Dynamics and Geometry of Moduli Spaces

Lecture 10. Masur–Veech volumes of the moduli space of quadratic differentials through Witten–Kontsevich correlators. Random square-tiled surfaces of large genus. Random multicurves on surfaces of large genus.

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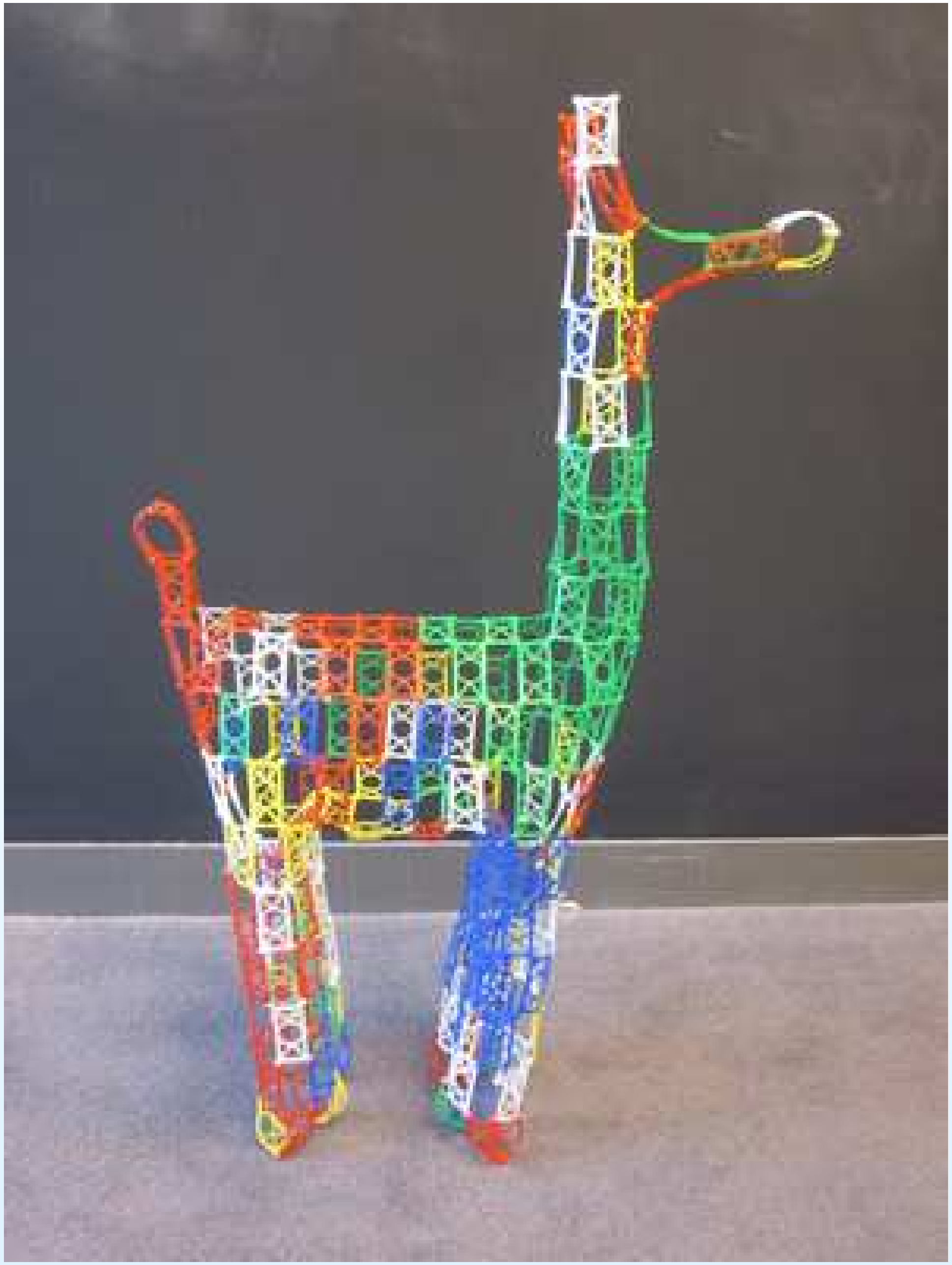
Count of square-tiled surfaces. (Masur–Veech volume of the moduli space of quadratic differentials)

A square-tiled surface



Square-tiled surfaces: formal definition

Take a finite set of copies of identical oriented squares for which two opposite sides are chosen to be horizontal and the remaining two sides are declared to be vertical. Identify pairs of sides of the squares by isometries in such way that horizontal sides are glued to horizontal sides and vertical sides to vertical. We get a topological surface S without boundary. We consider only those surfaces obtained in this way which are connected and oriented. The form dz^2 on each square is compatible with the gluing and endows S with a complex structure and with a non-zero quadratic differential $q = dz^2$ with at most simple poles. We call such a surface a *square-tiled surface*. Any square-tiled surface is a connected ramified cover over a standard flat square pillow with ramification points only over the four corners of the pillow.

Fix the genus g of the surface and the number n of corners with cone angle π (the ones adjacent to exactly two squares). The question on the asymptotic number of such square-tiled surfaces tiled with at most $N \gg 1$ squares (pillowcase covers) is equivalent to evaluation of the Masur–Veech volume of the moduli space of quadratic differentials.

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g - 3 + n$.

Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$. The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

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Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}\,.$$

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The famous *Witten's conjecture* claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in 2-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}} \,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

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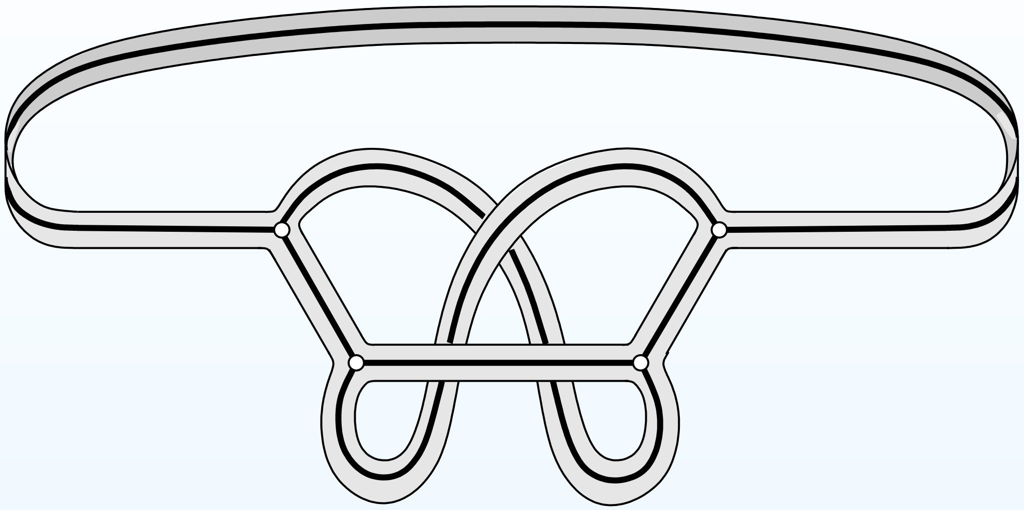
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big) \,,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 1$ with $n = 2$ boundary components. Assigning lengths to all edges of the core graph, we endow each boundary component with an induced length defined as the sum of the lengths of edges which it follows.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorems of Kontsevich and Norbury count them.

Count of metric ribbon graphs

Theorem (Kontsevich'92; in this stronger form — Norbury'10). *Consider a collection of positive integers* $b_1, \dots, b_n$ *such that* $\sum_{i=1}^n b_i$ *is even. The weighted count of genus* g *connected trivalent metric ribbon graphs* Γ *with integer edges and with* n *labeled boundary components of lengths* $b_1, \dots, b_n$ *is equal to* $N_{g,n}(b_1, \dots, b_n)$ *up to the lower order terms:*

$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \dots, b_n) = N_{g,n}(b_1, \dots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ *denote the set of (nonisomorphic) trivalent ribbon graphs* Γ *of genus* g *and with* n *boundary components.*

