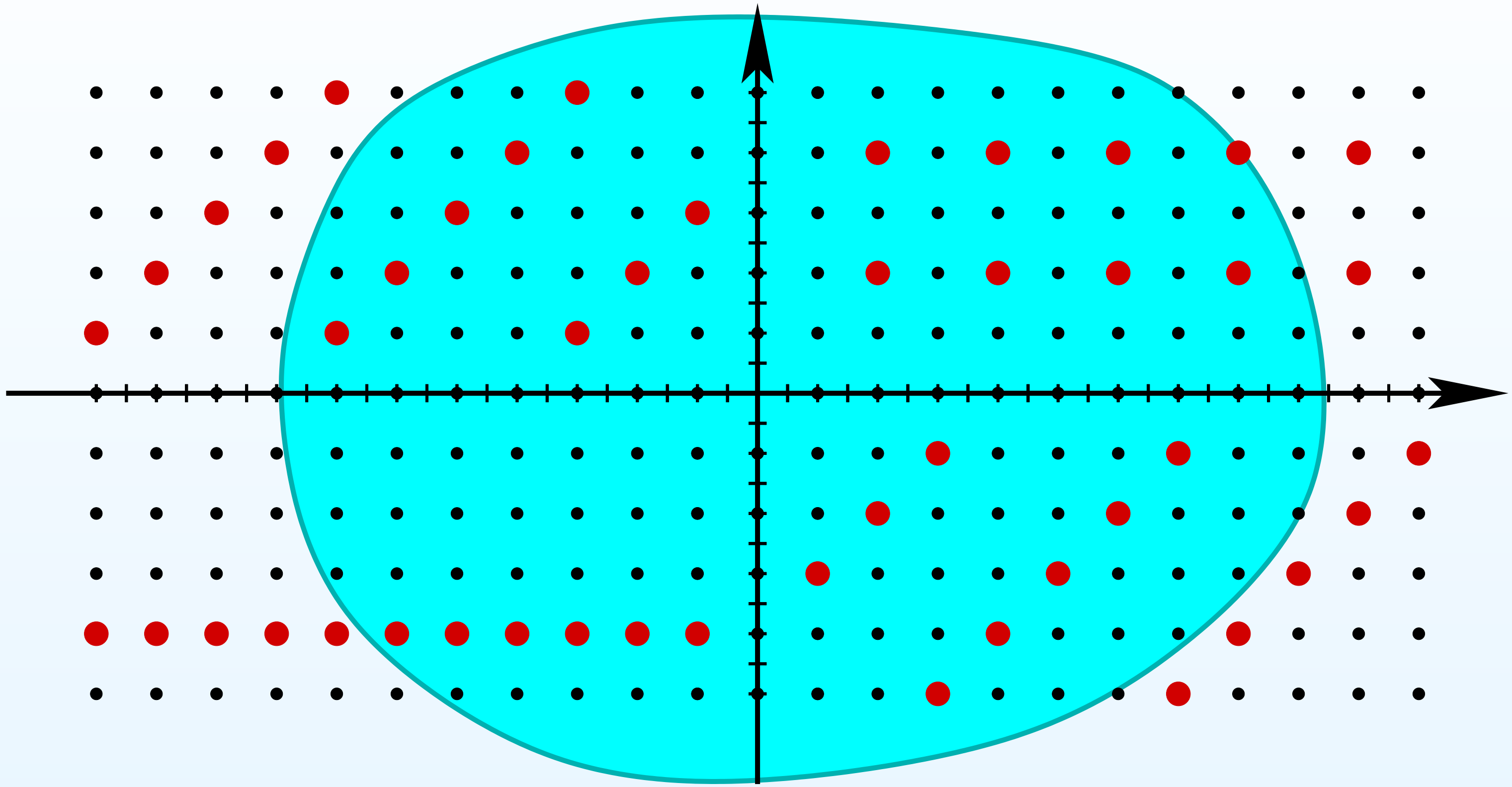

