Bridges between flat and hyperbolic enumerative geometry

Anton Zorich (with V. Delecroix, E. Goujard and P. Zograf)

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Decomposition of a random compound object into primitive blocks.

Statistics of prime decompositions: integer numbers

Let m be a positive integer, $m \in \mathbb{N}$, and let $m = p_1 \cdot p_2 \cdot \dots \cdot p_k$ be its decomposition into product of primes. Considering integers m in a large interval $[1, N]$, where $N \gg 1$, as equiprobable, one can collect statistics of their prime decompositions, say, number of distinct prime factors, their multiplicities, etc, and in this sense speak of "probability" of this or that property of the decomposition. The Prime Number Theorem gives an example of such statistics.

Prime Number Theorem for Numbers. *A randomly selected integer m of size $m \sim N$ will be prime with probability of the order $\frac{1}{\log N}$ when N is large.*

(Actually, one can tell more. For example, T. Tao proved that $\frac{\log p_k}{\log N}$ appropriately interpreted as a random variable follows the Poisson–Dirichlet process.)

Statistics of prime decompositions: permutations

Let $\sigma \in \mathfrak{S}_N$ be a permutation of N elements. and let $C_j(\sigma)$ be the number of cycles of length j in its decomposition into a product of cycles. By construction, $\sum_{j=1}^N j \cdot C_j(\sigma) = N$. We denote by $C(\sigma) = \sum_{j=1}^N C_j(\sigma)$ the total number of cycles.

Considering permutations $\sigma \in \mathfrak{S}_N$ as equiprobable, one can collect statistical properties of their cyclic decompositions: total number of cycles, number of cycles of given length j, etc, and in this sense speak of “probability” of this or that property of a random cyclic decomposition.

Prime Number Theorem for Permutations. *A randomly selected permutation $\sigma \in \mathfrak{S}_N$ will be composed of a single cycle with probability $\frac{1}{N}$ for any $N \in \mathbb{N}$.*

Proof. There are exactly $\frac{1}{N} \cdot N!$ such permutations in $\mathfrak{S}_N$.

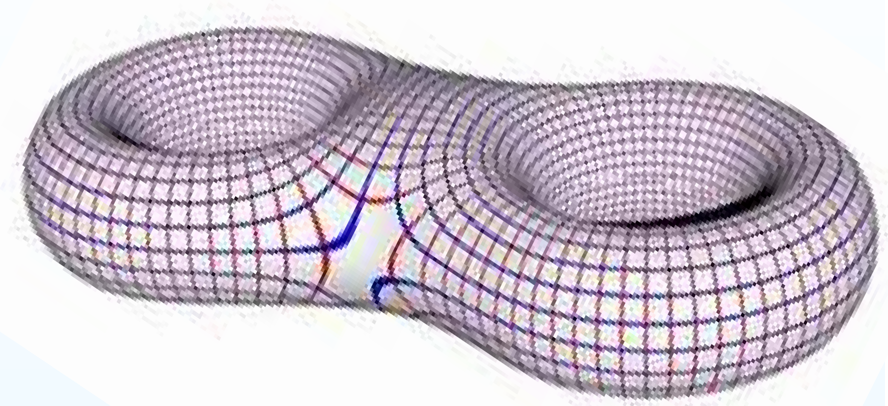
Theorem (in this formulation: T. Tao). *For any $1 \le j \le N$, one has $\mathrm{E}\, C_j = \frac{1}{j}$. In particular,*

$$\mathrm{E}\, C = 1 + \frac{1}{2} + \cdots + \frac{1}{N} = \log N + O(1)\,.$$

Shape of a random square-tiled surface

Squares of the tiling are *polarized*: horizontal sides are glued to horizontal sides and vertical sides to vertical sides.

A square-tiled surface has *translation structure* if the horizontal and vertical directions on each square are oriented and if gluing the squares together the orientation is respected.



Picture created by Jian Jiang

Questions.

- *With what probability a random square-tiled translation surface has* $k = 1, 2, \ldots, 3g - 3$ *maximal horizontal cylinders? How often it has a single horizontal cylinder?*
- *What are the typical heights of cylinders?*
- *How many distinct singular horizontal layers it has?*
- *What is the shape of a random square-tiled surface of large genus?*

Example of explicitly computed probabilities

Consider translation square-tiled surfaces of genus 3 with two conical singularities of angles 4π and 8π. The probabilities p_k that a random square-tiled surface of this type has $k = 1, 2, 3, 4$ maximal horizontal cylinders have the following values:

$$0.19 \approx p_1 = \frac{3\,\zeta(7)}{16\,\zeta(6)} \qquad \leftarrow \text{ the only quantity which is easy to compute}$$

$$0.47 \approx p_2 = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx p_3 = \frac{1}{32\,\zeta(6)}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\
&+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\
&- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\
&+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\
&- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\
&- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx p_4 = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big).$$

Moral: it is desperately difficult to compute individual contributions of k-cylinder surfaces. Conjecture: these contributions are rational polynomials in mzv.

Prime Number Theorem for Square-Tiled Surfaces

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Proof. Our proof with V. Delecroix, E. Goujard and P. Zograf uses large genus asymptotics of Masur–Veech volumes of moduli spaces of Abelian differentials recently proved by A. Aggarwal and independently by D. Chen, M. Möller, A. Sauvaget, D. Zagier. An alternative prove is due to Ph. Engel.

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Conjecture. *When g grows, the probability that a random translation square-tiled surface as above has k cylinders tends to the probability that a random permutation of $N = 2g + n$ elements has $k = 1, \dots, N$ cycles in its cyclic decomposition. In particular, a random translation square-tiled surface has about $\log(2g + n)$ cylinders.*

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Conjecture. *A random square-tiled translation surface of genus $g \gg 1$ with no conical points of angles π has single horizontal cylinder with probability $\sim \sqrt{\frac{\pi}{24g}}$.*

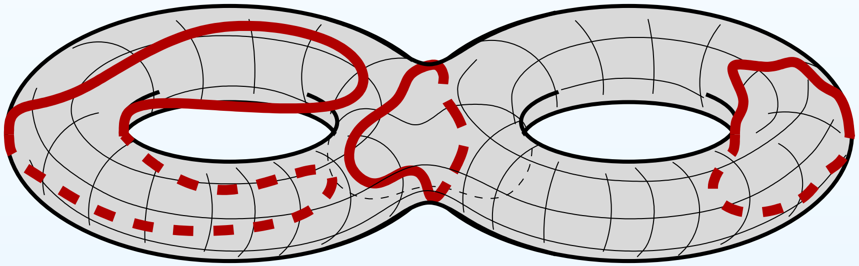
Mirzakhani's count of simple closed geodesics

Picture by François Labourie taken at CIRM

Multicurves

Consider a finite collection of pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g.n}$ of genus g with n punctures.

For any hyperbolic metric X on $S_{g,n}$ and for any simple closed curve γ_i there exists a unique geodesic representative in the free homotopy class of γ_i.



Fact. *For any hyperbolic metric X and any collection $\gamma_1, \ldots, \gamma_k$ of pairwise non-intersecting simple closed curves, their geodesic representatives do not self-intersect and do not pairwise intersect either.*

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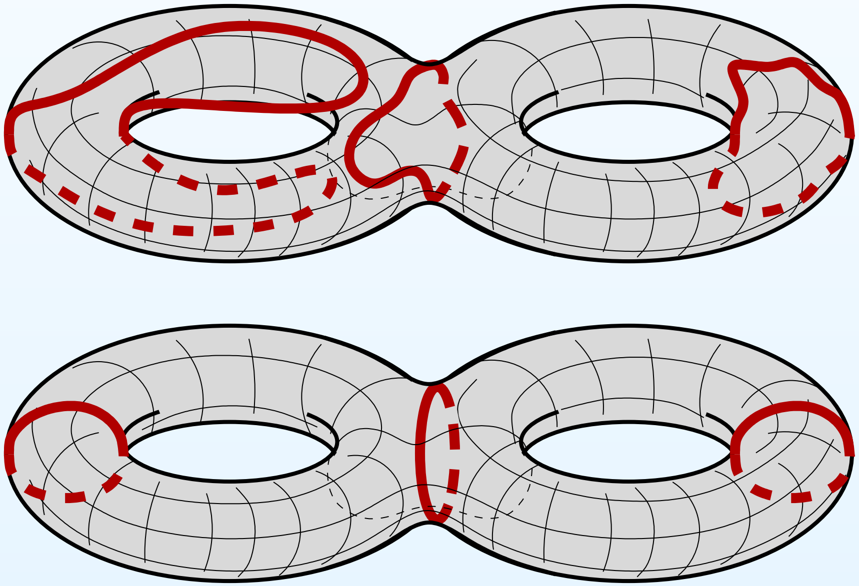

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Multicurves

We can consider formal linear combinations $\gamma := \sum_{i=1}^{k} m_i \gamma_i$ of such simple closed curves with positive coefficients. When all coefficients m_i are integer (respectively rational), we call such γ integral (respectively rational) *multicurve*. In the presence of a hyperbolic metric X we define the hyperbolic length of a multicurve γ as $\ell_\gamma(X) := \sum_{i=1}^{k} m_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

We say that two multicurves γ, ρ have the same *topological type* $[\gamma] = [\rho]$ *if and only if they belong to the same orbit of the mapping class group:* $\rho \in \mathrm{Mod}_{g,n} \cdot \gamma$.