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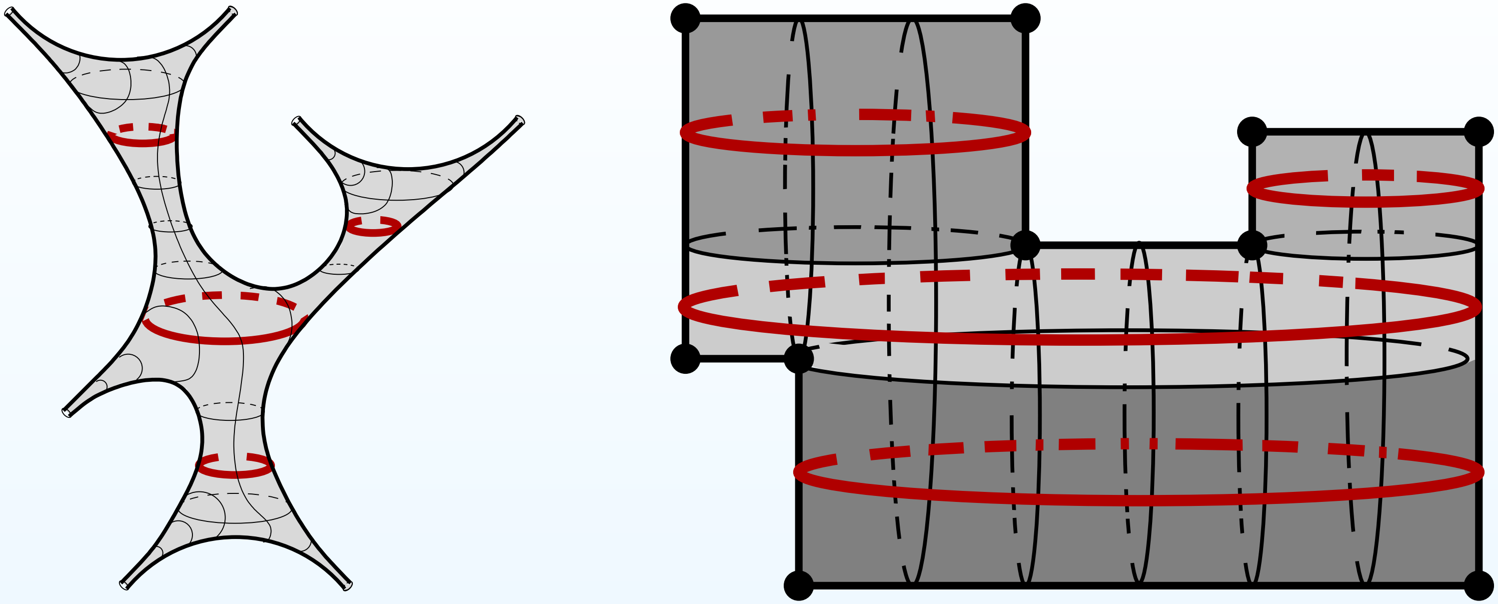
where $\mathcal{R}_{g,n}$ denote the set of (nonisomorphic) trivalent ribbon graphs Γ of genus g and with n boundary components.

(Formal statement justifying the notion of "lower order terms": the right-hand side is a *quasipolynomial* in the integers $b_1, \ldots, b_n$ depending on the number k of odd b_i. The top homogeneous part is zero when k is odd.)

A version of this Theorem is an important part of Kontsevich's proof of Witten's conjecture.

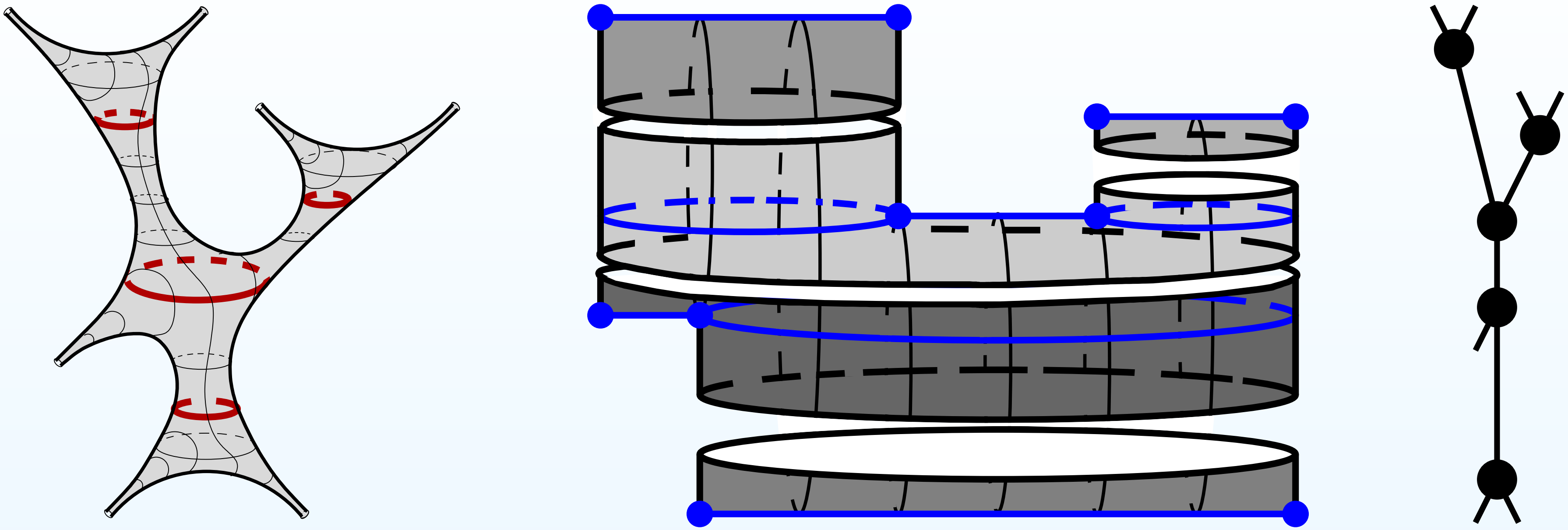
Multicurve associated to a square-tiled surface



Having a square-tiled surface we associate to it a topological surface S on which we mark all “corners” with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points.

We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. We encode the number of horizontal bands of squares in each cylinder by taking the components of the multicurve with integer weights.

Ribbon graph decomposition of a square-tiled surface



Leaves of the horizontal foliation on the square-tiled surface passing through singular points (in blue) are called *critical*. Considering tubular neighborhoods of these critical leaves we get metric ribbon graphs. Cylinders (represented by the multicurve in red) are joining boundary components of these ribbon graphs.

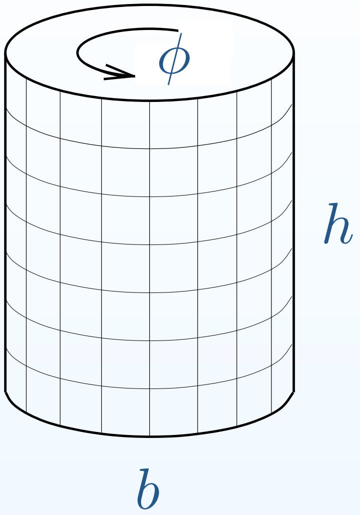
A dual graph to the multicurve is called *stable graph* Γ. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by “legs” — half-edges of the dual graph.

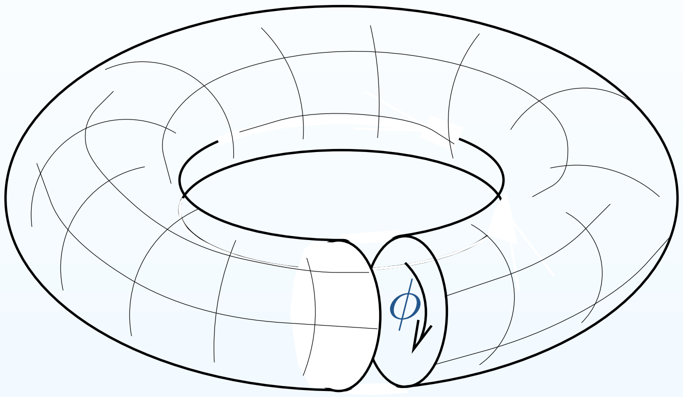
Count of square-tiled surfaces: an algorithm

1. Fix a genus g and a number n of corners (conical points) of angle π.
2. Consider a finite collection of *stable graphs* encoding all possible admissible decompositions of a hyperbolic surface of genus g with n cusps (equivalently, all complex stable curves of genus g with n marked points).
3. For each stable graph with k edges associate formal variables $b_1, \dots, b_k$ to its edges and associate metric ribbon graphs to the vertices.
4. Using the Kontsevich–Norbury count of metric ribbon graphs, count the number of ways to join them by square-tiled cylinders.

Masur–Veech volume of the moduli space of quadratic differentials. This moduli space is the total space of the cotangent bundle to the moduli space $\mathcal{M}_{g,n}$ of complex curves with n marked points and, hence, has a canonical symplectic structure, and a volume element. Square-tiled surfaces represent integer points in this space. Those, which are tiled with at most N squares, are integer points in a "bundle of balls of radius N" over $\mathcal{M}_{g,n}$. Thus, asymptotics of the number of square-tiled surfaces of genus g with n conical points of angle π tiled with at most $N \to +\infty$ squares gives us the Masur–Veech volume.