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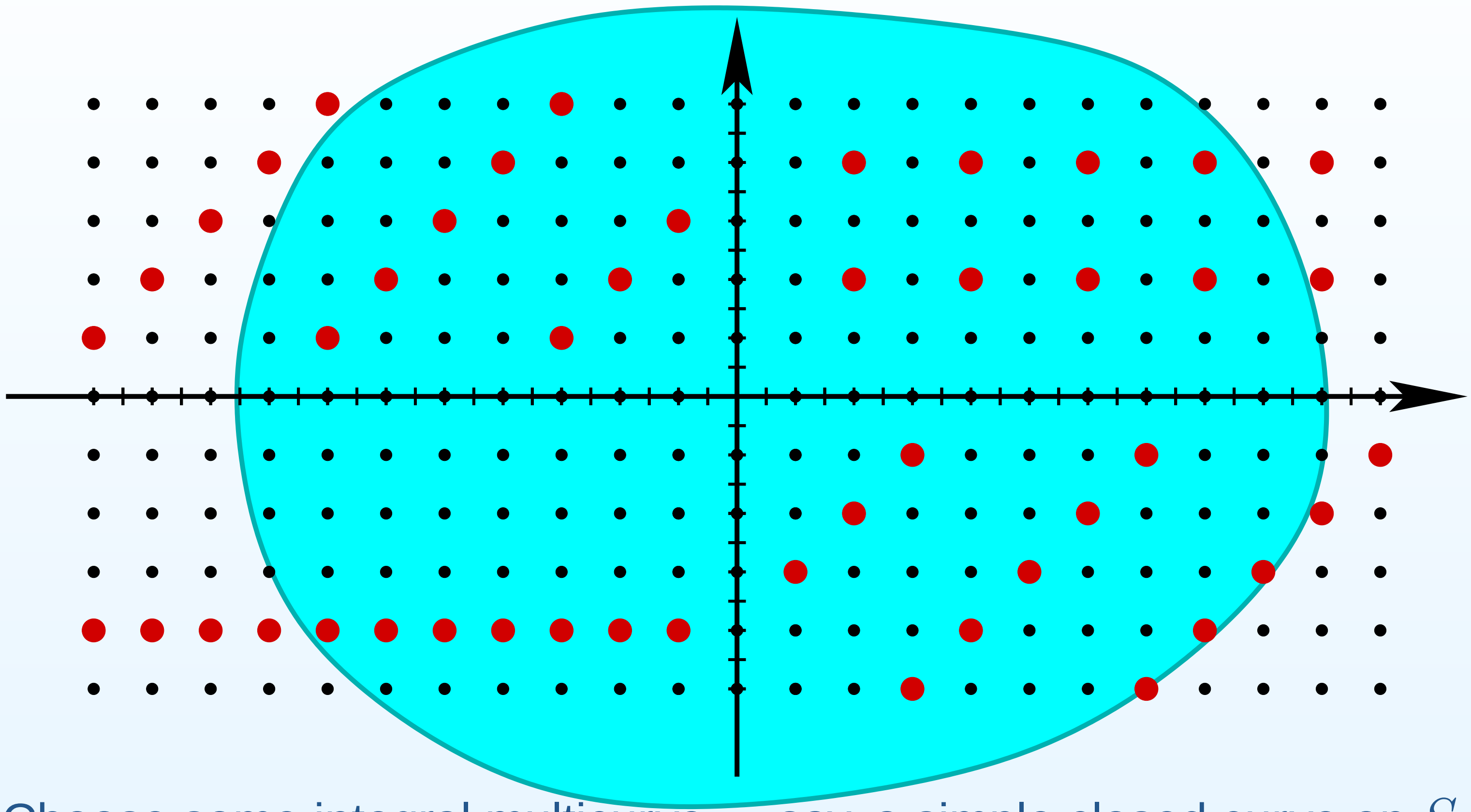

