Dynamics and Geometry of Moduli Spaces

Lecture 9. Enumeration of meanders and Masur–Veech volumes of moduli spaces of quadratic differentials

Anton Zorich
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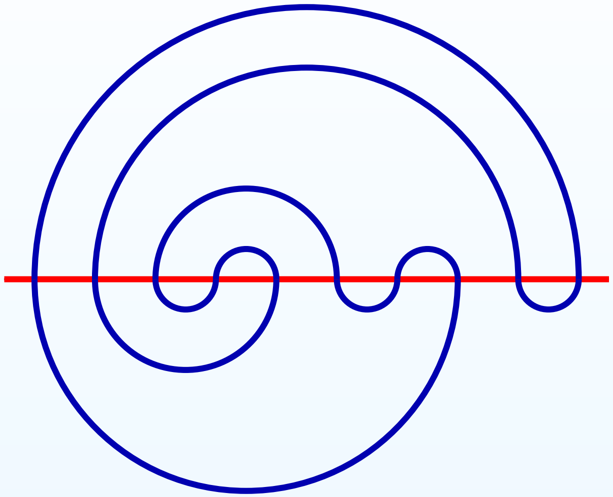
April 13, 2023

Meanders

Meanders and arc systems

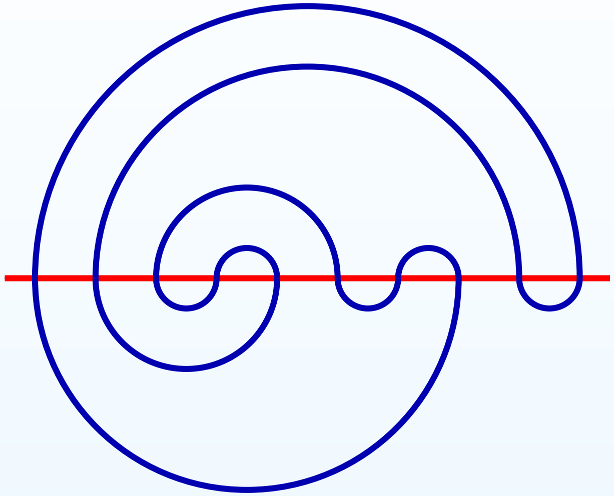


A closed *meander* is a smooth simple closed curve in the plane transversally intersecting the horizontal line.

According to S. Lando and A. Zvonkin the notion “meander” was suggested by V. Arnold though meanders were studied already by H. Poincaré.

Meanders appear in various contexts, in particular in mathematics, physics and biology.

Meanders and arc systems



Conjecture (S. Lando and A. Zvonkin, 1993). *The number of meanders with $2N$ crossings is asymptotic to*

$$const \cdot R^{2N} \cdot N^{\alpha} \quad \text{for} \quad N \to \infty \,,$$

where $R^2 \approx 12.26$ (value is due to I. Jensen) and $\alpha = -\frac{29+\sqrt{145}}{12}$ (conjectural value due to P. Di Francesco, O. Golinelli, E. Guitter, 1997).

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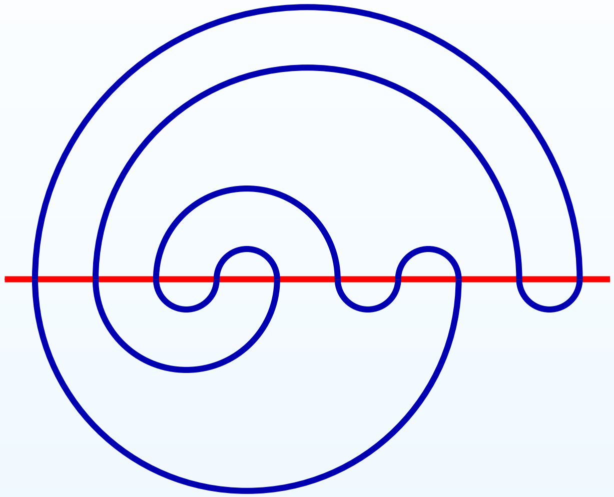
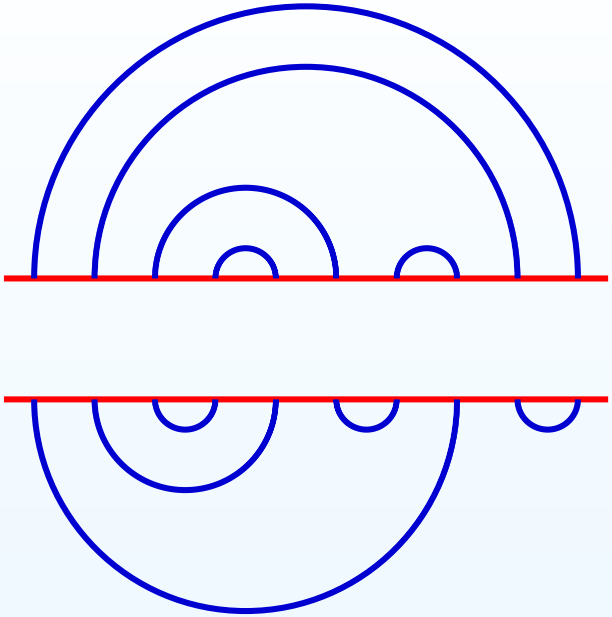
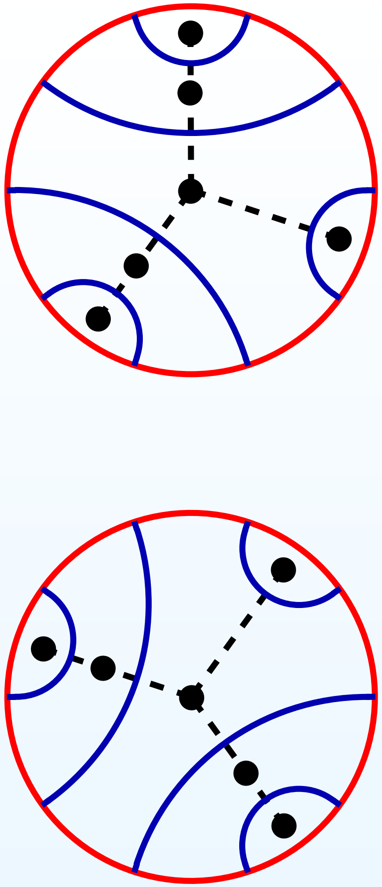
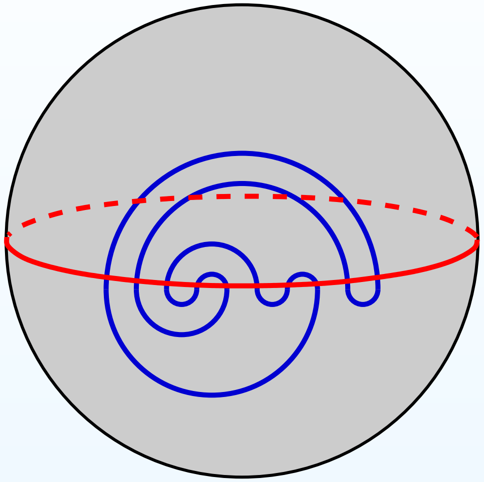


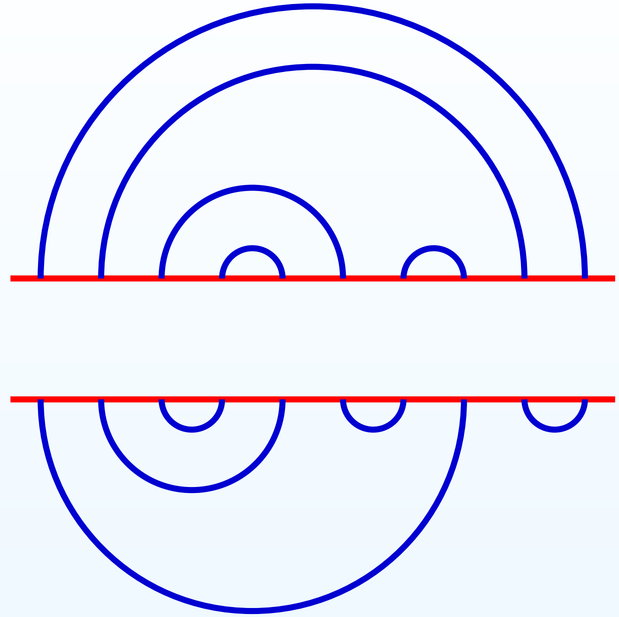


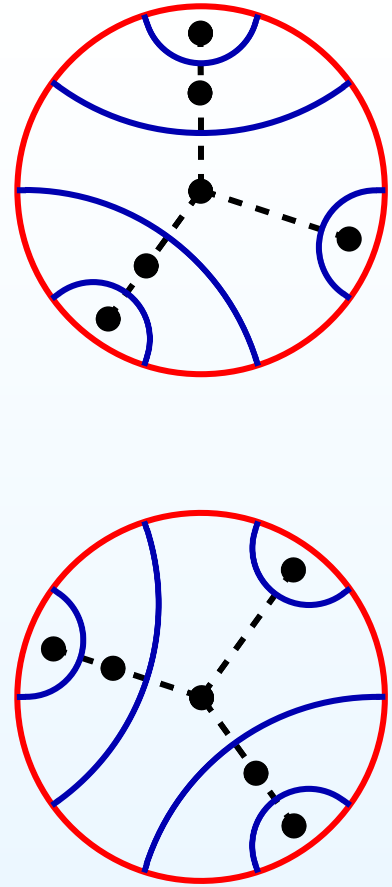


A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Meanders and arc systems





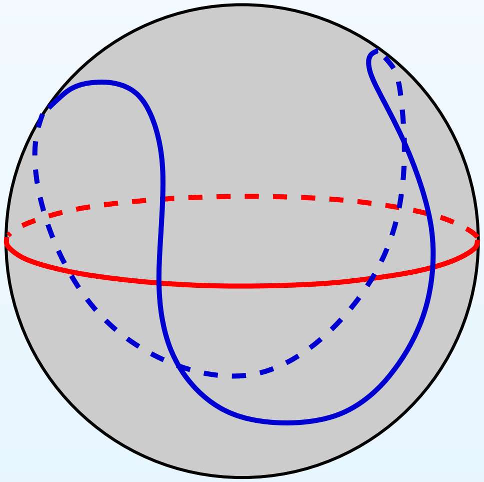


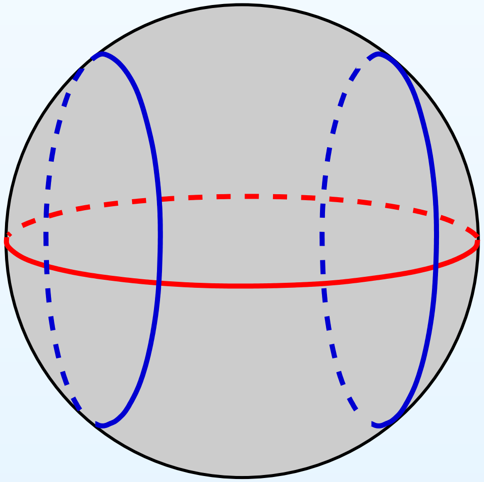
A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual graphs on the right. We usually erase vertices of valence 2 from dual trees.

Compactifying the plane (left picture) with one point at infinity, or gluing together arc systems on the two discs (right picture) we get an ordered pair of smooth simple transversally intersecting closed curves on the sphere.

Meanders versus multicurves

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a *multicurve*, i.e. a curve with several connected components.





