Dynamics and Geometry of Moduli Spaces

Lecture 3. Square-tiled surfaces

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University Paris Cité

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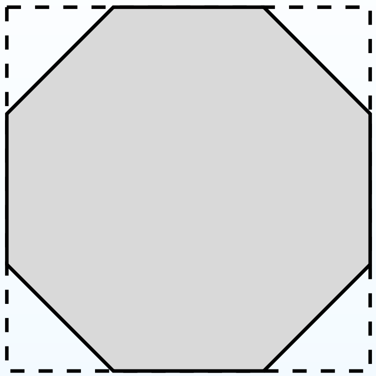
Masur–Veech volumes of the moduli spaces of Abelian differentials. Square-tiled surfaces

Very flat surface of genus two



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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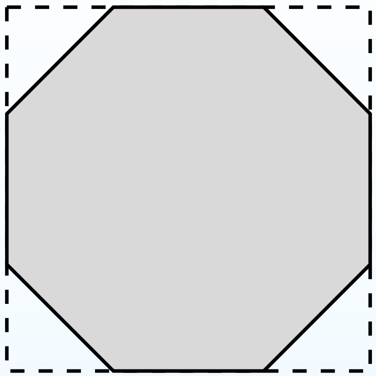

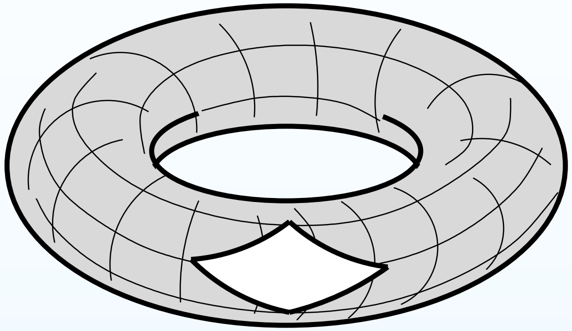

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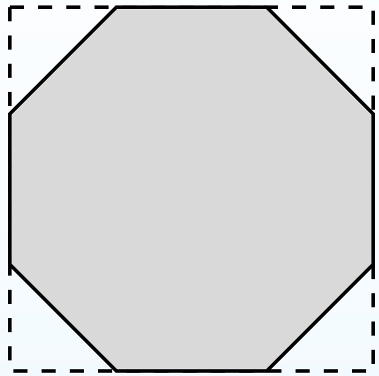

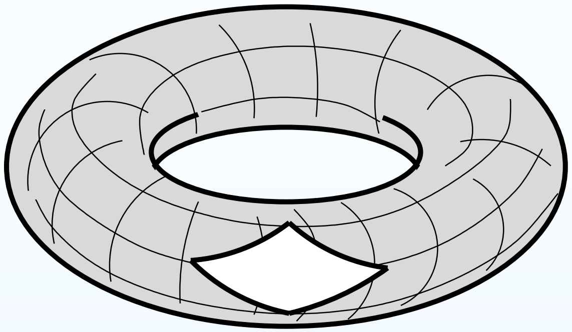

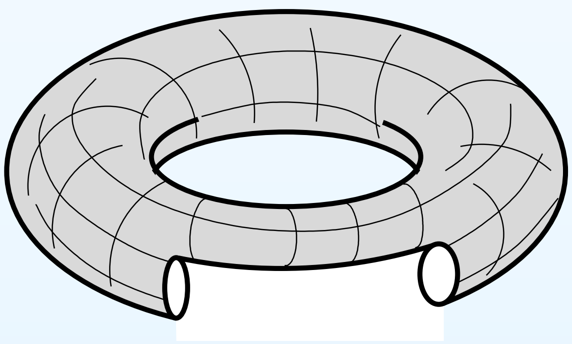

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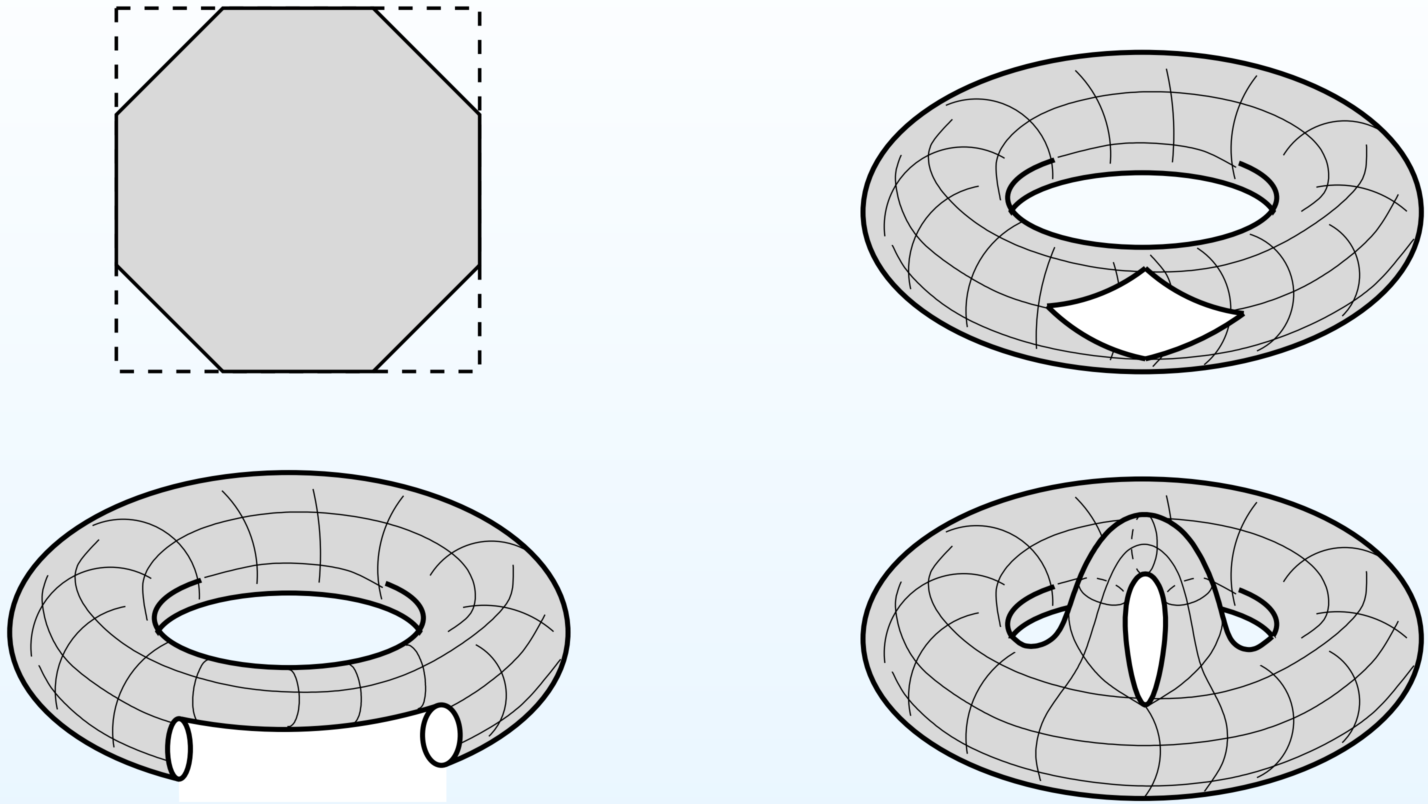

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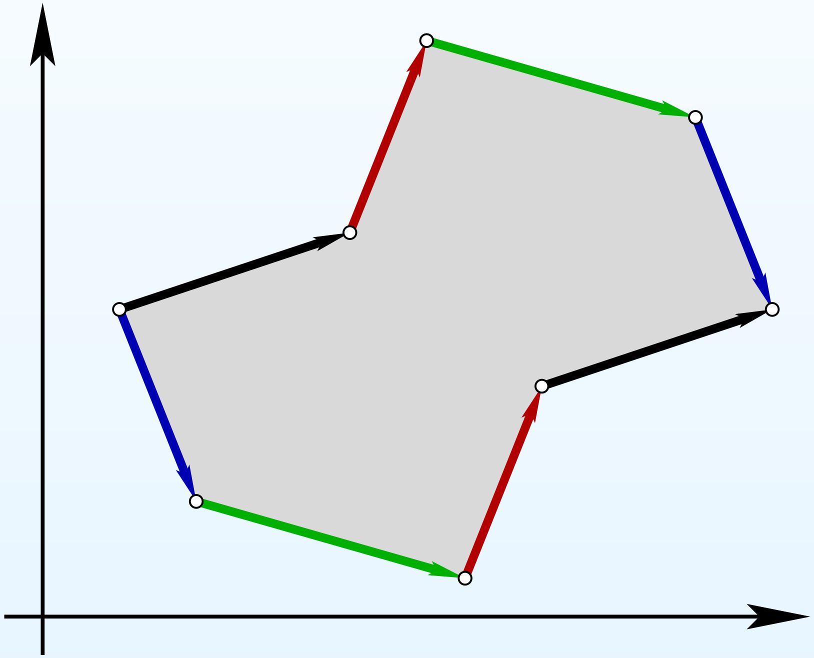
Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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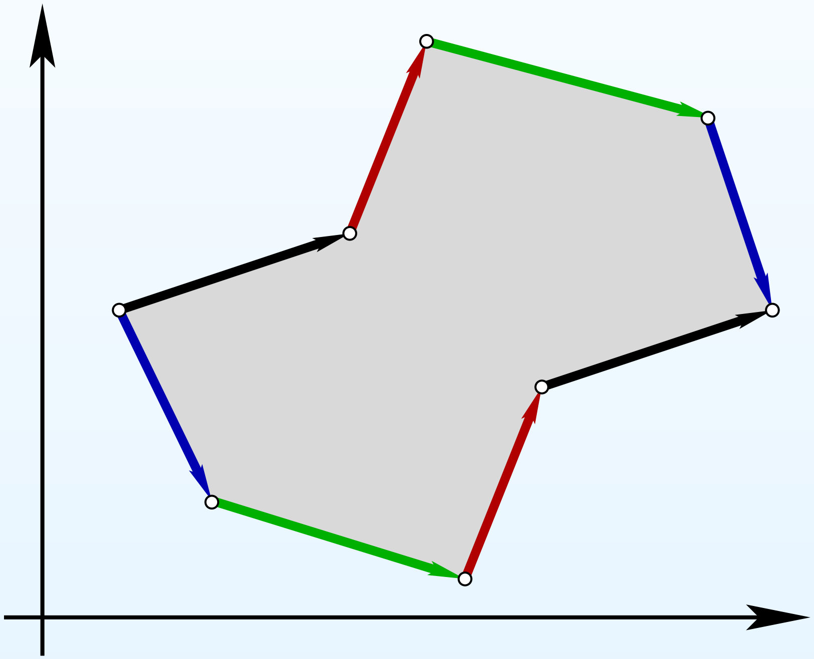