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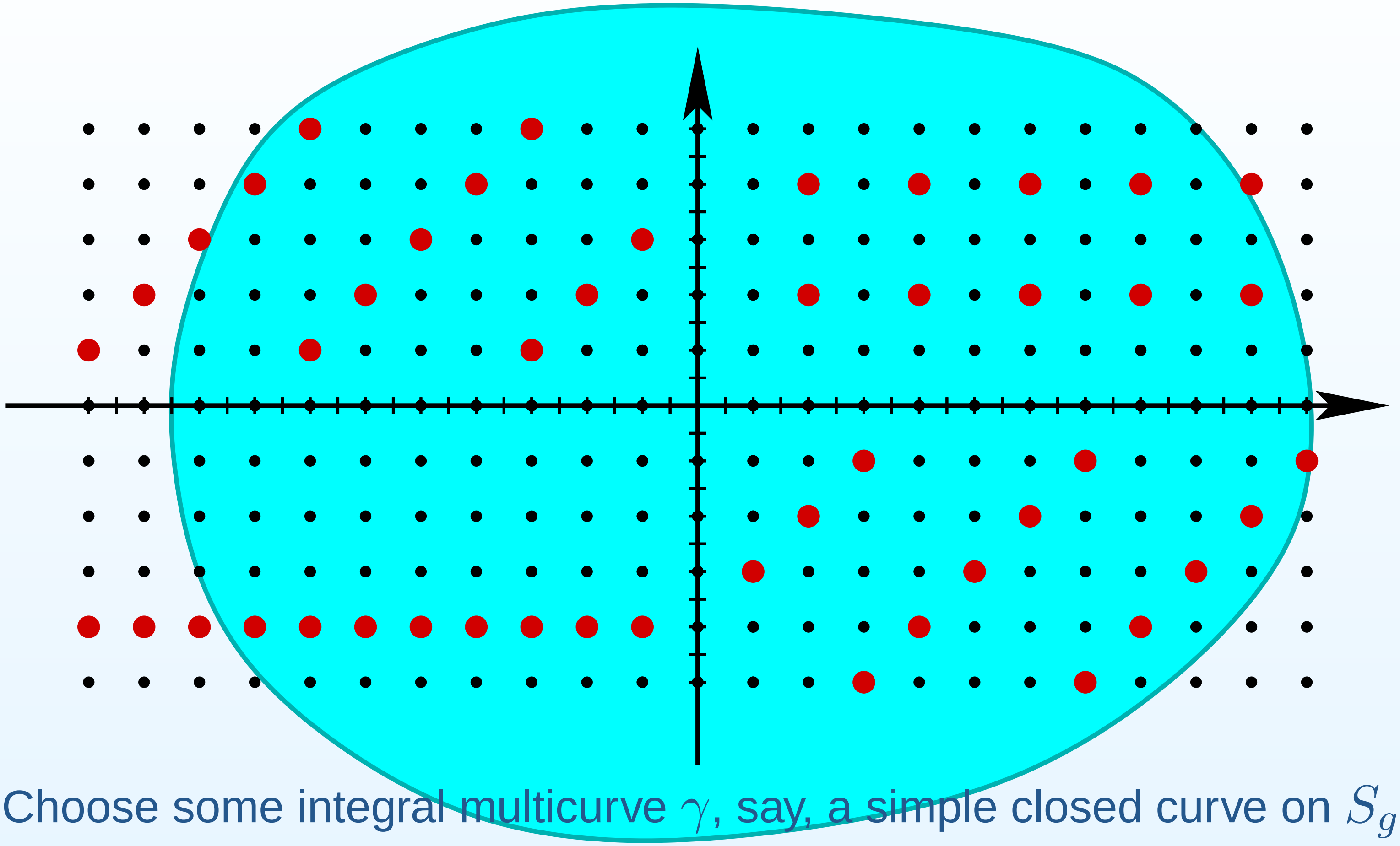