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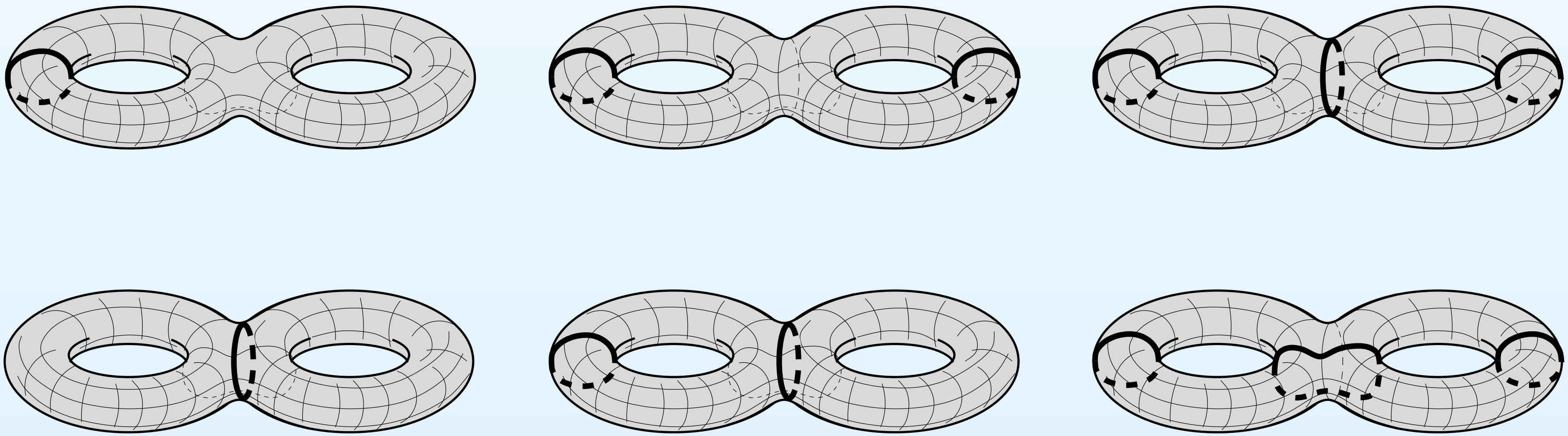
Example: primitive multicurves on a surface of genus two

The picture below illustrates all possible types of primitive multicurves on a surface of genus two without punctures.

Note that contracting all components of a multicurve we get a “stable curve” — a Riemann surface degenerated in one of the several regular ways. In this way the “topological types of primitive multicurves” on a smooth surface $S_{g,n}$ of genus g with n punctures are in the natural bijective correspondence with boundary classes of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of pointed complex curves.



Frequencies of multicurves

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *the number* $s_X(L,\gamma)$ *of simple closed geodesic multicurves on* X *of topological type* $[\gamma]$ *and of hyperbolic length at most* L *has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

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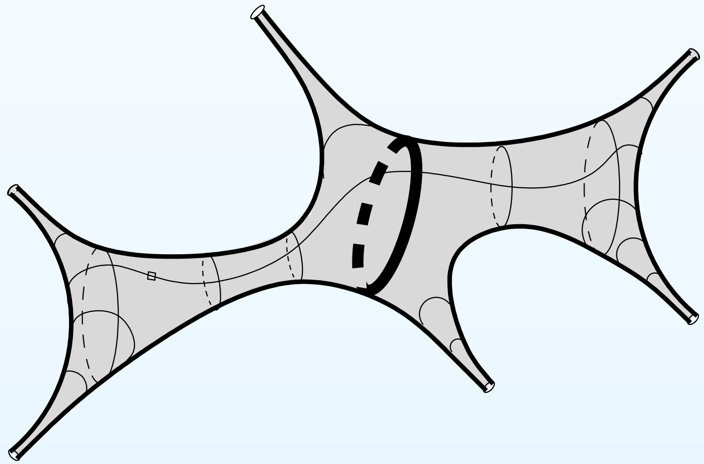
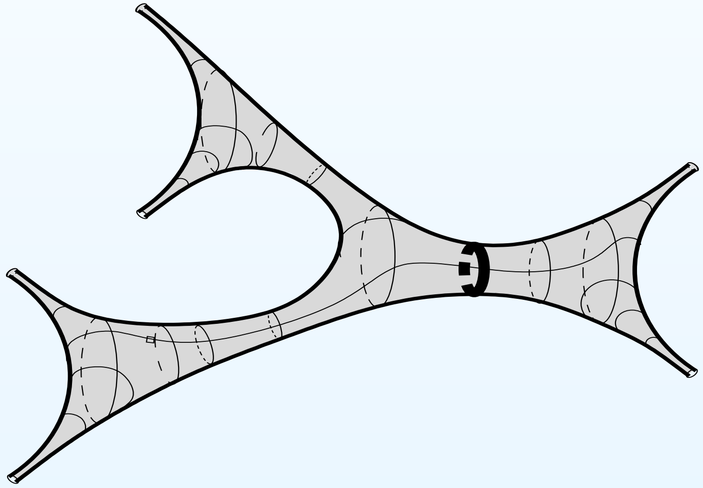
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1,γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty}\frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)}=\frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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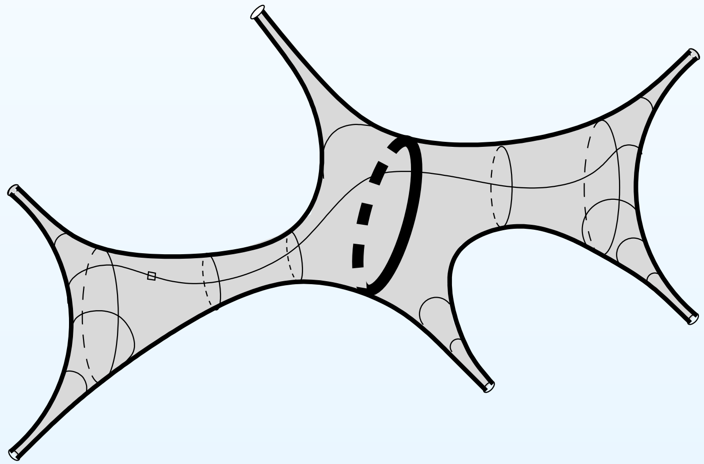
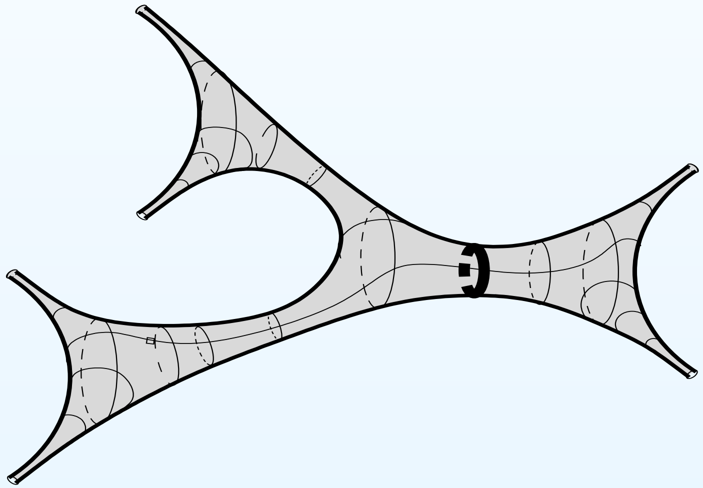
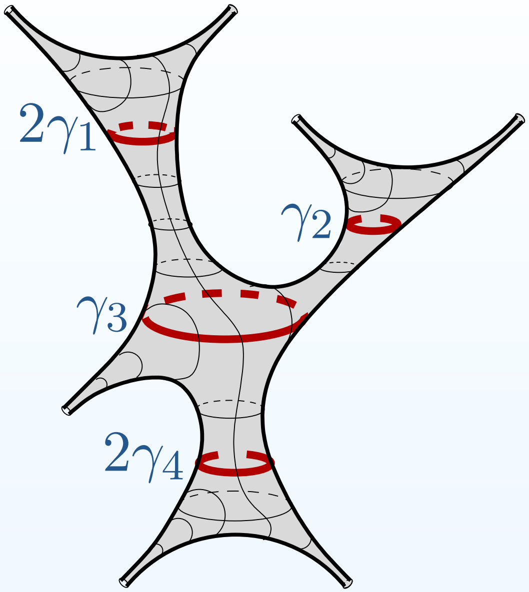