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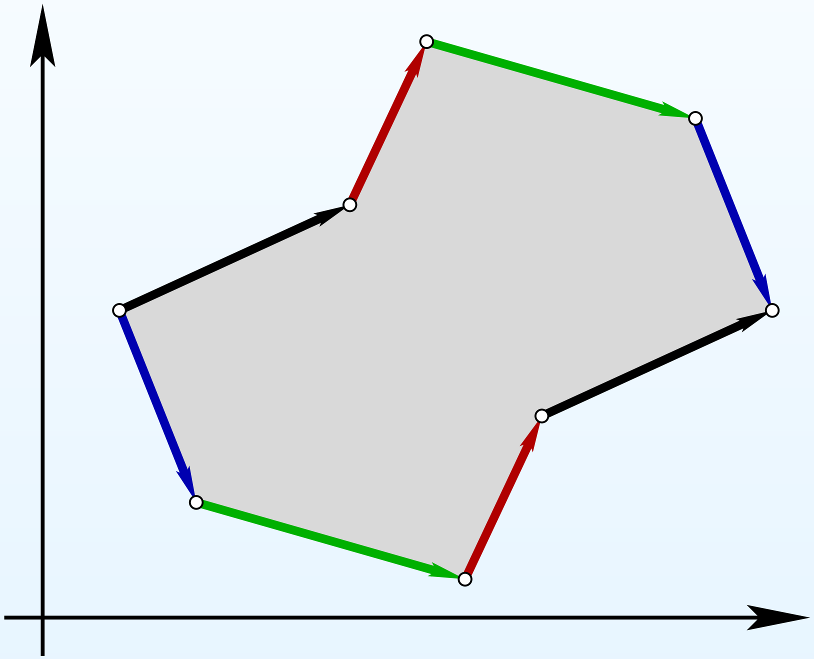

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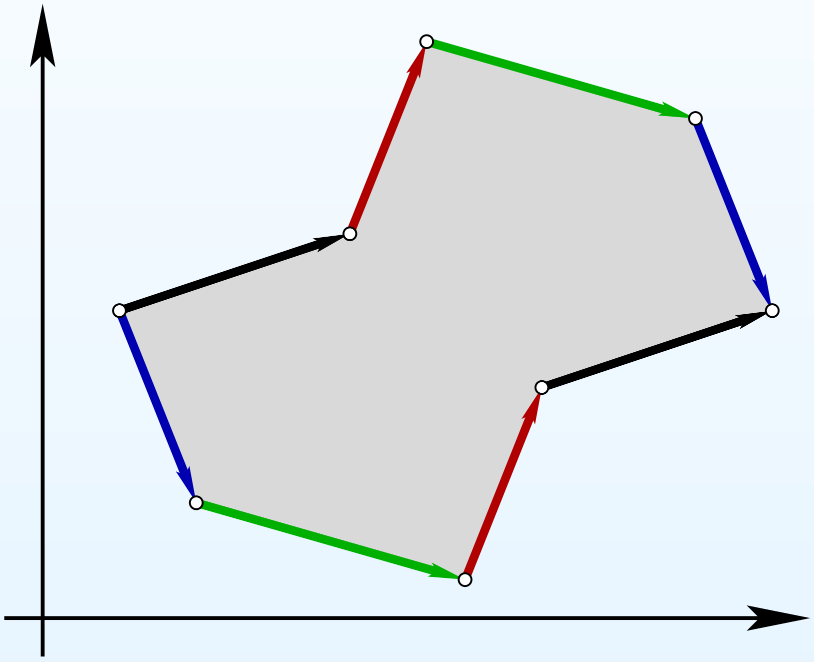
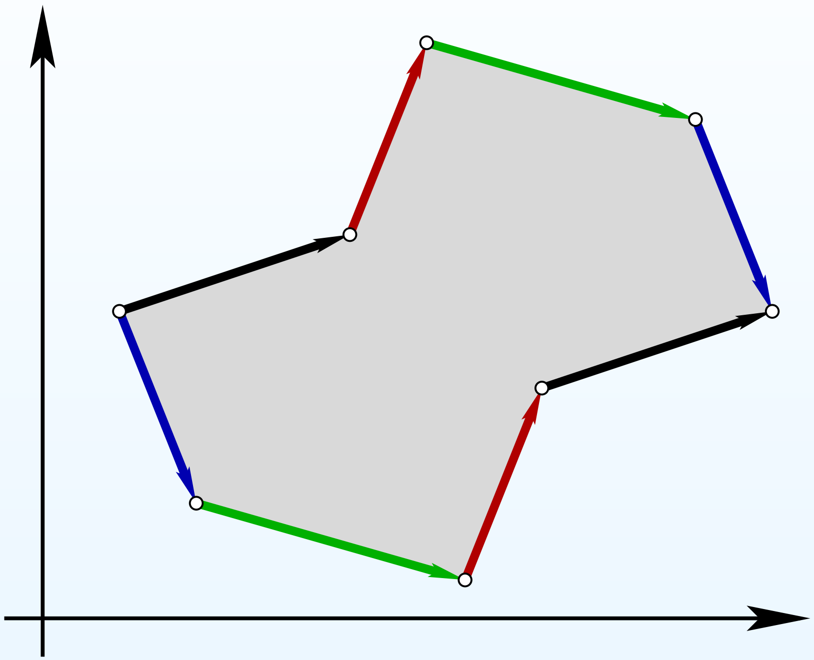


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

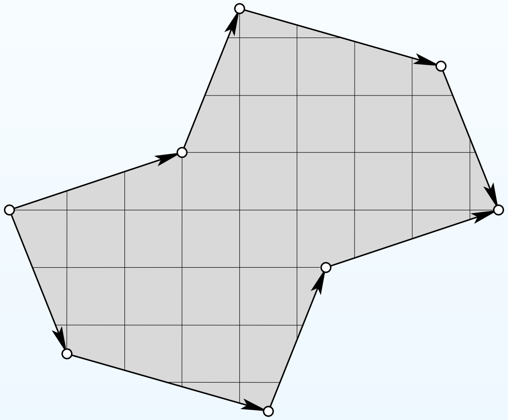
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g - 2$, is modeled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any moduli space $\mathcal{H}(m_1,\dots,m_n)$: given a positive integer $r > 0$ we can rescale a flat surface by factor r. The flat area of the surface gets rescaled by the factor r^2.



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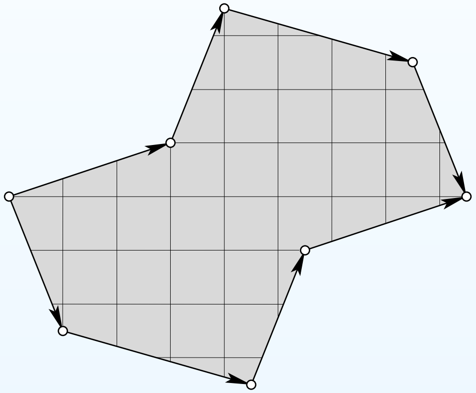

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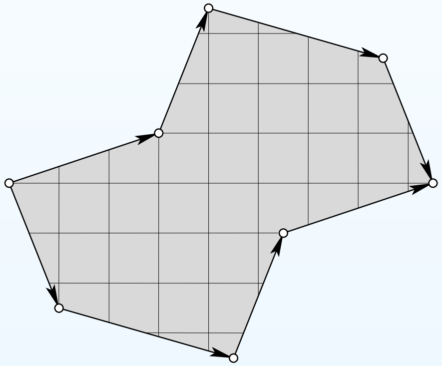

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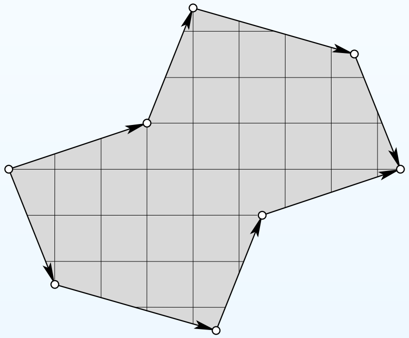

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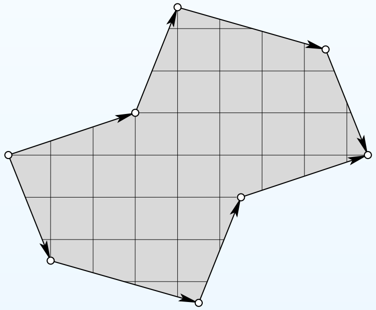

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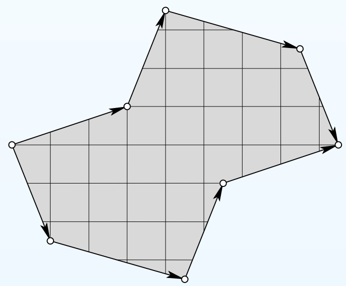

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1, \dots, m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega \wedge \bar{\omega} = \sum_{i=1}^{g}(A_i \bar{B}_i - \bar{A}_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S = (C, r \cdot \omega)$, where $r > 0$ and $(C, \omega) \in \mathcal{H}_1(m_1, \dots, m_n)$. In these "polar coordinates" the volume element disintegrates as $d\nu = r^{2d-1} dr\, d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n)$.

Period coordinates and Masur–Veech volume element

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs (C,ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

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The area function defined on every stratum $\mathcal{H}(m_1,\dots,m_n)$

$$\text{area}(C,\omega) = \frac{i}{2}\int_C \omega\wedge\bar\omega = \frac{i}{2}\sum_{i=1}^{g}(A_i\bar B_i - \bar A_i B_i)\,.$$

allows to define an analog of a “unit ball” $\mathcal{H}_{\le 1}$ in any stratum as a subset of those (C,ω) in $\mathcal{H}(m_1,\dots,m_n)$, where $\operatorname{area}(C,\omega)\le 1$. (Note that in period coordinates the “unit ball” is rather the interior of a “unit hyperboloid”.)

Definition.

$$\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n) := 2d\cdot\int_{\mathcal{H}_{\le 1}} d\nu\,,$$

where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$ is just a conventional factor.

