Random square-tiled surfaces, Masur-Veech volumes and meanders

Anton Zorich

(after a joint work with V. Delecroix, E. Goujard and P. Zograf)

Field Theory, Geometry and Statistical Mechanics
in honor of Alexander Gorsky and Senya Shlosman
Independent University of Moscow and Poncelet Center

October 21, 2021

Formula for the Masur–Veech volume of the moduli space of quadratic differentials

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \dots, P_n \in C$ is a complex orbifold of complex dimension $3g - 3 + n$.

Choose index i in $\{1, \dots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$. The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g - 3 + n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in* 2*-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g-3+n$.

Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$. The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in 2-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

Recursive relations

Initial data: $\langle\tau_0^3\rangle = 1, \qquad \langle\tau_1\rangle = \frac{1}{24}\,.$

String equation:

$$\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = \langle\tau_{d_1-1}\ldots\tau_{d_n}\rangle_{g,n} + \cdots + \langle\tau_{d_1}\ldots\tau_{d_n-1}\rangle_{g,n}\,.$$

Dilaton equation:

$$\langle\tau_1\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = (2g-2+n)\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n}\,.$$

Virasoro constraints (in Dijkgraaf–Verlinde–Verlinde form; $k \geq 1$):

$$\langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\left[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g\right.$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}$$

$$\left.+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\sum_{\{1,\ldots,n\}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\right].$$

Uniform large genus asymptotics

We stated in August 2019 a conjecture which was proved by Amol Aggarwal already in April 2020.

Theorem (Aggarwal). *The following **uniform** asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1 + \dots + d_n = 3g-3+n$*, as* $g \to +\infty$*.*

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}}\,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}}\,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b)\,.$$

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

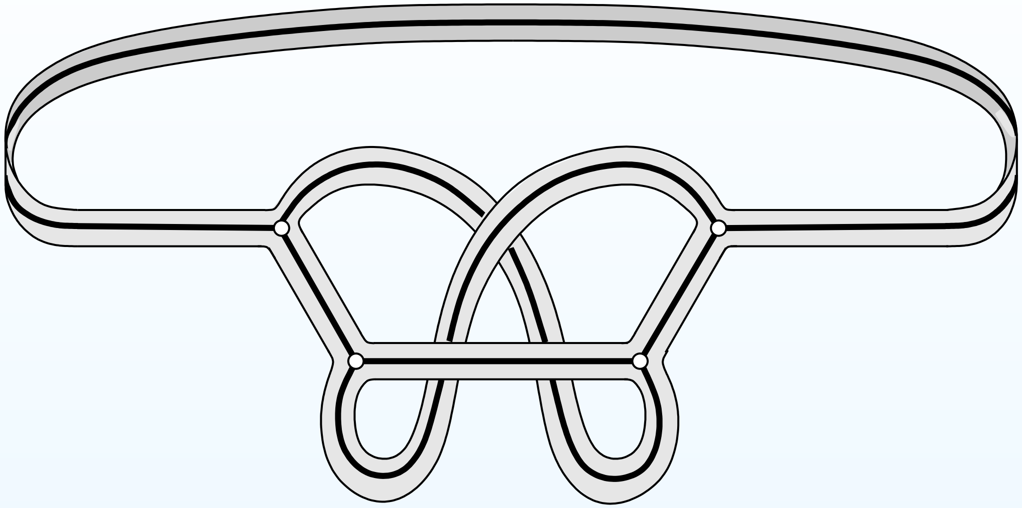
$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \ : \quad \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 1$ with $n = 2$ boundary components. If we assigned lengths to all edges of the core graph, each boundary component gets induced length, namely, the sum of the lengths of the edges which it follow.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorem of Kontsevich counts them.

Kontsevich's count of metric ribbon graphs

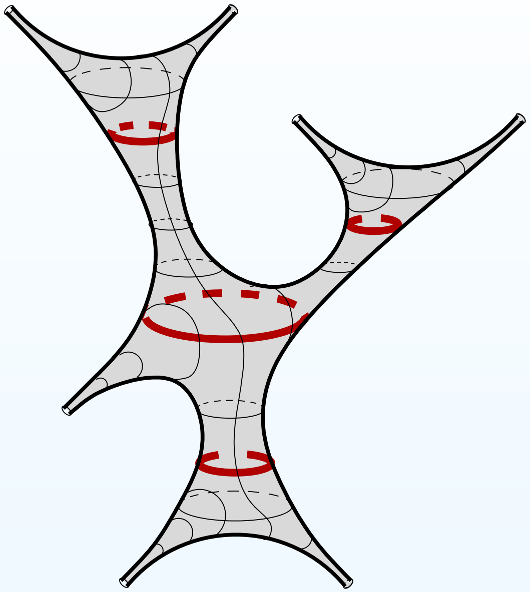
Theorem (M. Kontsevich; in this form — P. Norbury). *Consider a collection of positive integers $b_1, \ldots, b_n$ such that $\sum_{i=1}^n b_i$ is even. The weighted count of genus g connected trivalent metric ribbon graphs Γ with integer edges and with n labeled boundary components of lengths $b_1, \ldots, b_n$ is equal to $N_{g,n}(b_1, \ldots, b_n)$ up to the lower order terms:*

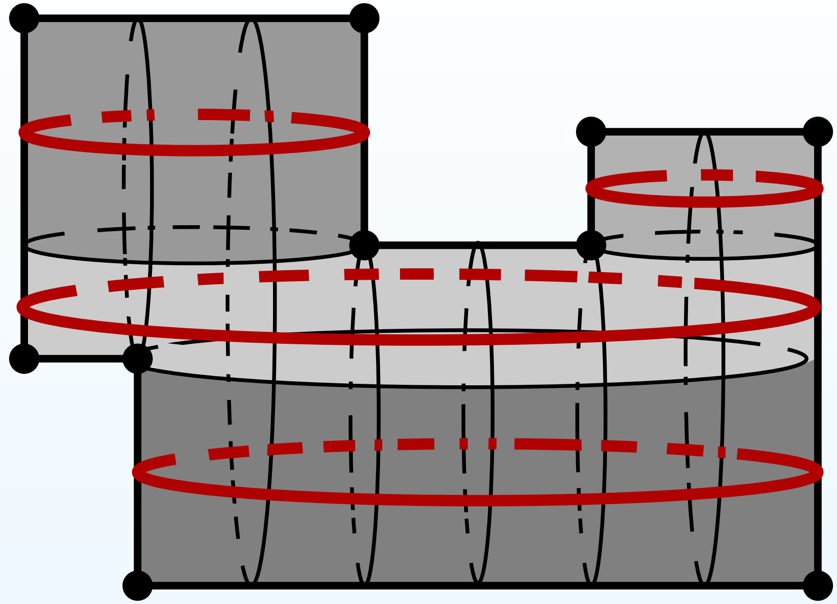
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n) = N_{g,n}(b_1, \ldots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ denote the set of (nonisomorphic) trivalent ribbon graphs Γ of genus g and with n boundary components.

This Theorem is an important part of Kontsevich's proof of Witten's conjecture.

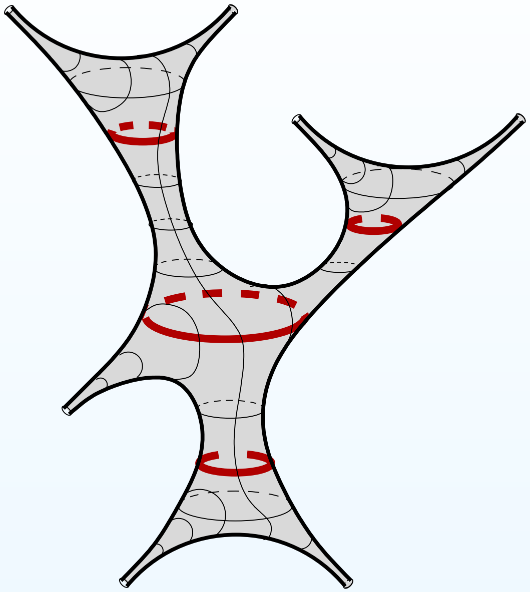
Stable graph associated to a square-tiled surface

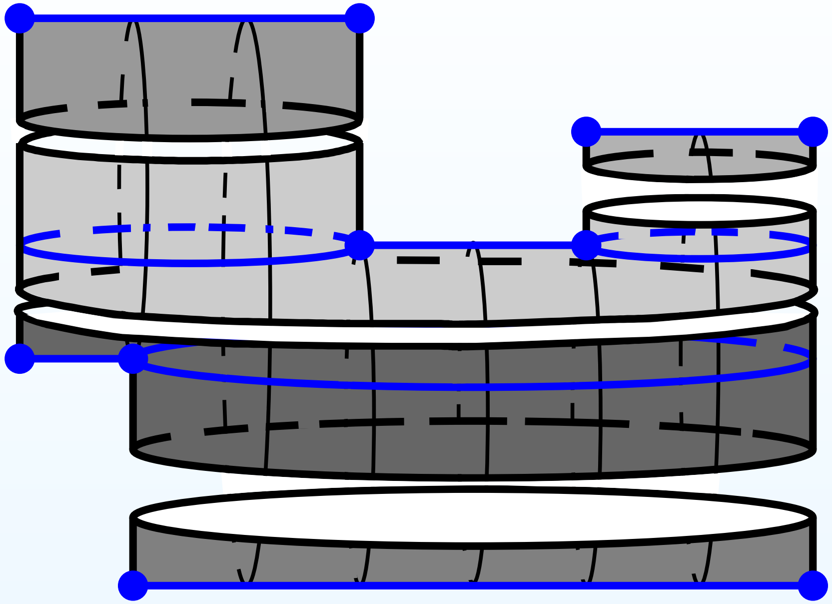


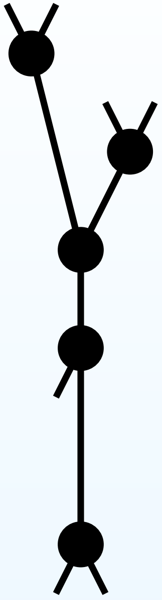


Having a square-tiled surface we associate to it a topological surface S on which we mark all "corners" with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders.

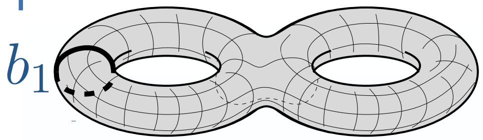
Stable graph associated to a square-tiled surface



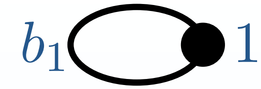


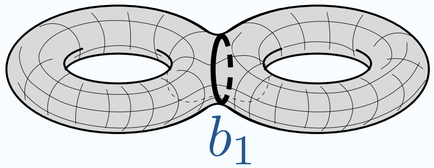


Having a square-tiled surface we associate to it a topological surface S on which we mark all "corners" with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. The associated *stable graph* Γ is the dual graph to the multicurve. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by "legs" — half-edges of the dual graph.