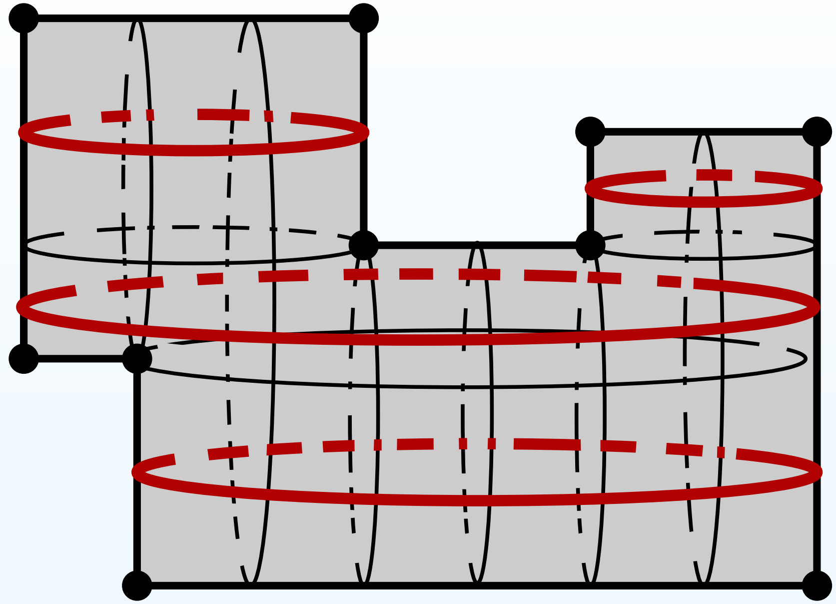


Example (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

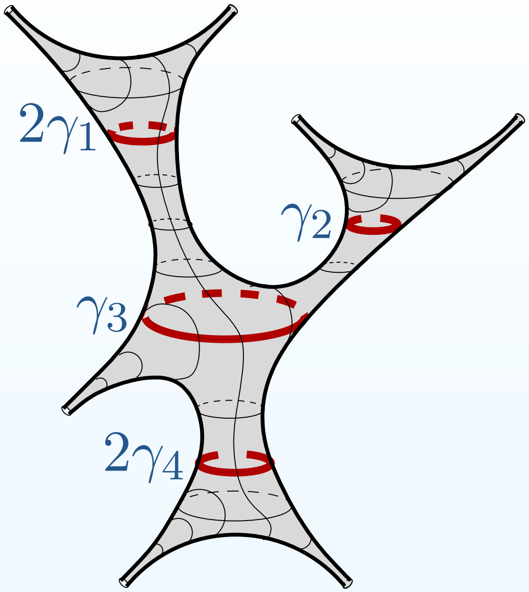
Hyperbolic and flat geodesic multicurves

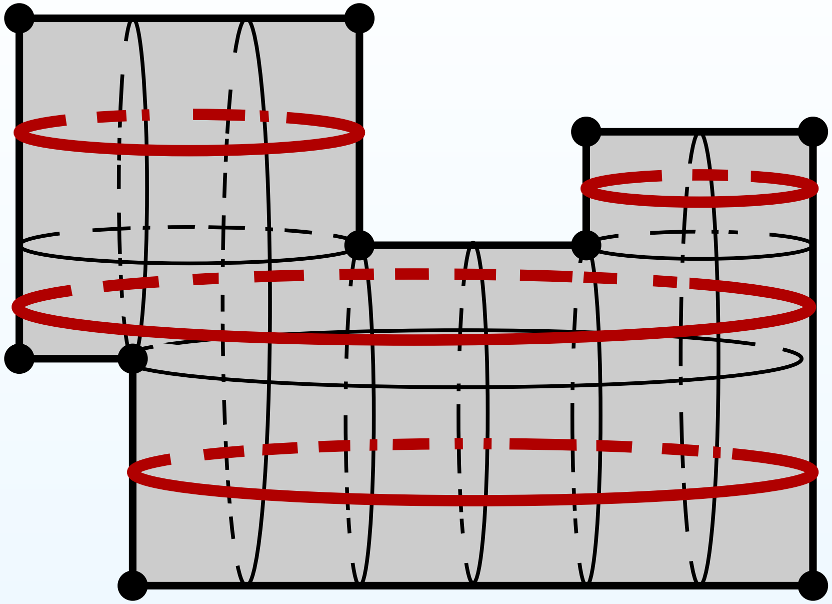




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves





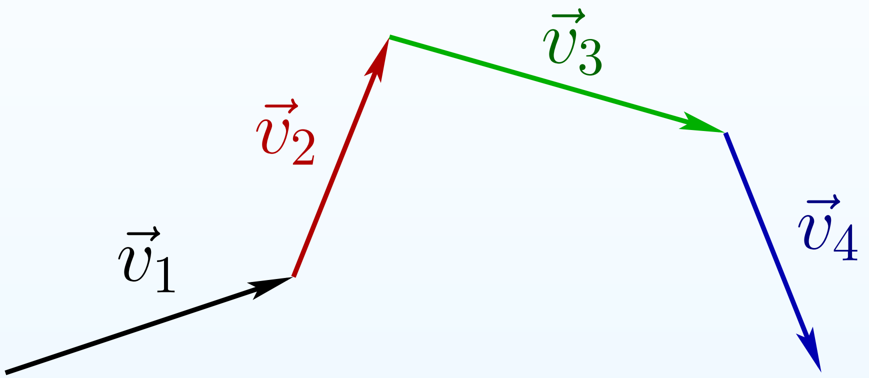
Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of **hyperbolic** multicurves coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials

Very flat surfaces: construction from a polygon

Consider a broken line constructed from vectors $\vec{v}_1, \ldots, \vec{v}_k$.



and another one constructed from the same vectors taken in another order.

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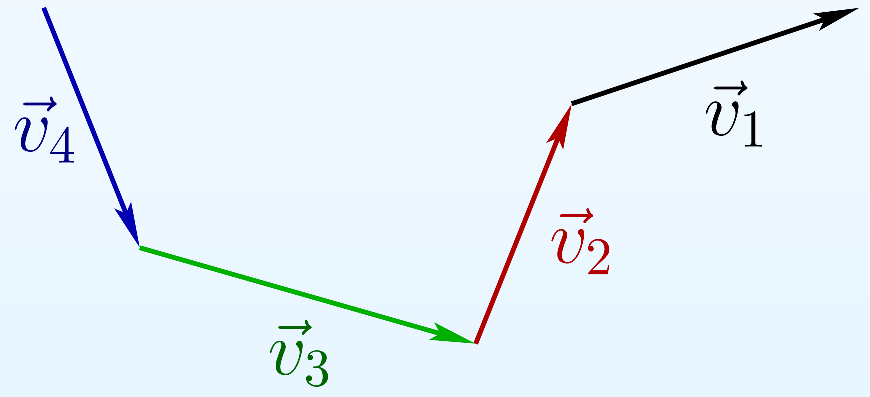