Example: number of square-tiled tori

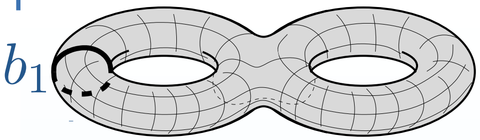




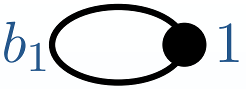
The number of square-tiled tori tiled with at most N squares has asymptotics

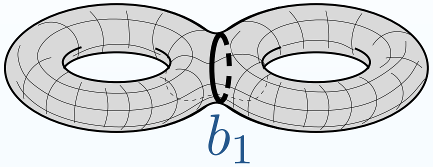
$$\sum_{\substack{b,h\in\mathbb{N}\\ b\cdot h\leq N}} b = \sum_{\substack{b,h\in\mathbb{N}\\ b\leq\frac{N}{h}}} b \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\zeta(2) =$$

$$= \frac{N^2}{2}\mathcal{Z}(b) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

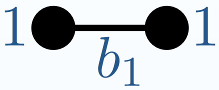


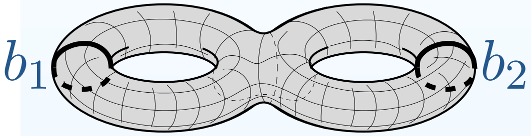
$$\frac{1}{2}\cdot 1\cdot b_1\cdot N_{1,2}(b_1,b_1)$$



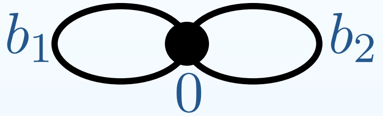


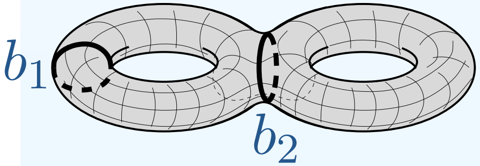
$$\frac{1}{2}\cdot\frac{1}{2}\cdot b_1\cdot N_{1,1}(b_1)\cdot N_{1,1}(b_1)$$



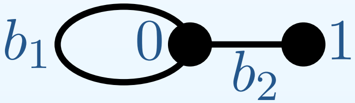


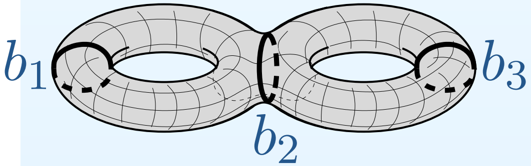
$$\frac{1}{8}\cdot 1\cdot b_1b_2\cdot N_{0,4}(b_1,b_1,b_2,b_2)$$



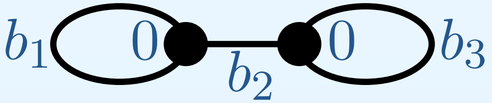


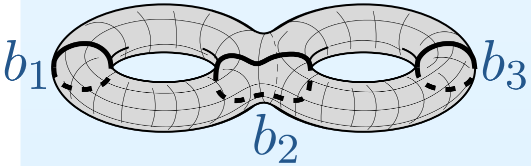
$$\frac{1}{2}\cdot\frac{1}{2}\cdot b_1b_2\cdot N_{0,3}(b_1,b_1,b_2)\cdot$$
$$\cdot N_{1,1}(b_2)$$



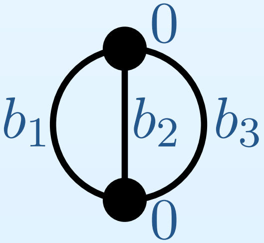


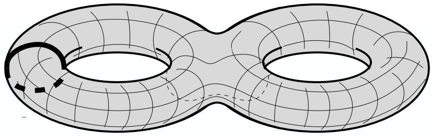
$$\frac{1}{8}\cdot\frac{1}{2}\cdot b_1b_2b_3\cdot N_{0,3}(b_1,b_1,b_2)\cdot$$
$$\cdot N_{0,3}(b_2,b_3,b_3)$$



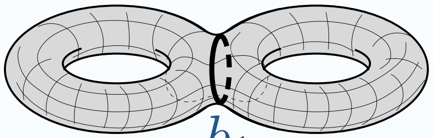


$$\frac{1}{12}\cdot\frac{1}{2}\cdot b_1b_2b_3\cdot N_{0,3}(b_1,b_2,b_3)\cdot$$
$$\cdot N_{0,3}(b_1,b_2,b_3)$$

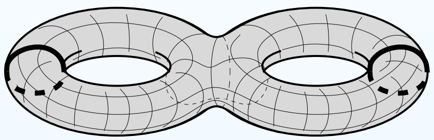




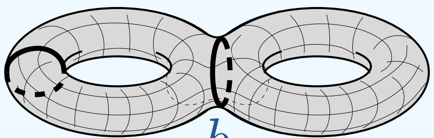
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1) = \frac{1}{2} \cdot b_1 \left(\frac{1}{384}(2b_1^2)(2b_1^2) \right)$$



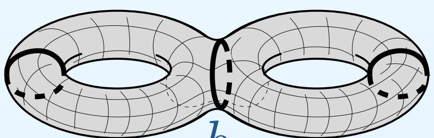
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1) = \frac{1}{4} \cdot b_1 \left(\frac{1}{48} b_1^2 \right) \left(\frac{1}{48} b_1^2 \right)$$



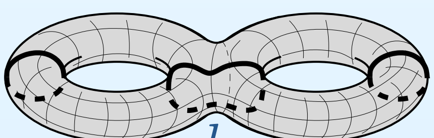
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2) = \frac{1}{8} \cdot b_1 b_2 \cdot \left(\frac{1}{4}(2b_1^2 + 2b_2^2) \right)$$



$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2) = \frac{1}{4} \cdot b_1 b_2 \cdot (1) \cdot \left(\frac{1}{48} b_2^2 \right)$$

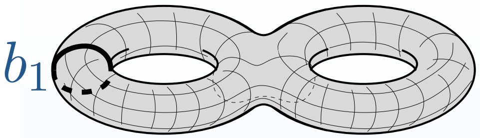


$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3) = \frac{1}{16} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

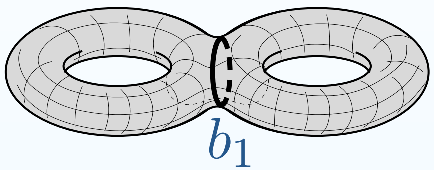


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3) = \frac{1}{24} \cdot b_1 b_2 b_3 \cdot (1) \cdot (1)$$

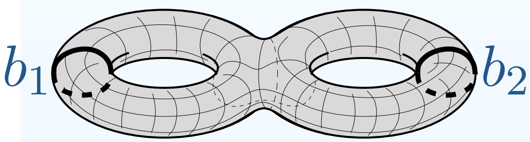
Volume of $\mathcal{Q}_2$



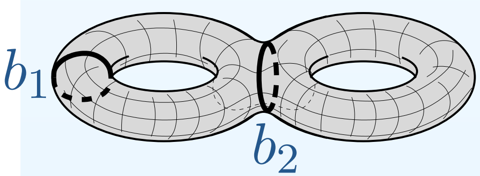
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



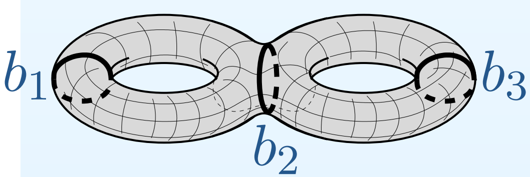
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



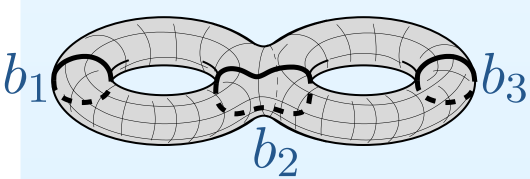
$$\frac{1}{16}(b_1^3 b_2 + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$

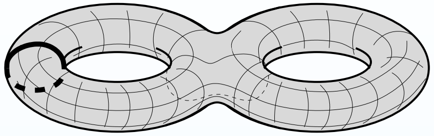


$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$

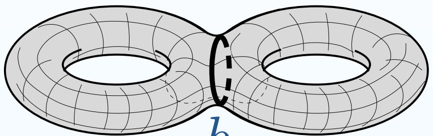


$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

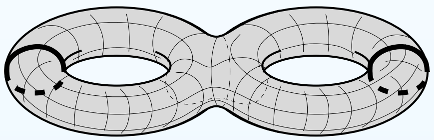
Volume of $\mathcal{Q}_2$



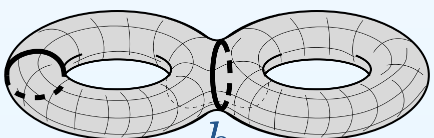
$$\frac{1}{192}\cdot b_1^5 \xmapsto{\mathcal{Z}} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



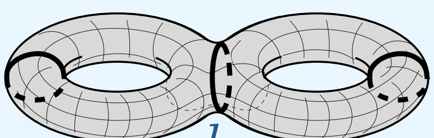
$$\frac{1}{9216}\cdot b_1^5 \xmapsto{\mathcal{Z}} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



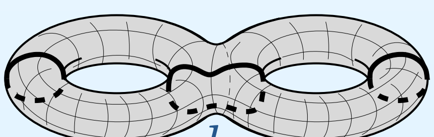
$$\frac{1}{16}(b_1^3b_2 + b_1b_2^3) \xmapsto{\mathcal{Z}} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1b_2^3 \xmapsto{\mathcal{Z}} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16}b_1b_2b_3 \xmapsto{\mathcal{Z}} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$



$$\frac{1}{24}b_1b_2b_3 \xmapsto{\mathcal{Z}} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\Big(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\Big)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix–Goujard–Zograf–Zorich'21). *The Masur–Veech volume* $\operatorname{Vol}\mathcal{Q}_{g,n}$ *of the moduli space of meromorphic quadratic differentials with* n *simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot\sum_{\substack{\text{Weighted graphs }\Gamma\\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of }\Gamma-1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e\cdot\prod_{\text{Vertices } v \text{ of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right),$$

where g_v *is the "genus decoration" of the vertex* v*,* n_v *is the part of valency of* v *corresponding to true edges (i.e. the valency of* v *without counting legs, and* p_v *is the number of legs attached to* v*.*