Attaching arc systems on a pair of hemispheres along the common equator we might get a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

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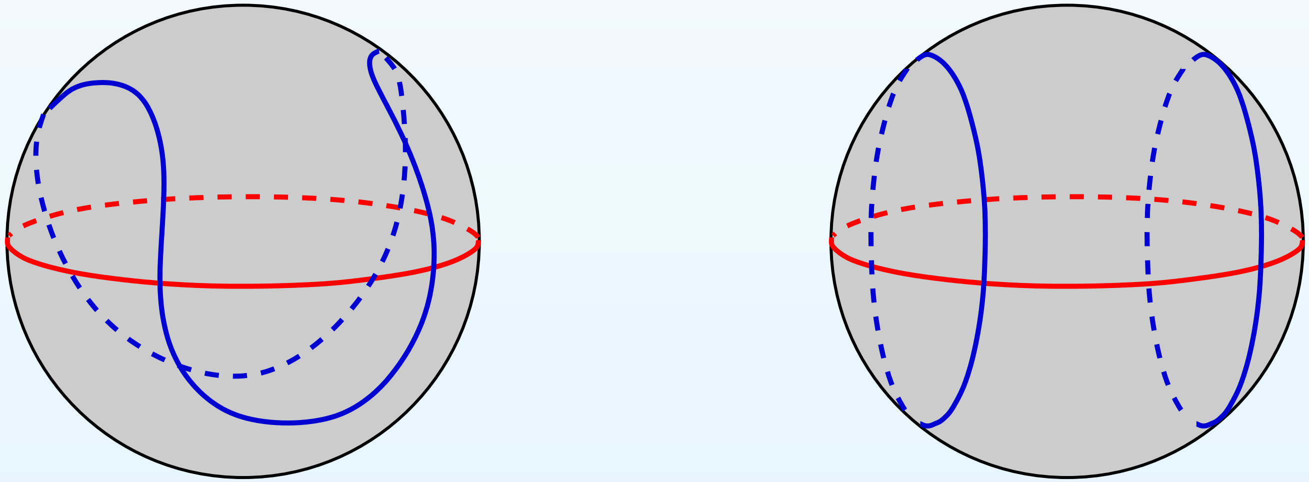

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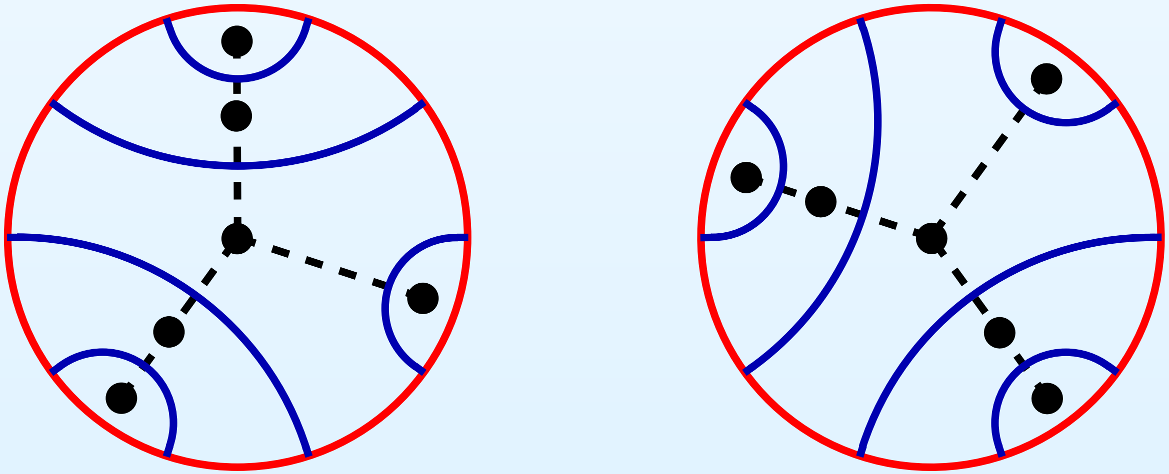
Asymptotic frequency of meanders

Fix any connected planar tree $\mathcal{T}_{North}$ on the northern hemisphere and any connected planar tree $\mathcal{T}_{South}$ on the southern hemisphere, each tree having no vertices of valence 2. Consider all possible pairs of arc systems with the same number $n \leq N$ of arcs having $\mathcal{T}_{North}$ and $\mathcal{T}_{South}$ as reduced dual trees. There are $2n$ ways to identify isometrically the two hemispheres into the sphere in such way that the endpoints of the arcs match. Consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{North};\, n\text{-arc system of type } \mathcal{T}_{South};\, \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,.$$



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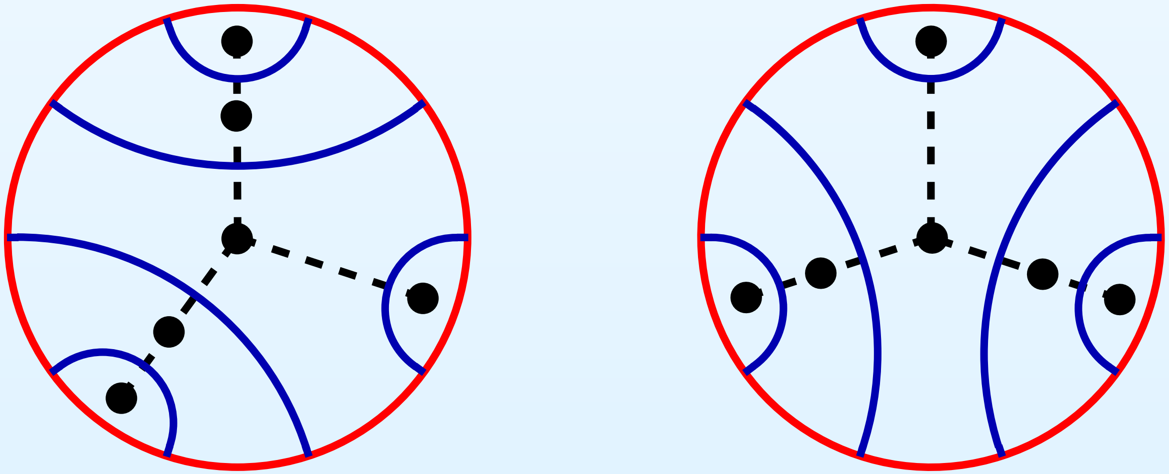

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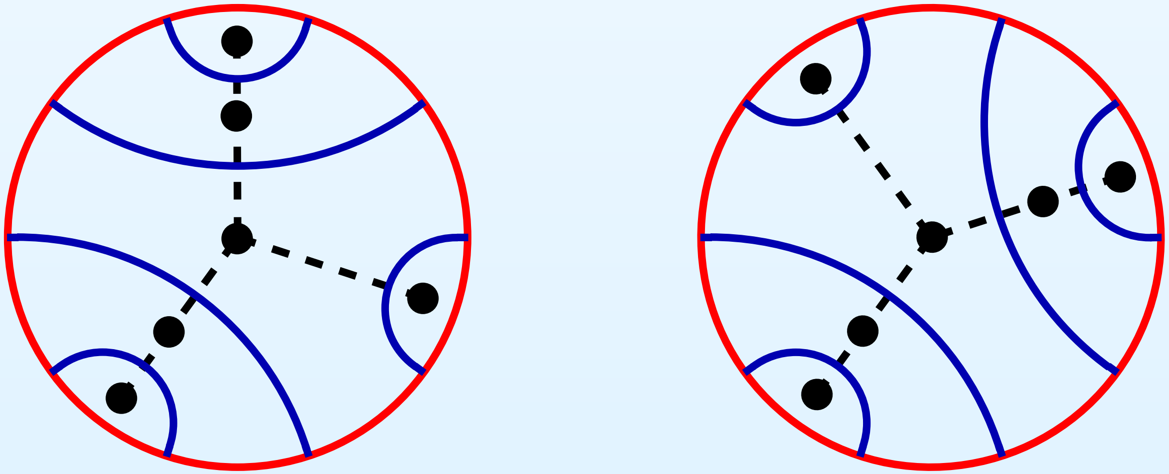

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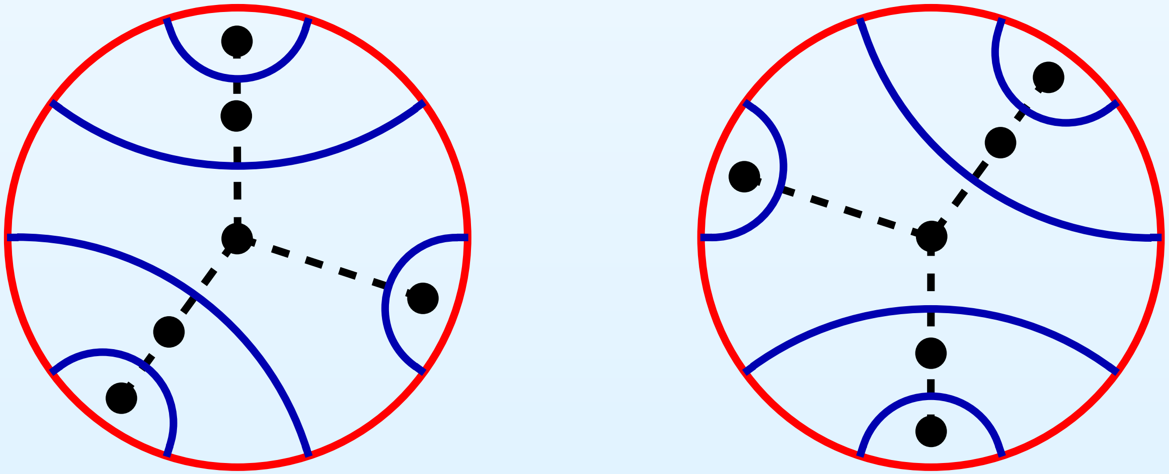

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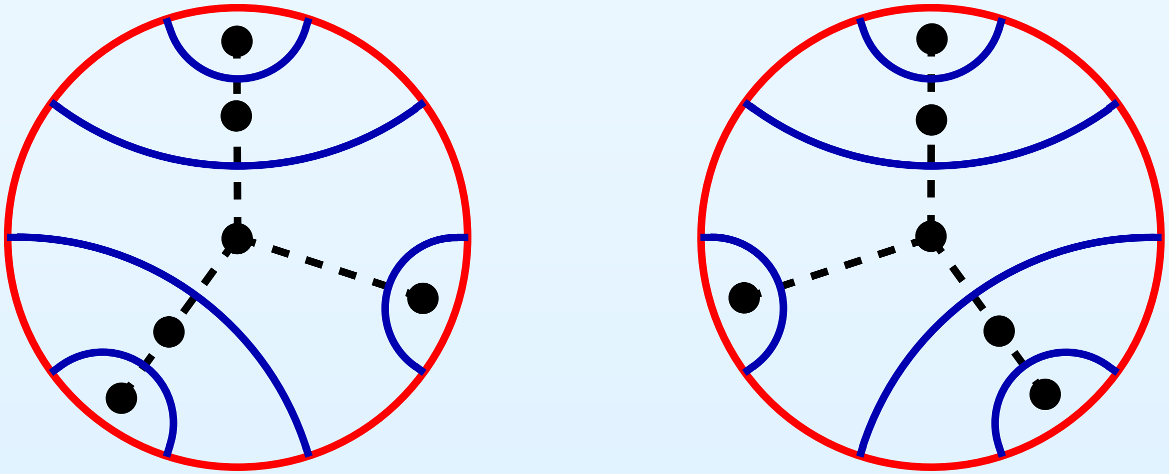

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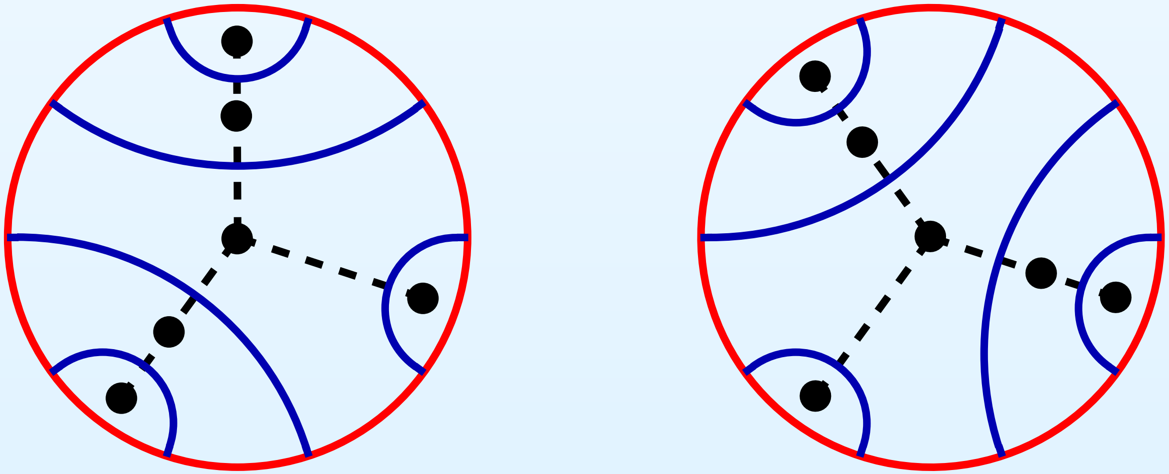