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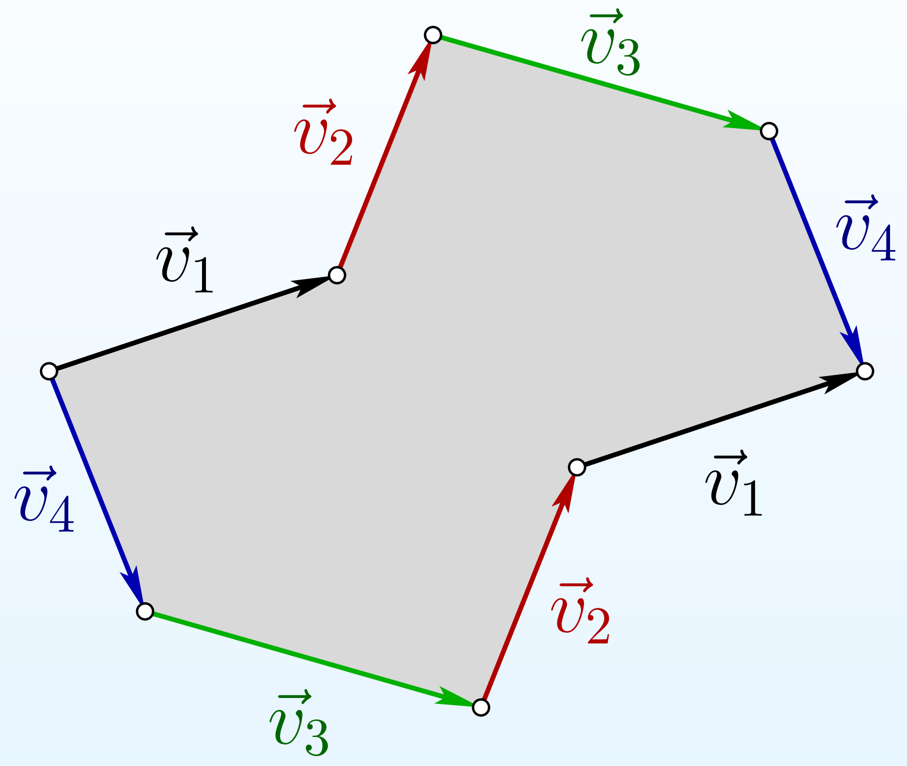

and another one constructed from the same vectors taken in another order. If we are lucky enough the two broken lines do not intersect and form a polygon.

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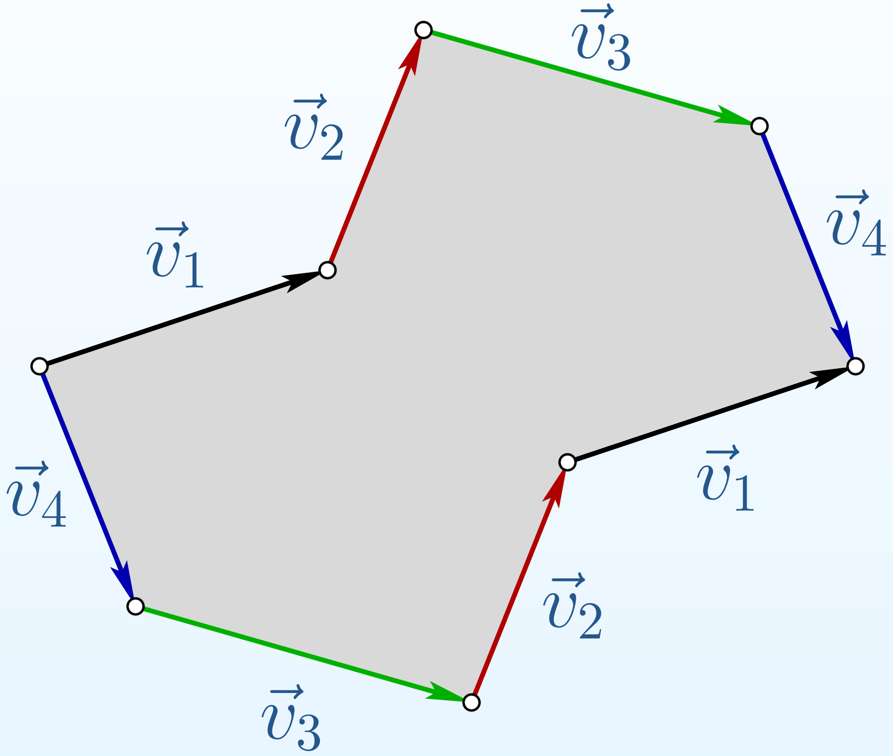
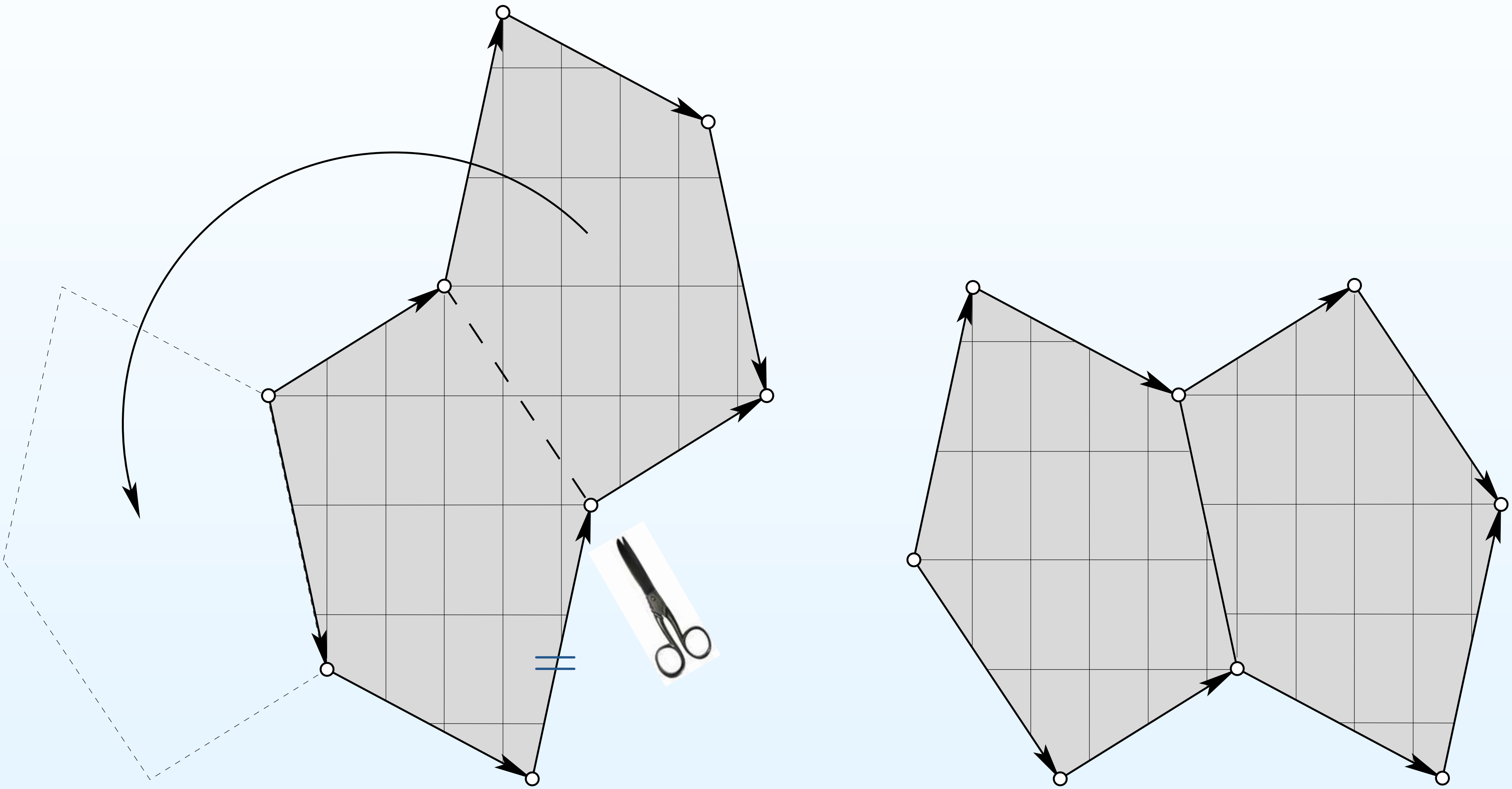


Identifying the corresponding pairs of sides by parallel translations we get a closed surface endowed with a flat metric.