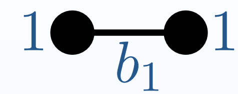


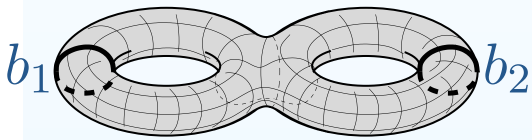
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



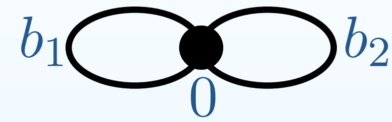


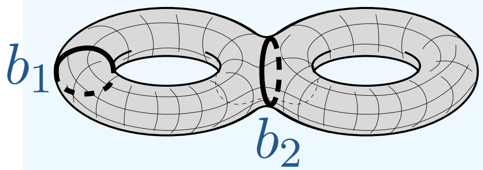
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



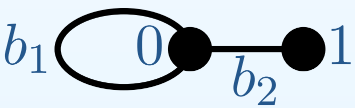


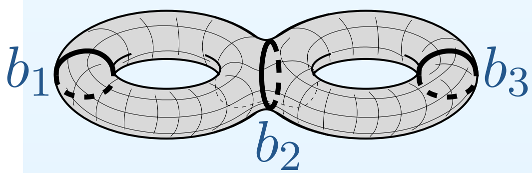
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



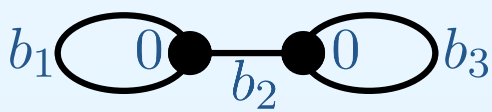


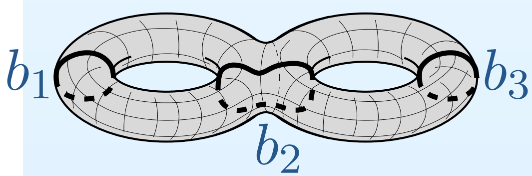
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



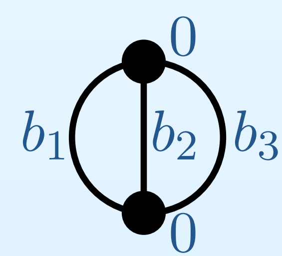


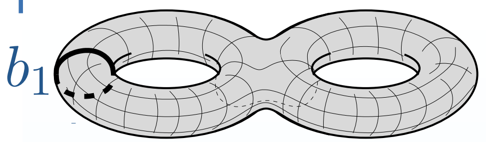
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



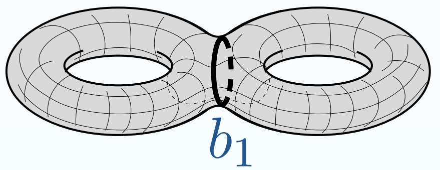


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$

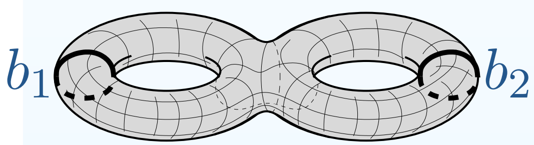




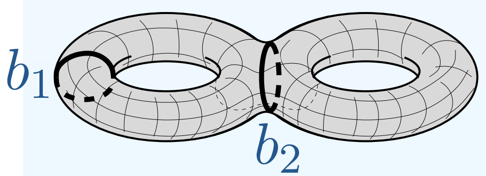
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) \qquad = \frac{1}{2} \cdot b_1 \left(\frac{1}{384} (2b_1^2)(2b_1^2) \right)$$



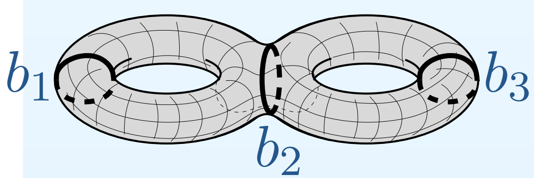
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) \qquad = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$



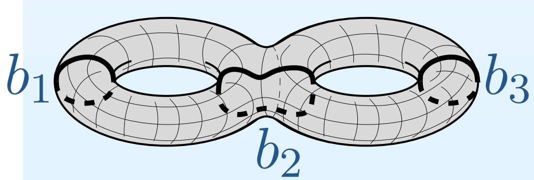
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) \qquad = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4} (2b_1^2 + 2b_2^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) \qquad = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$

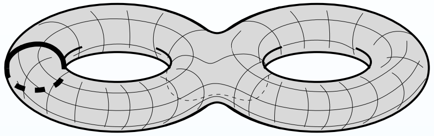


$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) \qquad = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

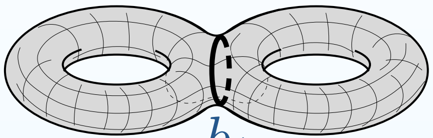


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) \qquad = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

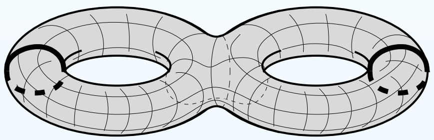
Volume of $\mathcal{Q}_2$



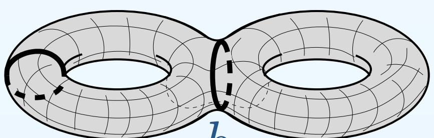
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



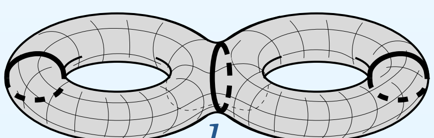
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



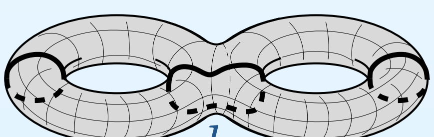
$$\frac{1}{16}(b_1^3 b_2 + +b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16}b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$

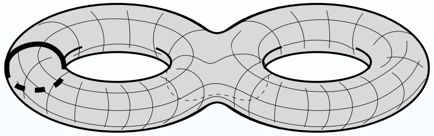


$$\frac{1}{24}b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

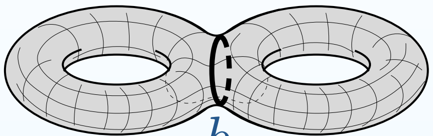
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\Big(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\Big)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

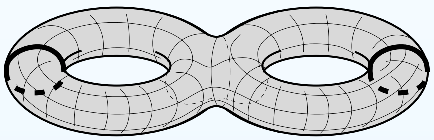
Volume of $\mathcal{Q}_2$



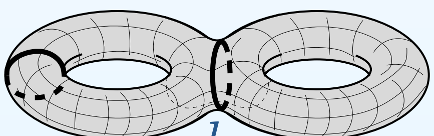
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



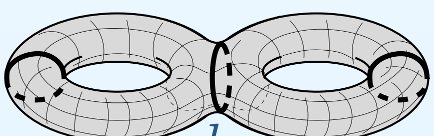
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



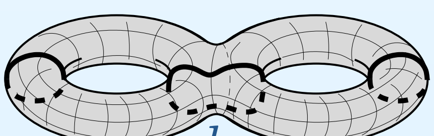
$$\frac{1}{16}(b_1^3 b_2 + {} + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix, Goujard, Zograf, Zorich). *The Masur–Veech volume $\operatorname{Vol}\mathcal{Q}_{g,n}$ of the moduli space of meromorphic quadratic differentials with n simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs } \Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of } \Gamma - 1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e\cdot\prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\;,$$

The partial sum for fixed number k of edges gives the contribution of k-cylinder square-tiled surfaces.

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix, Goujard, Zograf, Zorich). *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot\sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}}\frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e\cdot\prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right),$$

Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

Mirzakhani's count of simple closed geodesics

Frequencies of multicurves

Theorem (M. Mirzakhani, 2008). *For any integral multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ the number $s_X(L,\gamma)$ of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

Frequencies of multicurves

Theorem (M. Mirzakhani, 2008). *For any integral multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ the number $s_X(L,\gamma)$ of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

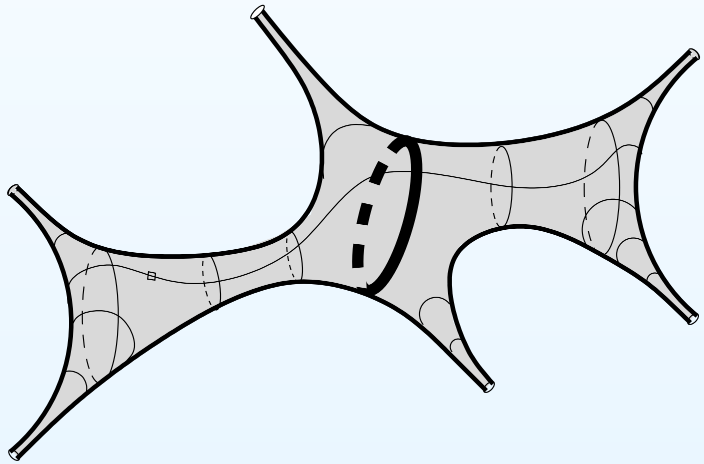
Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

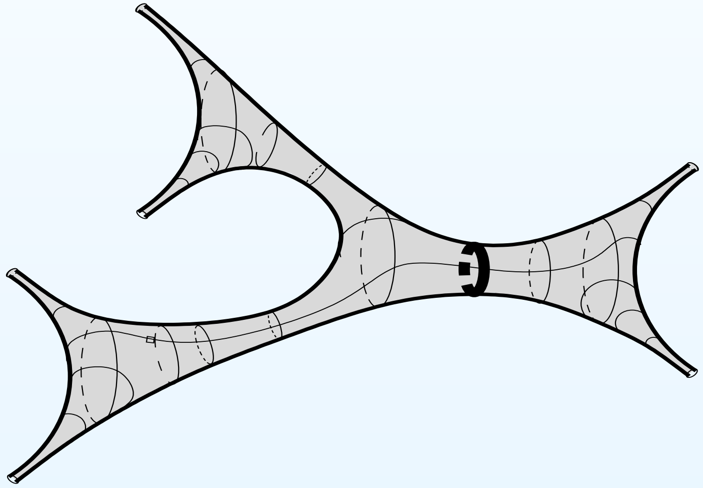
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

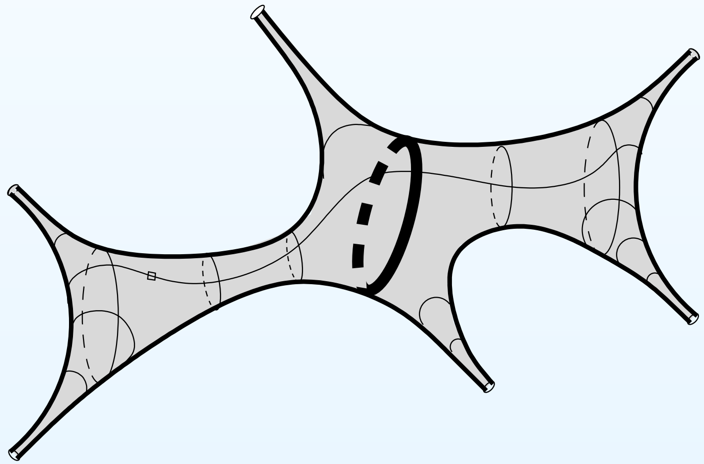
A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.

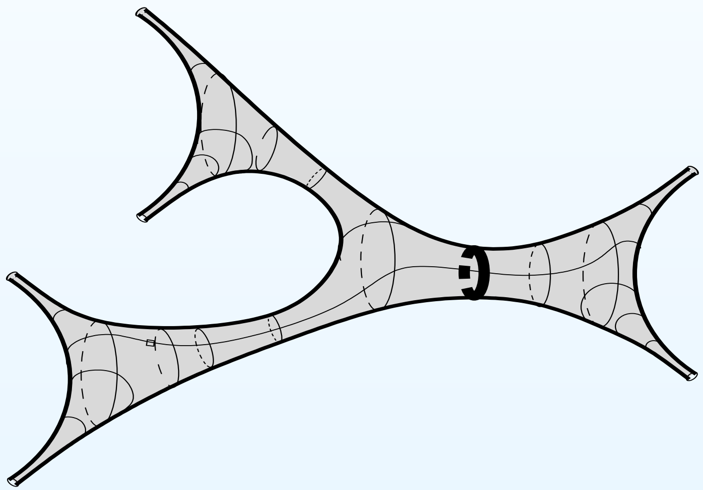




Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.