The partial sum for fixed number k *of edges gives the contribution of* k*-cylinder square-tiled surfaces.*

Volume of $\mathcal{Q}_{g,n}$

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$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices } v \text{ of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

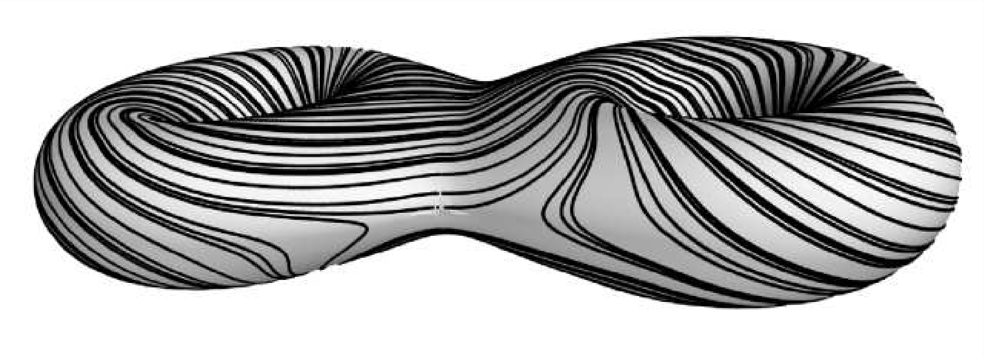
Mirzakhani's count of simple closed geodesics

Picture taken by F. Labourie at CIRM, Luminy

Simple closed multicurve, its topological type and underlying primitive multicurve

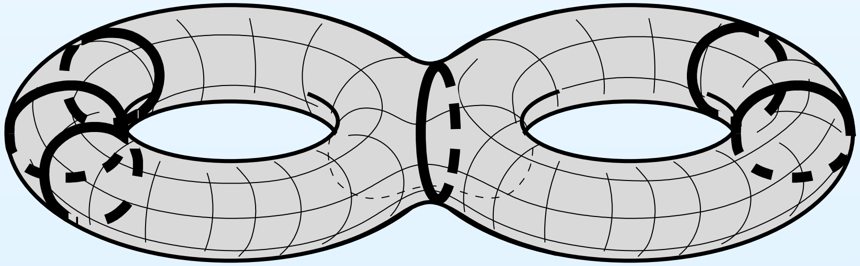
Having an arbitrary collection of complicated non self-intersecting and non pairwise intersecting curves (called a *multicurve*), one can apply an appropriate diffeomorphism of the surface which "unwraps" the multicurve to a simple canonical representative.

A general multicurve ρ:

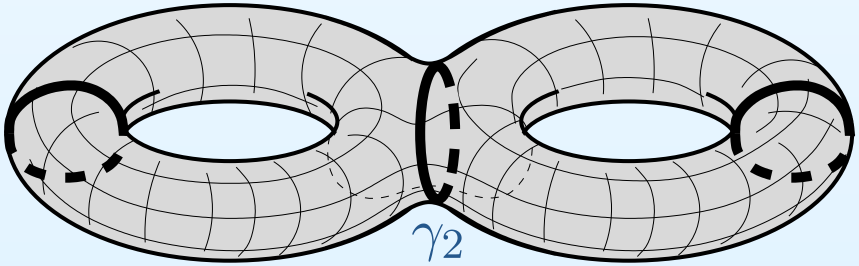


the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

$$\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3.$$



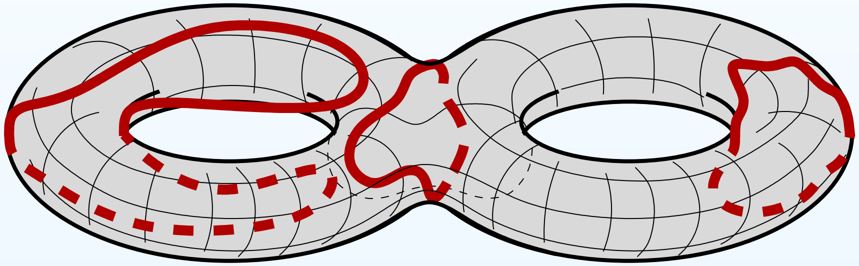
$$\gamma_{reduced} = \gamma_1 + \gamma_2 + \gamma_3$$

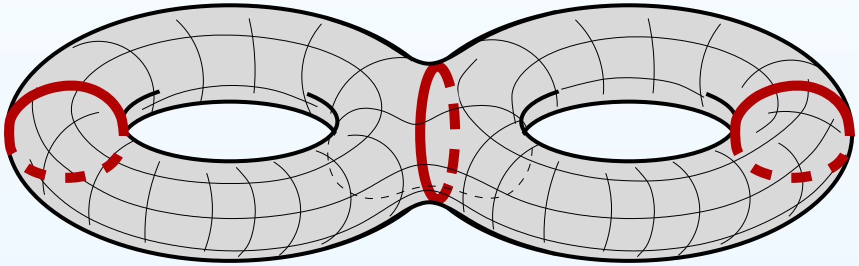


(You can practice in unwinding curves at `https://aharalab.sakura.ne.jp/teruaki.html`)

Geodesic representatives of multicurves

Consider several pairwise nonintersecting essential simple closed curves $\gamma_1, \dots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. In the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves $\gamma_1, \dots, \gamma_k$ contract to simple closed geodesics.





Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \dots, \gamma_k$ do not have pairwise intersections.*

We define the hyperbolic length of a multicurve $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

Frequencies of multicurves

Theorem (Mirzakhani'08). *For any integral multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ the number $s_X(L,\gamma)$ of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

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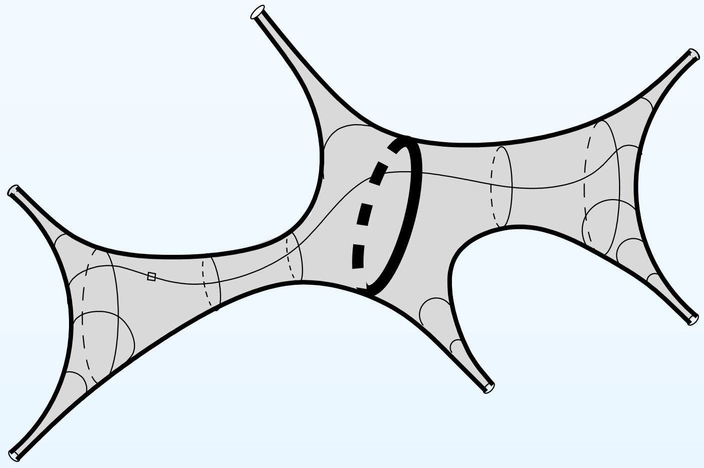
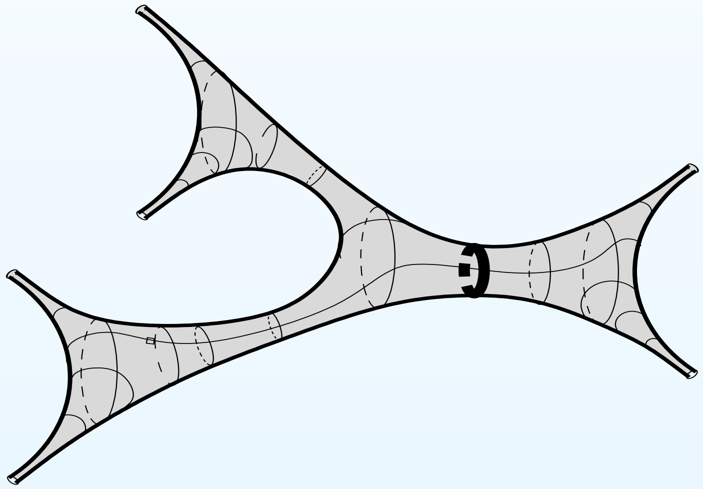
Corollary (Mirzakhani'08). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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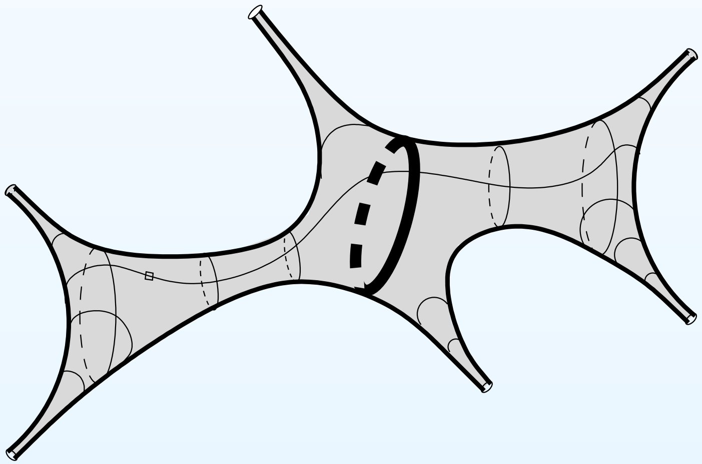