Polygonal patterns of the same translation surface



Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane, where lives the polygon. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points inherits the complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

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Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs (C,ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1=\mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1=\text{area}(S)=\frac{i}{2}\int_C \omega\wedge\bar\omega=\sum_{i=1}^{g}(A_i\bar B_i-\bar A_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu=r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian or quadratic differentials is finite.*

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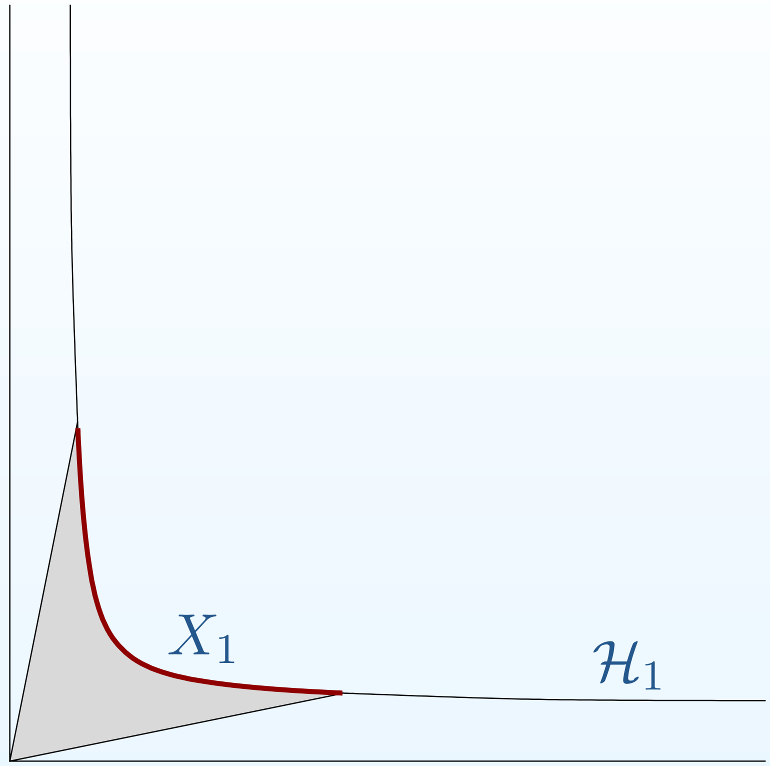
Any flat surface S can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu = r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian or quadratic differentials is finite.*

Counting volume by counting integer points in a large cone



ν_1-volume of a domain X_1 in a unit hyperboloid $\mathcal{H}_1$ is related to ν-volume of a cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ over X_1 as $\nu_1(X_1) = 2d \cdot \nu(C(X_1))$.

To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it.

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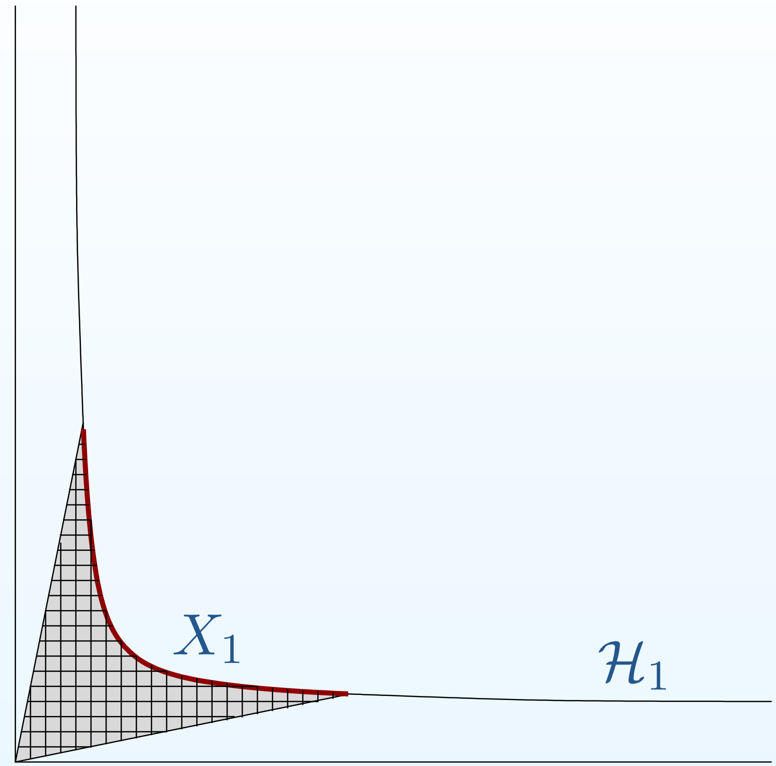

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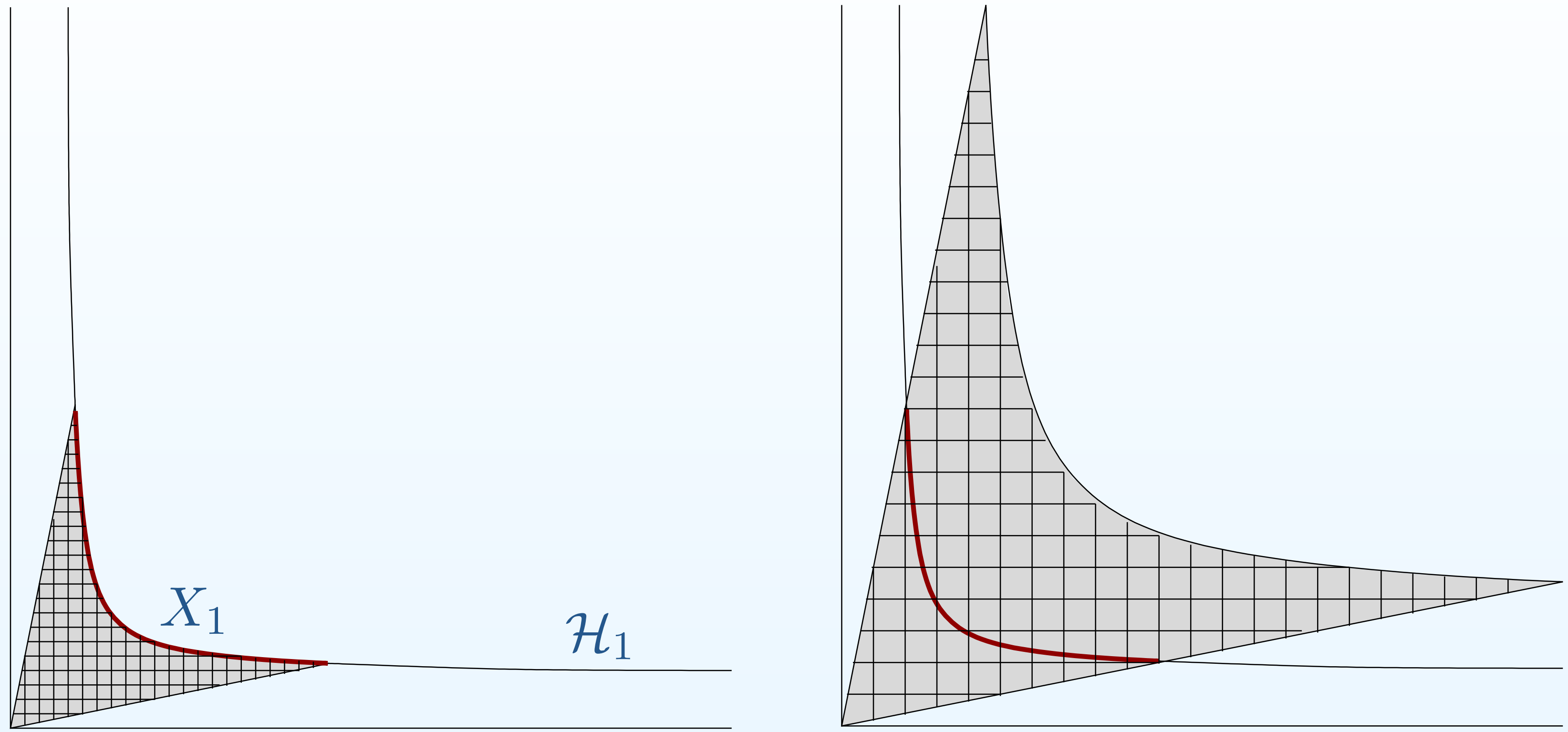

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To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it. Counting points of the $\frac{1}{N}$-grid in the cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ is the same as counting integer points in the larger proportionally rescaled cone $C_N(X_1) = \{r \cdot S | S \in X_1, r \leq N\}$.

Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z}\right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}, \text{ where } P_1 \text{ is a zero of } \omega\,.$$

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Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous “pillowcase covers” over $\mathbb{CP}^1$ branched at four points. Thus, counting volumes of the strata is similar to counting analogs of Hurwitz numbers.