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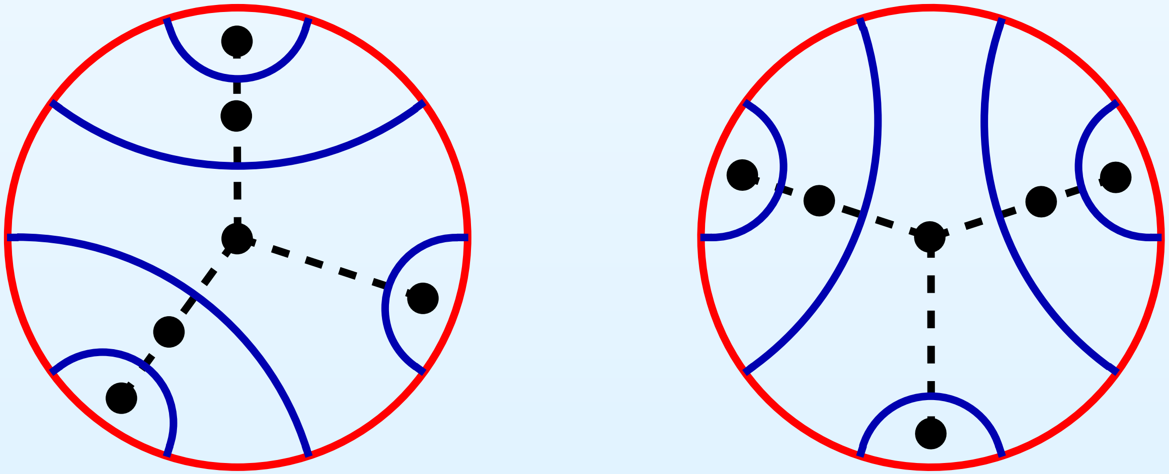

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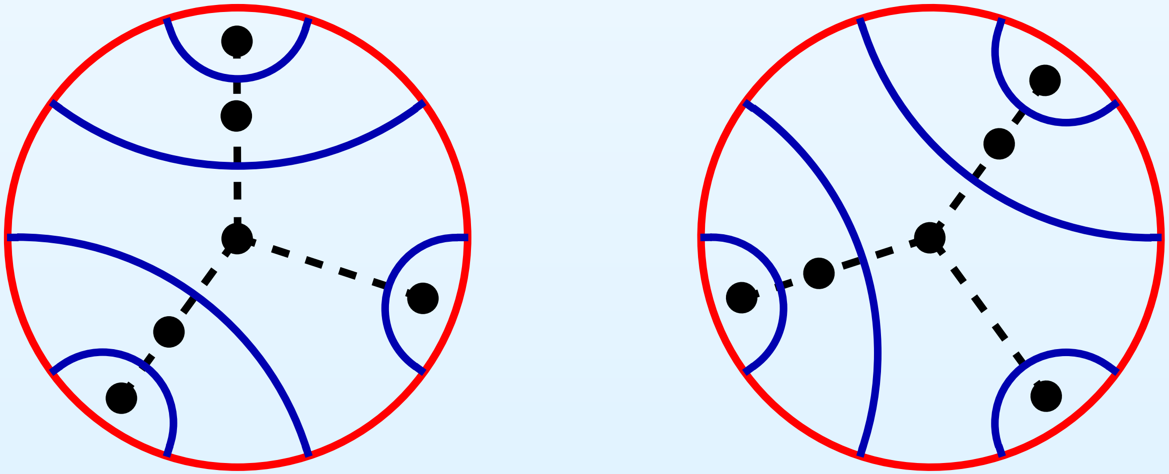

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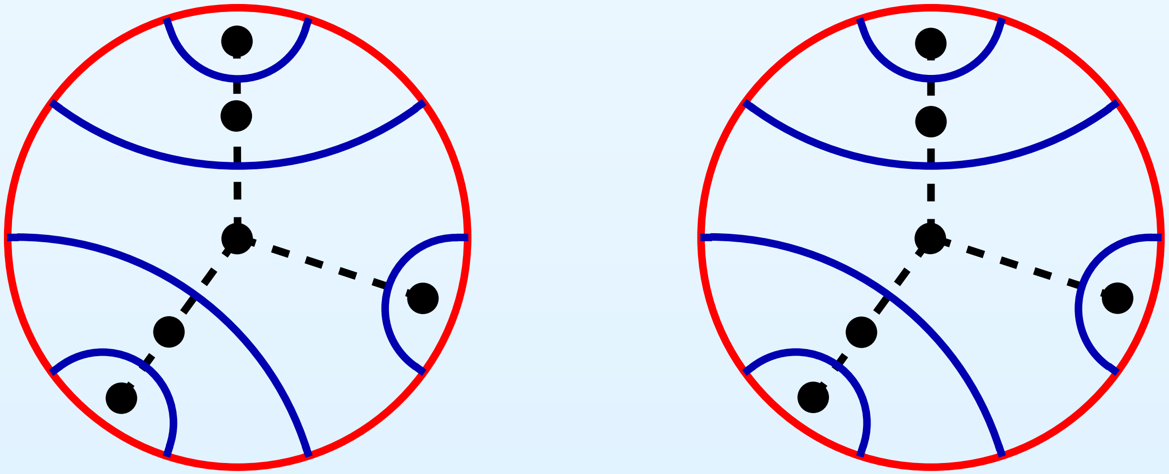

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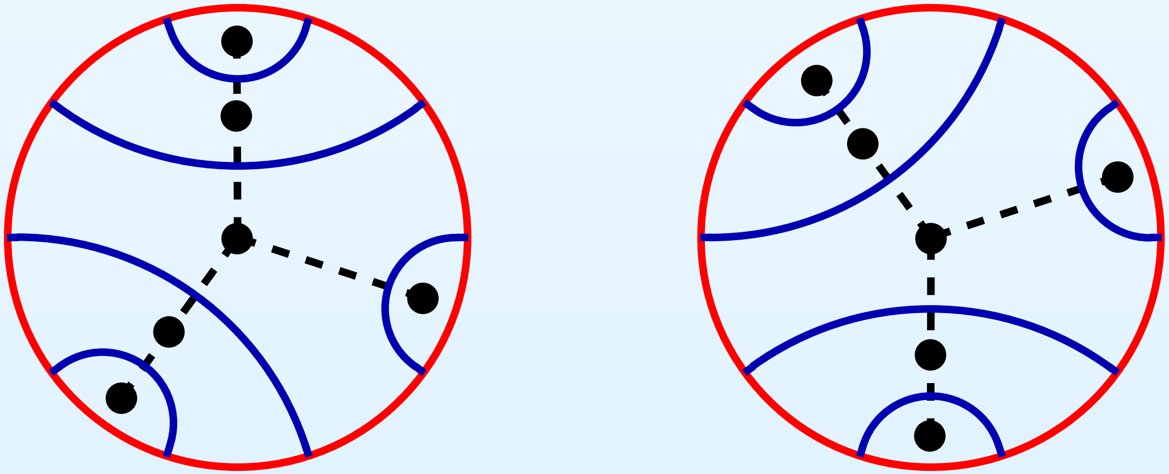

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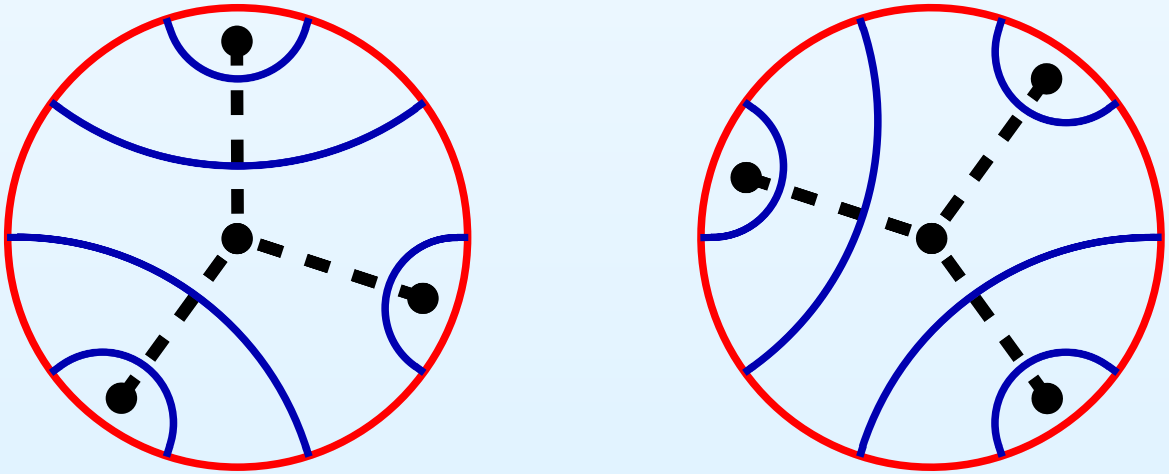

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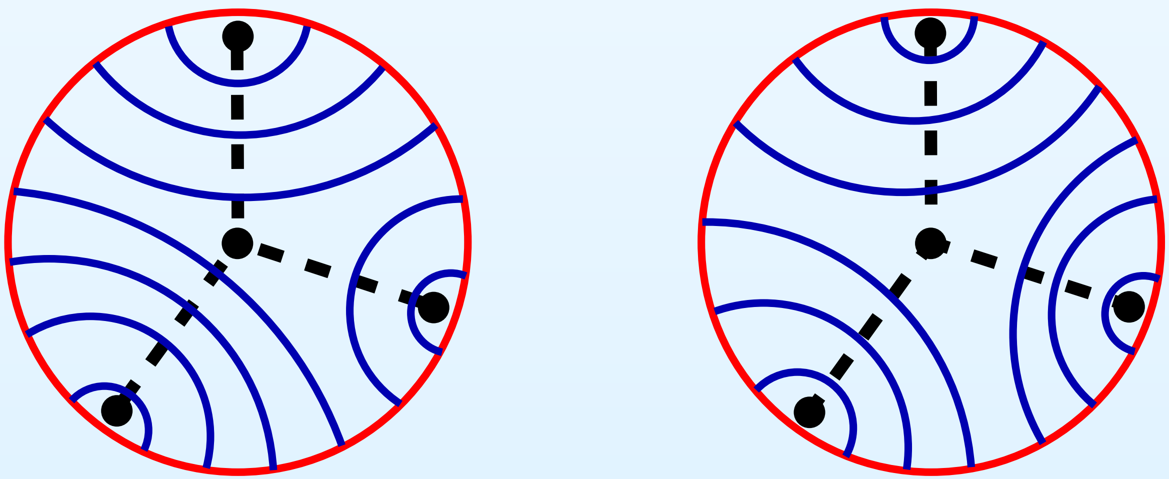

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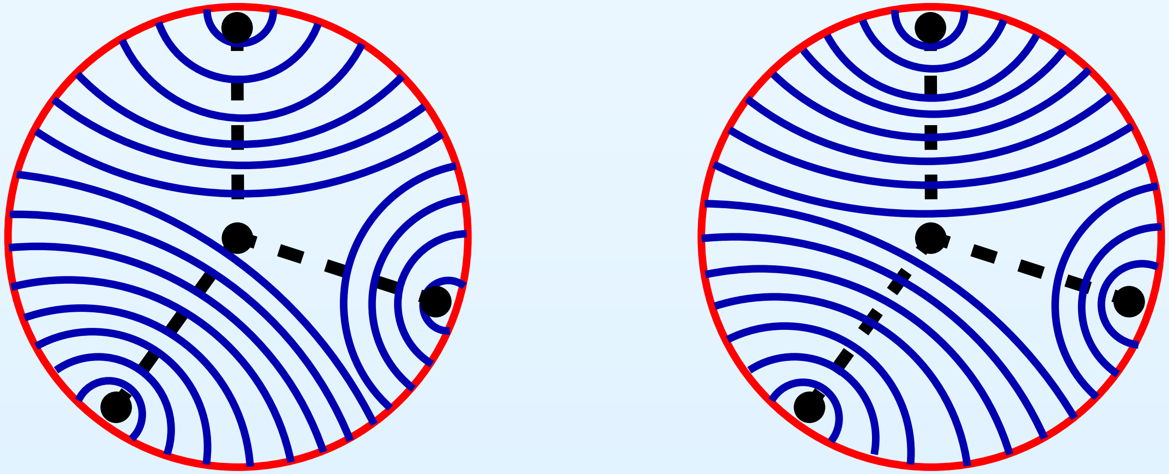

Asymptotic frequency of meanders

Question. *What is the asymptotic probability*

$$\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N) \sim ? \quad \text{as } N \to +\infty$$

to get a meander (i.e. a connected curve) by a random gluing of a random pair of arc systems as above with $n \leq N$ arcs?
Does it behave like N^{-a}? Like $\exp(-bN)$? If so, describe how a (respectively b) depend on $\mathcal{T}_{North}, \mathcal{T}_{South}$.