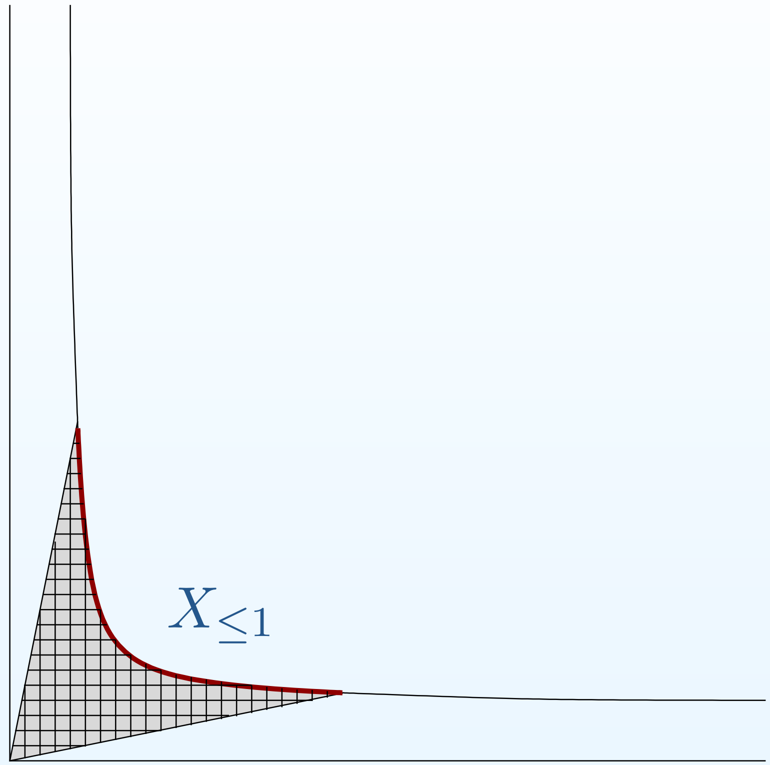
Masur–Veech volume

Summary. Every stratum of Abelian differentials admits

- A local structure of a vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$;
- An integer lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

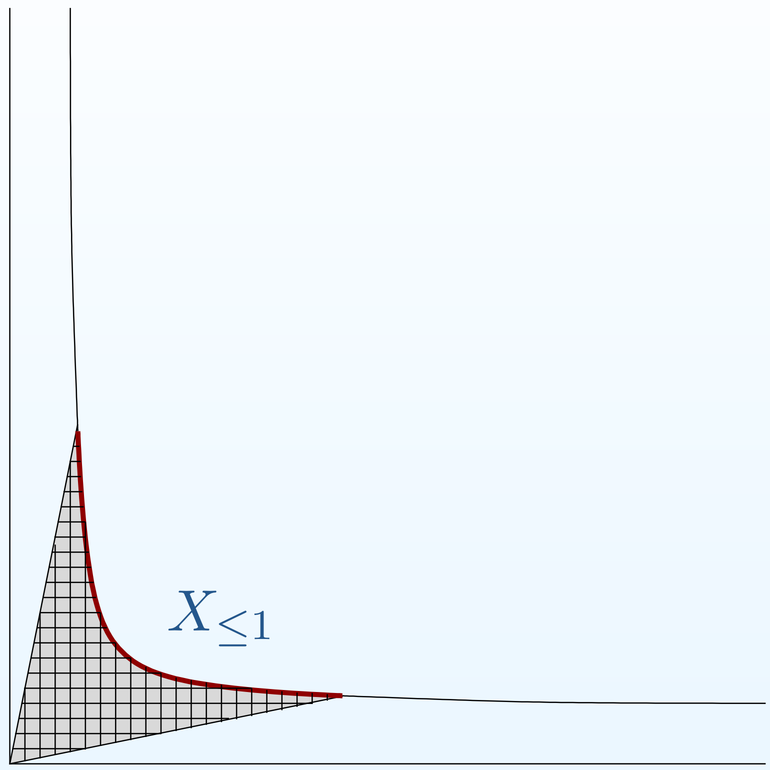
Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1, \dots, m_n)$ or $\mathcal{Q}_1(m_1, \dots, m_n)$ of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

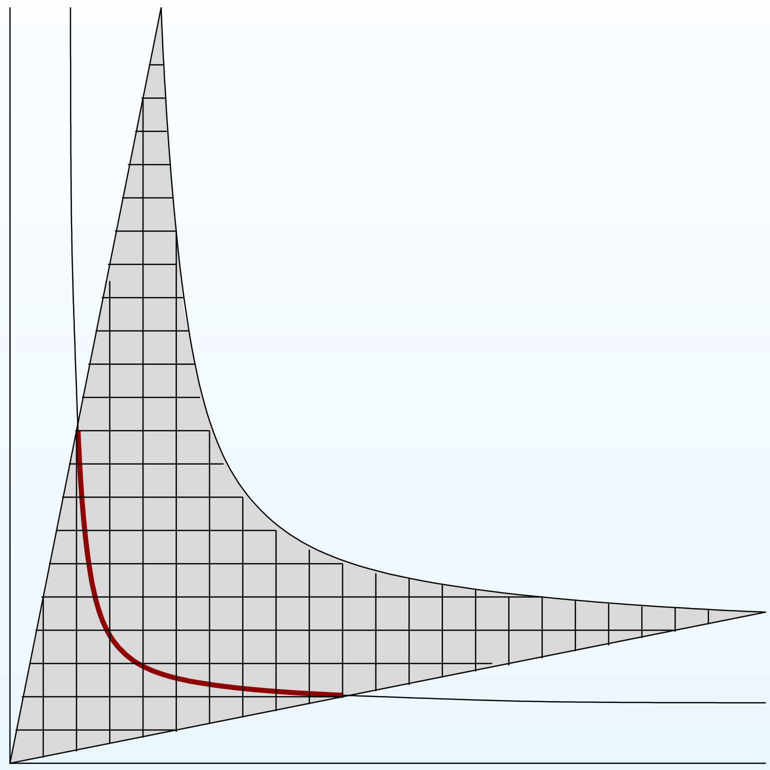
Counting volume by counting integer points in a large cone



To count volume of the cone $X_{\leq 1}$ one can take an ε-grid and count the number of lattice points inside it.

Counting volume by counting integer points in a large cone





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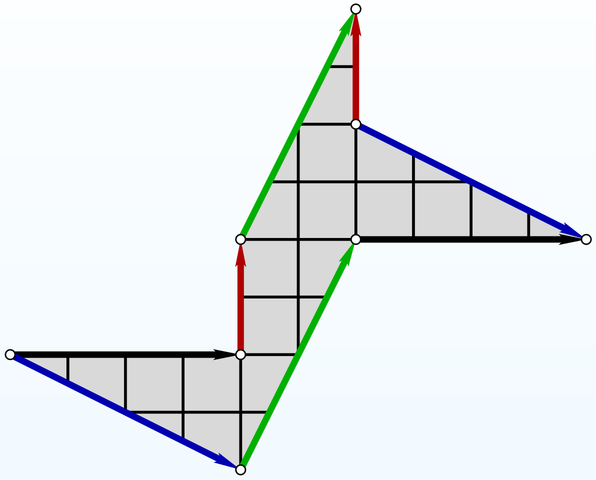
Counting points of the ε-grid in the cone $X_{\leq 1}$ is the same as counting integer points in the proportionally rescaled cone $X_{\leq 1/\varepsilon}$.

Integer points as square-tiled surfaces



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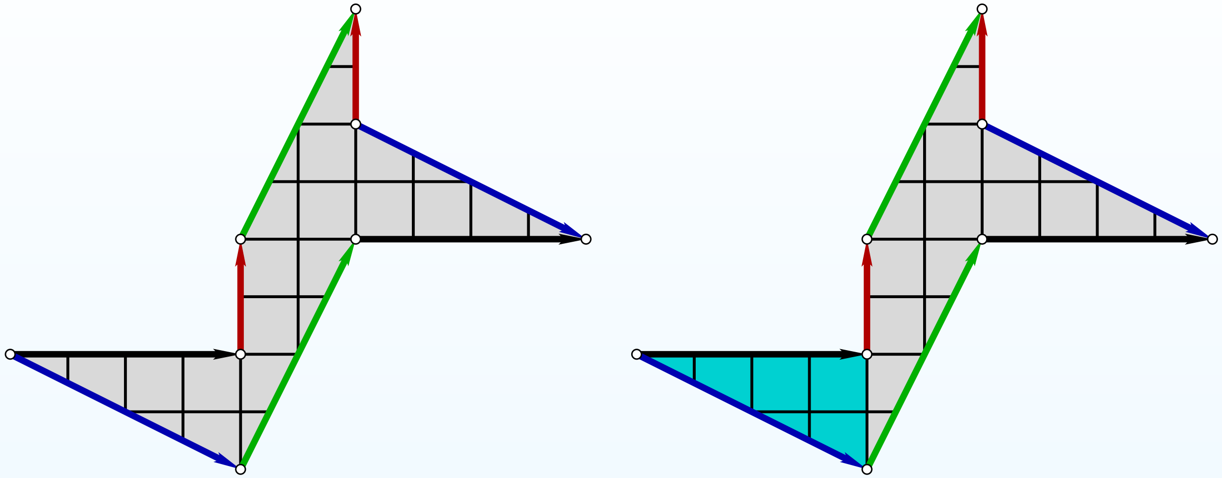

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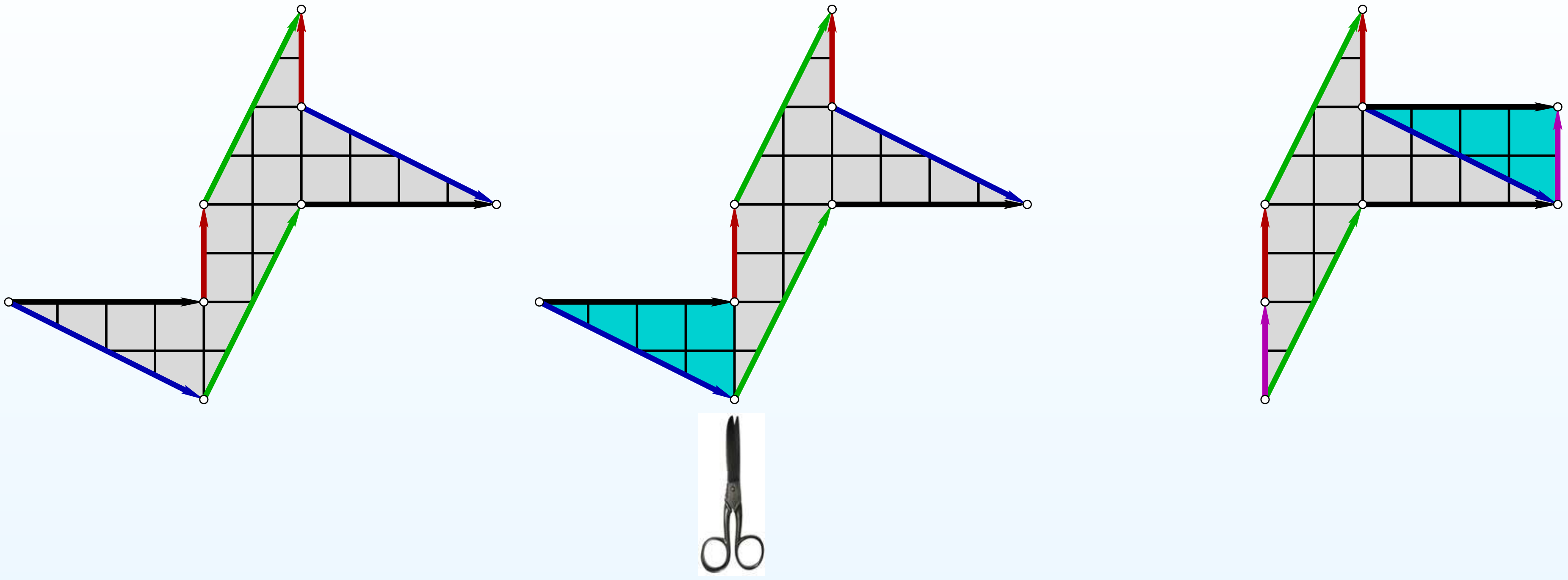