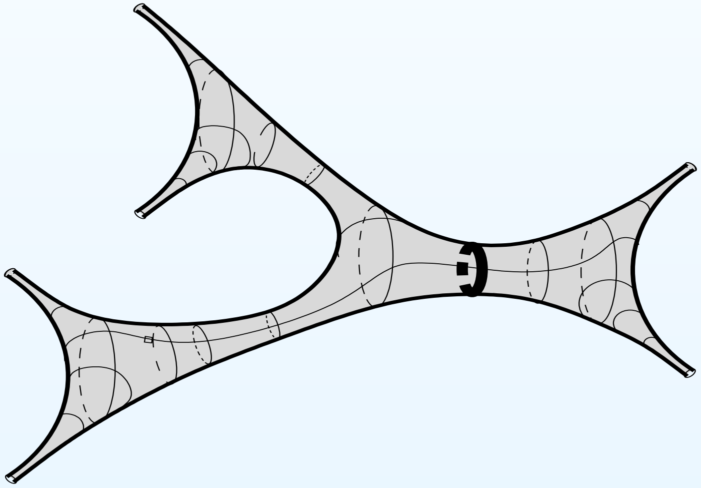

Example (Mirzakhani'08); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

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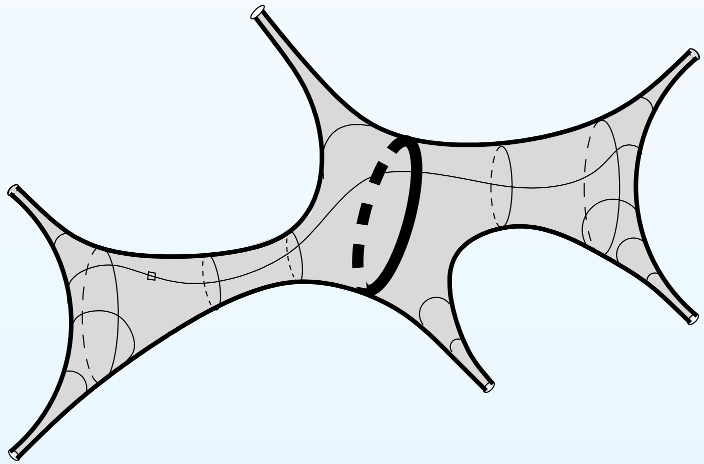
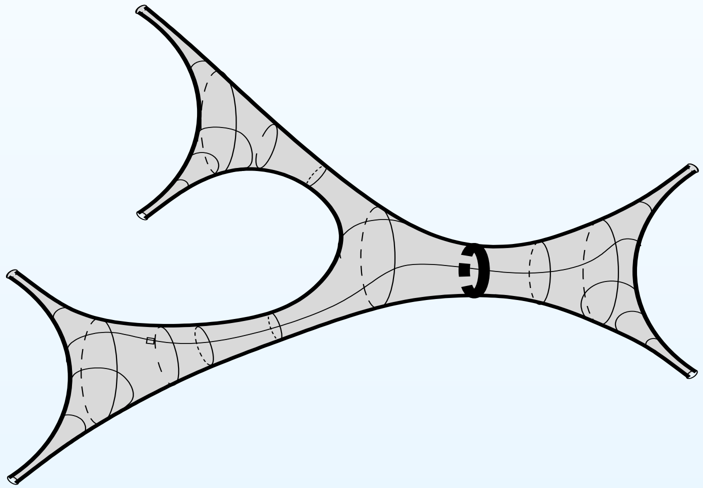


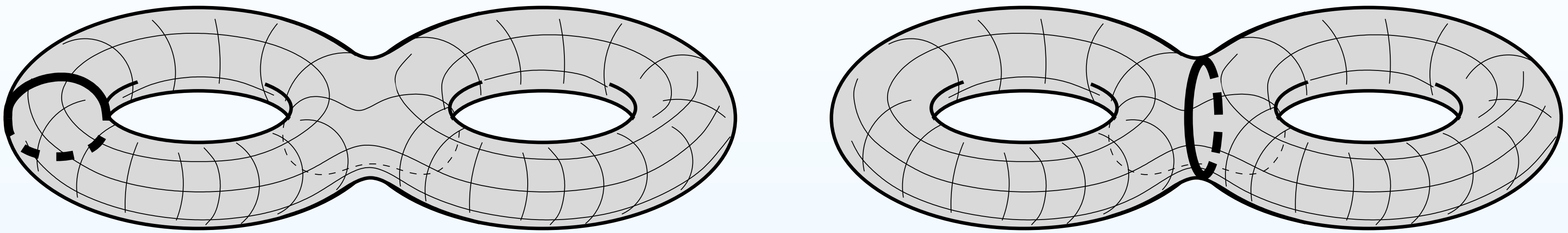


In this sense one can say that for any hyperbolic metric X on a sphere with 6 cusps, a long simple closed geodesic separates the cusps as $(3+3)$ with probability $\frac{4}{7}$ and as $(2+4)$ with probability $\frac{3}{7}$.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}\,.$$

Separating versus non-separating simple closed curves in $g = 2$

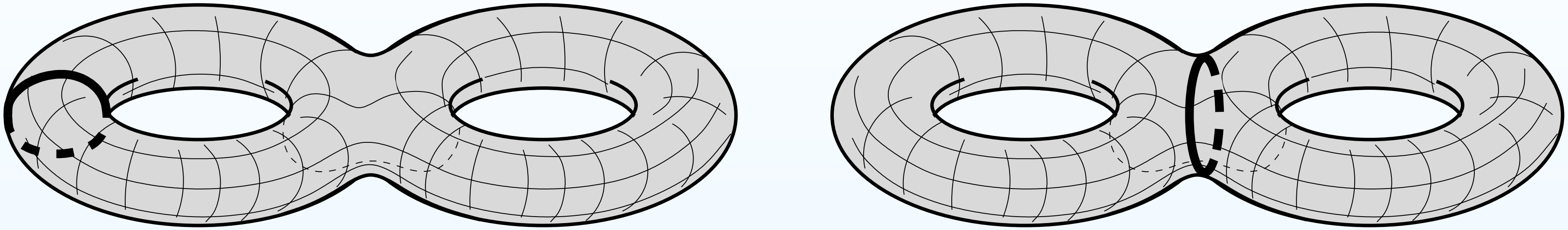
Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$



$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{6}$$

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$

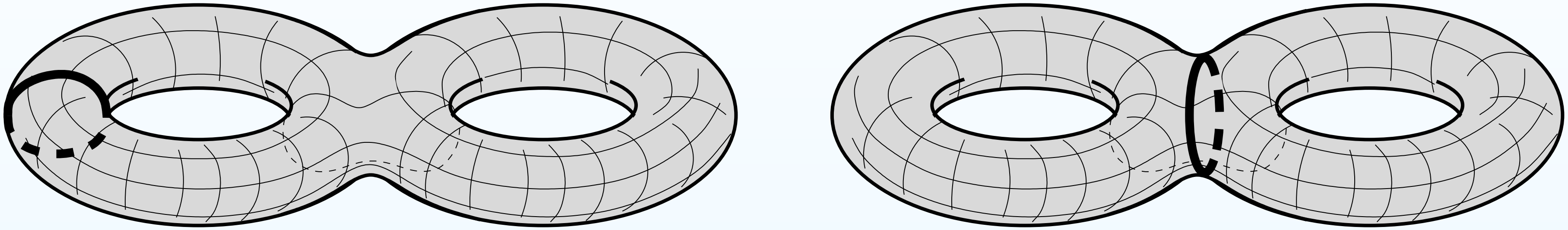


$$\lim_{L \to +\infty} \frac{\text{Number of \textbf{separating} simple closed geodesics of length at most } L}{\text{Number of \textbf{non-separating} simple closed geodesics of length at most } L} = \frac{1}{24}$$

after correction of a tiny bug in Mirzakhani's calculation.