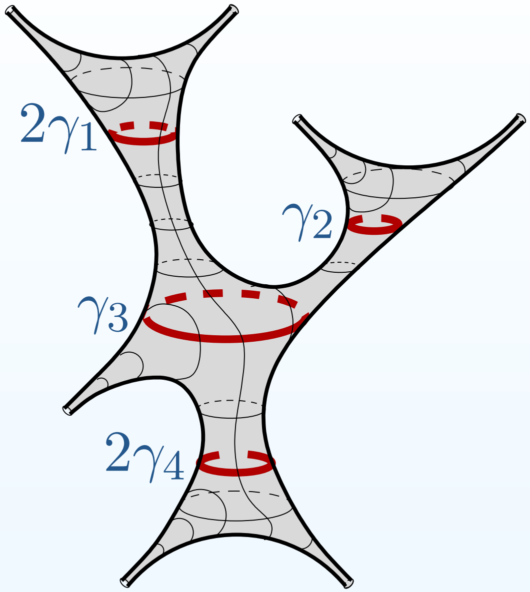


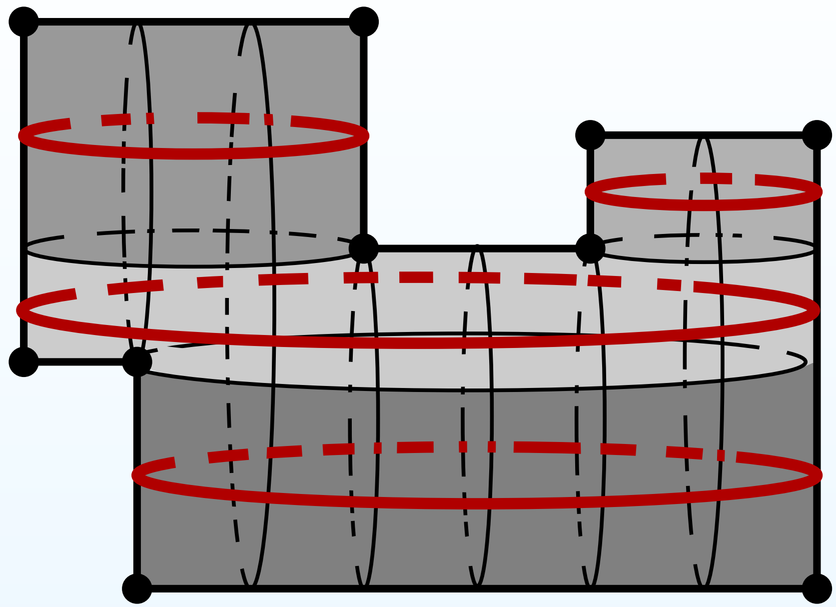


Example (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

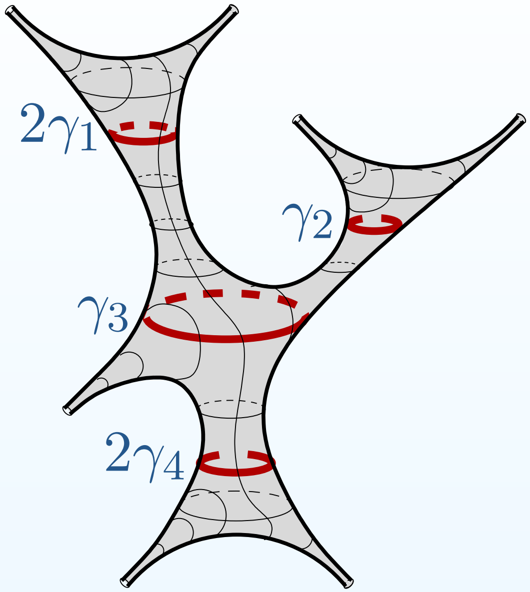
Hyperbolic and flat geodesic multicurves

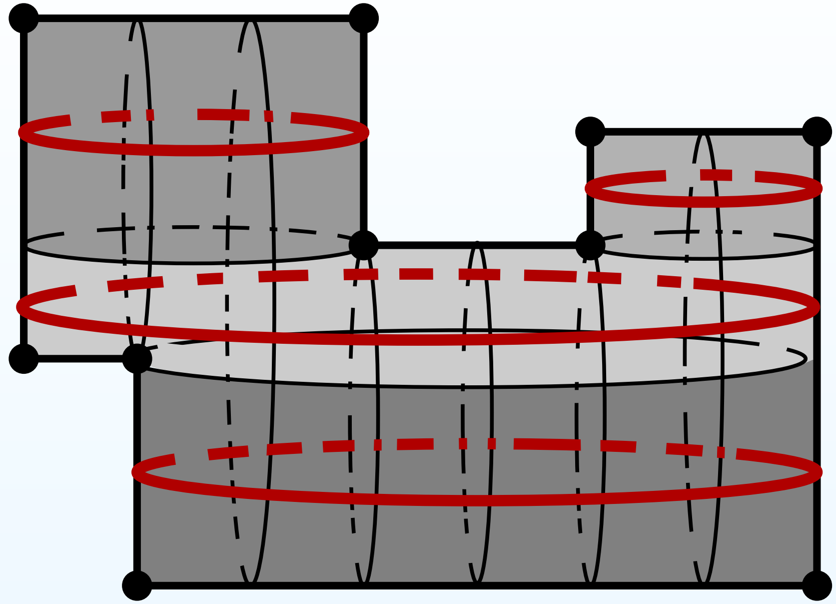




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves



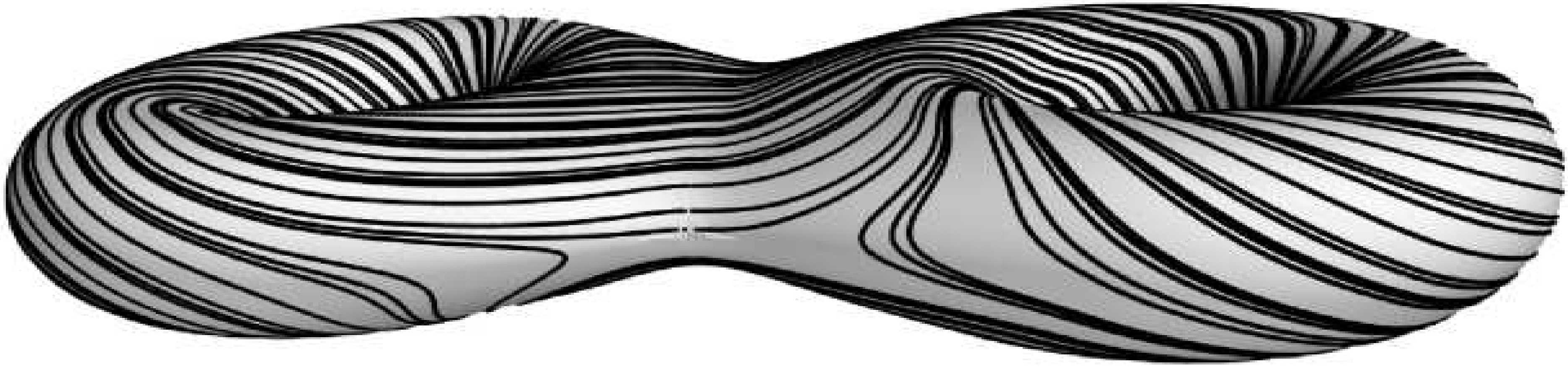


Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2018). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of* ***hyperbolic*** *multicurves coincides with the asymptotic frequency of simple closed* ***flat*** *geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Shape of a random multicurve on a surface of genus two

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

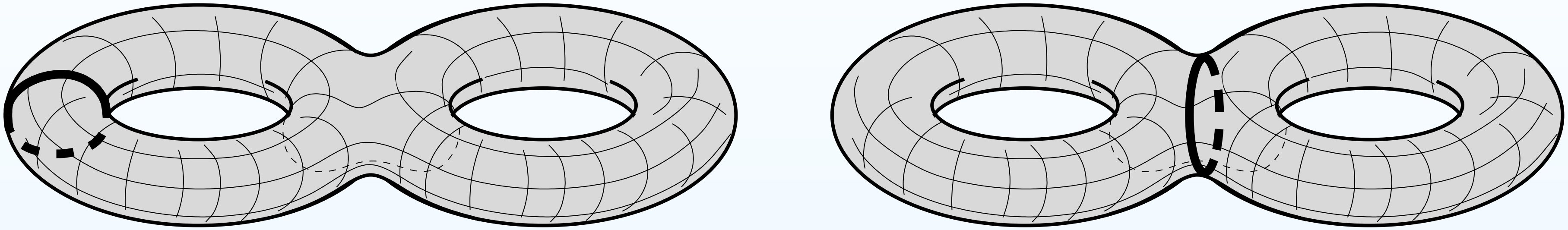
- *Which simple closed geodesics are more frequent: separating or non-separating?*

Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

- *With what probability that* $\gamma_{reduced}$ *slices the surface into* $1, ..., 2g-2$ *connected components?*
- *With what probability* $\gamma_{reduced}$ *has* $k = 1, 2, \ldots, 3g-3$ *primitive connected components* $\gamma_1, \ldots, \gamma_k$*?*

Separating versus non-separating simple closed curves in $g = 2$

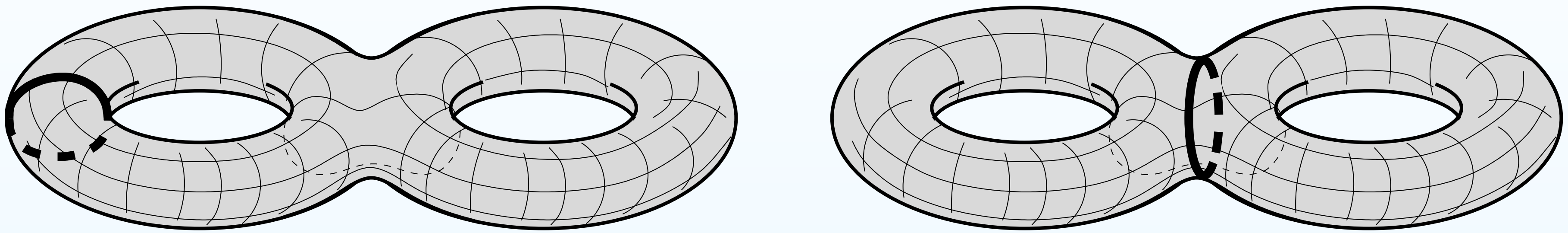
Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$



$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$

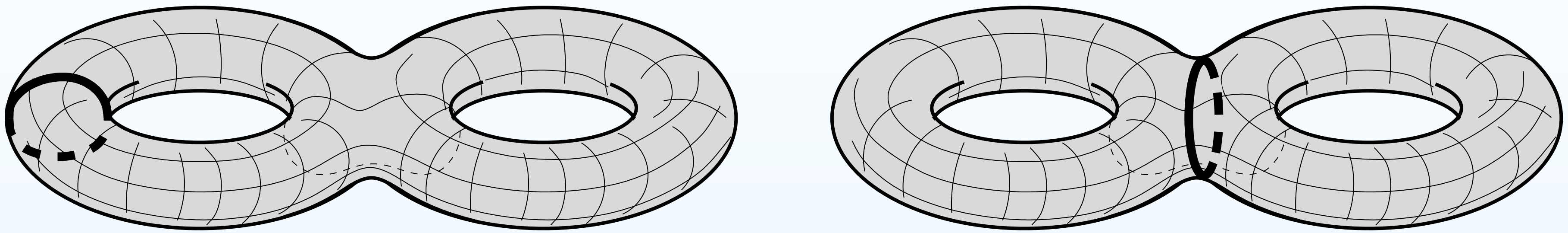


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Mirzakhani's calculation.

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (M. Mirzakhani, 2008). *Genus* $g = 2$

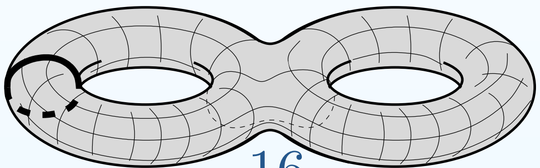


$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

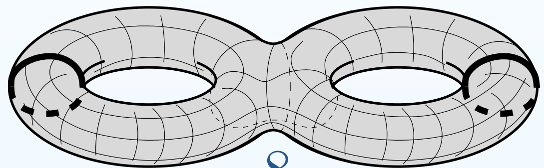
after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell. Most recently it was independently confirmed by V. Erlandsson, K. Rafi, J. Souto and by A. Wright by methods independent of ours.