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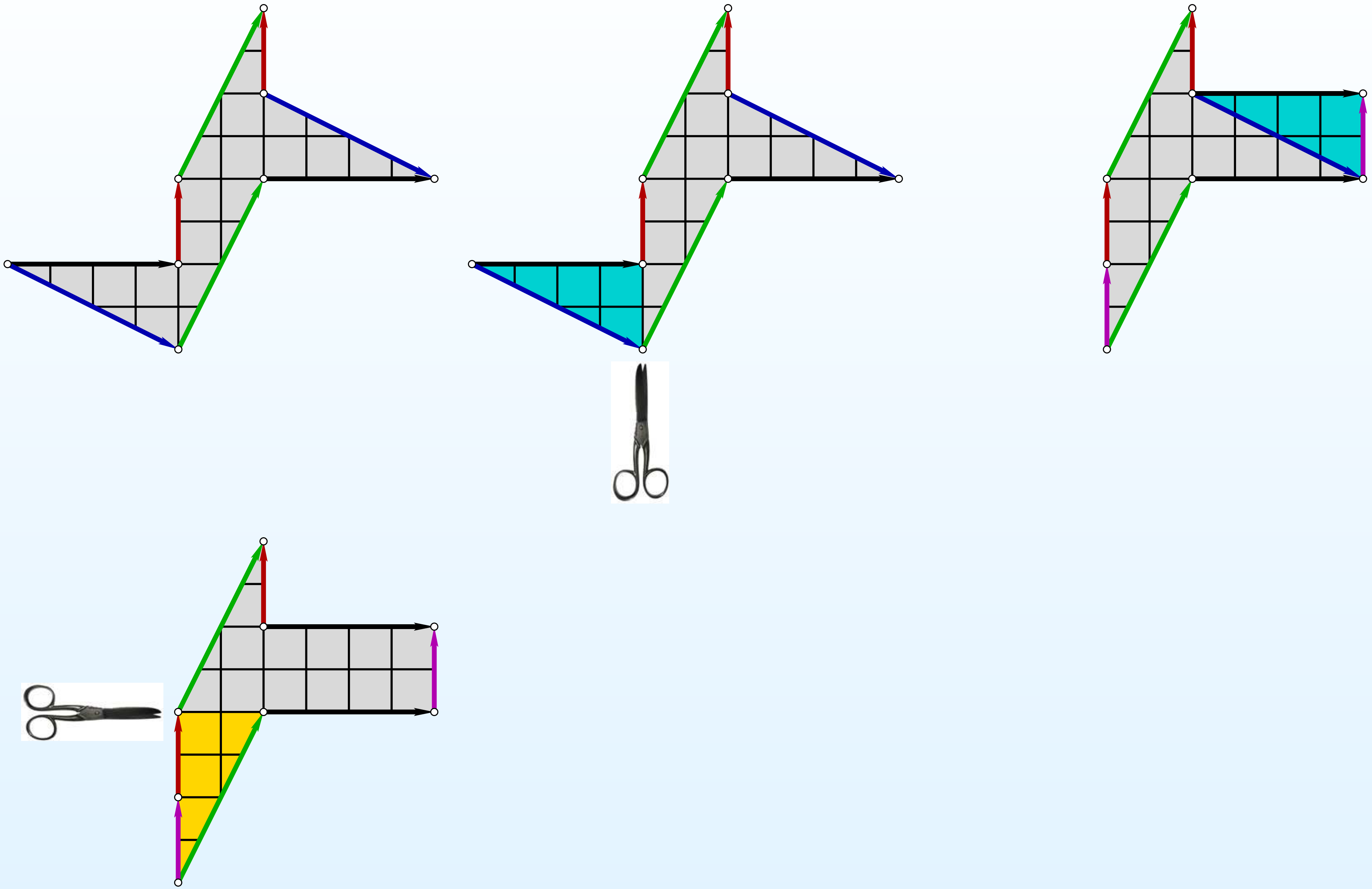

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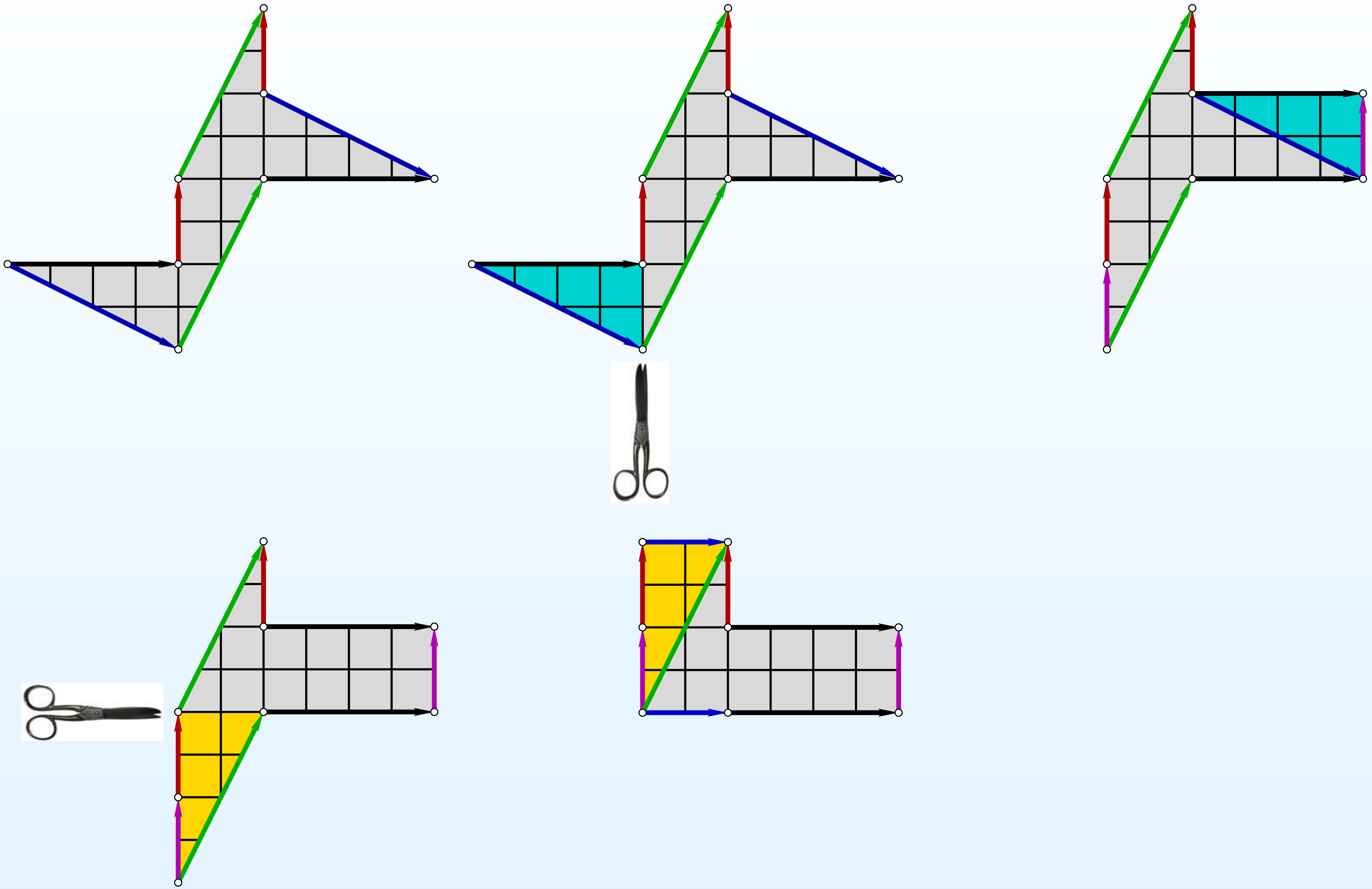

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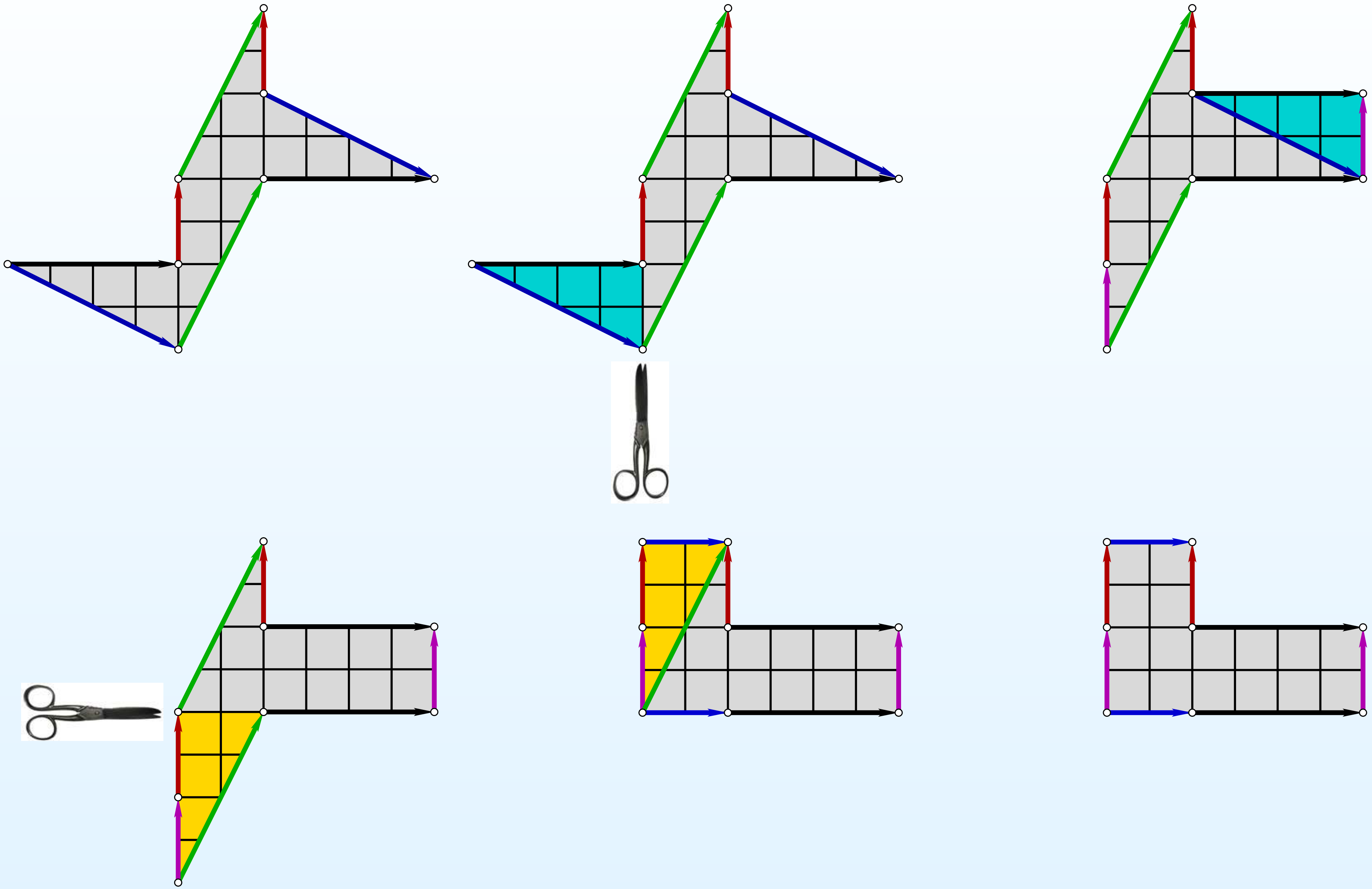