Separating versus non-separating simple closed curves in $g=2$

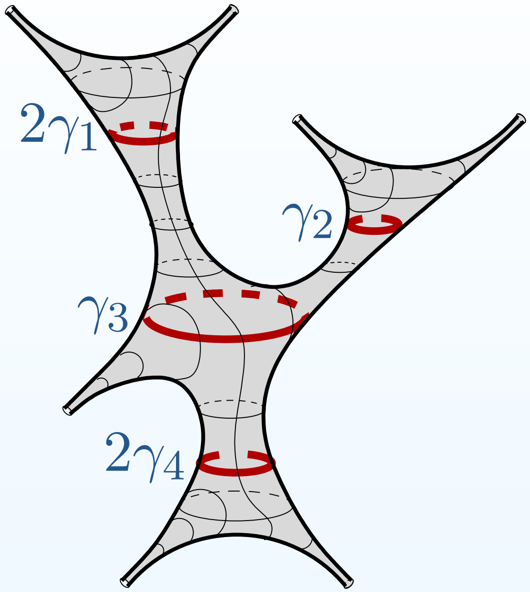
Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g=2$

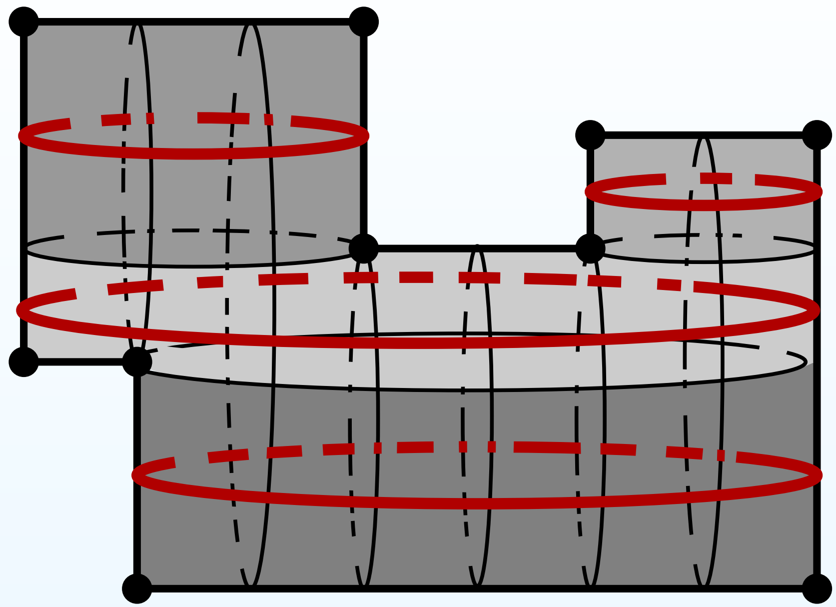


$$\lim_{L\to+\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

after further correction of another trickier bug in Mirzakhani's calculation. Confirmed by crosscheck with Masur–Veech volume of $\mathcal{Q}_2$ computed by E. Goujard using the method of Eskin–Okounkov. Confirmed by calculation of M. Kazarian; by independent computer experiment of V. Delecroix; by extremely heavy and elaborate recent experiment of M. Bell. Most recently it was independently confirmed by V. Erlandsson, K. Rafi, J. Souto and by A. Wright by methods independent of ours.

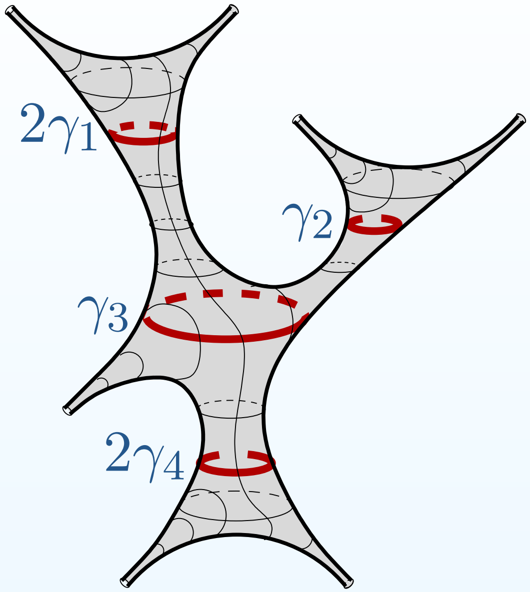
Hyperbolic and flat geodesic multicurves

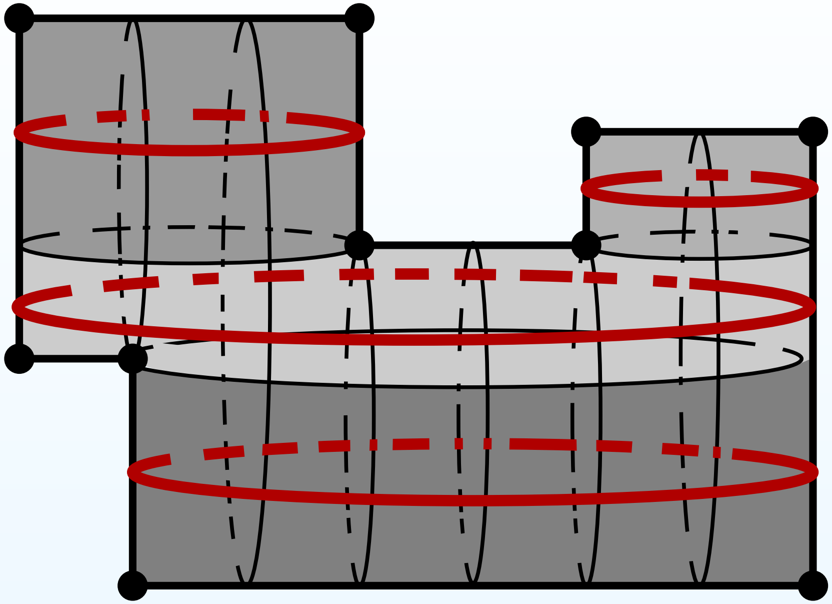




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves



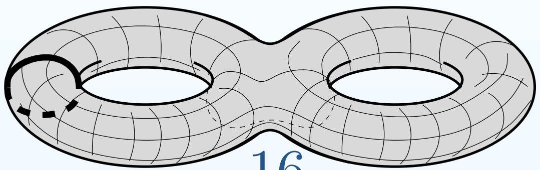


Theorem (Delecroix–Goujard–Zograf–Zorich'21). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of* ***hyperbolic*** *multicurves coincides with the asymptotic frequency of simple closed* ***flat*** *geodesic multicurves of type γ represented by associated square-tiled surfaces.*

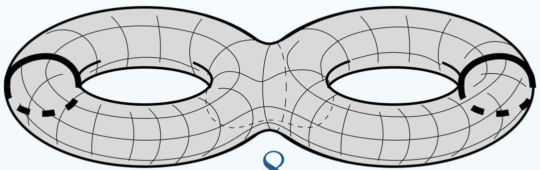
Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Multicurves on a surface of genus two and their frequencies

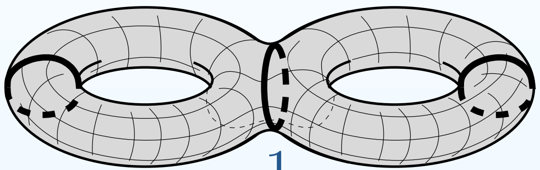
The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



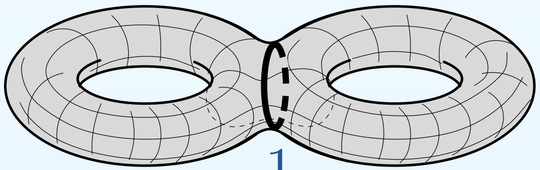
$\frac{16}{63}$



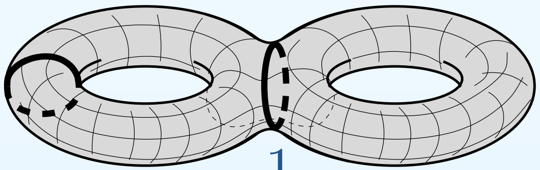
$\frac{8}{15}$



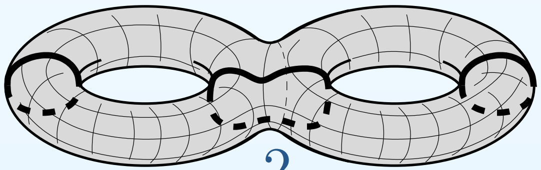
$\frac{1}{9}$



$\frac{1}{189}$



$\frac{1}{45}$



$\frac{2}{27}$

In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Shape of a random multicurve on a surface of large genus. Shape of a random square-tiled surface of large genus.

Statistics of prime decompositions: random integer numbers

The Prime Number Theorem states that an integer number n taken randomly in a large interval $[1, N]$ is prime with asymptotic probability $\frac{\log N}{N}$.

Actually, one can tell much more about prime decomposition of a large random integer. Denote by $\omega(n)$ the number of prime divisors of an integer n counted without multiplicities. In other words, if n has prime decomposition $n = p_1^{m_1} \dots p_k^{m_k}$, let $\omega(n) = k$. By the Erdős–Kac theorem, the centered and rescaled distribution prescribed by the counting function $\omega(n)$ tends to the normal distribution:

Erdős–Kac Theorem (1939)

$$\lim_{N\to+\infty} \frac{1}{N} \operatorname{card}\left\{ n \le N \,\middle|\, \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt\,.$$

The subsequent results of A. Selberg (1954) and of A. Rényi and P. Turán (1958) describe the rate of convergence.