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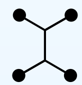
Theorem. *For any pair of trees $\mathcal{T}_{North}, \mathcal{T}_{South}$ the quantity $\mathrm{P}_{connected}(\mathcal{T}_{North}, \mathcal{T}_{South}; N)$ admits a strictly positive limit as $N \to +\infty$.*
We have an explicit formula for this limit in terms of the total number of vertices of valence $1, 3, 4, \ldots$ of the two trees.

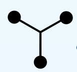
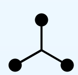
I have to confess that the fact that this asymptotic frequency is nonzero was unexpected to me.

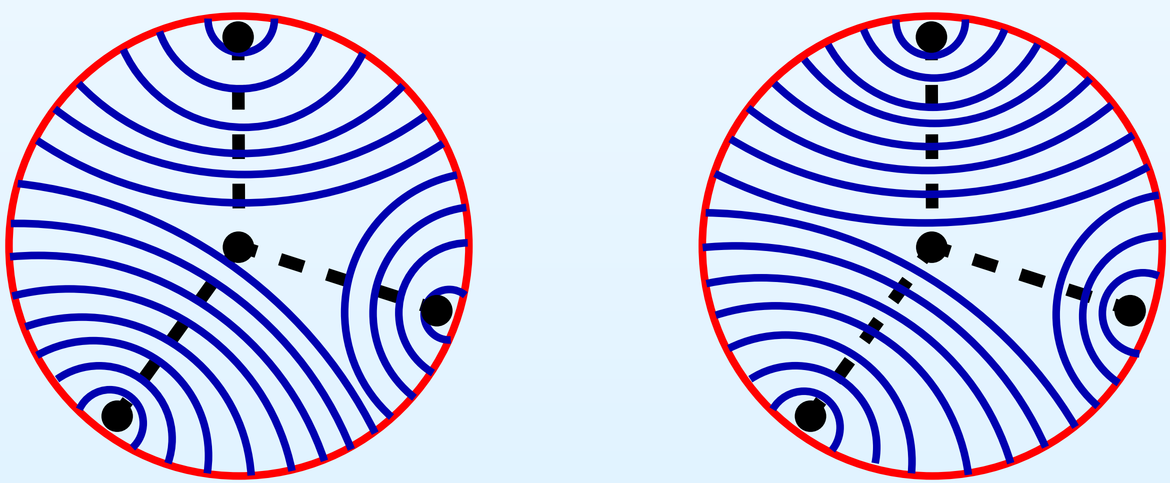
Asymptotic frequency of meanders

Theorem. *Let $p_{North}, p_{South} \geq 2$. Let $p = p_{North} + p_{South}$. The frequency $\mathrm{P}_{connected}(p_{North}, p_{South}; N)$ of meanders obtained by all possible identifications of all arc systems with at most N arcs represented by all possible pairs of plane trees having p_{North}, p_{South} of leaves (vertices of valence one) has the following limit:*

$$\lim_{N\to+\infty} \mathrm{P}_{connected}(p_{North}, p_{South}; N) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}\,.$$

Example. $\lim_{N\to+\infty} \mathrm{P}_{connected}($  $,$  $, N) =$

$= \lim_{N\to+\infty} \mathrm{P}_{connected}($  $,$  $, N) = \frac{280}{\pi^6} \approx 0.291245\,.$



Elementary estimates on the asymptotic number of meanders

The number of the arc diagrams of order N is the N-th Catalan number

$$C_N = \frac{1}{N+1}\binom{2N}{N} \sim \frac{1}{\sqrt{\pi}} \cdot 4^N \cdot N^{-3/2}\,.$$

Any upper arc diagram may be completed to a meander by an appropriate lower arc diagram (up to the best o f my knowledge, this observation is due to S. Lando and A. Zvonkin). Thus, we obtain trivial upper and lower bounds:

$$C_N \le M_N \le C_N^2\,.$$

The conjectural asymptotics is in between:

Conjecture (S. Lando and A. Zvonkin, 1993). *The number M_N of meanders with $2N$ crossings is asymptotic to*

$$M_N \sim const \cdot R^{2N} \cdot N^{\alpha} \quad \text{for} \quad N \to \infty\,,$$

where $R^2 \approx 12.26$ (value is due to I. Jensen) and $\alpha = -\frac{29+\sqrt{145}}{12}$ (conjectural value due to P. Di Francesco, O. Golinelli, E. Guitter, 1997).

Twisting a pair of arc systems

The conjecture claims that M_N grows much slower than C_N^2. This indicates that for a random pair of arc systems, twisting cyclically one of them we never get a meander, whatever twist we chose.

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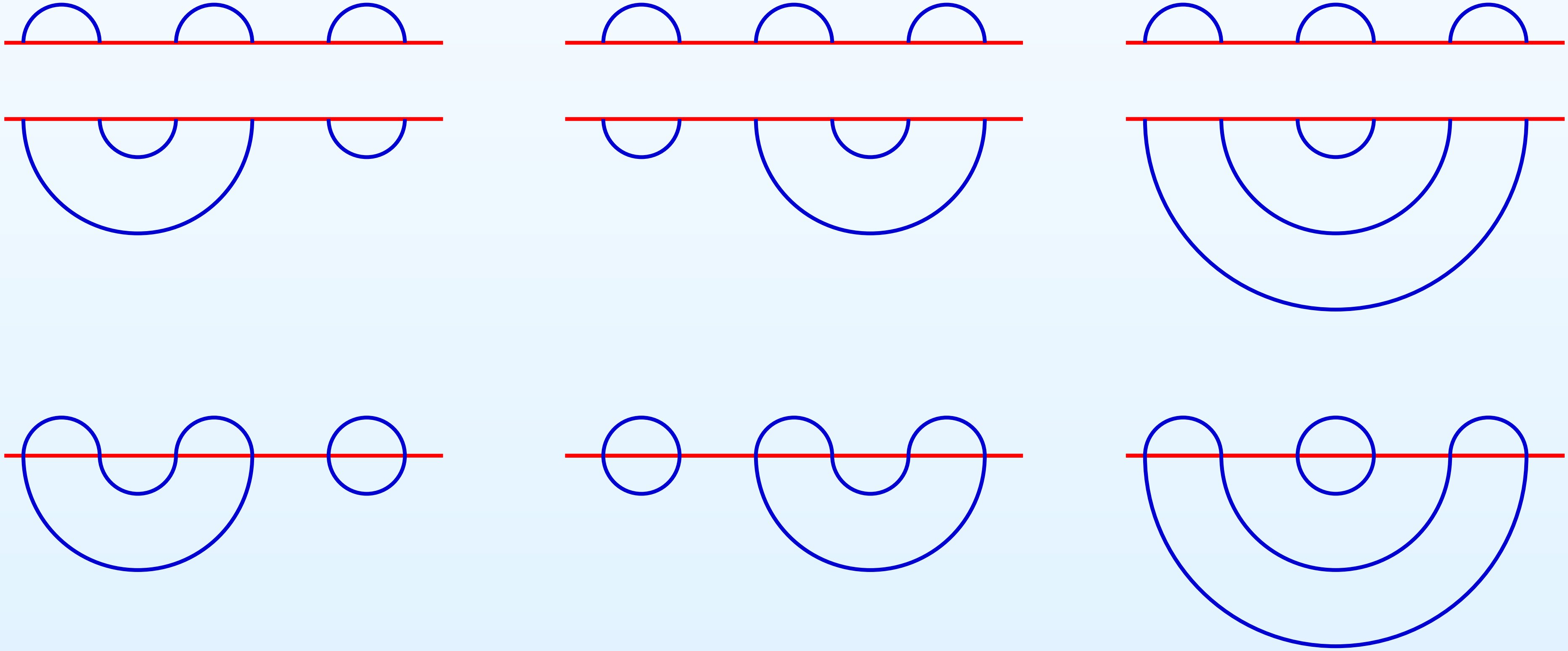
Here is a concrete example:



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The conjecture claims that M_N grows much slower than C_N^2. This indicates that for a random pair of arc systems, twisting cyclically one of them we never get a meander, whatever twist we chose.

However, if we change the setting and choose a random pair of arc systems *having a fixed total number* p *of minimal arcs* and identify them by a random twist, then by our result with V. Delecroix, E. Goujard and P. Zograf we get a meander with probability

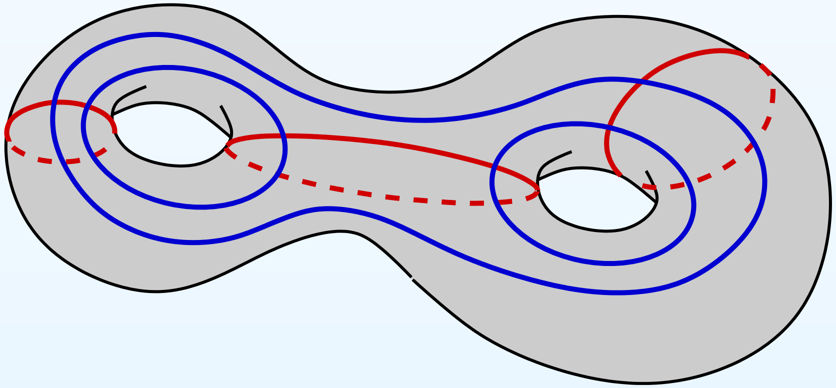
$$\frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2} \sim \frac{2}{\sqrt{\pi p}} \cdot \left(\frac{8}{\pi^2}\right)^{p-3} \text{ for } p \gg 1$$

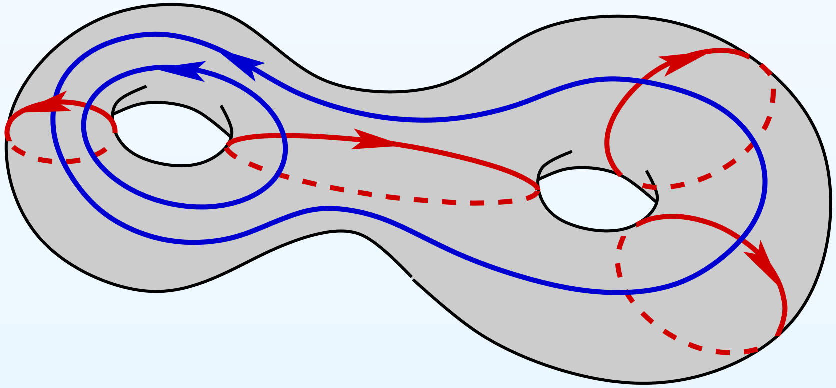
even without further twisting.

There is no contradiction. General pairs of random arc systems with N arcs on each side have about N minimal arcs in total, while in our conditional setting we have only a fixed number p of minimal arcs, while $N \to +\infty$.

Meanders in higher genera

A pair of smooth simple closed transverse oriented multicurves is called *positively intersecting* if each connected component of each multicurve is oriented in such way that all intersections match the orientation of the surface. A pair of transverse multicurves is called *orientable* if it admits an orientation with positive intersection and *non-orientable* otherwise.





Exercise. Verify that the pair of transverse multicurves on the right is positively intersecting and that the pair of multicurves on the left is non-orientable.

Definition. A *meander* on a surface of genus g is an ordered pair of smooth transverse simple closed curves considered up to a diffeomorphism of the surface. Similarly we define *positive meanders*.

Results: general (non-orientable) case

Fix the genus g of the surface. Fix a nonnegative integer p denoting the number of bigons produced by intersections of pairs of multicurves.

Observation. *The following quantities have polynomial asymptotics:*

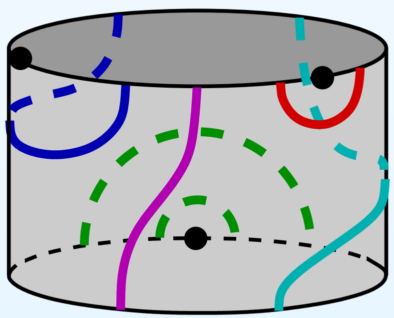
- *Number of pairs of transverse multicurves with at most N intersections and with exactly p bigons $= c(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*
- *Number of pairs (simple closed curve, transverse multicurve) with at most N intersections and p bigons $= c_1(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*
- *Number of meanders with at most N intersections and with exactly p bigons $= c_{1,1}(g,p) \cdot N^{6g-6+2p} + o(N^{6g-6+2p})$.*

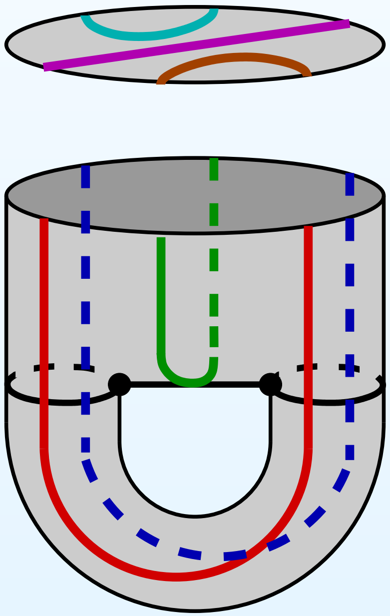
Theorem. *The coefficients $c(g,p), c_1(g,p), c_{1,1}(g,p)$ satisfy the following relation:*

$$\frac{c_1(g,p)}{c(g,p)} = \frac{c_{1,1}(g,p)}{c_1(g,p)}\,.$$

Asymptotic frequency of meanders

As in genus 0, we can construct multicurves from systems of arcs on a surface of genus $g-1$ with two boundary components or on a pair of surfaces of genera g_1 and g_2, where $g_1+g_2=g$, each with a single boundary component. As before we fix the total number p of bigons. We assume that there are exactly n arcs landing to each of the two boundary components, and that $n \le N$.