Multicurves on a surface of genus two and their frequencies

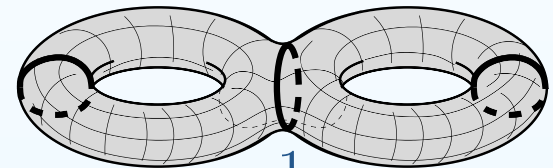
The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



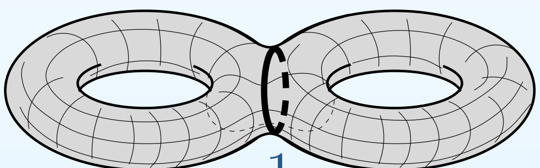
$\frac{16}{63}$



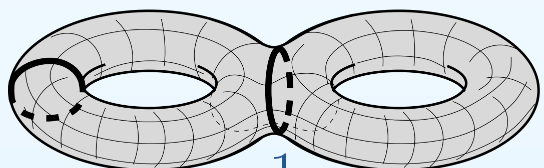
$\frac{8}{15}$



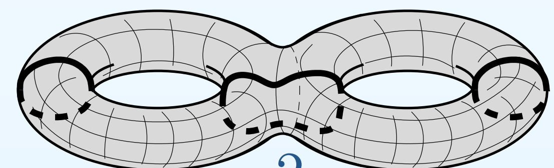
$\frac{1}{9}$



$\frac{1}{189}$



$\frac{1}{45}$

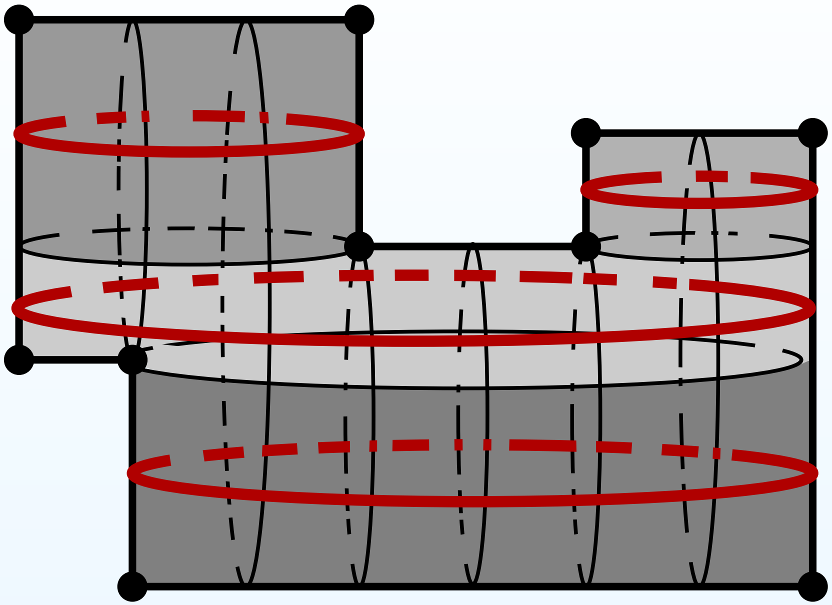


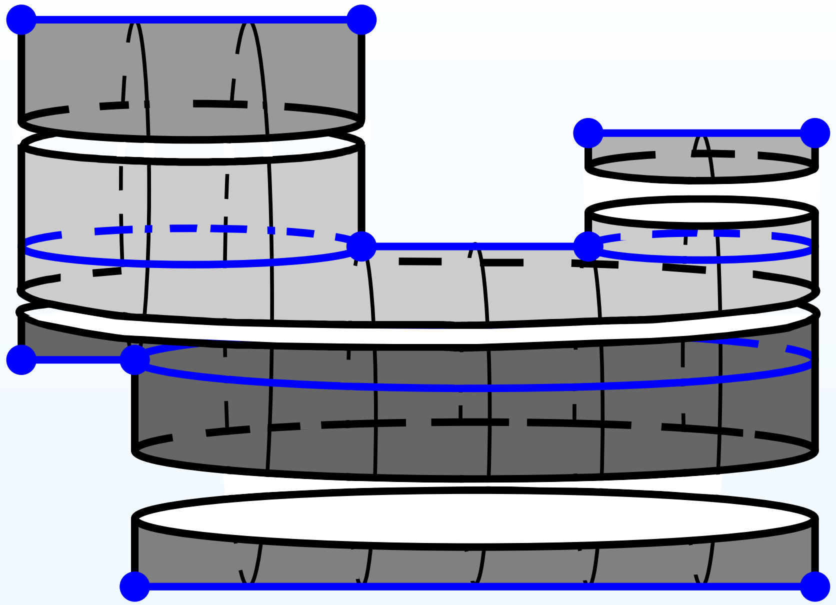
$\frac{2}{27}$

In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Shape of a random multicurve on a surface of large genus. Shape of a random square-tiled surface of large genus.

Shape of a random square-tiled surface of large genus?





Questions.

● *With what probability a random square-tiled surface* S *of genus* g *has* $1, 2, 3, \ldots$ *singular horizontal leaves (in blue on the right picture)?*

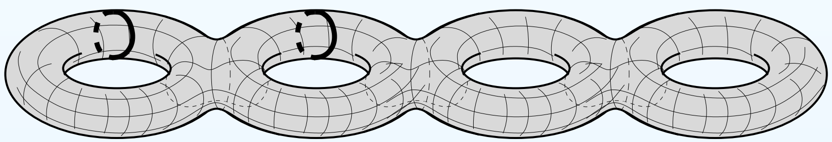
● *With what probability a random square-tiled surface* S *of genus* g *has* $K_g(S) = 1, 2, 3, \ldots, 3g-3$ *maximal horizontal cylinders (represented by red waist curves on the left picture)?*

● *What are the typical heights* $h_1, \ldots, h_k$ *of the cylinders?*

● *What is the shape of a random square-tiled surface of large genus?*

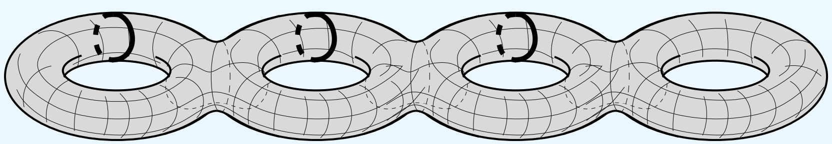
Shape of a random multicurve (random square-tiled surface) on a surface of large genus in simple words

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *The reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ associated to a random integral multicurve $m_1\gamma_1 + \ldots m_k\gamma_k$ separates the surface with probability which tends to zero as genus g grows. For large g, $\gamma_{reduced}$ has about $(\log g)/2$ components and has one of the following topological types*



$0.09\log(g)$ components

$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$



$0.62\log(g)$ components

$$\mathbb{P}\Big(0.09\log g < K_g(\gamma) < 0.62\log g\Big) = 1 - O\left((\log g)^{24} g^{-1/4}\right)\,.$$

A random square-tiled surface (without conical points of angle π) of large genus has about $\frac{\log(g)}{2}$ cylinders, and all its conical points sit at the same level.

Weights of a random multicurve (heights of cylinders of a ransom square-tiled surface)

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *A random integer multicurve $m_1\gamma_1 + \cdots + m_k\gamma_k$ with bounded number k of primitive components is reduced (i.e., $m_1 = \cdots = m_k = 1$) with probability which tends to 1 as $g \to +\infty$. In other terms, if we consider a random square-tiled surface with at most K cylinders, the heights of all cylinders would be very likely equal to 1 for $g \gg 1$.*

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *A general random integer multicurve $m_1\gamma_1 + \cdots + m_k\gamma_k$ is reduced (i.e., $m_1 = \cdots = m_k = 1$) with probability which tends to $\frac{\sqrt{2}}{2}$ as genus grows. More generally, all weights $m_1, \dots m_k$ of a random multicurve are bounded from above by an integer m with probability which tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

In other words, for more 70% of square-tiled surfaces of large genus, the heights of all cylinders are equal to 1.

However, the mean value of $m_1 + \ldots + m_k$ is infinite in any genus g.

Main Theorem (informally)

Main Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z.). *As g grows, the probability distribution p_g rapidly becomes, basically, indistinguishable from the distribution of the number of cycles in a (very explicitly nonuniform) random permutation. In particular, for any $k \in \mathbb{N}$ the difference of the k-th moments of the two distributions is of the order $O(g^{-1})$.*

Actually, we have an explicit asymptotic formula for all cumulants. For example

$$\mathbb{E}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 + o(1)\,,$$
$$\mathbb{V}(K_g) = \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1)\,,$$

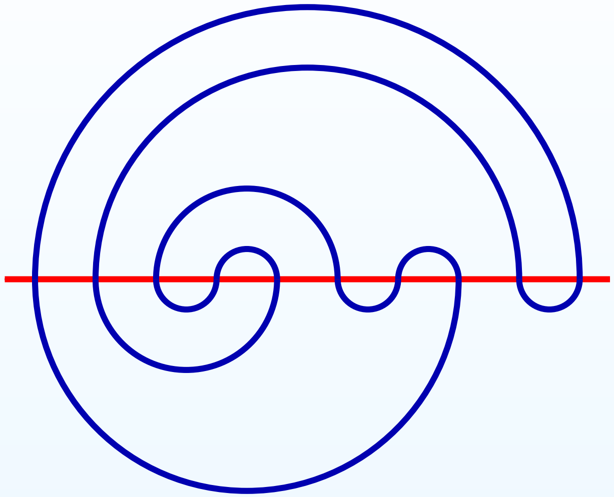
where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

Let $\lambda_{3g-3} = \log(6g-6)/2$. We have uniformly in $0 \le k \le 1.233 \cdot \lambda_{3g-3}$

$$\mathbb{P}\big(K_g(\gamma) = k{+}1\big) = e^{-\lambda_{3g-3}} \cdot \frac{\lambda_{3g-3}^k}{k!} \cdot \left(\frac{\sqrt{\pi}}{2\Gamma\left(1+\frac{k}{2\lambda_{3g-3}}\right)} + O\left(\frac{k}{(\log g)^2}\right)\right)\,.$$

Meanders count

Meanders and arc systems

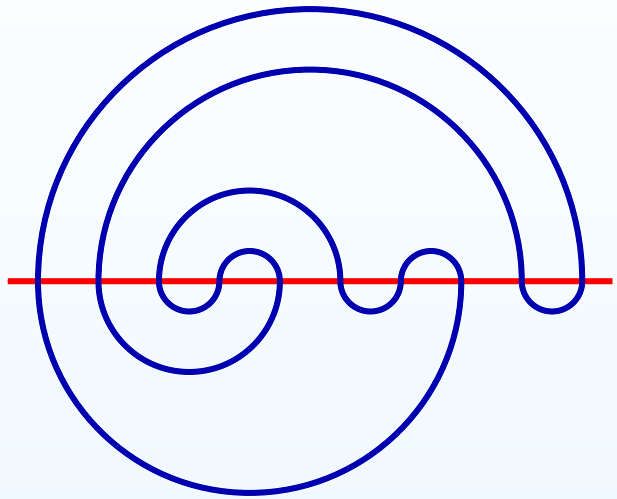


A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line.

According to S. Lando and A. Zvonkin the notion “meander” was suggested by V. Arnold though meanders were studied already by H. Poincaré.

Meanders appear in various contexts, in particular in mathematics, physics and biology.

Meanders and arc systems

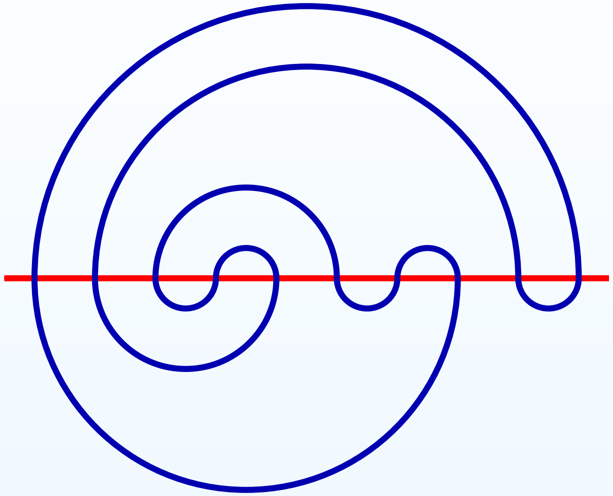
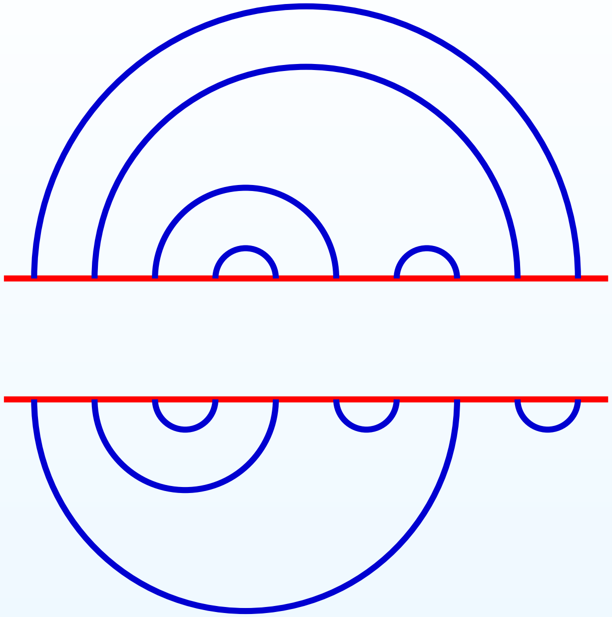
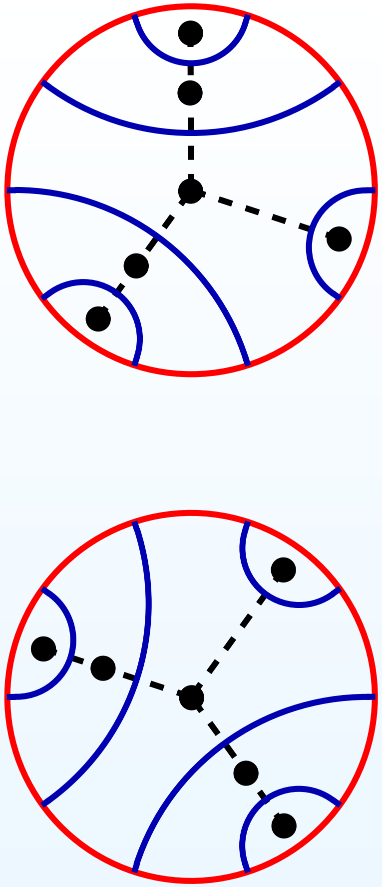