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Let $\mathcal{H} = \mathcal{H}(m_1, \dots, m_n)$; let $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n) = 2g + n - 1$. We get:

$$\operatorname{Vol} \mathcal{H}_1 = 2d \cdot \lim_{N \to +\infty} \frac{\begin{pmatrix} \text{number of square-tiled surfaces in } \mathcal{H} \\ \text{tiled with at most } N \text{ identical squares} \end{pmatrix}}{N^d}\,.$$

Evaluation of volumes for strata of Abelian differentials

Theorem (A. Eskin, A. Okounkov, R. Pandharipande). *For every connected component $\mathcal{H}^c(d_1, \dots, d_n)$ of every stratum, the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled} \\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form. The Masur–Veech volume of every connected component of every stratum is a rational multiple of π^{2g}, where g is the genus.

- A. Eskin implemented this theorem to an algorithm allowing to compute volumes for all strata in sufficiently small genera, which allowed us to state a conjecture on a simple asymptotic formula for volumes in large genera.
- D. Chen, M. Möller, D. Zagier can compute the volume of the principal stratum up to genus 2000 and more; in 2017 they proved our conjecture with Eskin on large genus volume asymptotics for the principal stratum.
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All these approaches treat the entire Masur–Veech volume as a single block.

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Masur–Veech versus Weil–Petersson volume

Intersection numbers (correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complexx dimension $3g-3+n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} .$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in 2-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; one of alternative proofs belongs to M. Mirzakhani.

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Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1,\dots,d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1+\dots+d_n=3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1,\dots,d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1}\cdot\dots\cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1,\dots,b_n)$ of degree $6g-6+2n$ in variables $b_1,\dots,b_n$:

$$N_{g,n}(b_1,\dots,b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}}\,,$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b)\,.$$

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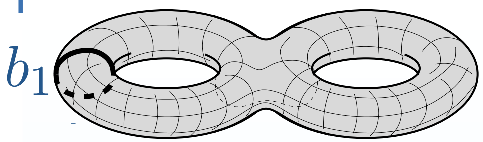
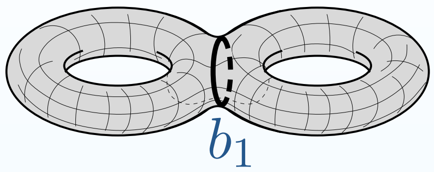
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\quad \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

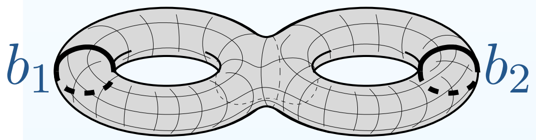
and extend it to symmetric polynomials in b_i by linearity.



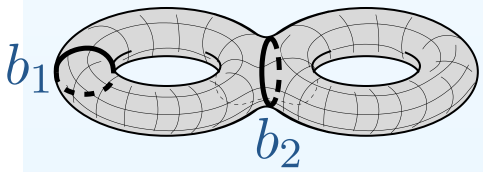
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



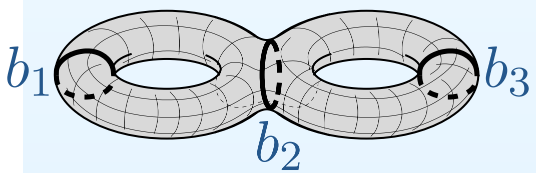
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



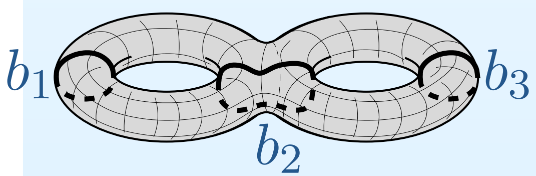
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



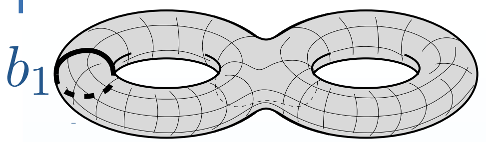
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



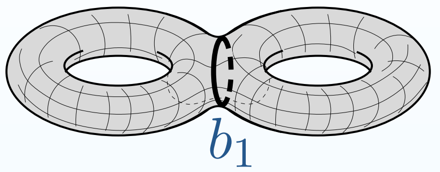
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



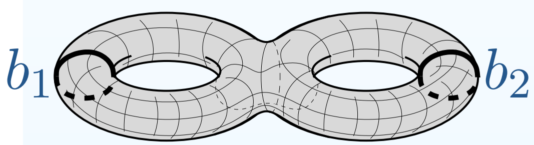
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$



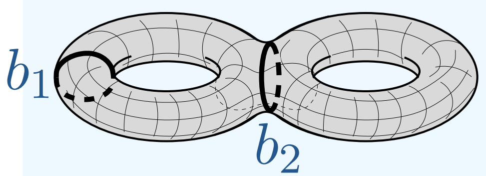
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) \qquad = \frac{1}{2} \cdot b_1 \left(\frac{1}{384}(2b_1^2)(2b_1^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) \qquad = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$

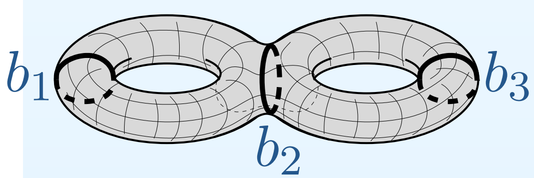


$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) \qquad = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4}(2b_1^2 + 2b_2^2) \right)$$



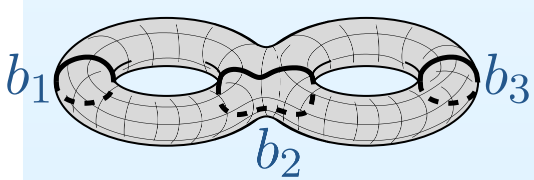
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

$$\cdot N_{1,1}(b_2) \qquad = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$



$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$

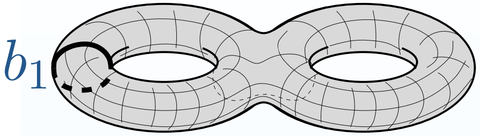
$$\cdot N_{0,3}(b_2, b_3, b_3) \qquad = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$



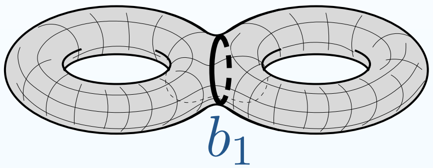
$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$

$$\cdot N_{0,3}(b_1, b_2, b_3) \qquad = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

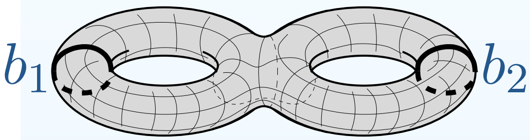
Volume of $\mathcal{Q}_2$



$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



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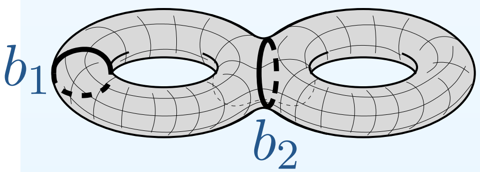

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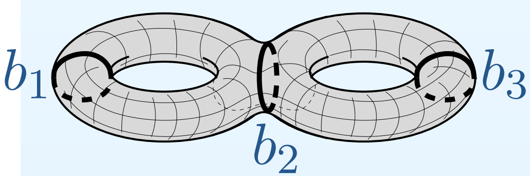

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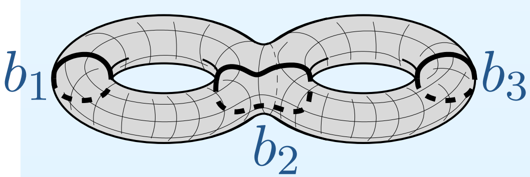

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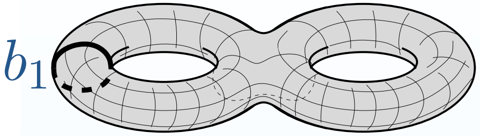
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\Big(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\Big)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_2$



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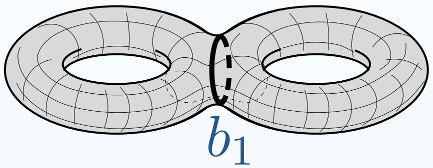

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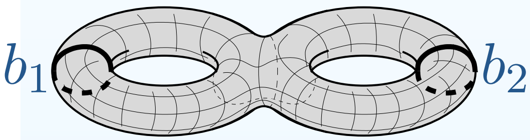