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$$\begin{array}{rccl} p: & P & \mapsto & \int_{P_1}^{P} \omega \pmod{\mathbb{Z} \oplus i\mathbb{Z}} \\ p: & C & \to & \mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \end{array}$$

The ramification points of the cover p are exactly the zeroes of ω.

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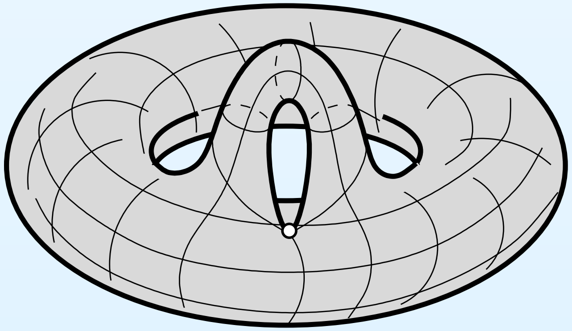
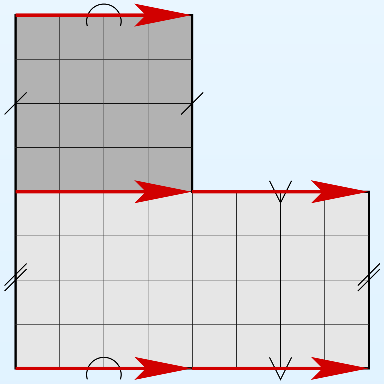
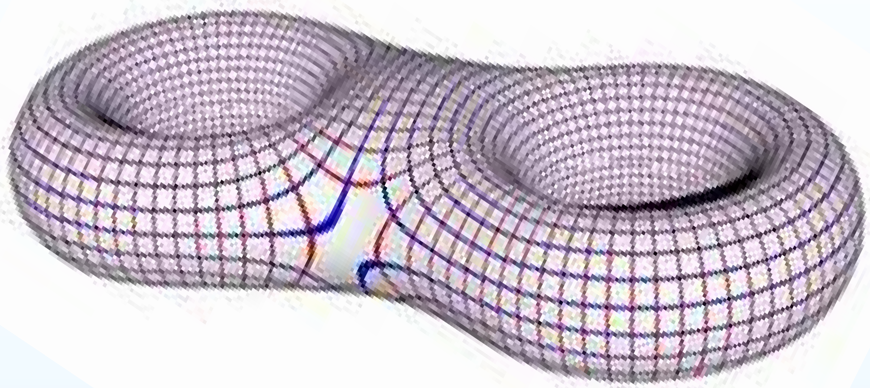
Choosing the standard unit square pattern for $\mathbb{T}$ we get induced tiling of (C, ω) by unit squares which form horizontal and vertical cylinders. The square-tiled surface of genus two in the picture has 2 maximal horizontal cylinders filled with periodic geodesics.





Count of square-tiled surfaces



Picture created by Jian Jiang

We reduced evaluation of the Masur–Veech volumes $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ to a combination of the following two related problems:

- Describe all combinatorial types of square-tiled surfaces in any given stratum $\mathcal{H}(m_1,\dots,m_n)$.
- Count the leading term in the asymptotics of the number of square-tiled surfaces of any given combinatorial type tiled with at most N squares when $N\to+\infty$.

Count of square-tiled surfaces through separatrix diagrams

Multiple zeta-values

Define

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1,\ldots,n_k \geq 1} \frac{1}{n_1^{s_1}(n_1+n_2)^{s_2}\ldots(n_1+\cdots+n_k)^{s_k}}\,.$$

Multiple zeta-values (MZV) are values of $\zeta(s_1, s_2, \ldots, s_k)$ at positive integers $s_j \in \mathbb{N}$, where $s_k \geq 2$. For example

$$\zeta(2) = \frac{\pi^2}{6}\,(Euler); \qquad \zeta(4) = \frac{\pi^4}{90}; \quad \ldots\,\zeta(2n) = \frac{p}{q}\pi^{2n}, \quad \text{where } p, q \in \mathbb{N}\,.$$

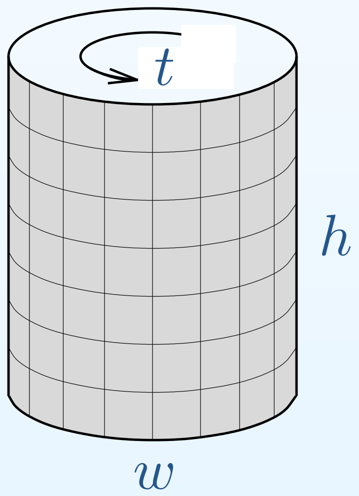
Conjecturally $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

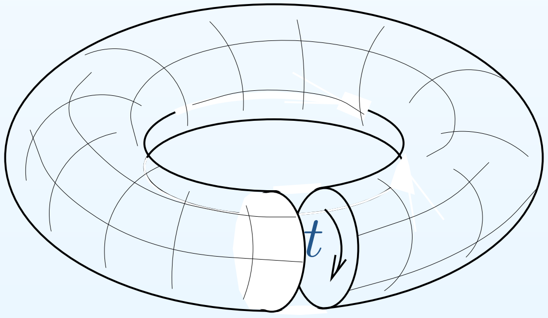
Multiple zeta values satisfy numerous relations. For example

$$\zeta(1,3) = \frac{1}{4}\zeta(4); \qquad \zeta(2,2) = \frac{3}{4}\zeta(4) \quad (Euler)\,.$$

Baby case: decomposition of a square-tiled torus

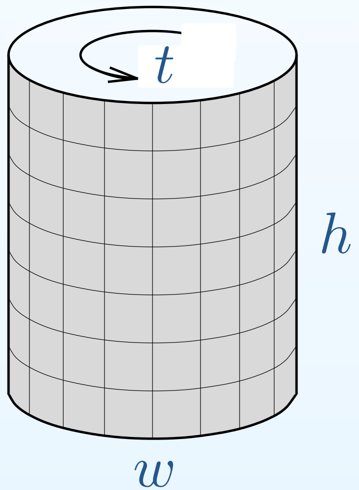
Let us count the number of square-tiled tori tiled by at most $N \gg 1$ squares. Cutting our flat torus by a horizontal waist curve we get a cylinder with a waist curve of length $w \in \mathbb{N}$ and a height $h \in \mathbb{N}$. The number of squares in the tiling equals $w \cdot h$.

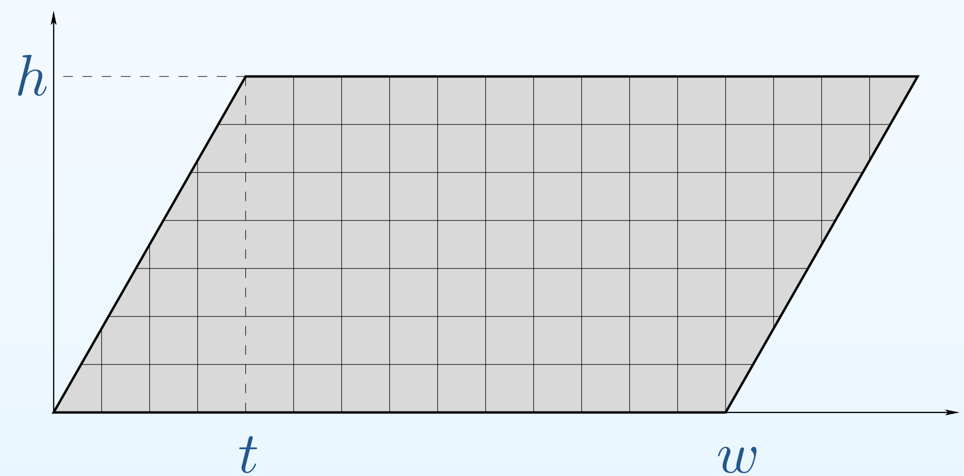




Baby case: counting square-tiled tori

We can glue a torus from a cylinder with some integer twist t. Making an appropriate Dehn twist along the waist curve we can reduce the value of the twist t to one of the values $0, 1, \dots, w-1$. Fixing the integer perimeter w and height h of a cylinder we get w square-tiled tori.



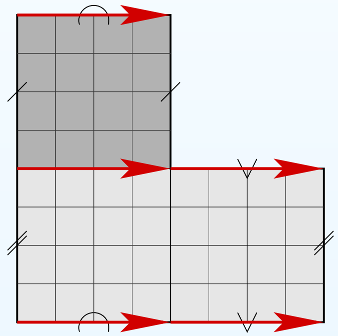


Number of square-tiled tori constructed with at most N squares $=$

$$= \sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w = \sum_{\substack{w,h\in\mathbb{N}\\ w\le \frac{N}{h}}} w \approx \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\zeta(2) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

Critical graphs (separatrix diagrams)

Note that all leaves of the horizontal (vertical) foliation on a square-tiled surface are closed. The *critical graph* Γ (*separatrix diagram*) is the union of all horizontal critical leaves. Vertices of Γ are represented by the conical points; the edges of Γ are formed by horizontal saddle connections.