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Mirzakhani's measures on $\mathcal{ML}_{g,n}$

More formally: the Thurston measure of a subset $B \subset \mathcal{ML}_{g,n}$ is defined as

$$\mu_{\mathrm{Th}}(B) := \lim_{t \to +\infty} \frac{\mathrm{card}\{tB \cap \mathcal{ML}_{g,n}(\mathbb{Z})\}}{t^{6g-6+2n}}.$$

Mirzakhani defines a new measure μ_γ as

$$\mu_\gamma(B) := \lim_{t \to +\infty} \frac{\mathrm{card}\{tB \cap \mathcal{O}_\gamma\}}{t^{6g-6+2n}}.$$

Clearly, for any B we have $\mu_\gamma(B) \leq \mu_{\mathrm{Th}}(B)$ since $\mathcal{O}_\gamma \subset \mathcal{ML}_{g,n}(\mathbb{Z})$, so μ_γ belongs to the Lebesgue measure class. By construction μ_γ is $\mathrm{Mod}_{g,n}$-invariant. Ergodicity of μ_{Th} implies that $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ where $k_\gamma = const$.

Length function and unit ball

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space. One can extend the length function to simple closed multicurves $\ell_{\sum a_i \gamma_i} = \sum a_i \ell_(\gamma_i)(X)$ by linearity. By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t \cdot \lambda}(X) = t \cdot \ell_\lambda(X)$. (Ask S. Kerckhoff for details: that's his results.)

Notations. Each hyperbolic metric X defines its own "unit ball" B_X in $\mathcal{ML}_{g,n}$:

$$B_X := \{\lambda \in \mathcal{ML}_{g,n} \,|\, \ell_\lambda(X) \le 1\} \,.$$

By definition of μ_{Th}, the Thurston volume of the unit ball is equal to the normalized number of integral points in a "ball of radius L" associated to X:

$$\mu_{\mathrm{Th}}(B_X) = \lim_{L \to +\infty} \frac{\operatorname{card}\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \,|\, \ell_\lambda(X) \le L\}}{L^{6g-6+2n}} \,.$$

Denote by $\quad b_{g,n} := \int_{\mathcal{M}_{g,b}} \mu_{\mathrm{Th}}(B(X))\, dX \quad$ the average volume of unit balls .

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Completion of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the “unit ball” B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ is computed as follows:

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so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzahani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \leq F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

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Average volume of unit balls

Recall that

$$b_{g,n} := \int_{\mathcal{M}_{g,b}} \mu_{\mathrm{Th}}(B(X))\, dX$$

denotes the average volume of “unit balls” measured in Thurston measure.

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Theorem (M. Mirzakhani, 2008). *Weil–Petersson volume of the moduli space of boarded hyperbolic surfaces is a polynomial in lengths of boundary components $b_1^2, \ldots, b_n^2$. Its term of top degree $3g-3+n$ has the form:*

$$\mathrm{Vol}_{WP}\left(\mathcal{M}_{g,n}\right)(b_1^2,\ldots,b_n^2) = \frac{2}{2^{5g-6+2n}} \sum_{|d|=3g-3+n} \frac{\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \rangle}{d_1! \ldots d_n!} b^{2d_1} \ldots b^{2d_n} +$$

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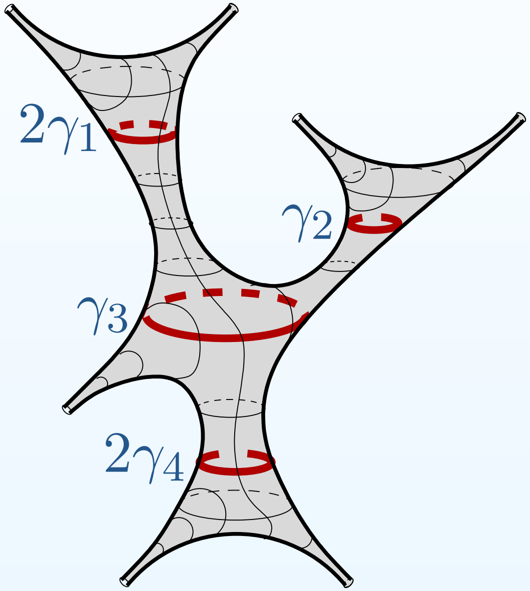
Application: horizontal geodesics on square-tiled surfaces

Hyperbolic and flat geodesic multicurves

Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$.



Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders.

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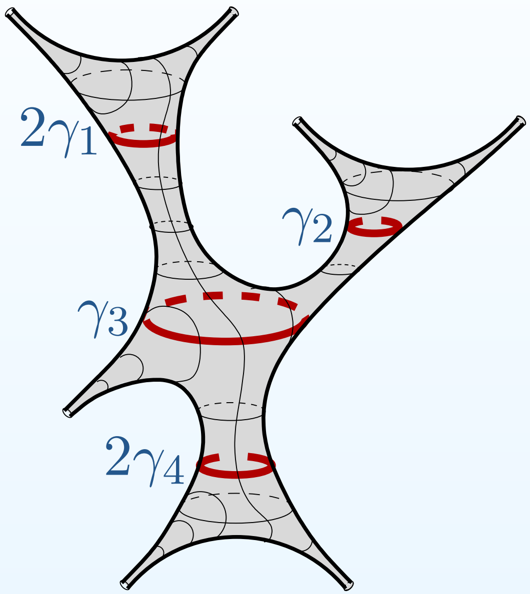

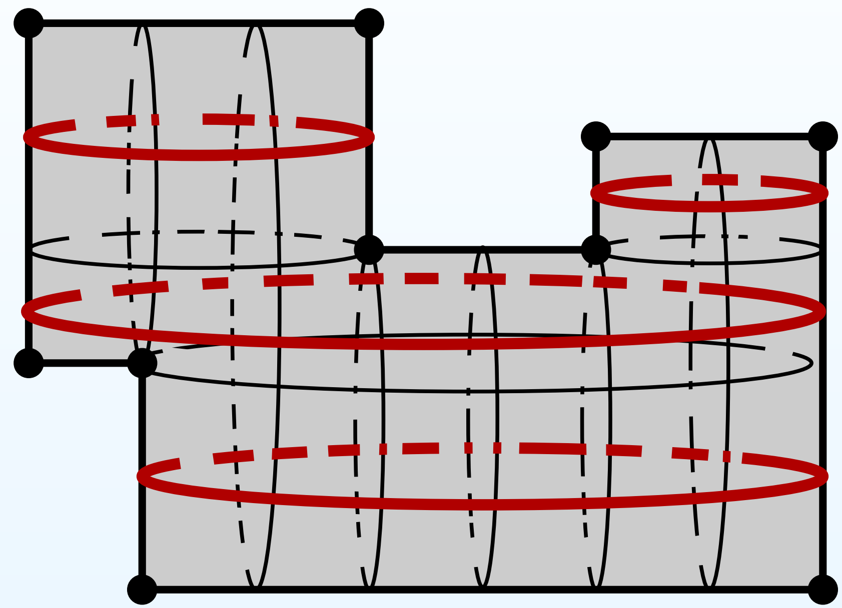

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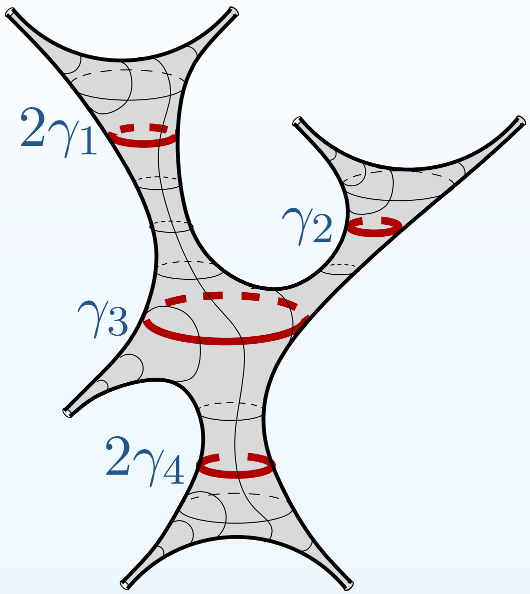