Theorem. *The asymptotic probability to get a meander after a random gluing of a random system of arcs as above is* $\frac{c_1(g,p)}{c(g,p)}$.

Results on positively intersecting pairs of multicurves

The case of positively intersecting pairs of multicurves is analogous. However, the power of N and all the coefficients in the polynomial asymptotics do change. Fix the genus g of the surface.

Observation. *The following quantities have polynomial asymptotics:*

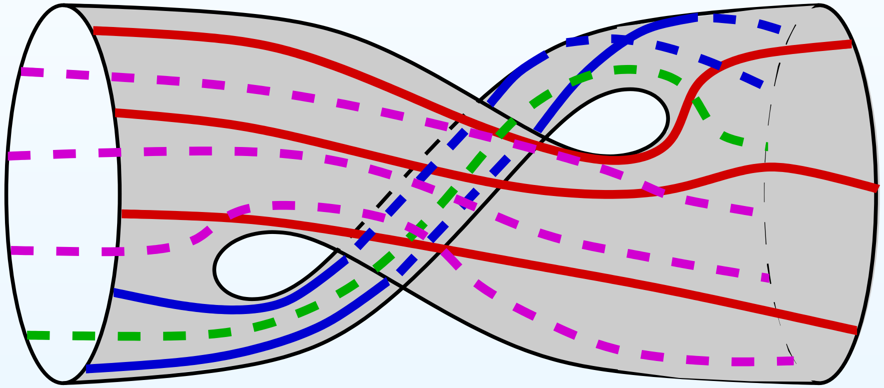
- *Number of pairs of transverse positively intersecting multicurves with at most N intersections $= c^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*
- *Number of positively intersecting pairs (simple closed curve, transverse multicurve) with at most N intersections $= c_1^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*
- *Number of positive meanders with at most N intersections $= c_{1,1}^+(g) \cdot N^{4g-3} + o(N^{4g-3})$.*

Theorem. *The coefficients $c^+(g), c_1^+(g), c_{1,1}^+(g)$ satisfy the relation:*

$$\frac{c_1^+(g)}{c^+(g)} = \frac{c_{1,1}^+(g)}{c_1^+(g)}\,.$$

Asymptotic frequency of positive meanders

As before we can glue systems of arcs on a surface of genus $g-1$ with two boundary components. This time we assume that each of n arc goes from one boundary component to the other, and that $n \leq N$.

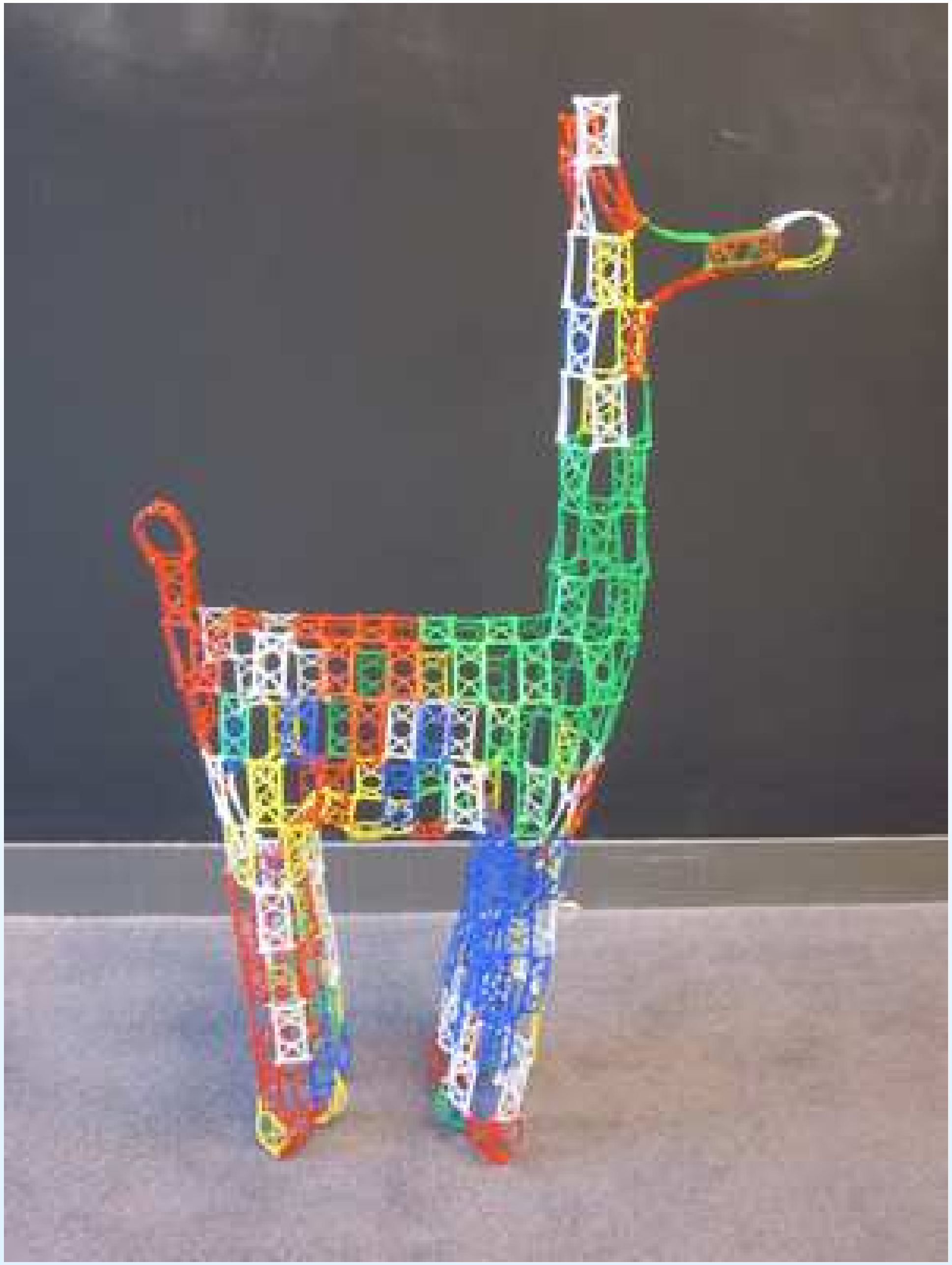


Theorem. *The asymptotic probability to get a positive meander after a random gluing of a system of arcs as above is* $\frac{c_1^+(g)}{c_g^+}$*. We have*

$$\frac{c_1^+(g)}{c_g^+} = \frac{1}{4g} + o\left(\frac{1}{g}\right) \text{ as } g \to +\infty\,.$$

Square-tiled surfaces

An example of a square-tiled surface



Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres.

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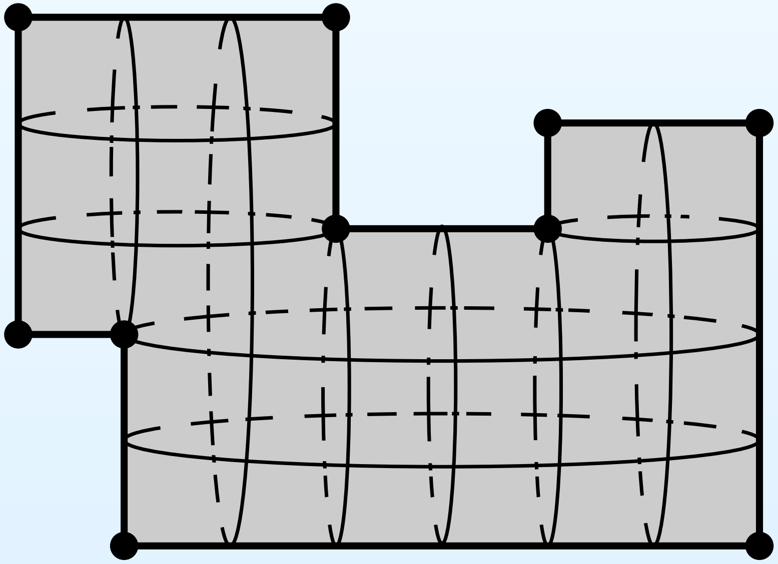

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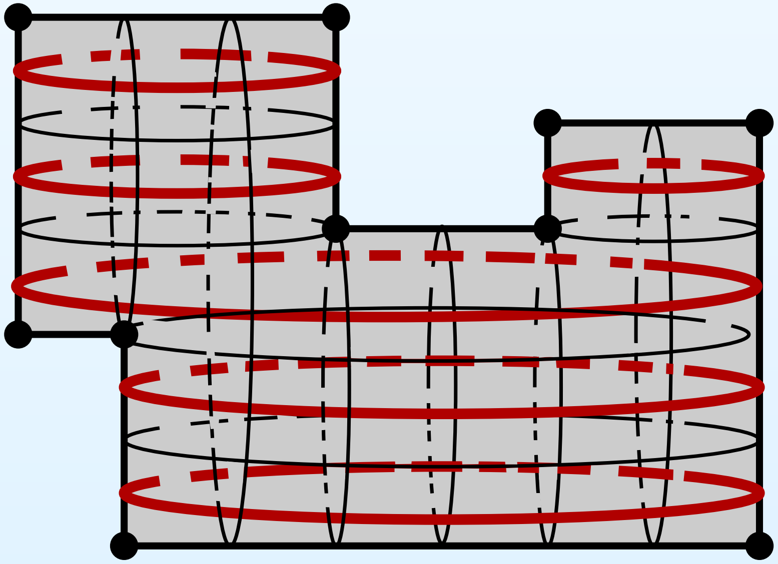
Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve.



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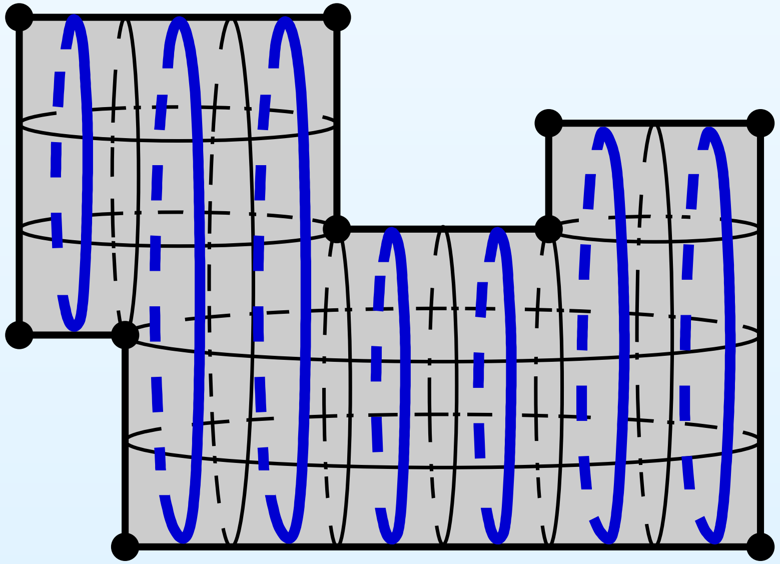
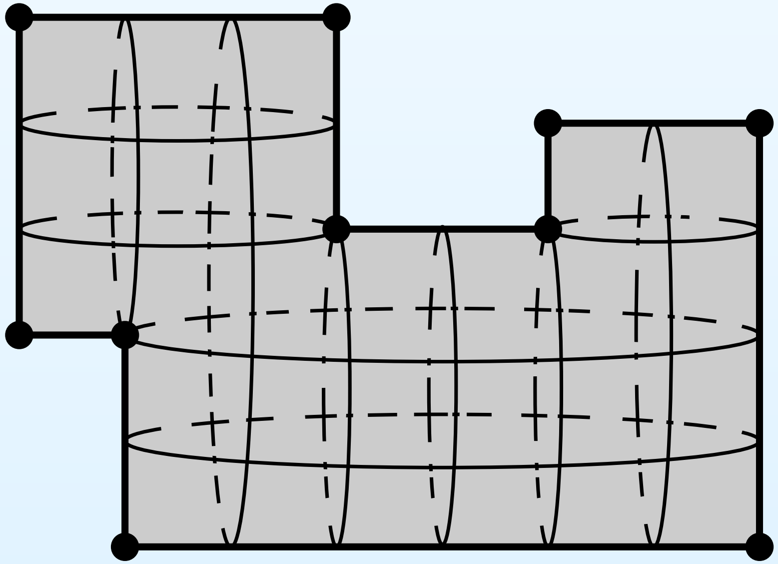
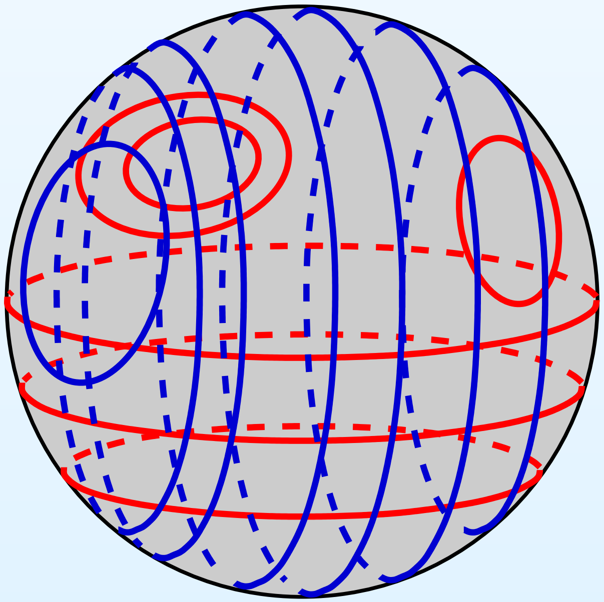
Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve. Consider the maximal collection of vertical lines passing through the centers of the squares. Color them in blue. This is the vertical multicurve.