Conjecture (P. Di Francesco, O. Golinelli, E. Guitter, 1997). *The number of meanders with $2N$ crossings is asymptotic to*

$$const \cdot R^{2N} \cdot N^{\alpha} \,,$$

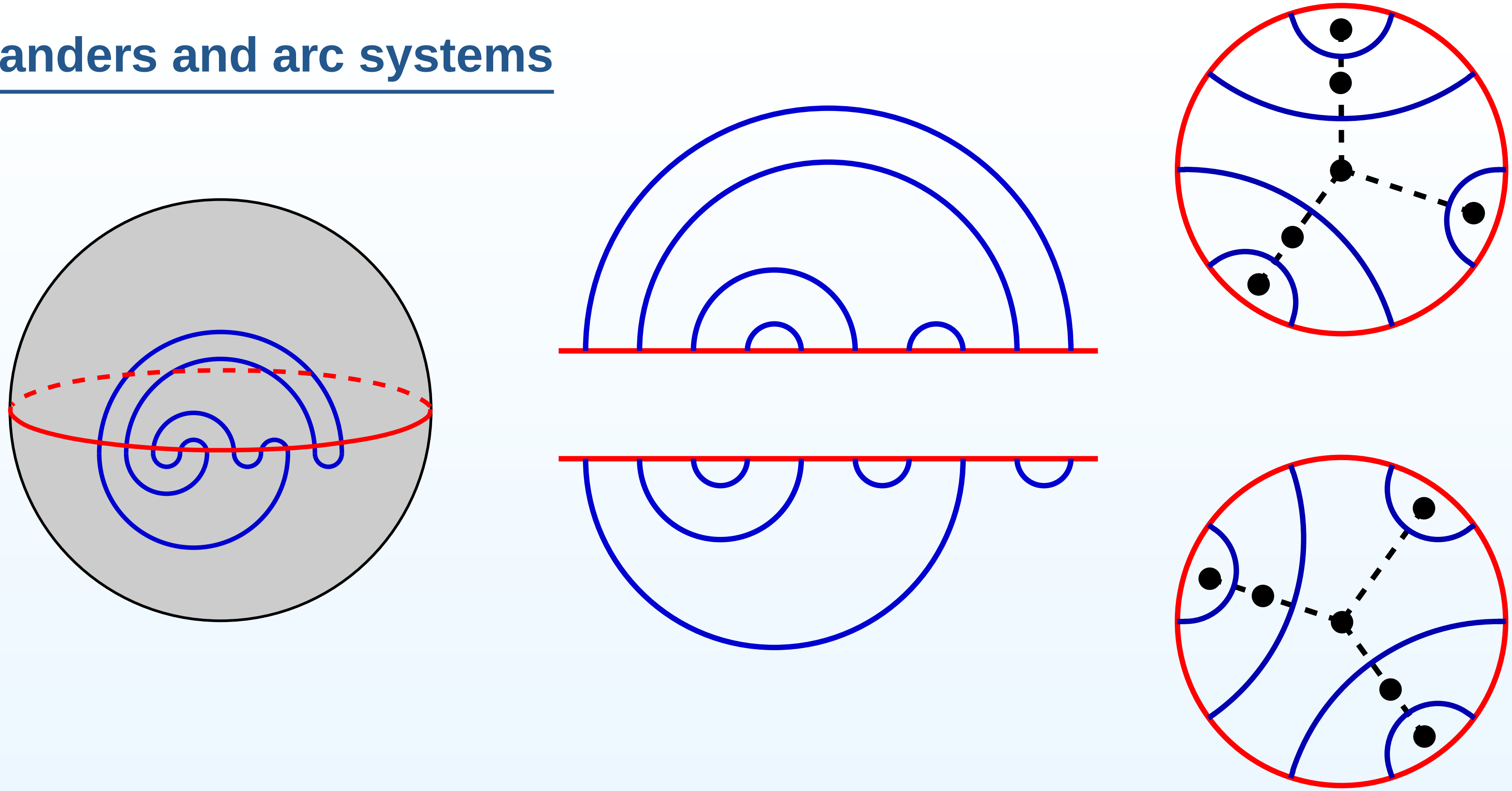
where $R^2 \approx 12.26$ (value is due to I. Jensen) and $\alpha = -\frac{29+\sqrt{145}}{12}$.

Meanders and arc systems

A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Meanders and arc systems

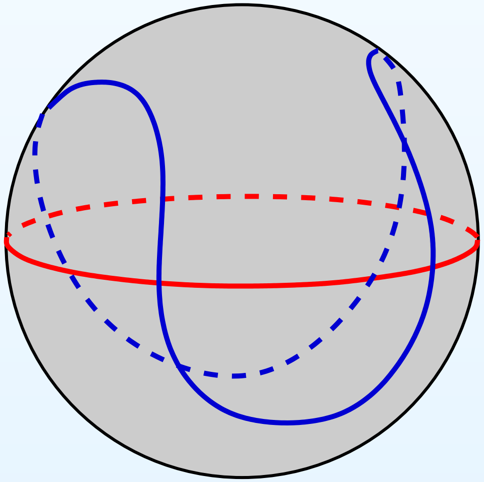


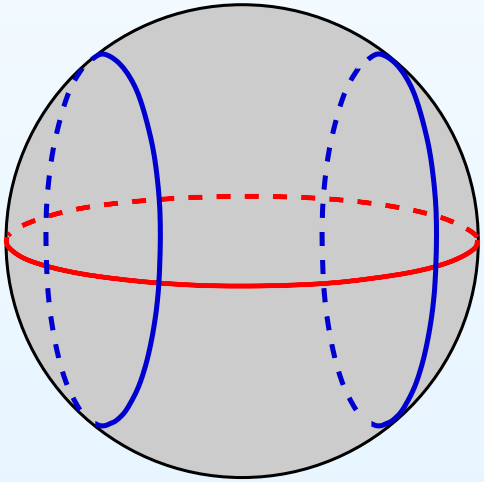
A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Compactifying the plane (left picture) with one point at infinity, or gluing together arc systems on the two discs (right picture) we get an ordered pair of smooth simple transversally intersecting closed curves on the sphere.

Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a *multicurve*, i.e. a curve with several connected components.

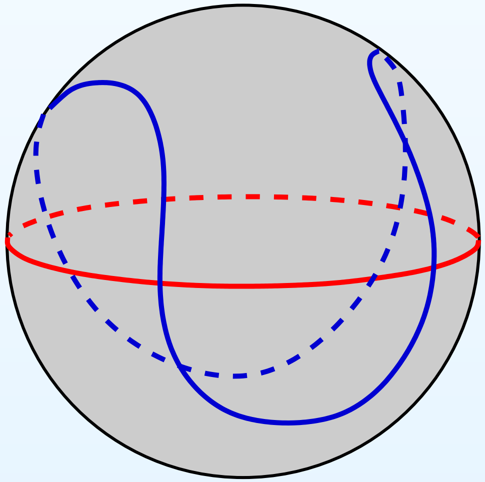


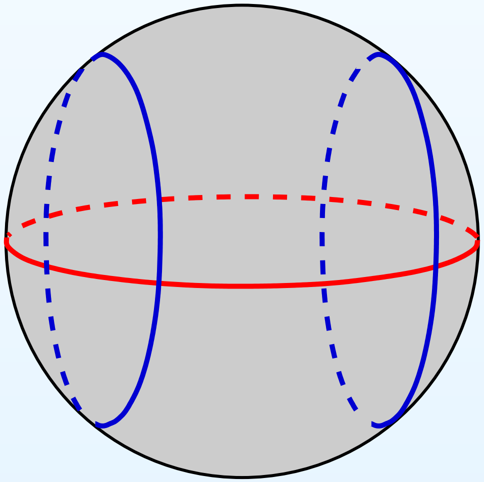


Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a *multicurve*, i.e. a curve with several connected components.





Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

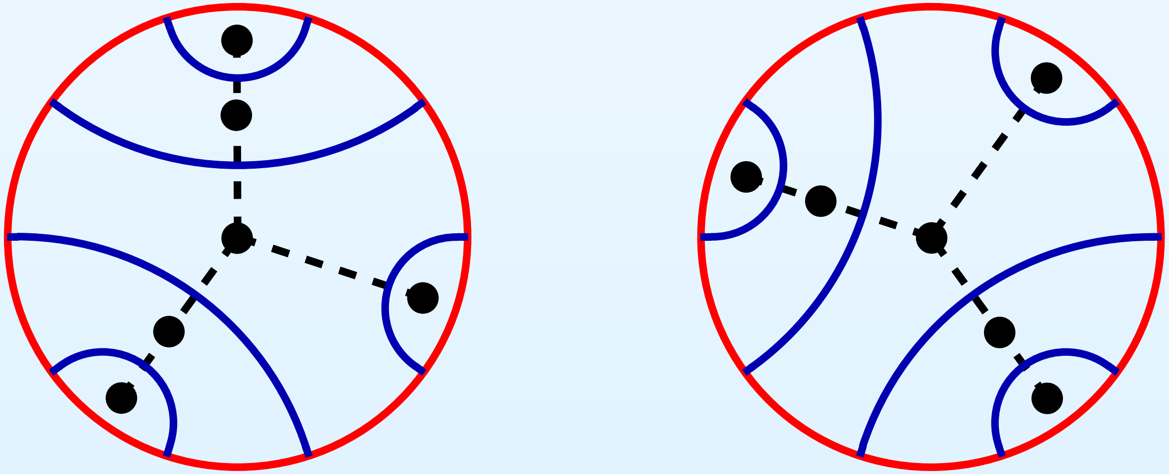
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



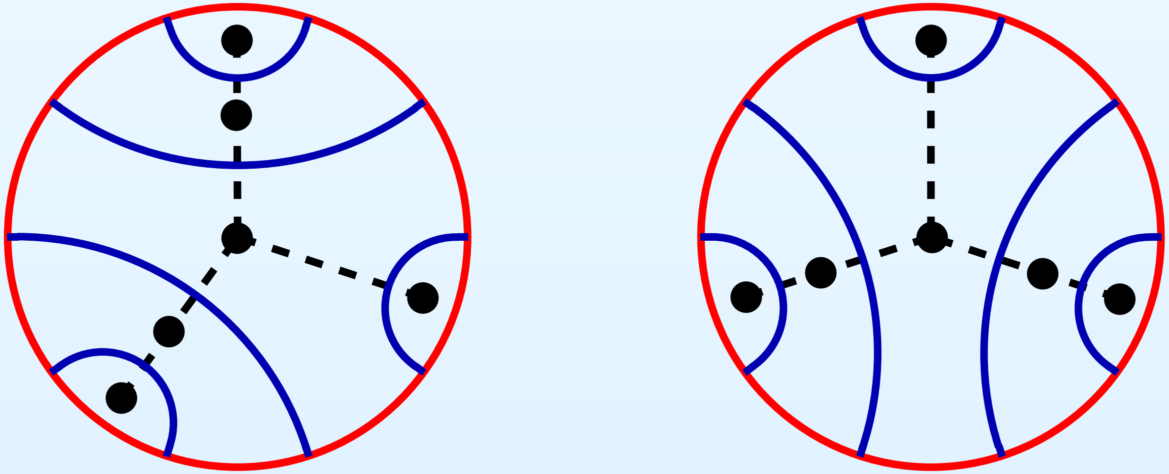
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