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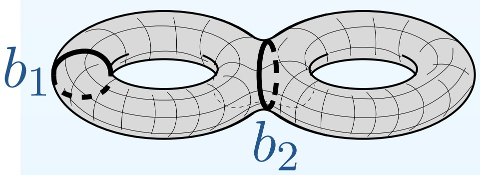

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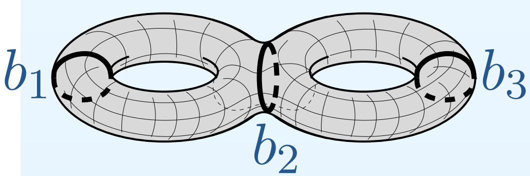

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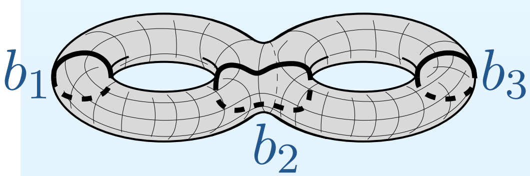

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These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_{g,n}$

Theorem. (Delecroix, Goujard, Zograf, Zorich) *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right),$$

The partial sum for fixed number k *of edges gives the contribution of* k*-cylinder square-tiled surfaces.*

Note that in contrast to the approach based on quasimodularity of the generating function, the Masur–Veech volume in our formula is the sum of simpler parts, where each individual part is geometrically meaningful. This allows to study statistical properties of the contributions.

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- A. Eskin conceptually understood this 20 years ago but has chosen other way.
- One can obtain this formula developing results of M. Mirzakhani.
- In May 2019 this formula was reproved by J. Andersen, G. Borot, S. Charbonnier, V. Delecroix, A. Giacchetto, D. Lewanski, C. Wheeler through topological recursion.

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Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

Volume of $\mathcal{Q}_{g,n}$

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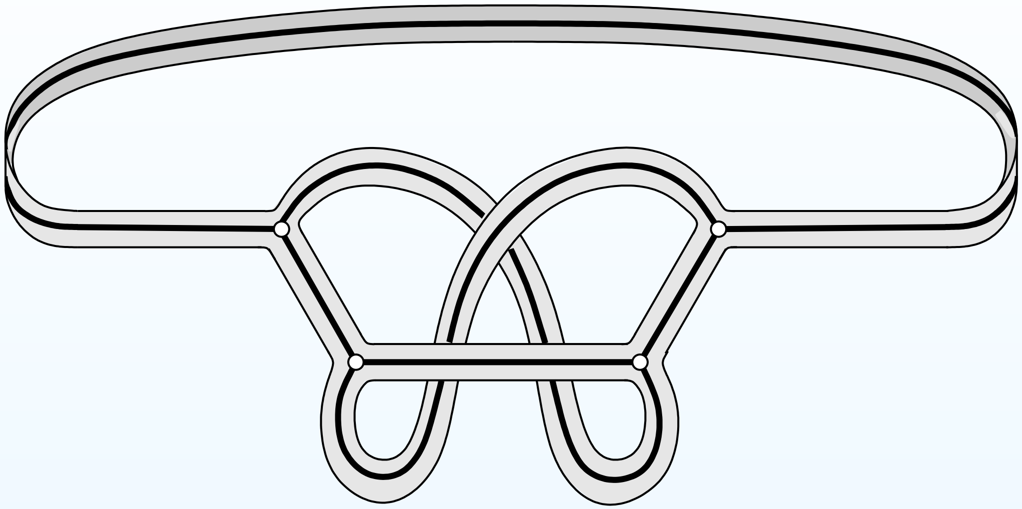
When any of g, n grow, the number of graphs grows very fast. Also the correlators are computed only inductively. Thus, the formula gives effective answer only in a limited number of cases.

Miracles: two exceptions which admit simple closed answer: when $g = 0$ and n is arbitrary (rigorous); when $g \gg 1$ and $n = 0$ (conjectural asymptotic value).

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 2$ with $n = 2$ boundary components. If we assigned lengths to all edges of the core graph, each boundary component gets induced length, namely, the sum of the lengths of the edges which it follow.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorem of Kontsevich counts them.

Idea of the proof: Kontsevich's count of metric ribbon graphs

Each horizontal layer containing zeroes or poles of a square-tiled surface can be seen as a metric ribbon graph. When the associate quadratic differential has only simple zeroes, the metric ribbon graph is trivalent.

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Theorem (Kontsevich). *Consider a collection of positive integers* $b_1, \dots, b_n$ *such that* $\sum_{i=1}^n b_i$ *is even. The weighted count of genus* g *connected trivalent metric ribbon graphs* Γ *with integer edges and with* n *labeled boundary components of lengths* $b_1, \dots, b_n$ *is equal to* $N_{g,n}(b_1, \dots, b_n)$ *up to the lower order terms:*

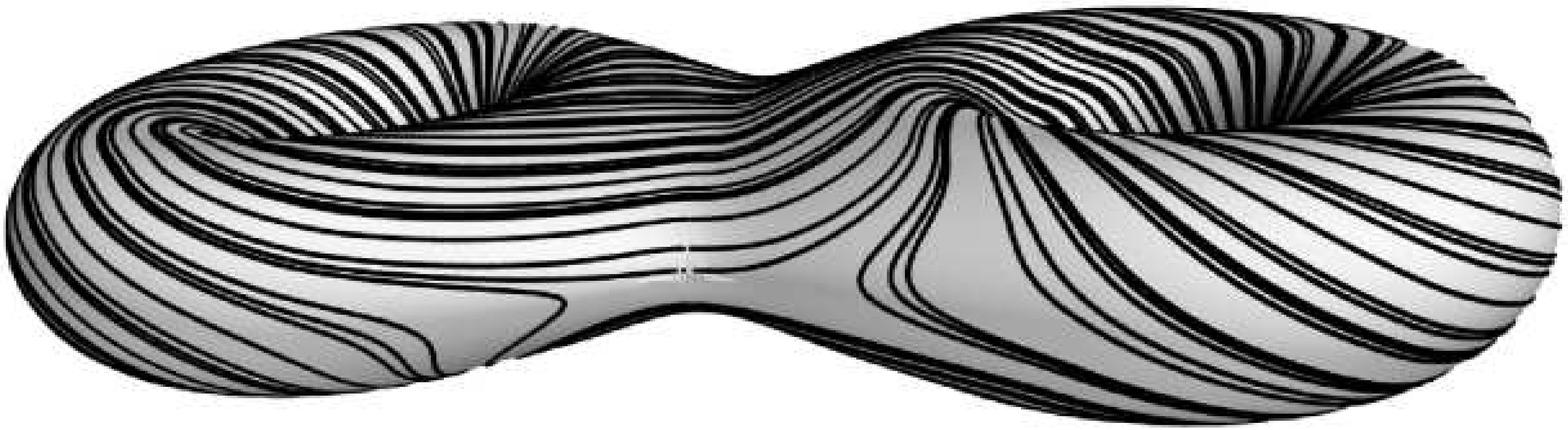
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \dots, b_n) = N_{g,n}(b_1, \dots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ *denote the set of (nonisomorphic) trivalent ribbon graphs* Γ *of genus* g *and with* n *boundary components.*

This Theorem is an important part of Kontsevich's proof of Witten's conjecture.

Shape of a random multicurve on a surface of large genus.

What shape has a random simple closed multicurve?



Picture from the book of Danny Calegari

Questions.