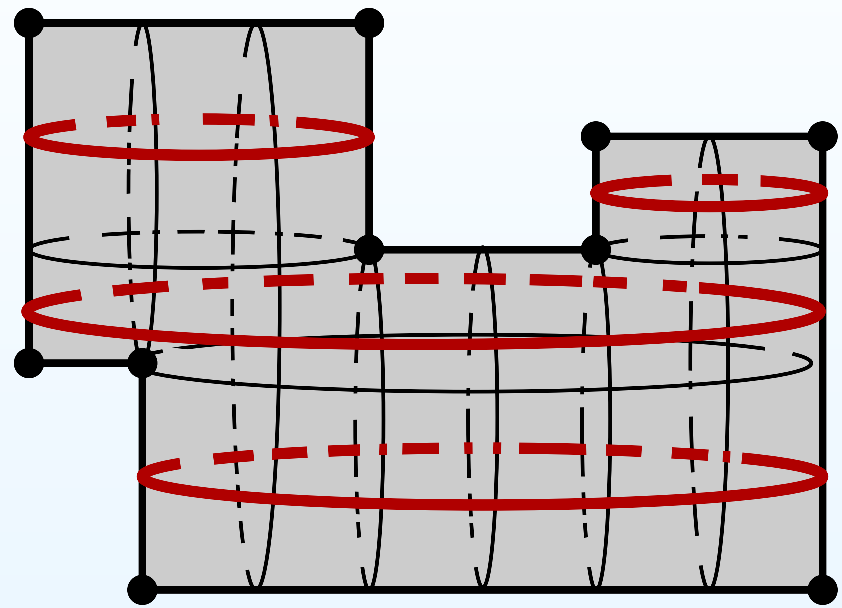

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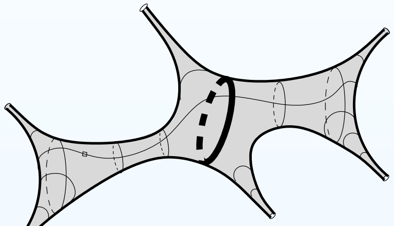
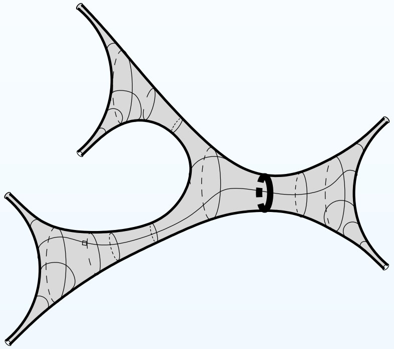
Clearly, there are plenty of square-tiled surface realizing this multicurve.

Frequencies of hyperbolic and flat simple closed geodesics

Theorem (M. Mirzakhani, 2008). *The asymptotic frequency of simple closed hyperbolic geodesics of fixed topological type does not depend on the choice of particular hyperbolic metric.*





Example 1 (M. Mirzakhani; confirmed experimentally by M. Bell and S. Schleimer)