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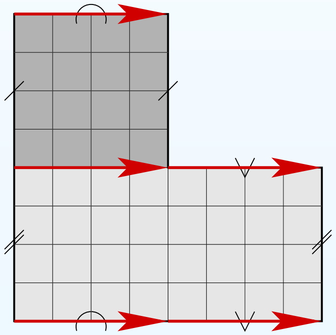
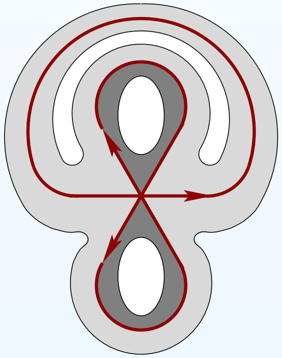




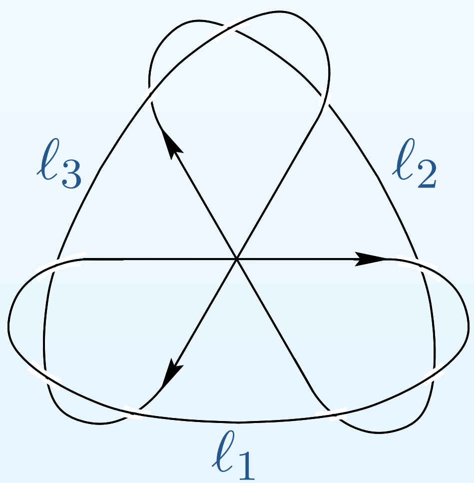
A critical graph Γ is an *oriented ribbon graph* endowed with the following structure:

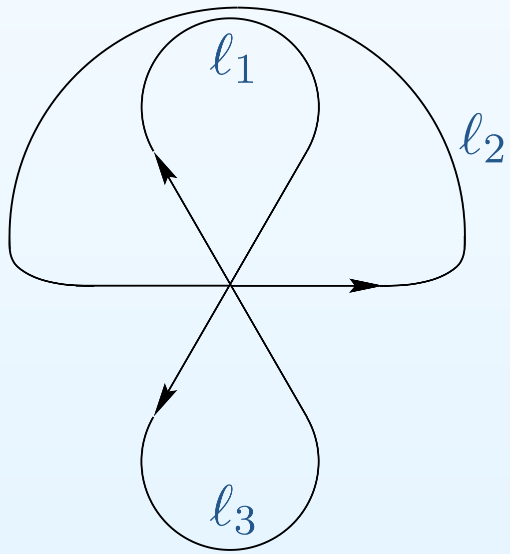
1. The orientation of edges at any vertex is alternated with respect to the cyclic order of edges at this vertex.

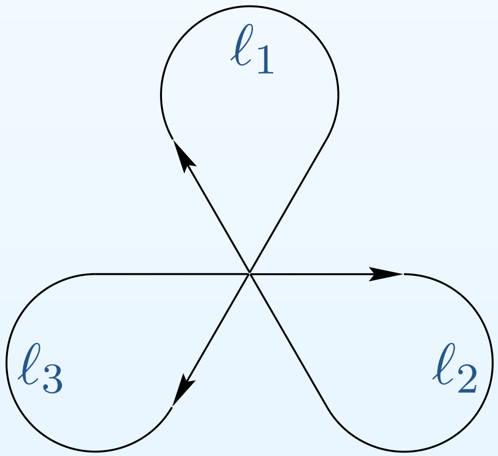
2. The complement $S - \Gamma$ is a finite disjoint union of flat cylinders foliated by oriented circles. Thus, the set of boundary components of the ribbon graph is decomposed into pairs: to each pair of boundary components we glue a cylinder, and there is one positively oriented and one negatively oriented boundary component in each pair.

Realizable separatrix diagrams

Note, however, that not all ribbon graphs as above correspond to actual flat surfaces. A flat metric endows saddle connections with positive lengths ℓ_i. The left graph is realizable for any lengths ℓ_1, ℓ_2, ℓ_3. The middle one — only when $\ell_1 = \ell_3$. The rightmost one is never realizable: pairs of boundary components bounding the same cylinder have to have equal length, and we cannot find a pair for the component of length $\ell_1 + \ell_2 + \ell_3$.

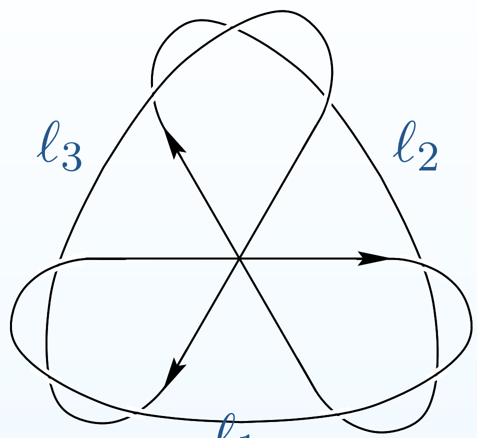






Lemma. *The set of all square-tiled surfaces (respectively pillowcase covers) sharing the same realizable separatrix diagram provides a nontrivial contribution to the volume of the corresponding stratum.*

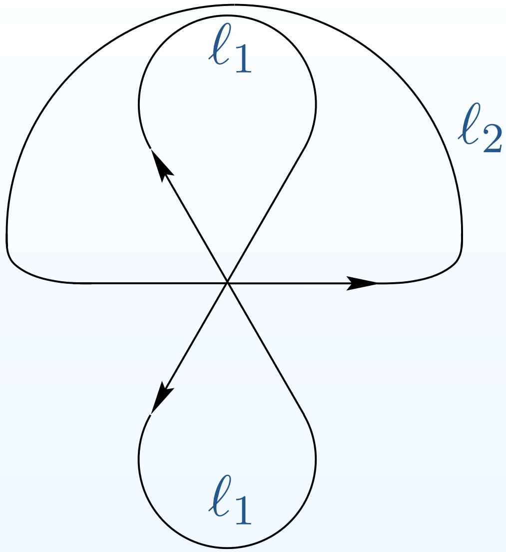
Volume computation for $\mathcal{H}(2)$: the 1-cylinder diagram

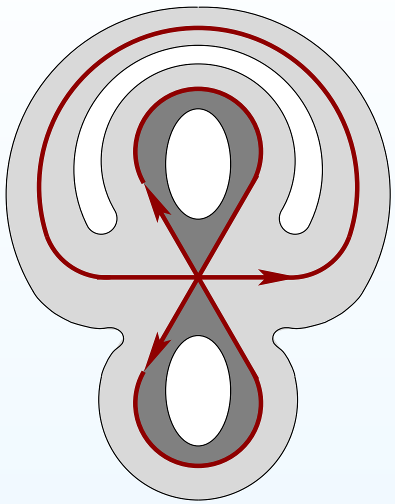


Single cylinder

$$\begin{aligned}
\frac{1}{3} \sum_{\substack{\ell_1,\ell_2,\ell_3,h\in\mathbb{N}\\ (\ell_1+\ell_2+\ell_3)h\le N}} (\ell_1+\ell_2+\ell_3) &\approx \frac{1}{3}\sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w\cdot\frac{w^2}{2} = \frac{1}{6}\sum_{\substack{w,h\in\mathbb{N}\\ w\le\frac{N}{h}}} w^3 \\
&\approx \frac{1}{6}\sum_{h\in\mathbb{N}}\frac{1}{4}\cdot\left(\frac{N}{h}\right)^4 = \frac{N^4}{24}\cdot\sum_{h\in\mathbb{N}}\frac{1}{h^4} \\
&= \frac{N^4}{24}\cdot\zeta(4) = \frac{N^4}{24}\cdot\frac{\pi^4}{90}\,.
\end{aligned}$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram





$$\sum_{\substack{\ell_1,\ell_2,h_1,h_2\in\mathbb{N}\\ \ell_1 h_1+(\ell_1+\ell_2)h_2\le N}} \ell_1(\ell_1+\ell_2) = \sum_{\substack{\ell_1,\ell_2,h_1,h_2\in\mathbb{N}\\ \ell_1(h_1+h_2)+\ell_2 h_2\le N}} (\ell_1^2+\ell_1\ell_2) =$$

$$= \sum_{h_1,h_2\in\mathbb{N}} \sum_{\substack{\ell_1,\ell_2\in\mathbb{N}\\ \frac{\ell_1(h_1+h_2)}{N}+\frac{\ell_2 h_2}{N}\le 1}} (\ell_1^2+\ell_1\ell_2)\,.$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram

For any fixed h_1, h_2 we can replace the sum with respect to ℓ_1, ℓ_2 by the integral. Let $x_1 := \ell_1 \cdot \dfrac{h_1+h_2}{N}$ and $x_2 := \ell_2 \cdot \dfrac{h_2}{N}$ be the new variables, where h_1, h_2 are considered as parameters. After this change of variables our sums with respect to ℓ_1, ℓ_2 become the integral with respect to x_1, x_2, where we integrate over the simplex $\Delta = \{x_1 + x_2 \le 1 : x_1 \ge 0; x_2 \ge 0\}$:

$$\sum_{\substack{\ell_1,\ell_2\in\mathbb{N}\\ \frac{\ell_1(h_1+h_2)}{N}+\frac{\ell_2 h_2}{N}\le 1}} (\ell_1^2 + \ell_1\ell_2) \approx$$

$$\approx \int_\Delta \left[\left(\frac{x_1 N}{h_1+h_2}\right)^2 + \left(\frac{x_1 N}{h_1+h_2}\right)\left(\frac{x_2 N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2}dx_1\right)\left(\frac{N}{h_2}dx_2\right) .$$

Volume computation for $\mathcal{H}(2)$: the 2-cylinders diagram

$$\sum_{h_1,h_2}\int_\Delta \left[\left(\frac{x_1N}{h_1+h_2}\right)^2+\left(\frac{x_1N}{h_1+h_2}\right)\left(\frac{x_2N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2}dx_1\right)\left(\frac{N}{h_2}dx_2\right)$$

$$= N^4\left[\int_\Delta x_1^2\,dx_1dx_2\cdot\sum_{h_1,h_2\in\mathbb{N}}\frac{1}{h_2(h_1+h_2)^3}\right.$$