Statistics of prime decompositions: random permutations

Denote by $\mathrm{K}_n(\sigma)$ the number of disjoint cycles in the cycle decomposition of a permutation σ in the symmetric group S_n. Consider the uniform probability measure on S_n. A random permutation σ of n elements has exactly k cycles in its cyclic decomposition with probability $\mathbb{P}\big(\mathrm{K}_n(\sigma) = k\big) = \frac{s(n,k)}{n!}$, where $s(n,k)$ is the unsigned Stirling number of the first kind. It is immediate to see that $\mathbb{P}\big(\mathrm{K}_n(\sigma) = 1\big) = \frac{1}{n}$. V. L. Goncharov computed the expected value and the variance of K_n as $n \to +\infty$:

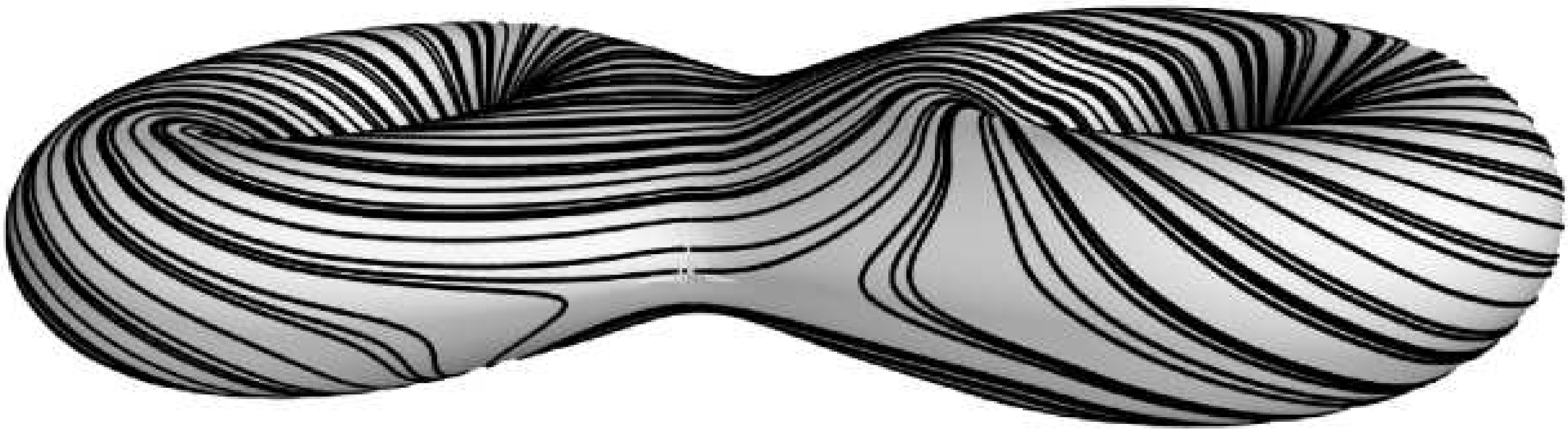
$$\mathbb{E}(\mathrm{K}_n) = \log n + \gamma + o(1)\,, \qquad \mathbb{V}(\mathrm{K}_n) = \log n + \gamma - \zeta(2) + o(1)\,,$$

and proved the following central limit theorem:

Theorem (V. L. Goncharov, 1944)

$$\lim_{n \to +\infty} \frac{1}{n!} \operatorname{card} \left\{ \sigma \in S_n \,\middle|\, \frac{\mathrm{K}_n(\sigma) - \log n}{\sqrt{\log n}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt\,.$$

What shape has a random simple closed multicurve on a surface of large genus?

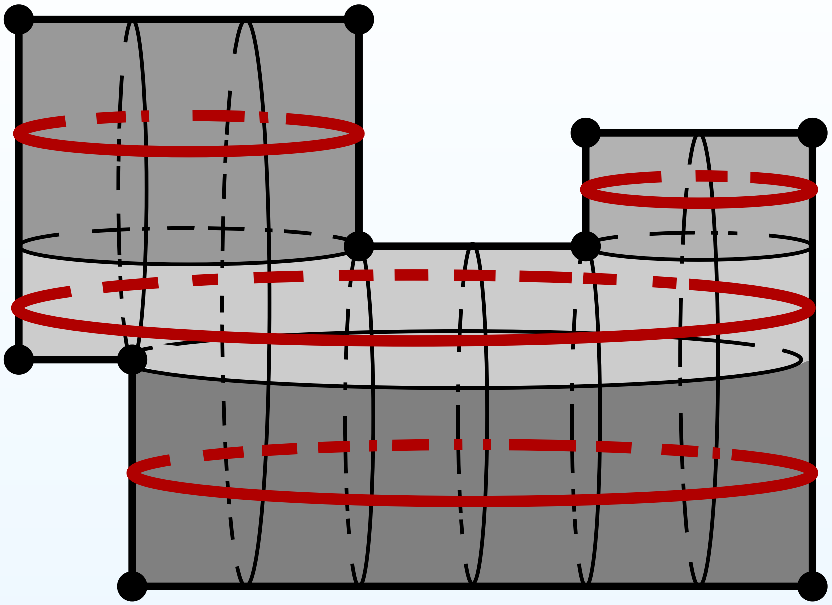


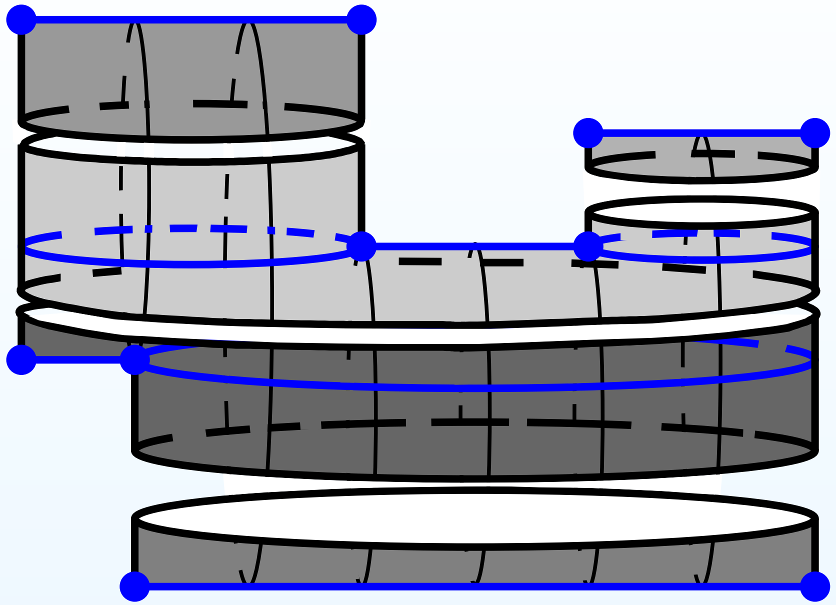
Picture from a book of Danny Calegari

Questions.

- *With what probability a random primitive multicurve γ on a surface of genus g slices the surface into $1, 2, 3, \ldots$ connected components?*
- *With what probability a random multicurve $m_1\gamma_1 + m_2\gamma_2 + \cdots + m_k\gamma_k$ has $k = 1, 2, \ldots, 3g-3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*
- *What are the typical weights $m_1, \ldots, m_k$?*
- *What is the shape of a random multicurve on a surface of large genus?*

Shape of a random square-tiled surface of large genus





Questions.

- *How many singular horizontal leaves (in blue on the right picture) has a random square-tiled surface of genus g?*
- *Find the probability distribution for the number $K_g(S) = 1, 2, 3, \ldots, 3g-3$ of maximal horizontal cylinders (represented by red waist curves on the left picture)*
- *What are the typical heights $h_1, \ldots, h_k$ of the cylinders?*
- *What is the shape of a random square-tiled surface of large genus?*

Partly suggested to Leonid Monin and to Peter Pushkar Jr. about ten years ago as a 3-weeks student's project

Random multicurves and random square-tiled surfaces

Denote by $K_{g,n}(\gamma)$ the number k of components of a multicurve $\gamma = \sum_{i=1}^{k} m_i\gamma_i$ (counted *without* multiplicities m_i) on a surface of genus g with n cusps.

Denote by $K_{g,n}(S)$ the number of maximal horizontal cylinders in the cylinder decomposition of a square-tiled surface S of genus g with n cone-angles π

Theorem (Delecroix–Goujard–Zograf–Zorich'21.). *For any genus $g \geq 2$ and for any $k \in \mathbb{N}$, the probability $p_g(k)$ that a random multicurve γ on a surface of genus g has exactly k components counted without multiplicities coincides with the probability that a random square-tiled surface S of genus g has exactly k maximal horizontal cylinders:*

$$\mathbb{P}\big(K_{g,n}(\gamma) = k\big) = \mathbb{P}\big(K_{g,n}(S) = k\big)\,.$$

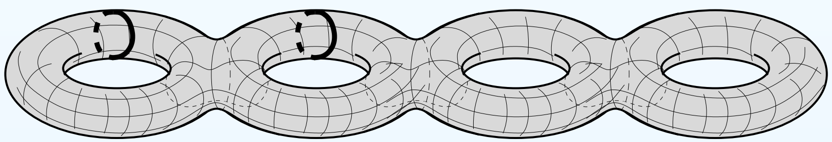
In other words, $K_{g,n}(\gamma)$ and $K_{g,n}(S)$, considered as random variables, determine the same probability distribution for any g, n, $3g+n \geq 4$.