Pairs of transverse multicurves as square-tiled surfaces

There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and square-tiled spheres. Consider a square-tiled sphere. Consider the maximal collection of horizontal lines passing through the centers of the squares. Color them in red. This is the horizontal multicurve. Consider the maximal collection of vertical lines passing through the centers of the squares. Color them in blue. This is the vertical multicurve. Reciprocally, any transverse connected pair of multicurves on a sphere defines a square-tiling given by the graph dual to the graph formed by the pair of multicurves.





Masur–Veech volumes of the moduli spaces of Abelian and quadratic differentials

Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs (C,ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega\wedge\bar\omega = \sum_{i=1}^{g}(A_i\bar B_i - \bar A_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S=(C,r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu = r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1,\dots,m_n)$ or $\mathcal{Q}_1(m_1,\dots,m_n)$ of Abelian or quadratic differentials is finite.*

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Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) \in \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) = \mathbb{T}\,,$$

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Integer points in the strata $\mathcal{Q}(d_1, \ldots, d_n)$ of quadratic differentials are represented by analogous “pillowcase covers” over $\mathbb{C}\mathrm{P}^1$ branched at four points. Thus, counting Masur–Veech volumes of strata $\operatorname{Vol}\mathcal{H}$ or $\operatorname{Vol}\mathcal{Q}$ is equivalent to counting the coefficient c in the polynomial asymptotics $c \cdot N^d$ for the number of square-tiled surfaces in the stratum $\mathcal{H}$ or $\mathcal{Q}$ respectively, where d is the complex dimension of the stratum.

Non-correlation of vertical and horizontal foliations on square-tiled surfaces

Density and uniform density

Given a subset $D_{\mathbb{Z}} \subset \mathcal{H}_{\mathbb{Z}}$ in the set $\mathcal{H}_{\mathbb{Z}}$ of all square-tiled surfaces in some stratum $\mathcal{H}$ in the moduli space of Abelian (or quadratic) differentials and a subset $V \subset \mathcal{H}$ in the stratum define the following counting function:

$$\mathcal{N}_{D_{\mathbb{Z}}}(V,N) := \operatorname{Card}\{V \cap D_{\mathbb{Z}} :\ \operatorname{Area}(S) \le N\}\,.$$

The Masur–Veech volume $\operatorname{Vol}(\mathcal{H})$ is defined as the leading term of the asymptotic of $\mathcal{N}_{\mathcal{H}_{\mathbb{Z}}}(\mathcal{H},N)$ normalized by $d = \dim_{\mathbb{C}}(\mathcal{H})$:

$$\mathcal{N}_{\mathcal{H}_{\mathbb{Z}}}(\mathcal{H},N) = \frac{1}{2d}\cdot \operatorname{Vol}(\mathcal{H})\cdot N^d + o(N^d) \quad \text{as } N \to +\infty\,.$$

We say that $D_{\mathbb{Z}} \subset \mathcal{H}_{\mathbb{Z}}$ *has a density* $\delta(D_{\mathbb{Z}})$, if the following limit exists:

$$\delta(D_{\mathbb{Z}}) := \lim_{N\to+\infty} \frac{\mathcal{N}_{D_{\mathbb{Z}}}(\mathcal{H},N)}{\mathcal{N}_{\mathcal{H}_{\mathbb{Z}}}(\mathcal{H},N)}\,.$$

We say that a subset $D_{\mathbb{Z}} \subset \mathcal{H}_{\mathbb{Z}}$ *has a uniform density* if for any open cone C

$$\lim_{N\to+\infty} \frac{\mathcal{N}_{D_{\mathbb{Z}}}(C,N)}{\mathcal{N}_{\mathcal{H}_{\mathbb{Z}}}(C,N)} = \delta(D_{\mathbb{Z}})\,.$$

Analogy. The subset $\mathbb{N}\oplus\mathbb{N} \subset \mathbb{Z}\oplus\mathbb{Z}$ has density $1/4$ in $\mathbb{Z}\oplus\mathbb{Z}$ but not uniform density. The sublattice $2\mathbb{Z}\oplus 2\mathbb{Z} \subset \mathbb{Z}\oplus\mathbb{Z}$ has uniform density $1/4$.

Moore's Ergodicity Theorem

Consider the following horocyclic subgroups of the group $\mathrm{SL}(2,\mathbb{Z})$:

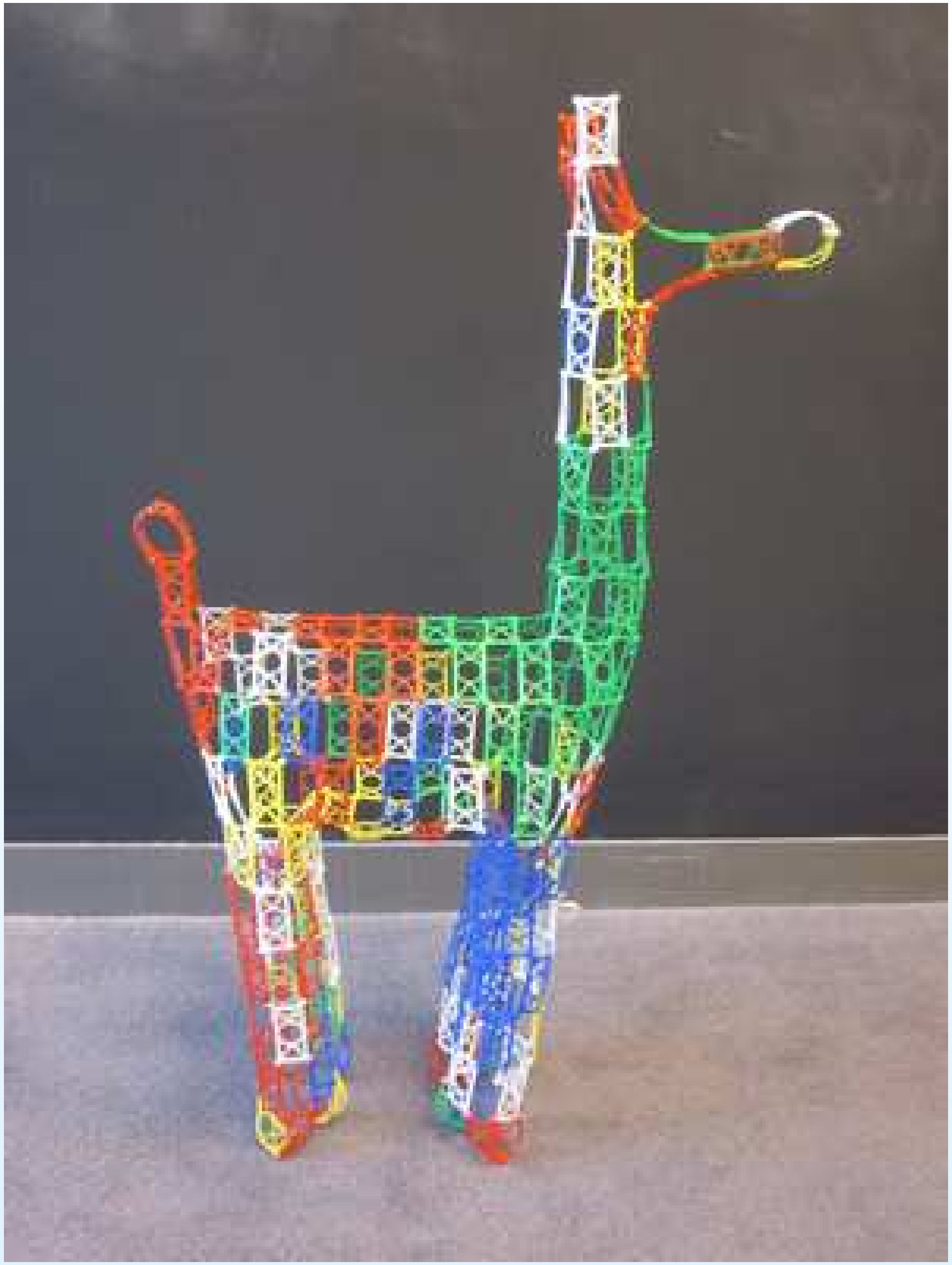
$$\mathrm{U}_h(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \quad \text{and} \quad \mathrm{U}_v(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

By definition, the $\mathrm{SL}(2,\mathbb{Z})$ action preserves the set of square-tiled surfaces. Let us prove that for $\mathrm{U}_h(\mathbb{Z})$ or $\mathrm{U}_v(\mathbb{Z})$-invariant sets, a density is always uniform.

Theorem. *Any finite $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic measure ν_1 on any unit hyperboloid of a stratum of Abelian differentials is ergodic with respect to the actions of the discrete parabolic subgroups $\mathrm{U}_h(\mathbb{Z})$ and $\mathrm{U}_v(\mathbb{Z})$.*

Proof. Let G be a simple Lie group, H be a closed non-compact subgroup of G and let G-action be ergodic with respect to a finite invariant measure. By a particular case of Moore's Ergodicity Theorem the H-action is also ergodic. In our case the simple Lie group is $\mathrm{SL}(2,\mathbb{R})$ and the closed non-compact subgroup H is $\mathrm{U}_h(\mathbb{Z})$. $\square$

Cylinder decomposition of a square-tiled surface



Equidistribution Theorem

Corollary. *Let $D_{\mathbb{Z}}$ be a density subset of square-tiled surfaces in $\mathcal{H}$. If $D_{\mathbb{Z}}$ is invariant under at least one of $\mathrm{U}_h(\mathbb{Z})$ or $\mathrm{U}_v(\mathbb{Z})$, then $D_{\mathbb{Z}}$ has uniform density.*

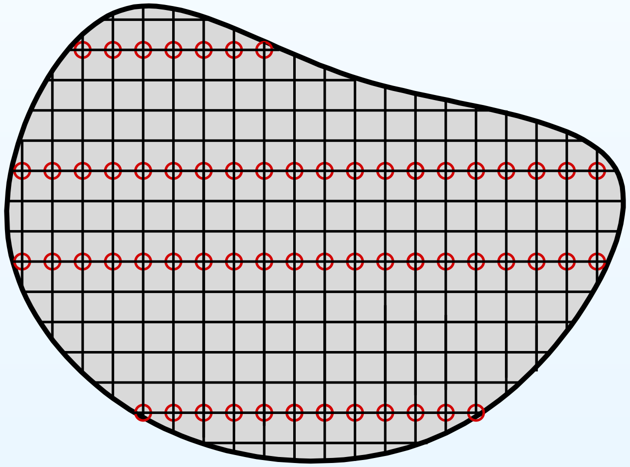
Proof Use the same argument as Mirzakhani. Putting Dirac masses at *all* square-tiled surfaces, and normalizing, we get the Masur–Veech measure. Putting Dirac masses only at square-tiled surfaces from $D_{\mathbb{Z}}$ and normalizing we get a measure dominated by the Masur–Veech measure. This measure is $\mathrm{U}_h(\mathbb{Z})$-invariant and by Moore's Ergodicity Theorem the Masur–Veech measure is $\mathrm{U}_h(\mathbb{Z})$-ergodic. Hence, the two measure are proportional. The coefficient of proportionality coincides with the value of the density $\delta(D_{\mathbb{Z}})$. $\square$

We have proved the following equidistribution Theorem. Let $c_k(\mathcal{H})$ be a contribution of k-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}$ of some stratum $\mathcal{H}$ of Abelian or quadratic differentials.