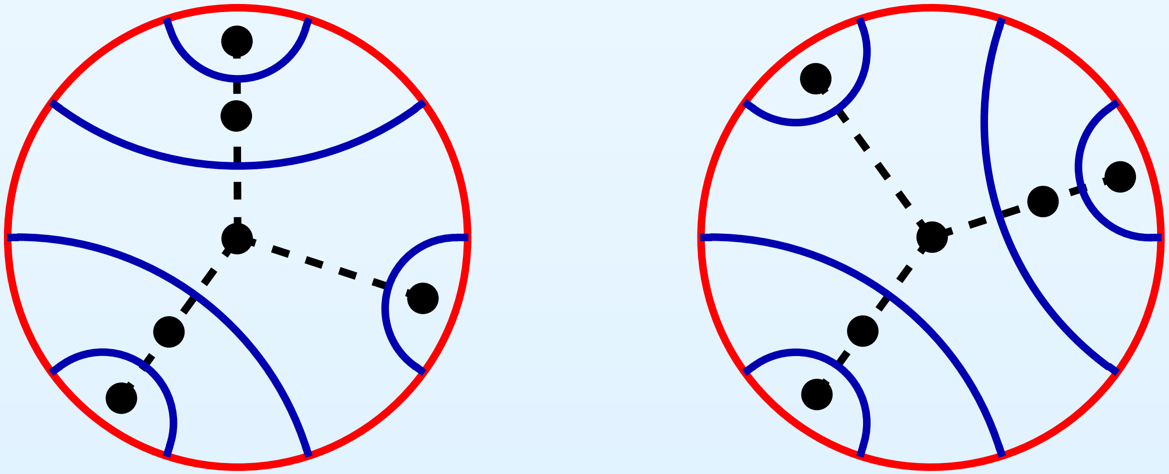
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



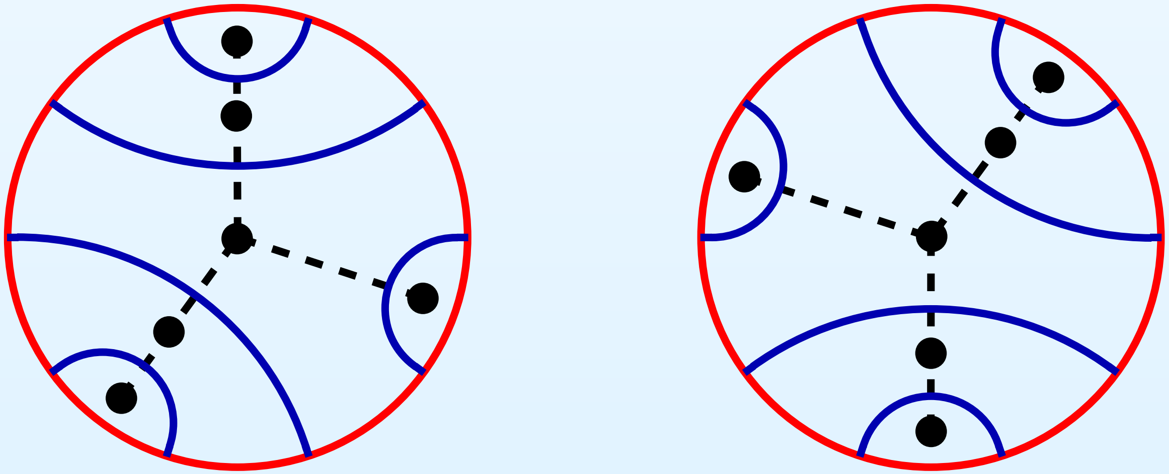
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\ n\text{-arc system of type } \mathcal{T}_{South};\ \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



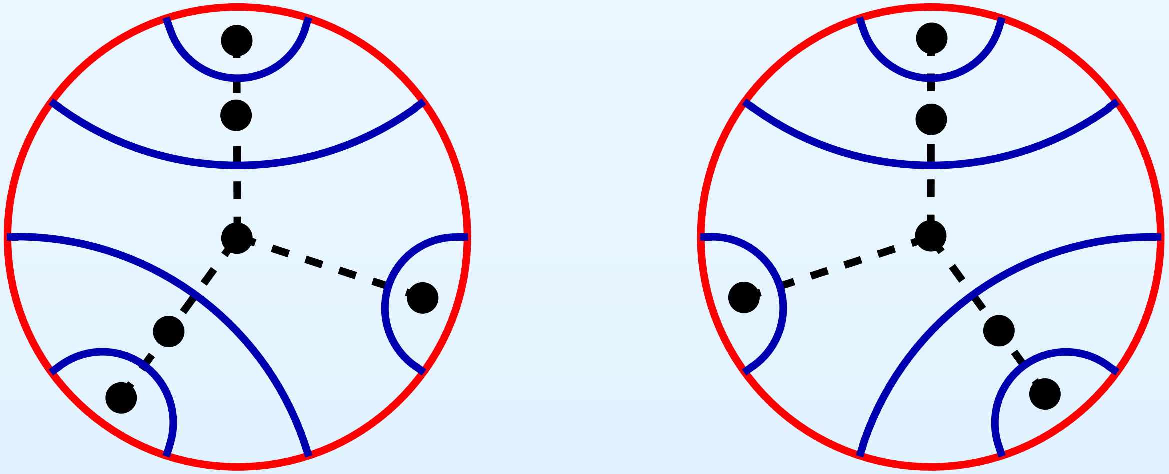
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}} .$$



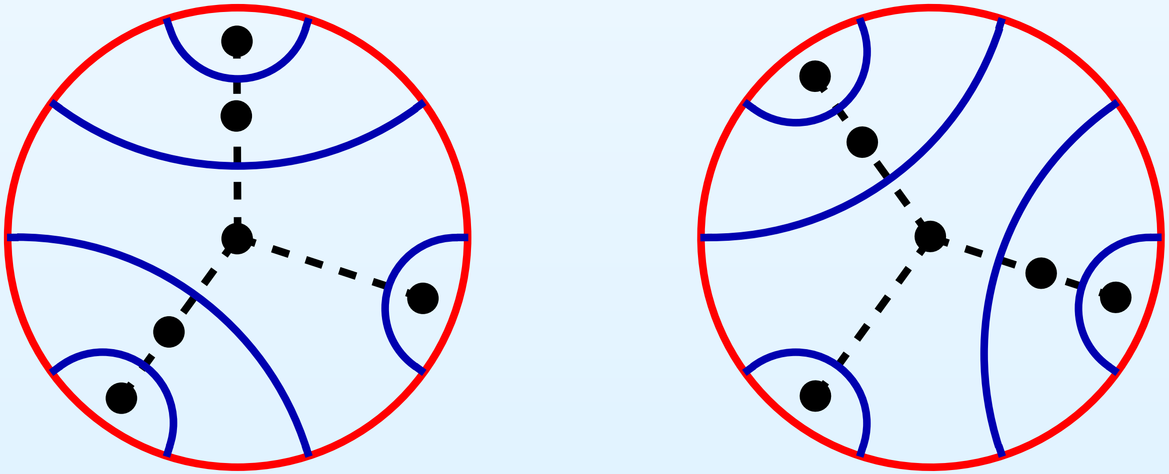
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\ n\text{-arc system of type } \mathcal{T}_{South};\ \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



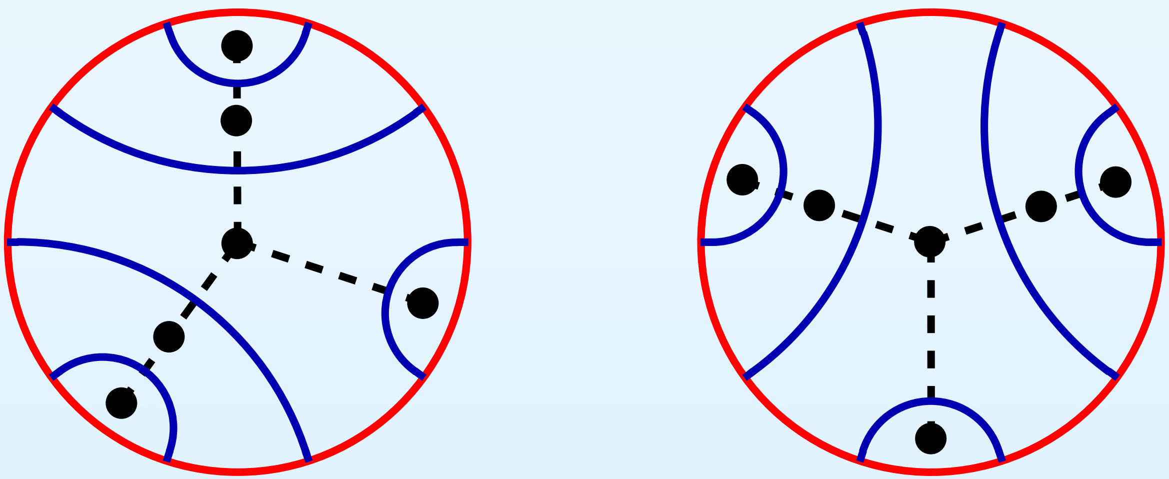
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



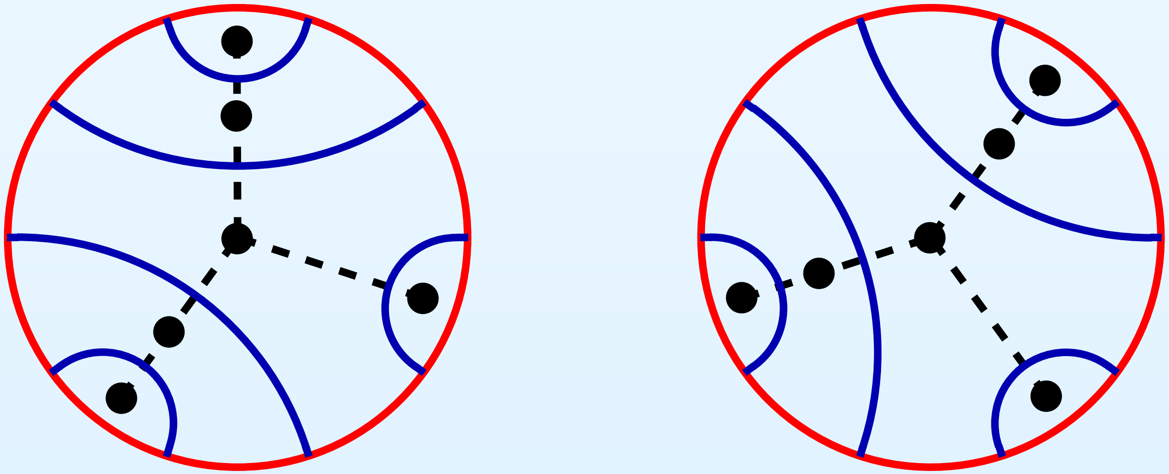
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



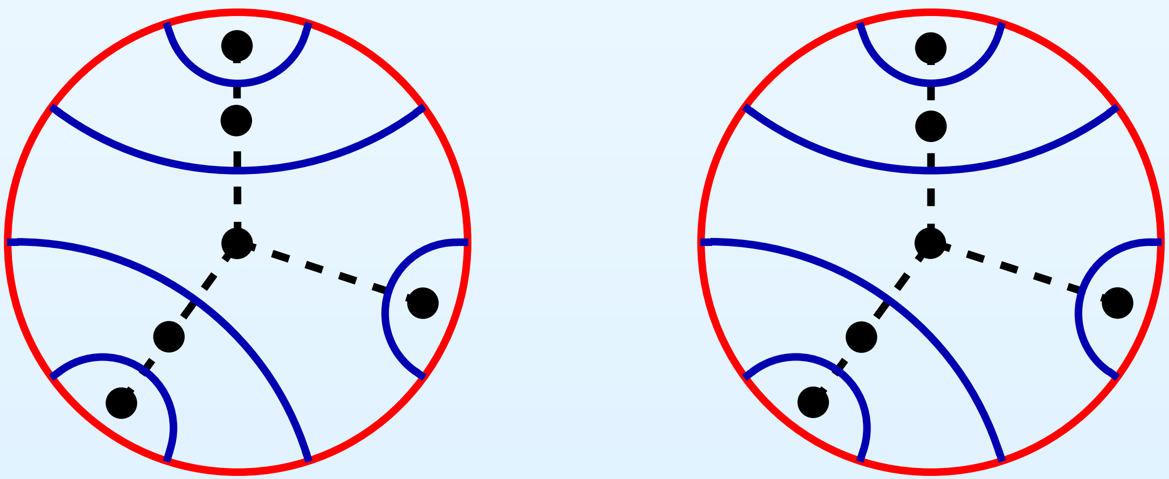
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