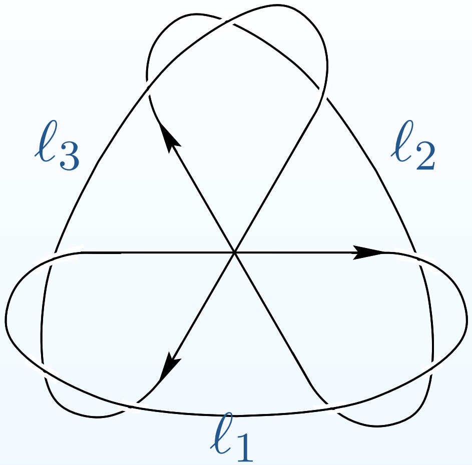
$$\left. + \int_\Delta x_1x_2\,dx_1dx_2\cdot\sum_{h_1,h_2\in\mathbb{N}}\frac{1}{h_2^2(h_1+h_2)^2}\right]$$

$$= \frac{N^4}{24}[2\cdot\zeta(1,3)+\zeta(2,2)] = \frac{N^4}{24}\left[2\cdot\frac{\zeta(4)}{4}+\frac{3\zeta(4)}{4}\right]$$

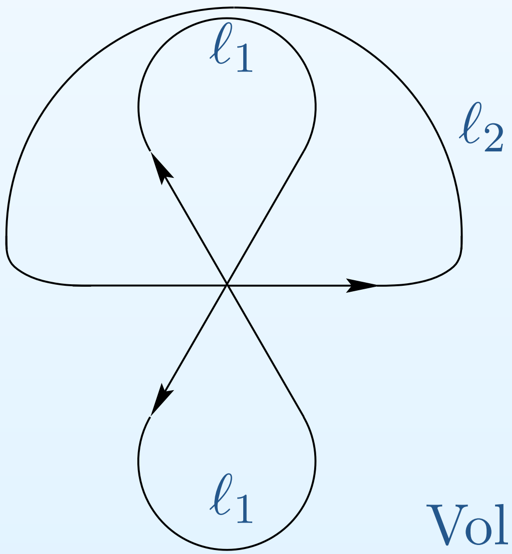
$$= \frac{N^4}{24}\cdot\frac{5}{4}\cdot\frac{\pi^4}{90}.$$

where we used the identities $\zeta(1,3)=\frac{1}{4}\zeta(4),\quad \zeta(2,2)=\frac{3}{4}\zeta(4)$ and the values $\int_\Delta x_1^2\,dx_1dx_2 = 2\int_\Delta x_1x_2\,dx_1dx_2 = 2\cdot\frac{1}{4!}$.

Volume computation for $\mathcal{H}(2)$: summary



$$\frac{1}{3}\sum_{\substack{\ell_1,\ell_2,\ell_3,h\in\mathbb{N}\\(\ell_1+\ell_2+\ell_3)h\le N}}(\ell_1+\ell_2+\ell_3)\approx\frac{N^4}{24}\cdot\zeta(4)$$



$$\sum_{\substack{\ell_1,\ell_2,h_1,h_2\\\ell_1h_1+(\ell_1+\ell_2)h_2\le N}}\ell_1(\ell_1+\ell_2)$$

$$=\frac{N^4}{24}\big[2\cdot\zeta(1,3)+\zeta(2,2)\big]=\frac{N^4}{24}\cdot\frac{5}{4}\cdot\zeta(4)$$

$$\operatorname{Vol}(\mathcal{H}_1(2))=\lim_{N\to\infty}\frac{2\cdot 4}{N^4}\cdot\text{(Number of surfaces)}=\frac{\pi^4}{120}$$

Contributions $\operatorname{Vol}_k \mathcal{H}(3,1)$ of k-cylinder surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$\operatorname{Vol}_1 \mathcal{H}(3,1) = \frac{\zeta(7)}{15}$$

$$\operatorname{Vol}_2 \mathcal{H}(3,1) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{45}$$

$$\begin{aligned}\operatorname{Vol}_3 \mathcal{H}(3,1) = \frac{1}{90}\Big(& 12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\ & + 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\ & - 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\ & + 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\ & - 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\ & - 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$\operatorname{Vol}_4 \mathcal{H}(3,1) = \frac{2\zeta(2)}{45}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big)\,.$$

After simplification

Multiple zeta values satisfy numerous relations. After simplification (which is now accessible through a SAGE package) we get

$$
\begin{aligned}
\operatorname{Vol}_1 \mathcal{H}(3,1) &= 1/15 \cdot \zeta(7)\\
\operatorname{Vol}_2 \mathcal{H}(3,1) &= -7/135 \cdot \zeta(1,6) + 1/135 \cdot \zeta(2,5) + 23/135 \cdot \zeta(7)\\
\operatorname{Vol}_3 \mathcal{H}(3,1) &= -2/15 \cdot \zeta(1,6) - 2/45 \cdot \zeta(2,5) + 1/5 \cdot \zeta(6) - 4/45 \cdot \zeta(7)\\
\operatorname{Vol}_4(\mathcal{H}(3,1) &= 5/27 \cdot \zeta(1,6) + 1/27 \cdot \zeta(2,5) + 7/45 \cdot \zeta(6) - 4/27 \cdot \zeta(7)
\end{aligned}
$$

Conjecturally, multiple zeta values involved in these simplified expressions are linearly independent over rational numbers. However, the total contribution is a rational multiple of π^{2g} in accordance with the general result by A. Eskin and A. Okounkov, 2001:

$$\operatorname{Vol} \mathcal{H}(3,1) = \operatorname{Vol}_1 \mathcal{H}(3,1) + \cdots + \operatorname{Vol}_4 \mathcal{H}(3,1) = \frac{16}{42525}\pi^6$$

Volumes of some low-dimensional strata

$$
\begin{aligned}
\operatorname{Vol}(\mathcal{H}_1(\emptyset)) &= 2\cdot\zeta(2) &&= \frac{1}{3}\cdot\pi^2\\
\operatorname{Vol}(\mathcal{H}_1(2)) &= \frac{2}{3!}\cdot\frac{9}{4}\cdot\zeta(4) &&= \frac{1}{120}\cdot\pi^4\\
\operatorname{Vol}(\mathcal{H}_1(1,1)) &= \frac{1}{4!}\cdot 4\cdot\zeta(4) &&= \frac{1}{135}\cdot\pi^4\\
\operatorname{Vol}(\mathcal{H}_1^{hyp}(4)) &= \frac{2}{5!}\cdot\frac{135}{16}\cdot\zeta(6) &&= \frac{1}{6720}\cdot\pi^6\\
\operatorname{Vol}(\mathcal{H}_1^{odd}(4)) &= \frac{2}{5!}\cdot\frac{70}{3}\cdot\zeta(6) &&= \frac{1}{2430}\cdot\pi^6\\
\operatorname{Vol}(\mathcal{H}_1(1,3)) &= \frac{2}{6!}\cdot 128\cdot\zeta(6) &&= \frac{16}{42525}\cdot\pi^6\\
\operatorname{Vol}(\mathcal{H}_1^{hyp}(6)) &= \frac{2}{7!}\cdot\frac{2625}{64}\cdot\zeta(8) &&= \frac{1}{580608}\cdot\pi^8
\end{aligned}
$$

Volumes through multiple zeta values

Conjecture. *Prove that for any connected component of any stratum the contribution to the Masur–Veech volume coming from square-tiled having exatly* k *horizontal cylinders is a linear combination with rational coefficients of multiple zeta values.*

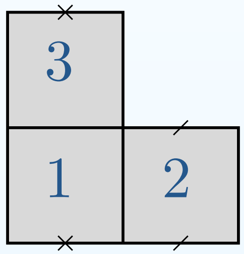
Stronger Conjecture. *Prove the that contribution to the Masur–Veech volume coming from square-tiled corresponding to any fixed separatrix diagram is a linear combination with rational coefficients of multiple zeta values.*

The latter statement is elementary for 1-cylinder separatrix diagrams, simple for 2-cylinder diagrams. It is already a nontrivial theorem (proved by B. Allombert and V. Delecroix) for 3-cylinder diagrams.

Approach of Eskin and Okounkov

Encoding square-tiled surfaces by pairs of permutations

Consider a square-tiled surface $S \in \mathcal{H}(d_1, \ldots, d_n)$. Enumerate the squares in some way. For the square number j let $\pi_h(j)$ be the number of its neighbor to the right and let $\pi_v(j)$ be the number of the square atop the square number j.



Example. Our favorite L-shaped surface tiled with 3 squares can be encoded by the following two permutations decomposed into cycles:

$$\pi_h = (1,2)(3) \qquad \pi_v = (1,3)(2)$$

Note that there is no canonical enumeration of squares, so the permutations π_h, π_v are defined up to a simultaneous conjugation.

Almost commuting permutations

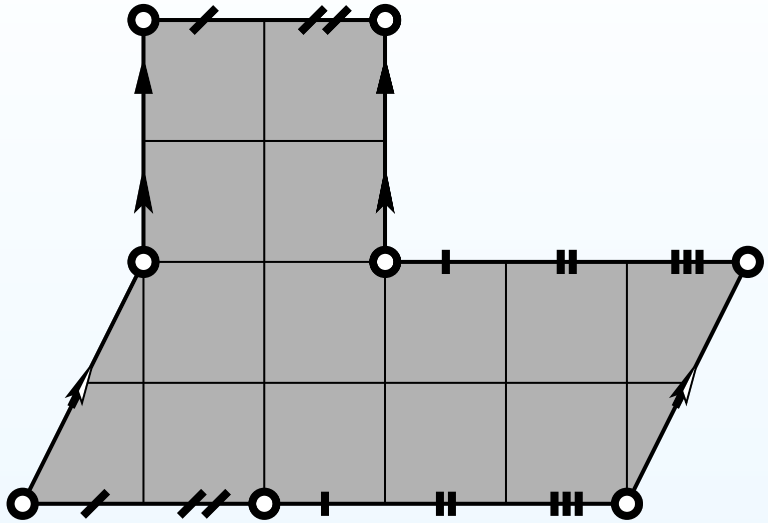
Consider the commutator $\pi_h\pi_v\pi_h^{-1}\pi_v^{-1}$ The resulting permutation corresponds to the following path: we start from a square number j, then we move one step right, one step up, one step left, one step down, and we arrive to $\pi'(j)$.

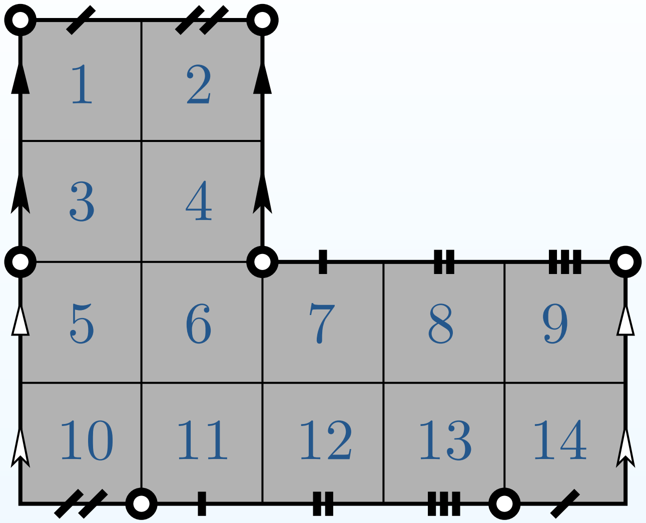
When the total number of squares is large, then for majority of the squares such path brings us back to the initial square; for such squares j we get $\pi'(j) = j$.