Theorem. *The asymptotic proportion $p_k(\mathcal{H}) = \frac{c_k(\mathcal{H})}{\operatorname{Vol}\mathcal{H}}$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{H}$ in a stratum $\mathcal{H}$ of Abelian or quadratic differentials does not depend on B.*

Non-correlation Theorem

Let $c_k(\mathcal{H})$ be the contribution of horizontally k-cylinder square-tiled surfaces to the Masur–Veech volume of the stratum $\mathcal{H}$, so that $c_1(\mathcal{H}) + c_2(\mathcal{H}) + \cdots = \operatorname{Vol}\mathcal{H}$, and $p_k(\mathcal{H}) = c_k(\mathcal{H})/\operatorname{Vol}(\mathcal{H})$. Let $c_{k,j}(\mathcal{H})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.



Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

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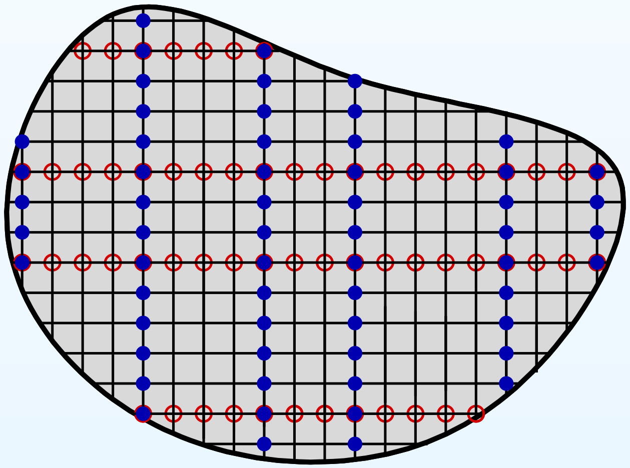

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Proof of the Non-correlation Theorem

A stratum of Abelian differentials of complex dimension d is endowed with a pair of transverse foliations of real dimension d induced from the canonical direct sum decomposition in period coordinates

$$H^1(S,\Sigma;\mathbb{C}) = H^1(S,\Sigma;\mathbb{R}) \oplus H^1(S,\Sigma;i\mathbb{R})\,.$$

The subsets $D_{\mathbb{Z}}$ of square-tiled surfaces having exactly k horizontal (respectively j vertical cylinders) are Re-*invariant* (respectively Im-*invariant*): for any point S in $D_{\mathbb{Z}}$ all square-tiled surfaces located in the leaf of the Re-foliation (respectively Im-foliation) in $\mathcal{H}$ passing through S also belong to $D_{\mathbb{Z}}$. This remark combined with uniform density of both subsets immediately implies that the intersection of the two subsets also has uniform density which is the product of densities $p_k \cdot p_j$. This completes the proof of the Non-Correlation Theorem.

Meanders count

Translation to the language of square-tiled surfaces

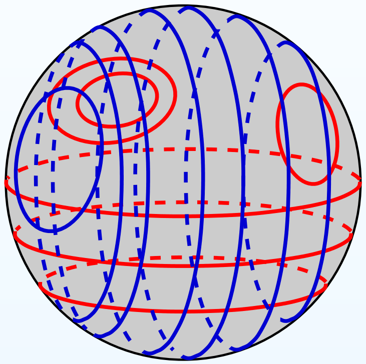
Every square-tiled surface defines a pair of transverse simple closed multicurves. The number of squares is the number of intersections of the two multicurves.

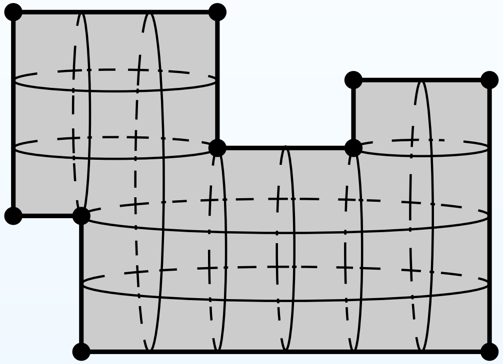
Reciprocal is not always true since in genera higher than 0 a pair of transverse multicurves might chop the surface into components more complicated than topological discs. However, it happens rarely in terms of the asymptotic count, so for the purposes of the count we can pretend that we have a bijection.

Bigons arising from intersection of transverse multicurves correspond to simple poles of the associated quadratic differentials. Thus, the count of pairs of transverse multicurves on a surface of genus g with at most N intersections and with p bigons corresponds to the count of square-tiled surfaces of genus g with p poles tiled by at most N squares, i.e. to evaluation of the Masur–Veech volume of the moduli space $\mathcal{Q}_{g,p}$. In this way we get the asymptotics $c(g,p) \cdot N^{6g-6+2p}$ for the number of multicurves and the constant $c(g,p)$.

How to count meanders

Step 1. There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





Step 2. Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height 1. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

Step 3. Fixing the number of minimal arcs ("pimples") we fix the number of simple poles p of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

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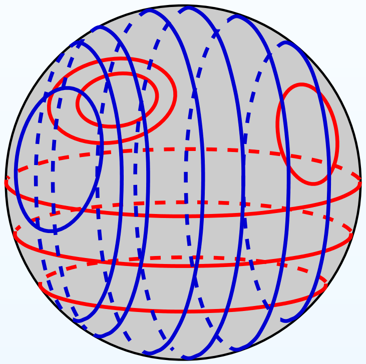
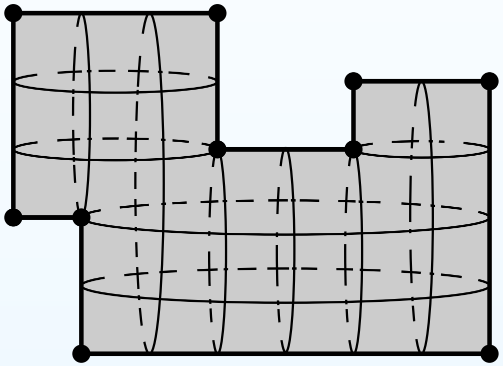


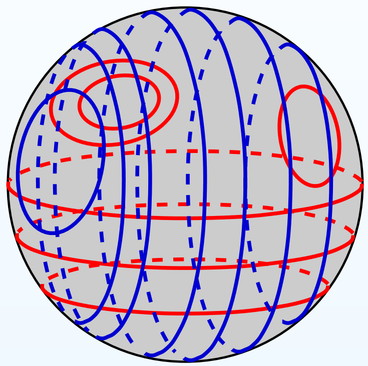


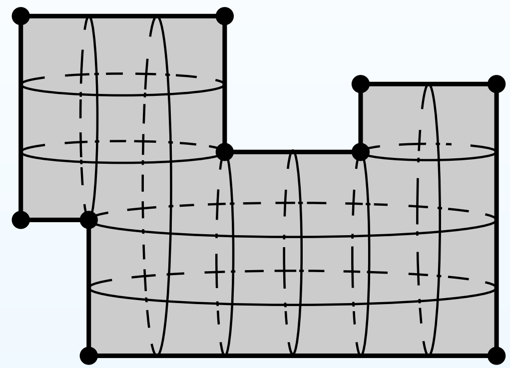
Step 2. Pairs of arc systems glued along common equator correspond to square-tiled surfaces having single horizontal cylinder of height 1. Meanders correspond to square-tiled surfaces having single horizontal cylinder and single vertical one; both of height one. So we can apply the formula $c_{1,1}(\mathcal{Q}) = \frac{c_1^2(\mathcal{Q})}{\mathrm{Vol}(\mathcal{Q})}$, where $c_1(\mathcal{Q})$ is easy to compute and $\mathrm{Vol}(\mathcal{Q})$ in genus zero is given by an explicit formula (obtained after 15 years of work of Athreya–Eskin–Zorich).

Step 3. Fixing the number of minimal arcs (“pimples”) we fix the number of simple poles p of the quadratic differential. All but negligible part of the corresponding square-tiled surfaces live in the only stratum $\mathcal{Q}(1^{p-4}, -1^p)$ of the maximal dimension.

How to count meanders

Step 1. There is a natural one-to-one correspondence between transverse connected pairs of multicurves on an oriented sphere and pillowcase covers, where the square tiling is given by the graph dual to the graph formed by the pair of multicurves.





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General philosophy

● Pairs of transverse multicurves correspond to square-tiled surfaces. Thus, count of all pairs of transverse multicurves is equivalent to count of Masur–Veech volumes.

● Count of arc systems, braids, ribbon graphs, pairs: simple closed curve plus transverse multicurve, one-cylinder square-tiled surfaces is another group of (somehow equivalent) problems, which usually admits a more efficient solution.

● Consider the following three counting problems:

1. count of all square-tiled surfaces (i.e. Masur–Veech volume Vol);
2. count of horizontally one-cylinder square-tiled surfaces (i.e. c_1);
3. count of horizontally and vertically square-tiled surfaces (i.e. $c_{1,1}$).

By non-correlation, $c_{1,1} = \dfrac{c_1^2}{\mathrm{Vol}}$. Count of c_1 usually admits a relatively efficient solution. Hence, as soon as we know the appropriate Masur–Veech volume, we know $c_{1,1}$, and hence we can count meanders, pairs of transverse simple closed curves etc.

How we count meanders

A pair of transverse multicurves associated to a square-tiled surface is orientable if and only if the square-tiled surface is Abelian. Thus, the count of positively intersecting pairs of transverse multicurves in genus g corresponds to the count of Abelian square-tiled surfaces in genus g, i.e. to the evaluation of the Masur–Veech volumes of the corresponding moduli space of Abelian differentials. In this way we get the asymptotics $c^+(g) \cdot N^{4g-3}$ and the constant $c_1^+(g)$ for the count of positively intersecting multicurves.

Pairs (simple closed curve, transverse multicurve) correspond to square-tiled surfaces having single horizontal band of squares. We found a way to count such square-tiled surfaces both in the Abelian and in the quadratic case and to evaluate the constants $c_1(g,p)$ and $c_1^+(g)$ in the corresponding asymptotics $c_1(g,p) \cdot N^{6g-6+2p}$ and $c_1^+(g) \cdot N^{4g-3}$ respectively.

Meanders correspond to square-tiled surfaces having single horizontal and single vertical band of squares. We apply our non-correlation theorem to get

$$c_{1,1}(g,p) = \frac{c_1^2(g,p)}{c(g,p)} \quad \text{and} \quad c_{1,1}^+(g) = \frac{\left(c_1^+(g)\right)^2}{c^+(g)}\,.$$

Masur–Veech volume in genus zero

In genus zero Masur–Veech volumes of the strata of meromorphic quadratic differentials admit alternative quite implicit computation through dynamics. An idea (which initially seemed somewhat crazy) of such computation belongs to M. Kontsevich, who stated about 2003 the conjecture on volumes in genus 0.