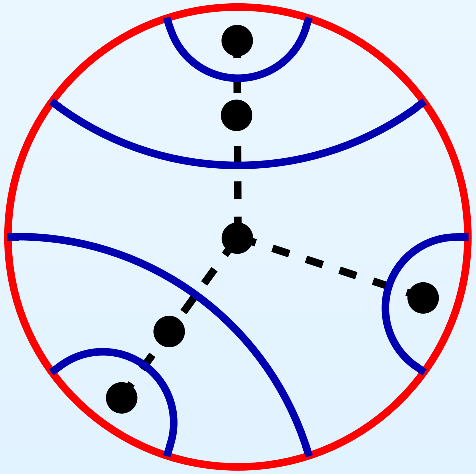
Asymptotic frequency of meanders

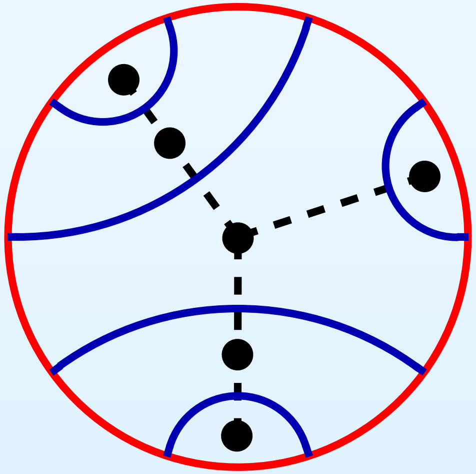
Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$





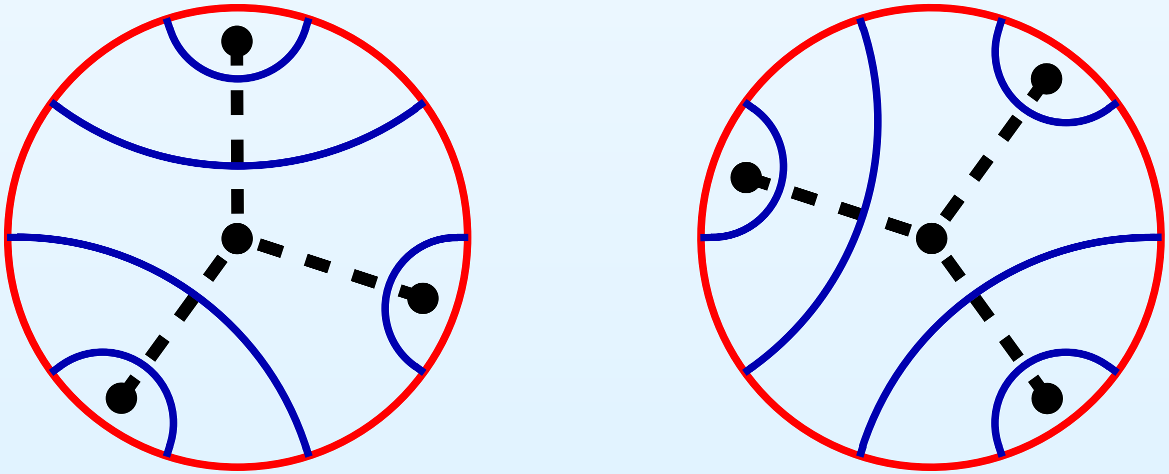
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \le N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \le N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



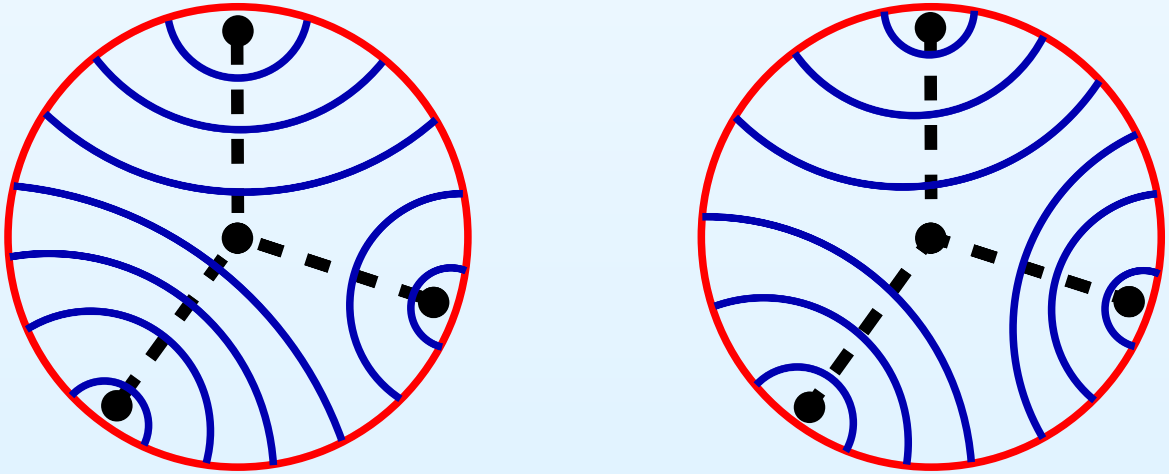
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



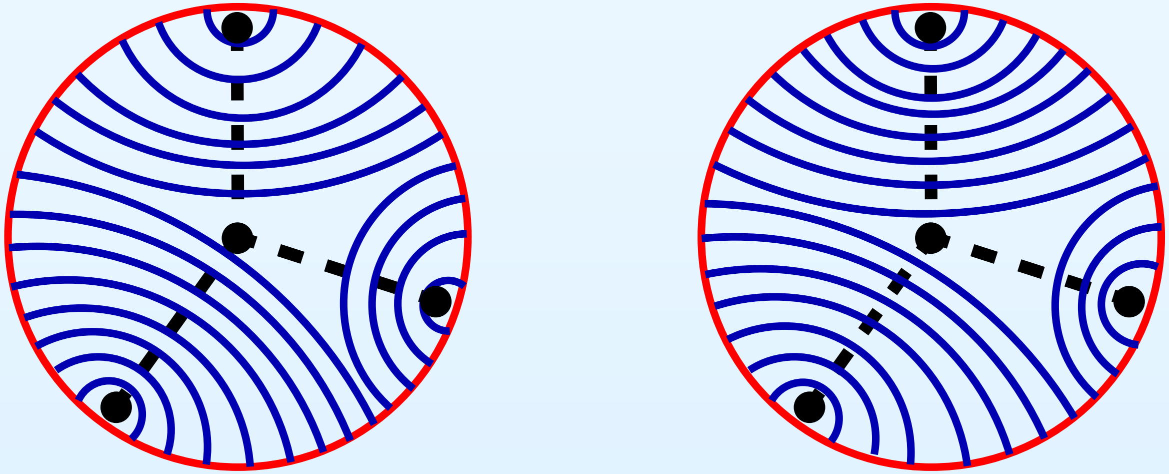
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



Asymptotic frequency of meanders

Question. *What is the asymptotic probability*

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) \sim ? \quad \text{as } N \to +\infty$$

to get a meander (i.e. a connected curve) by a random gluing of a random pair of arc systems as above with $n \leq N$ arcs?
Does it behave like $N^{-\alpha}$? Like $\exp(-\beta N)$? If so, describe how α (respectively β) depend on $\mathcal{T}_{North}, \mathcal{T}_{South}$.

Asymptotic frequency of meanders

Question. *What is the asymptotic probability*

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) \sim ? \quad \text{as } N \to +\infty$$

to get a meander (i.e. a connected curve) by a random gluing of a random pair of arc systems as above with $n \leq N$ *arcs?*

Does it behave like $N^{-\alpha}$*? Like* $\exp(-\beta N)$*? If so, describe how* α *(respectively* β*) depend on* $\mathcal{T}_{North}, \mathcal{T}_{South}$*.*

Theorem. *For any pair of trees* $\mathcal{T}_{North}, \mathcal{T}_{South}$ *the quantity* $\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N)$ *admits a strictly positive limit as* $N \to +\infty$*.*

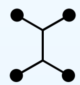
We have an explicit formula for this limit in terms of the total number of vertices of valence $1, 3, 4, \dots$ *of the two trees.*

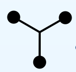
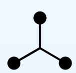
I have to confess that the fact that this asymptotic frequency is nonzero was unexpected to me.

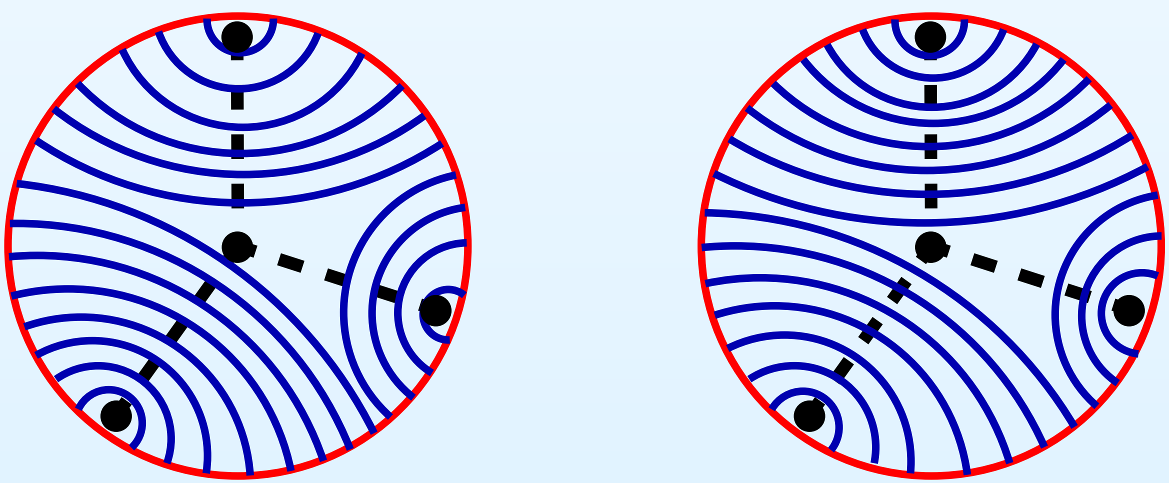
Asymptotic frequency of meanders

Theorem. *Let $p_{North}, p_{South} \geq 2$. Let $p = p_{North} + p_{South}$. The frequency $\mathrm{P}_{connected}(p_{North}, p_{South}; N)$ of meanders obtained by all possible identifications of all arc systems with at most N arcs represented by all possible pairs of plane trees having p_{North}, p_{South} of leaves (vertices of valence one) has the following limit:*

$$\lim_{N\to+\infty} \mathrm{P}_{connected}(p_{North}, p_{South}; N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}.$$

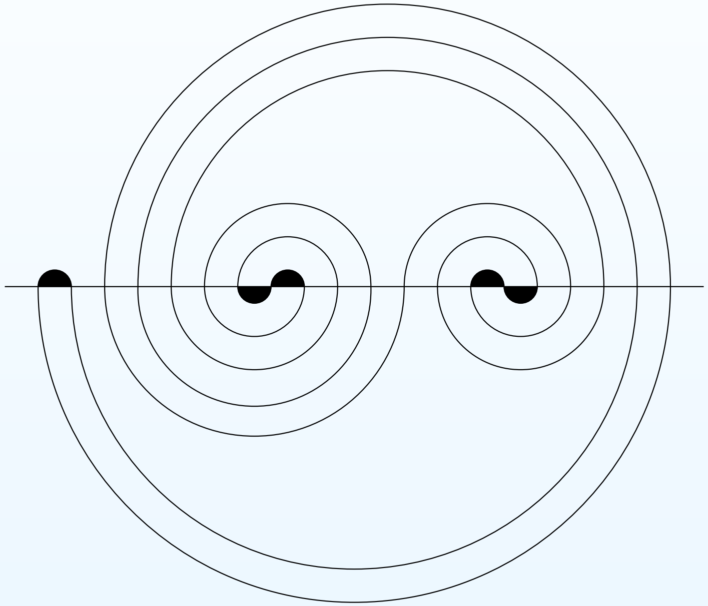
Example. $\lim_{N\to+\infty} \mathrm{P}_{connected}($  ,  $, N) =$