For squares having a singularity at the top right corner the path right-up-left-down does not bring us back to the initial square. The commutator $\pi_h\pi_v\pi_h^{-1}\pi_v^{-1}$ decomposes into a product of n cycles of lengths $(d_1 + 1), \ldots, (d_n + 1)$ correspondingly completed with cycles of length 1.

For example, for any square-tiled surface in $\mathcal{H}(2)$ the commutator is a single 3-cycle completed with plenty of fixed points.

Encoding square-tiled surfaces by pairs of permutations





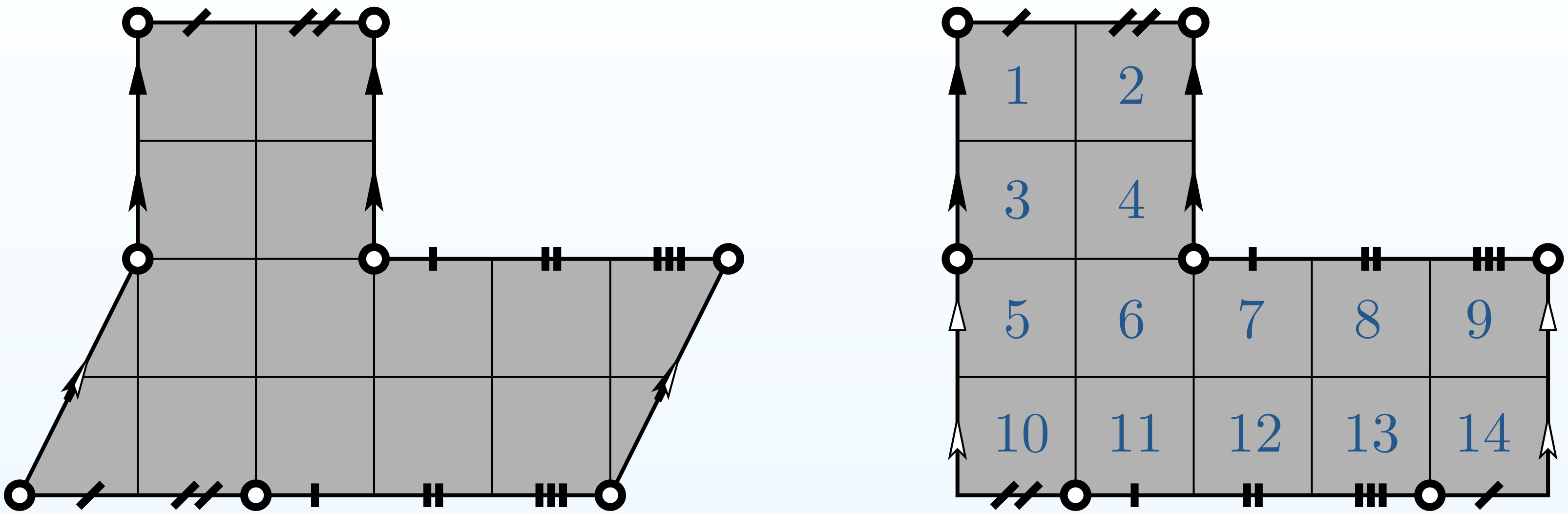
$$\pi_h = (1,2)\,(3,4)\,(5,6,7,8,9)\,(10,11,12,13,14)$$

$$\pi_v = (1,14,9,13,8,12,7,11,6,4,2,10,5,3)$$

$$\pi_h \pi_v \pi_h^{-1} \pi_v^{-1} = \mathbf{(2,9,6)}\,(1)\,(3)\,(4)\,(5)\,(7)\,(8)\,(10)\,(11)\,(12)\,(13)\,(14)$$

The commutator $\pi_h \pi_v \pi_h^{-1} \pi_v^{-1}$ decomposes into a single cycle of length 3 completed with cycles of length 1. The cycle of length 3 corresponds to 3 squares, for which the top right corner is located at the conical singularity. There are 4 times more corners of squares at the same singularity, so the cone angle is $3 \cdot 2\pi$, where 3 is the length of the cycle. Our surface lives in $\mathcal{H}(2)$.

Encoding square-tiled surfaces by pairs of permutations



$$\pi_h = (1,2)\,(3,4)\,(5,6,7,8,9)\,(10,11,12,13,14)$$

$$\pi_v = (1,14,9,13,8,12,7,11,6,4,2,10,5,3)$$

$$\pi_h\pi_v\pi_h^{-1}\pi_v^{-1} = \mathbf{(2,9,6)}\,(1)\,(3)\,(4)\,(5)\,(7)\,(8)\,(10)\,(11)\,(12)\,(13)\,(14)$$

We conclude that a square-tiled surface $S \in \mathcal{H}(d_1,\dots,d_n)$ tiled with N squares can be encoded by a pair of permutations π_h, π_r (defined up to a common conjugation) such that the commutator $\pi_h\pi_v\pi_h^{-1}\pi_v^{-1}$ decomposes into given number n of cycles of given lengths $(d_1+1),\dots,(d_n+1)$ and π_h, π_r do not have common nontrivial invariant subsets in $1,2,\dots,N$.

Count by A. Eskin, A. Okounkov, R. Pandharipande

Using a version of the description of square-tiled surfaces by pairs of almost commuting permutations and using results of S. Bloch and A. Okounkov, A. Eskin, A. Okounkov and R. Pandharipande proved the following assertion.

Theorem (A. Eskin, A. Okounkov, R. Pandharipande) *For every connected component of every stratum the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled}\\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form: it is a polynomial in Eisenstein series $G_2(q)$, $G_4(q)$, $G_6(q)$ of controllable complexity.

Corollary (A. Eskin, A. Okounkov, R. Pandharipande) *The Masur–Veech volume $\operatorname{Vol}\mathcal{H}_1^{comp}(d_1,\dots,d_n)$ of every connected component of every stratum is a rational multiple of π^{2g}, where $2g-2=d_1+\dots+d_n$.*

Masur–Veech volumes of strata of Abeliand differentials: a historical retrospective

● Around 1998. Masur–Veech volumes of several low-dimensional strata of Abelian differentials were evaluated by M. Kontsevich and A. Zorich through straightforward count of square-tiled surfaces.

● Around 2001. A. Eskin and A. Okounkov found a much more efficient approach based on quasimodularity of the relevant generating function. A. Eskin wrote a computer code giving volumes of all strata in genera at most 10 and of some strata in genera up to 200.

● 2020. D. Chen, M. Möller, A. Sauvaget and D. Zagier obtained very important advances based on recent BCGGM smooth compactification of the moduli space of Abelian differentials. They developed intersection theory of relevant moduli spaces and found a recursive formula for volumes.

● 2018–2020. D. Chen–M. Möller–A. Sauvaget–D. Zagier and independently A. Aggarwal obtained spectacular results on large genus asymptotics of Masur–Veech volumes uniform for all strata stratum of Abelian differentials proving a conjecture by A. Eskin and of A. Zorich based on their numerical experiments from 2003.

Masur–Veech volumes of strata of quadratic differentials: a brief historical retrospective

The knowledge of Masur–Veech volumes $\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)$ of strata of *quadratic* differentials is still limited.

● Around 1998-2000. Masur–Veech volumes of several low-dimensional strata of quadratic differentials were evaluated by A. Zorich through straightforward count of square-tiled surfaces.

● 2001. A. Eskin and A. Okounkov found a much more efficient approach based on quasimodularity of the generating function counting *pillowcase covers*. However, the resulting expressions contain huge tables of characters of the symmetric group, which makes the computation inefficient. The algorithm is more involved than for Abelian differentials.

● 2016. The algorithm of A. Eskin and A. Okounkov was implemented by E. Goujard. She wrote a code and computed volumes of all strata up to dimension 12.

Masur–Veech volumes of strata of quadratic differentials: a brief historical retrospective

- 2016. J. Athreya–A. Eskin–A. Zorich obtained a close expression (conjectured by M. Kontsevich) for the Masur–Veech volume of any stratum in genus zero through the formula of A. Eskin–M. Kontsevich–A. Zorich for the sum of Lyapunov exponents combined with some combinatorial considerations.
- 2019. V. Delecroix–E. Goujard–P. Zograf–A. Zorich computed volumes of the principal strata (the ones containing only simple zeroes and poles) in terms of Witten–Kontsevich correlators.
- 2019. D. Chen–M. Möller–A. Sauvaget expressed volumes of the principal strata in terms of certain Hodge integrals.
- 2019. J. Andersen–G. Borot–S. Charbonnier–V. Delecroix–A. Giacchetto–D. Lewanski–C. Wheeler used the DGZZ-formula to compute volumes through topological recursion.
- 2020. M. Kazarian and independently Di Yang–D. Zagier–Y. Zhang developed efficient recursion for the Hodge integrals involved in the CMS-formula.
- 2021. A. Aggarwal derived the large genus asymptotics for the volumes of principal strata conjectured by V. Delecroix–E. Goujard–P. Zograf–A. Zorich.

Open problem: volumes of strata of quadratic differentials

Let $\boldsymbol{d} = (d_1, \dots, d_n)$ be an unordered partition of a positive integer number $4g-4$ divisible by 4 into a sum $|\boldsymbol{d}| = d_1 + \dots + d_n = 4g-4$, where $d_i \in \{-1, 0, 1, 2, \dots\}$ for $i = 1, \dots, n$. Denote by $\hat{\Pi}_{4g-4}$ the set of those partitions as above, which satisfy the additional requirement that the number of entries $d_i = -1$ in $\boldsymbol{d}$ is at most $\log(g)$.

Open problem. *Find the Masur–Veech volume of strata $\mathcal{Q}(d_1, \dots, d_n)$ of meromorphic quadratic differentials with at most simple poles when at least one od d_i is even. Prove the following conjectural asymptotic formula (currently proved by A. Aggarwal only for the principal stratum): for any $\boldsymbol{d} \in \hat{\Pi}_{4g-4}$ one has*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_n) = \frac{4}{\pi} \cdot \prod_{i=1}^{n} \frac{2^{d_i+2}}{d_i+2} \cdot \big(1 + \varepsilon_1(\boldsymbol{d})\big)\,,$$

where

$$\lim_{g\to\infty} \max_{\boldsymbol{d}\in\hat{\Pi}_{4g-4}} |\varepsilon_1(\boldsymbol{d})| = 0\,.$$

For strata of dimension up to 12 the volumes are found by E. Goujard using Eskin–Okounkov algorithm.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