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

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Theorem (J. Athreya, A. Eskin, A. Z., 2014 ; conjectured by M. Kontsevich about 2003) *The volume of any stratum $\mathcal{Q}(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. when $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \dots, k$, and $\sum_{i=1}^k d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

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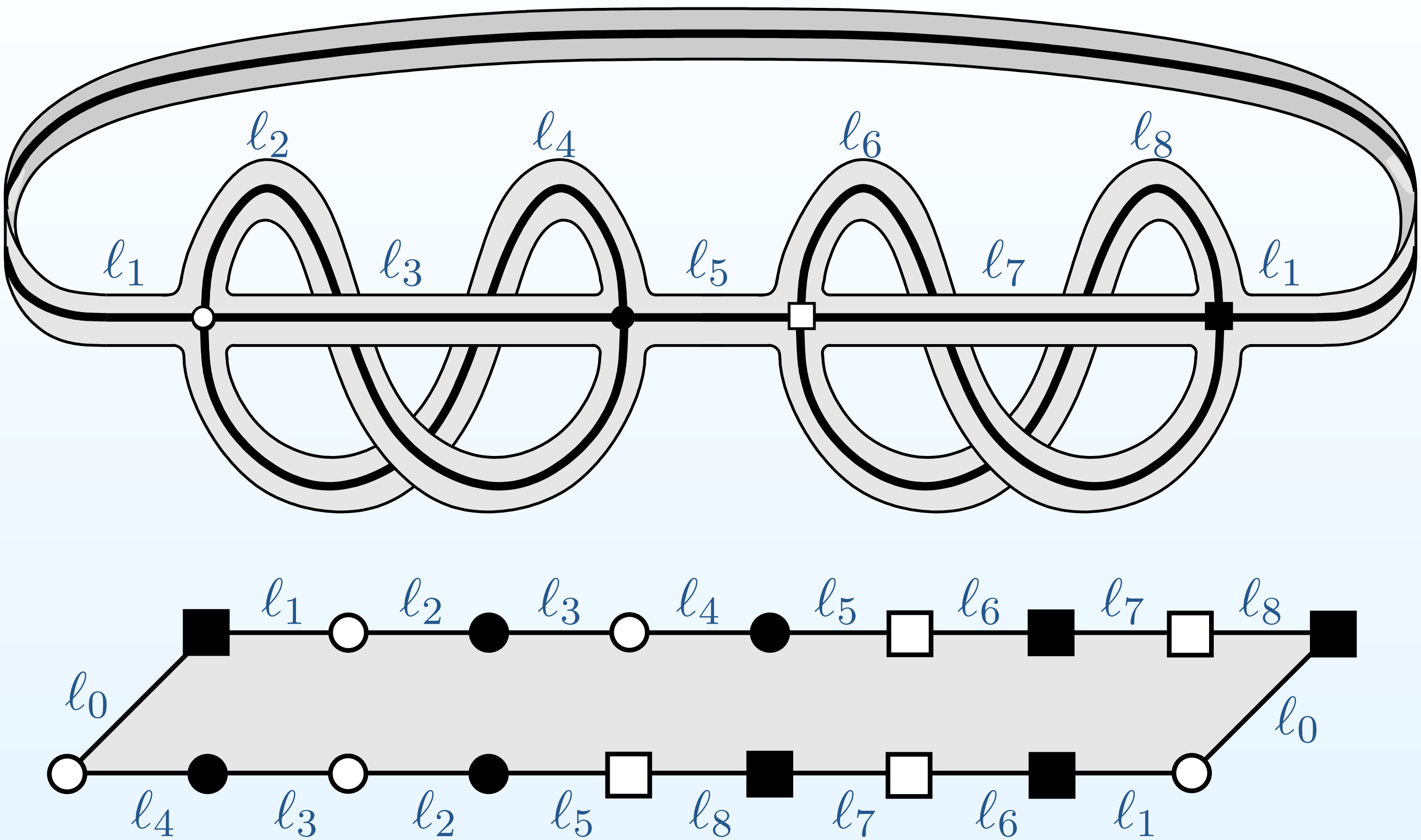
$$\operatorname{Vol} \mathcal{Q}_{0,n} = 2\pi \cdot \left(\frac{\pi^2}{2}\right)^{n-4} .$$

Applying formula based on Kontsevich polynomials one gets ENORMOUS sum over labeled trees, so this approach does not work. But this formula was reproved by Chen–Möller–Sauvaget through intersection theory.

1-cylinder square-tiled surfaces and permutations

1-cylinder surface as a pair of permutations



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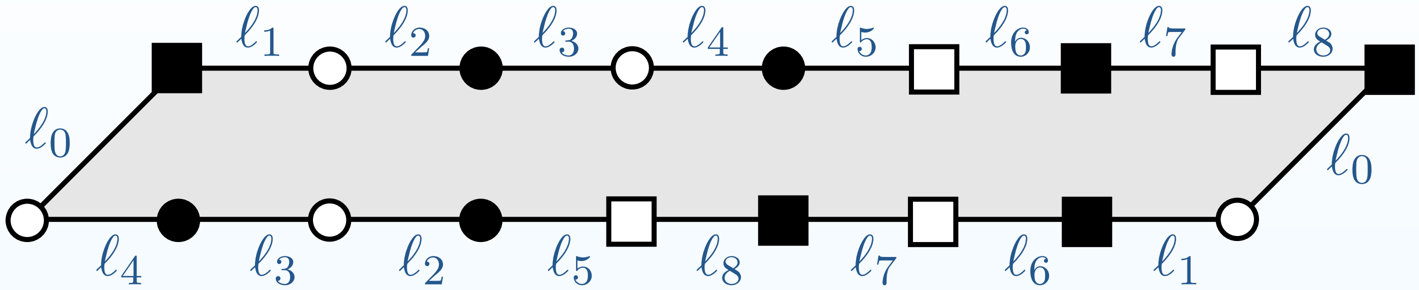

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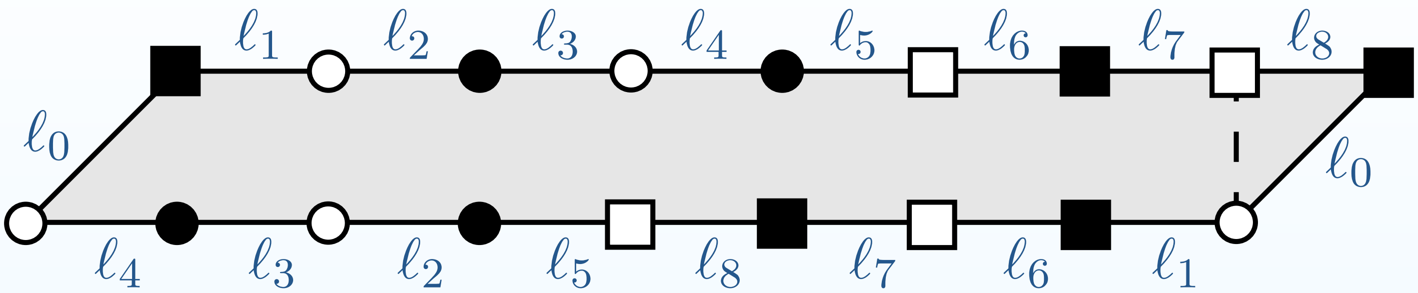

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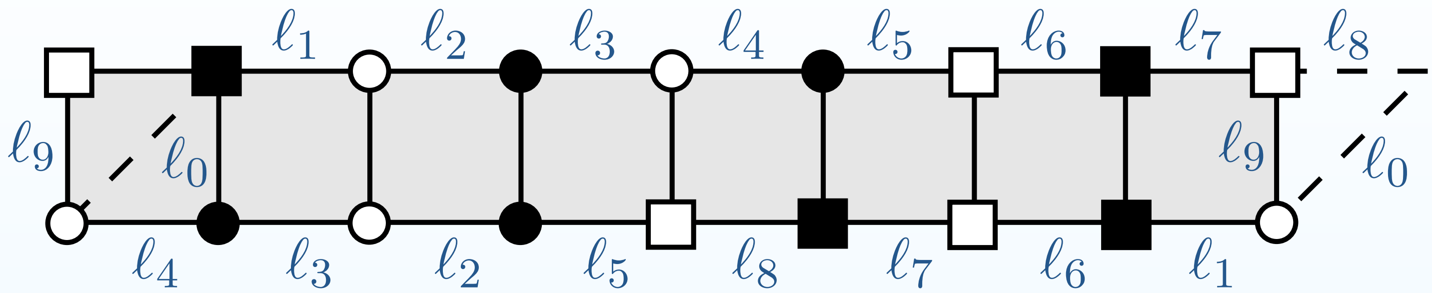

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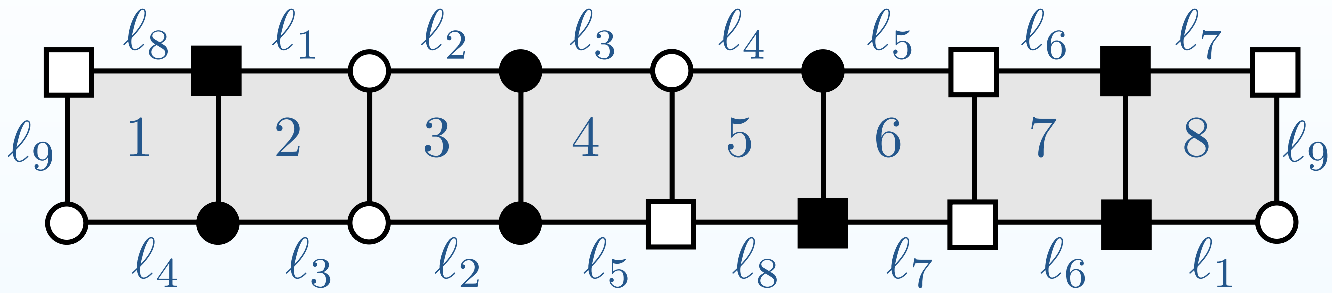

$$\pi_h = (1, 2, 3, 4, 5, 6, 7, 8)$$

$$\pi_v = (1, 5)\,(2, 8, 6, 4)\,(3)\,(7)$$

$$\pi_h \pi_v \pi_h^{-1} \pi_v^{-1} = (1, 7)\,(2, 4)\,(3, 5)\,(6, 8)$$

We see 4 cycles of length 2 which corresponds to $\mathcal{H}(1, 1, 1, 1)$.

Note that by construction, the permutation π_h of a square-tiled surface composed from a single band of squares is a long cycle $\pi_h = (1, \ldots, N)$. Thus, for any π_v, the permutation $\sigma = \pi_v \pi_h^{-1} \pi_v^{-1}$ is also a long cycle. For any pair of long cycles σ, π_h there are exactly N solutions π_v of the equation $\sigma = \pi_v \pi_h^{-1} \pi_v^{-1}$ (if π_v is a solution, $\pi_v (\pi_h)^k$, where $k = 0, 1, \ldots, N - 1$, is also a solution).

Frobenius formula

The count of 1-cylinder N-square-tiled surfaces in the stratum $\mathcal{H}(m_1,\dots,m_n)$ is reduced to the count of solutions of the following equation for permutations:

$(N-\text{cycle})\cdot(N-\text{cycle}) =$ product of cycles of lengths $m_1+1,\dots,m_n+1$,

completed with product of cycles of lengths 1.

Frobenius formula expresses this number in terms of characters of the exterior powers of the standard representation $\mathbf{St}_n$ of the symmetric group S_n:

$$\chi_j(g) := \operatorname{tr}(g,\pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0\le j\le n-1)\,.$$

Theorem. *The absolute contribution $c_1(\mathcal{H}(m_1,\dots,m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ equals*

$$c_1 = \frac{2}{(d-1)!}\cdot\prod_k \frac{1}{(k+1)^{\mu_k}}\cdot\sum_{j=0}^{d-2} j!\,(n-1-j)!\,\chi_j(\nu)\,.$$

Here $d=\dim\mathcal{H}(m_1,\dots,m_n)$; $\nu\in S_n$ is any permutation with decomposition into cycles of lengths $(m_1+1),\dots,(m_n+1)$; μ_i is the number of zeroes of order i, i.e. the multiplicity of the entry i in the multiset $\{m_1,\dots,m_n\}$.

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For permutations ν representing the principal and the minimal strata the characters $\chi_j(\nu)$ admit easier computation which leads to the following formulae:

$$c_1(\mathcal{H}(1^{2g-2})) = \frac{1}{4g-2}\cdot\frac{4}{2^{2g-2}}\,, \qquad c_1(\mathcal{H}(2g-2)) = \frac{1}{2g}\cdot\frac{4}{2g-1}\,.$$

Contribution of 1-cylinder diagrams.

Theorem. *The contribution c_1 of 1-cylinder square-tiled surfcaes to the volume $\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)$ of any nohyperelliptic stratum of Abelian differentials satisfies*

$$\frac{\zeta(d)}{d+1}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\le c_1\le\frac{\zeta(d)}{d-\frac{10}{29}}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\,,$$

where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

(Here we used a result of Zagier.)

Theorem (analog of the Prime Number Theorem). *The relative contribution of 1-cylinder square-tiled surfaces to the volume of the stratum is of the order $1/($dimension of the stratum$)$ when $g\gg 1$:*

$$d\cdot\frac{c_1(\mathcal{H}(m_1,\dots,m_n))}{\operatorname{Vol}(\mathcal{H}_1(m_1,\dots,m_n))}\to 1\ \text{ as }\ g\to+\infty\,,$$

where convergence is uniform for all strata in genus g.

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