• *With what probability a random primitive multicurve on a surface of genus g slices the surface into $1, 2, 3, \dots$ connected components?*

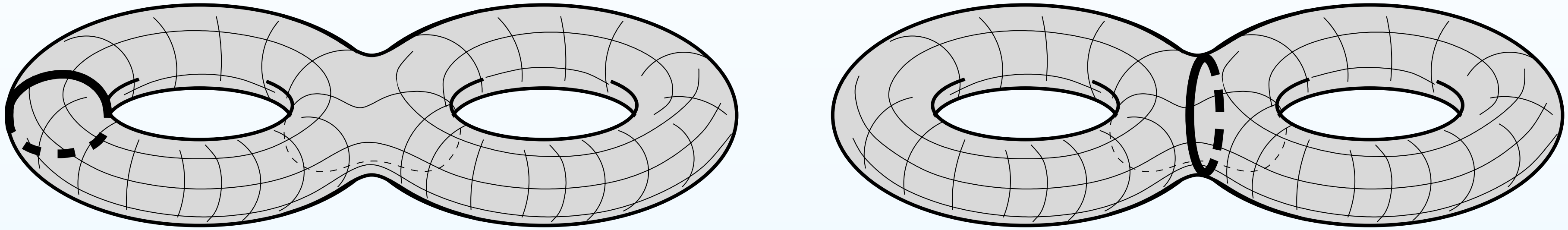
• *With what probability a random multicurve $m_1\gamma_1 + m_2\gamma_2 + \dots + m_k\gamma_k$ has $k = 1, 2, \dots, 3g-3$ primitive connected components $\gamma_1, \dots, \gamma_k$?*

• *What are the typical weights $m_1, \dots, m_k$?*

• *What is the shape of a random multicurve on a surface of large genus?*

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$



$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

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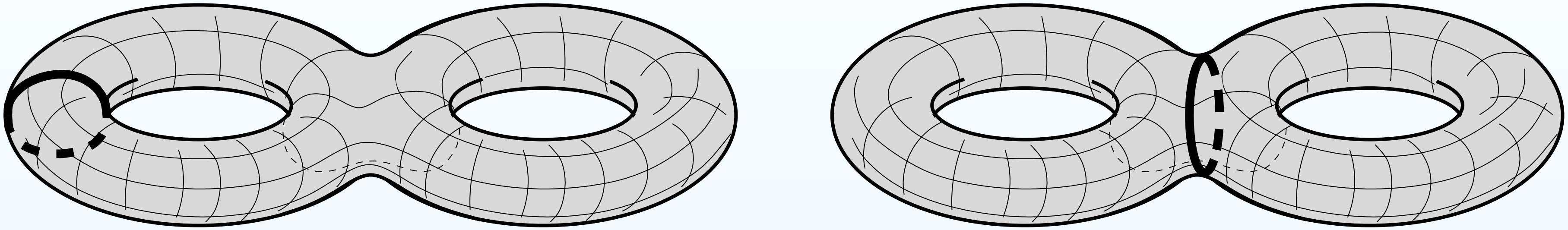
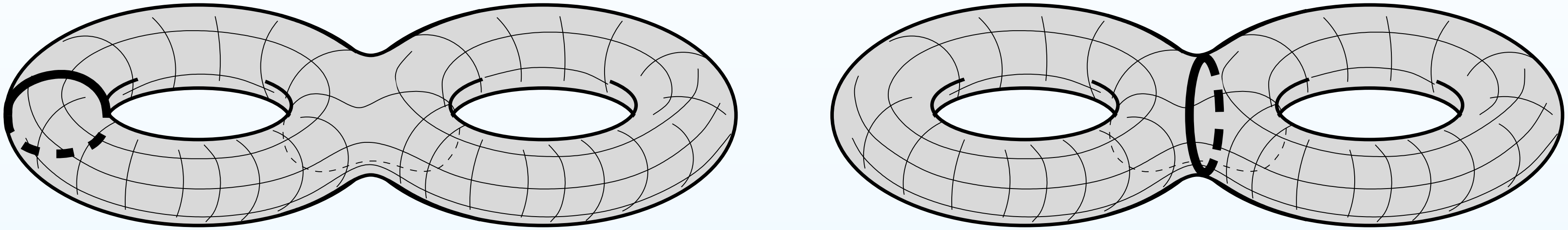


$$\lim_{L \to +\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Mirzakhani's calculation.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$



$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell; also nailed by C. Ball. Most recently it was independently confirmed independently by K. Rafi–J. Souto and by A. Wright by methods independent of ours.

Random simple closed curve rarely separates

Theorem. *A random simple closed curve on a surface of large genus separates the surface very rarely. Namely:*

$$\frac{c(\gamma_{sep})}{c(\gamma_{nonsep})} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g} \quad \text{as } g \to +\infty \,,$$

An integer multiple $m\gamma$ of a simple closed curve γ has weight m with probability $\frac{1}{m^{6g-6}} \cdot \frac{1}{\zeta(6g-6)}$. Thus, a random one-cylinder square-tiled surface of large genus has height 1 with probability very close to 1.

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Idea of the proof. Frequencies of separating simple closed curves are expressed in terms of the intersection numbers which admit closed expression:

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g\, g!}\,.$$

Frequencies of non-separating simple closed curves are expressed in terms of

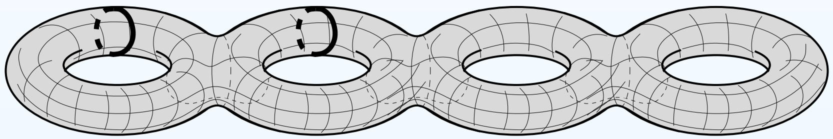
$$\int_{\overline{\mathcal{M}}_{g,2}} \psi_1^k \psi_2^{3g-1-k}$$

for which we obtain large genus asymptotics uniform for all k in fixed genus g.

The study of large genus asymptotics of correlators was initiated by M. Mirzakhani and P. Zograf.

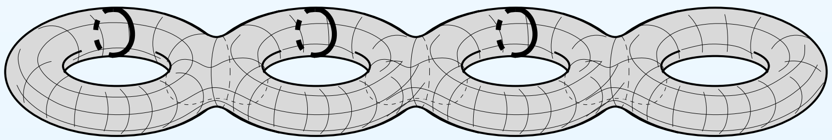
Conjectural shape of a random multicurve (random square-tiled surface) on a surface of large genus

Conditional Theorem. *The reduced multicurve* $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ *associated to a random integral multicurve* $m_1\gamma_1 + \ldots m_k\gamma_k$ *separates the surface with probability which tends to zero as genus* g *grows. Moreover, for* $g \gg 1$, $\gamma_{reduced}$ *has one of the following topological types with probability exceeding* 0.99*:*



$\frac{\log(g)}{2} - 3\sqrt{\frac{\log(g)}{2}}$ components

$\cdots$ $\cdots$



$\frac{\log(g)}{2} + 3\sqrt{\frac{\log(g)}{2}}$ components

More precisely: the distribution of probability that a random multicurve γ *has* k *components tends to the Poisson distribution* $\frac{1}{e^\lambda}\frac{\lambda^{k-1}}{(k-1)!}$ *with parameter* $\lambda = \frac{\log(6g-6)+\gamma}{2} + \log(2) - 1.$

A random square-tiled surface (without conical points of angle π*) of large genus has about* $\frac{\log(g)}{2}$ *cylinders, and all its conical points sit at the same level.*

Weights of a random multicurve (heights of cylinders of a ransom square-tiled surface)

Conditional Theorem. *A random integer multicurve* $m_1\gamma_1 + \cdots + m_k\gamma_k$ *with bounded number* k *of primitive components is reduced (i.e.,* $m_1 = \cdots = m_k = 1$*) with probability which tends to* 1 *as* $g \to +\infty$*.*

In other terms, if we consider a random square-tiled surface with at most K *cylinders, the heights of all cylinders would very likely be equal to* 1 *for* $g \gg 1$*.*

Remark. This generalizes the observation that when $k = 1$ the probability that a random integer multiple $m_1\gamma_1$ of a simple closed curve γ_1 is primitive (i.e. $m_1 = 1$) is exactly $\frac{1}{\zeta(6g-6)}$.

Conditional Theorem. *A general random integer multicurve* $m_1\gamma_1 + \cdots + m_k\gamma_k$ *type is reduced (i.e.,* $m_1 = \cdots = m_k = 1$*) with probability which tends to* $\frac{\sqrt{2}}{2}$ *as genus grows.*

In other words, for about 70% *of square-tiled surfaces of large genus, the heights of all cylinders are equal to* 1*.*

Two basic conjectures

These and other conjectures follow from the following two basic ones.

Basic Conjecture 1. *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol}\mathcal{Q}_g \stackrel{?}{\sim} \frac{4}{\pi}\cdot\left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g \to +\infty\,.$$

Basic Conjecture 2. *The following* ***uniform*** *asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\dots\psi_n^{d_n} =$$

$$= \frac{1}{24^g}\cdot\frac{(6g-5+2n)!}{g!\,(3g-3+n)!}\cdot\frac{d_1!\dots d_n!}{(2d_1+1)!\cdots(2d_n+1)!}\cdot\big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d})$ becomes ***uniformly*** *small for all $n \le 2\log(g)$ and all partitions $\boldsymbol{d}$, $d_1+\dots+d_n = 3g-3+n$, as $g \to +\infty$.*