$= \lim_{N\to+\infty} \mathrm{P}_{connected}($  ,  $, N) = \frac{280}{\pi^6} \approx 0.291245\,.$

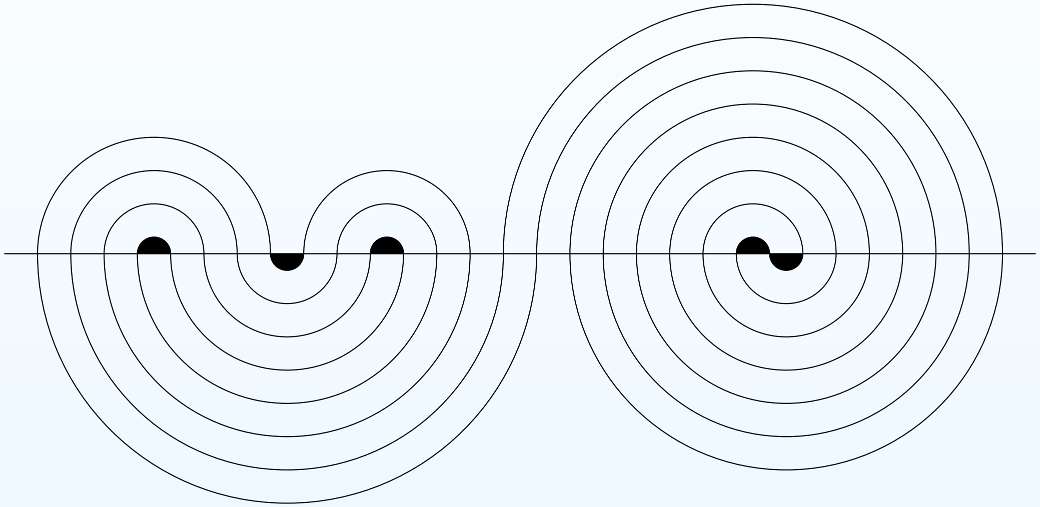


Meanders with and without maximal arc

These two meanders have 5 minimal arcs (“pimples”) each.



Meander with a maximal arc (“rainbow”) contributes to $\mathcal{M}_5^+(N)$



Meander without maximal arc contributes to $\mathcal{M}_5^-(N)$

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc (“rainbow”) and having at most $2N$ crossings with the horizontal line and exactly p minimal arcs (“pimples”). We consider p as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

Counting formulae for meanders

Theorem. *For any fixed p the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with p minimal arcs (pimples) and with at most $2N$ crossings have the following asymtotics as $N \to +\infty$:*

$$\mathcal{M}_p^+(N) = \frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4})\,.$$

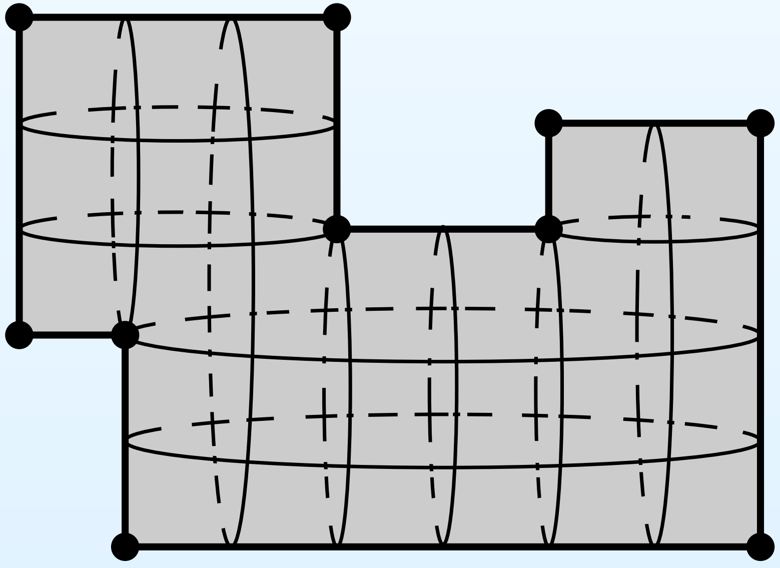
$$\mathcal{M}_p^-(N) = \frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,.$$

Note that $\mathcal{M}_p^+(N)$ grows as N^{2p-4} while $\mathcal{M}_p^-(N)$ grows as N^{2p-5}.

Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

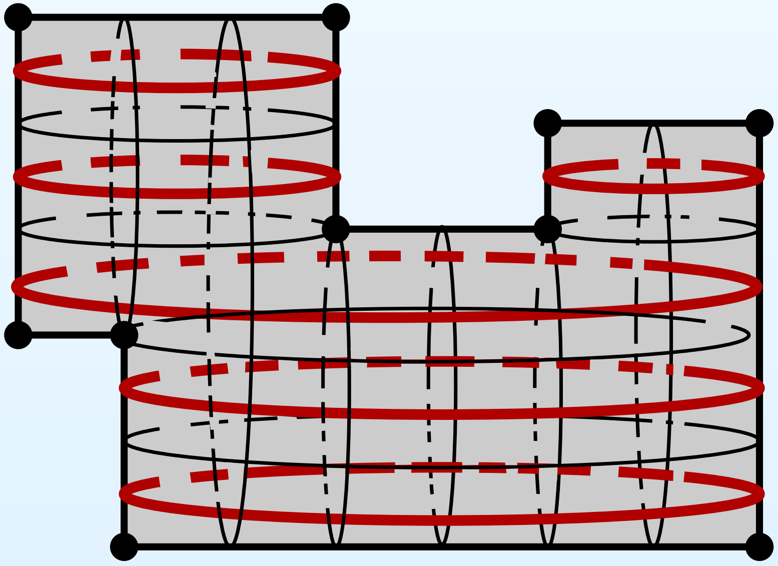
Consider a square-tiled sphere.



Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

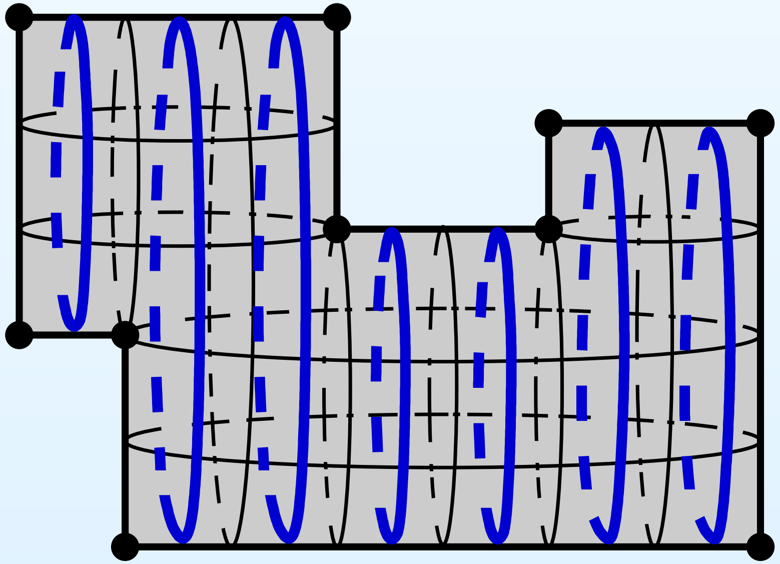
Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve.



Pairs of transverse multicurves as square-tiled surfaces

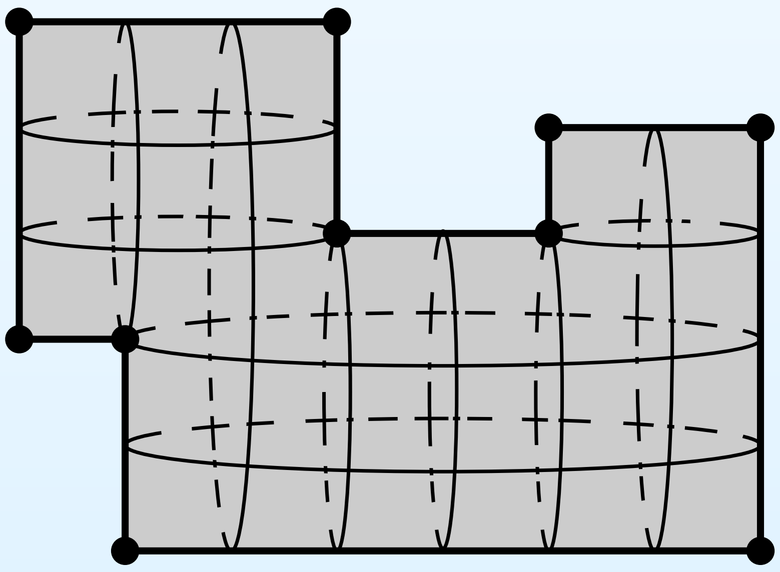
There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

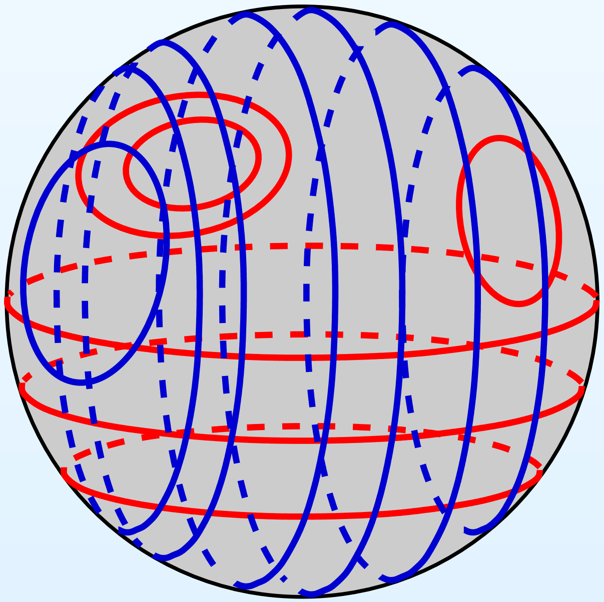
Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve. Consider the maximal collection of vertical lines passing through the centers of the squares. Color them in blue. This is the vertical multicurve.



Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres. Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve. Consider the maximal collection of vertical lines passing through the centers of the squares. Color them in blue. This is the vertical multicurve. Reciprocally, any transverse connected pair of multicurves on a sphere defines a square-tiling given by the graph dual to the graph formed by the pair of multicurves.





Horizontal and vertical foliations are not correlated

Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \dots = \operatorname{Vol}\mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L})/\operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

How we count meanders

A pair of transverse multicurves associated to a square-tiled surface is orientable if and only if the square-tiled surface is Abelian. Thus, the count of positively intersecting pairs of transverse multicurves in genus g corresponds to the count of Abelian square-tiled surfaces in genus g, i.e. to the evaluation of the Masur–Veech volumes of the corresponding moduli space of Abelian differentials. In this way we get the asymptotics $c^+(g) \cdot N^{4g-3}$ and the constant $c_1^+(g)$ for the count of positively intersecting multicurves.

Pairs (simple closed curve, transverse multicurve) correspond to square-tiled surfaces having single horizontal band of squares. We found a way to count such square-tiled surfaces both in the Abelian and in the quadratic case and to evaluate the constants $c_1(g,p)$ and $c_1^+(g)$ in the corresponding asymptotics $c_1(g,p) \cdot N^{6g-6+2p}$ and $c_1^+(g) \cdot N^{4g-3}$ respectively.

Meanders correspond to square-tiled surfaces having single horizontal and single vertical band of squares. We apply our non-correlation theorem to get

$$c_{1,1}(g,p) = \frac{c_1^2(g,p)}{c(g,p)} \quad \text{and} \quad c_{1,1}^+(g) = \frac{\left(c_1^+(g)\right)^2}{c^+(g)}\,.$$