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}$$

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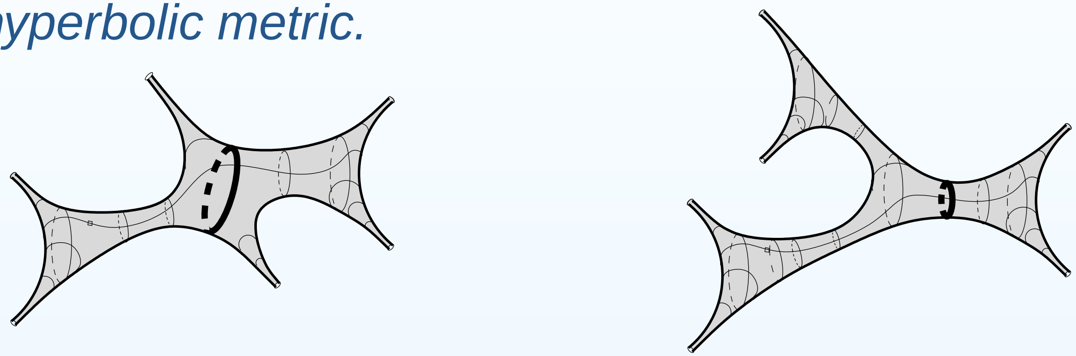

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Theorem (Delecroix, Goujard, Zograf, Zorich, 2018). *For any topological class Γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\Gamma)$ of simple closed **hyperbolic** multicurves of type Γ on any hyperbolic surface $X \in \mathcal{M}_{g,n}$ coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type Γ represented by associated square-tiled surfaces.*

Separating versus non-separating. Large genus asymptotic.

Example 2 (M. Mirzakhani, 2008). *Genus two; no cusps.*

$$\lim_{L\to+\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{6}$$

Separating versus non-separating. Large genus asymptotic.

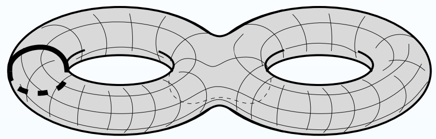
Example 2 (M. Mirzakhani, 2008). *Genus two; no cusps.*

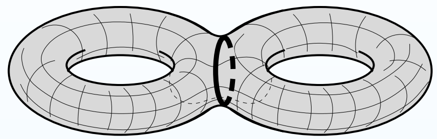
$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in the calculation of Mirzakhani.

Separating versus non-separating. Large genus asymptotic.

Example 2 (M. Mirzakhani, 2008). *Genus two; no cusps.*



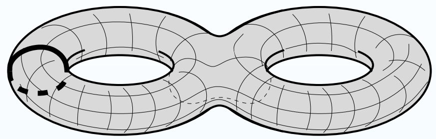


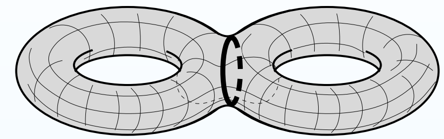
$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in the calculation of Mirzakhani. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell; also nailed by C. Ball.

Separating versus non-separating. Large genus asymptotic.

Example 2 (M. Mirzakhani, 2008). *Genus two; no cusps.*





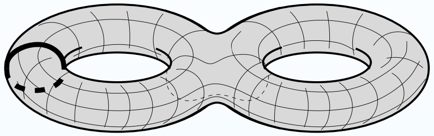
$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

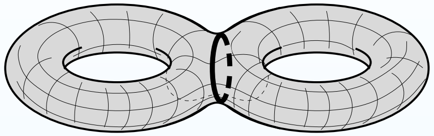
after further correction of another trickier bug in the calculation of Mirzakhani. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell; also nailed by C. Ball.

Question. *Which simple closed geodesics are more frequent on a closed hyperbolic surface of large genus: separating or not? What is the asymptotics of the ratio of their frequencies? Does this ratio stabilize when genus grows?*

Separating versus non-separating. Large genus asymptotic.

Example 2 (M. Mirzakhani, 2008). *Genus two; no cusps.*





$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in the calculation of Mirzakhani. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell; also nailed by C. Ball.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018–).

$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics}(L)}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics}(L)} \sim \frac{1}{4^g}$$

Random simple closed geodesic on a closed hyperbolic surface of large genus separates the surface *extremely rarely*!