From now on we consider only hyperbolic surfaces without cusps and only square-tiled surfaces without cone-angles π (i.e. the ones corresponding to *holomorphic* quadratic differentials).

Shape of a random multicurve (random square-tiled surface) on a surface of large genus in simple words

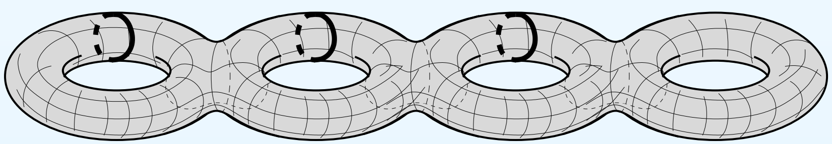
Theorem (Delecroix–Goujard–Zograf–Zorich'20.). *With probability which tends to 1 as $g \to \infty$,*

- *The reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$ associated to a random integral multicurve $\gamma = m_1\gamma_1 + \dots m_k\gamma_k$ does not separate the surface;*
- *$\gamma_{reduced}$ has about $(\log g)/2$ components and has one of the following types:*



$0.09\log(g)$ components

$\dots$ $\qquad$ $\dots \quad \dots \quad \dots$



$0.62\log(g)$ components

$$\mathbb{P}\Big(0.09\log g < K_g(\gamma) < 0.62\log g\Big) = 1 - O\left((\log g)^{24} g^{-1/4}\right).$$

A random square-tiled surface (without conical points of angle π) of large genus has about $\frac{\log(g)}{2}$ cylinders, and ***all conical points sit at the same horizontal and the same vertical level with probability which tends to 1 as $g \to \infty$.***

Heights of cylinders of a ransom square-tiled surface

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *If we fix any k and consider only k-cylinder square-tiled surfaces, then a (conditional) probability that every horizontal cylinder is composed of a single band of squares tends to 1 as $g \to +\infty$.*

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *If we do not fix the number of horizontal cylinders, then the probability that every horizontal cylinder of a random square-tiled surface is composed of a single band of squares tends to $\frac{\sqrt{2}}{2}$ as genus grows. More generally, each of the heights $m_1, \dots m_k$ of horizontal cylinders of a random square-tiled surface is bounded from above by an integer m with probability which tends to $\sqrt{\frac{m}{m+1}}$ as $g \to +\infty$.*

However, the mean value of $m_1 + \ldots + m_k$ is infinite in any genus g.

Weights of a random multicurve

Equivalently:

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *A random integer multicurve* $m_1\gamma_1 + \cdots + m_k\gamma_k$ *with bounded number* k *of primitive components is reduced (i.e.,* $m_1 = \cdots = m_k = 1$*) with probability which tends to* 1 *as* $g \to +\infty$*. In other terms, if we consider a random square-tiled surface with at most* K *cylinders, the heights of all cylinders would be very likely equal to* 1 *for* $g \gg 1$*.*

Theorem (Delecroix–Goujard–Zograf–Zorich'19). *A general random integer multicurve* $m_1\gamma_1 + \cdots + m_k\gamma_k$ *is reduced (i.e.,* $m_1 = \cdots = m_k = 1$*) with probability which tends to* $\frac{\sqrt{2}}{2}$ *as genus grows. More generally, all weights* $m_1, \ldots m_k$ *of a random multicurve are bounded from above by an integer* m *with probability which tends to* $\sqrt{\frac{m}{m+1}}$ *as* $g \to +\infty$*.*

Number of cycles in a random permutation

Given a permutation $\sigma \in S_n$ of cycle type $(1^{\mu_1} 2^{\mu_2} \dots n^{\mu_n})$ define its *weight* as

$$w_\theta(\sigma) := \theta_1^{\mu_1} \theta_2^{\mu_2} \cdots \theta_n^{\mu_n} \,,$$

where $\theta_j = \frac{\zeta(2j)}{2}$, $j \in \mathbb{N}$. Define a probability measure on S_n by setting

$$\mathbb{P}_\theta(\sigma) := \frac{w_\theta(\sigma)}{W_\theta} \,, \qquad \text{where} \qquad W_\theta := \sum_{\sigma \in S_n} w_\theta(\sigma).$$

Measures with $\theta_k = const$, $k \in \mathbb{N}$, are called *Ewens measures*; for $const = 1$ we get the uniform measure on S_n.

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The random variable $K(\sigma)$ counting the number of disjoint cycles in the cyclic decomposition of a random permutation is very well studied (Goncharov'44, ... Hwang'94–95, ... Kowalski–Nikeghbali'10,...). The corresponding probability distribution is given by the Poisson distribution with parameter depending on n, corrected by a convolution with certain explicit function independent of n.

Using this *Mod-Poisson convergence* technique we also get a very precise description of the law for the number of cycles $K(\sigma)$ in a random permutation for our nonuniform Ewens-like measure $\mathbb{P}_\theta$.

Probability that a random permutations has k cycles

The following Lemma identifies normalized weighted multi-variate harmonic sums as total contributions of permutations having exactly k cycles to the total sum $W_{\theta,n}$.

Lemma. *Let $\theta = \{\theta_j\}_{j\geq 1}$ be non-negative real numbers and consider the associated probability measure $\mathbb{P}_{\theta,n}$ on the symmetric group S_n for some n. Then*

$$\frac{k!}{n!} \cdot \sum_{\substack{\sigma\in S_n \\ \mathrm{K}_n(\sigma)=k}} w_\theta(\sigma) = \sum_{i_1+\cdots+i_k=n} \frac{\theta_{i_1}\theta_{i_2}\cdots\theta_{i_k}}{i_1\cdots i_k},$$

where $\mathrm{K}_n(\sigma)$ is the number of cycles in the cycle decomposition of σ and the sum in the right hand-side is taken over positive integers $i_1,\ldots,i_k$. In other words, we have the identity in the ring $\mathbb{Q}[[t,z]]$ of formal power series in t and z

$$\sum_{n\geq 1}\sum_{\sigma\in S_n} w_\theta(\sigma) t^{\mathrm{K}_n(\sigma)} \frac{z^n}{n!} = \exp\left(t\sum_{k\geq 1}\theta_k \frac{z^k}{k}\right).$$

Main Theorem (informally)

Main Theorem (Delecroix–Goujard–Zograf–Zorich'20). *As g grows, the probability distribution $\mathbb{P}(K_g = k)$ rapidly becomes, basically, indistinguishable from the distribution of the number $K_{3g-3}(\sigma)$ of disjoint cycles in a $\mathbb{P}_\theta$-random permutation σ of $3g-3$ elements. In particular, for any $j \in \mathbb{N}$ the difference of the j-th moments of the two distributions is of the order $O(g^{-1})$.*

We have an explicit asymptotic formula for all cumulants. It gives

$$
\begin{aligned}
\mathbb{E}(K_g) &= \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 + o(1)\,,\\
\mathbb{V}(K_g) &= \frac{\log(6g-6)}{2} + \frac{\gamma}{2} + \log 2 - \frac{3}{4}\zeta(2) + o(1)\,,
\end{aligned}
$$

where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

In practice, already for $g = 12$ the match of the graphs of the distributions is such that they are visually indistinguishable.

Mod-Poisson convergence (Hwang'94–95) . For any $x > 0$ the distribution of the number of cycles of a uniformly random permutation $\sigma \in S_n$ of n elements is uniformly well-approximated in a neighborhood of $x \log n$ by the Poisson distribution with parameter $\log n + a(x)$ with an explicit correction $a(x)$.

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where $\gamma = 0.5772\ldots$ denotes the Euler–Mascheroni constant.

Let $\lambda_{3g-3} = \log(6g-6)/2$. We have uniformly in $0 \le k \le 1.233 \cdot \lambda_{3g-3}$

$$\mathbb{P}\big(K_g(\gamma) = k{+}1\big) = e^{-\lambda_{3g-3}} \cdot \frac{\lambda_{3g-3}^k}{k!} \cdot \left(\frac{\sqrt{\pi}}{2\Gamma\left(1 + \frac{k}{2\lambda_{3g-3}}\right)} + O\left(\frac{k}{(\log g)^2}\right)\right)\,.$$

Idea of the proof and further conjectures

Schematic idea of the proof

- Observe that square-tiled surfaces corresponding to stable graphs with more than one vertex taken together contribute only $O\left(\frac{1}{g}\right)$ to the count of all square-tiled surfaces of genus g (this conjecture of ours was proved by A. Aggarwal).

- Using large genus asymptotics for the Witten–Kontsevich correlators (conjectured by us and proved by A. Aggarwal) compute the contribution of square-tiled surfaces of genus g represented by the stable graph with exactly one vertex and with k loops. Recognize in the resulting expression the multivariate harmonic sum as in the above Lemma corresponding to parameters $\theta_j = \zeta(2j)/2$, where $j = 1, 2, \dots$.

- Apply the analytic technique developed by H. Hwang for random permutations to prove mod-Poisson convergence of the resulting distribution of the number of cycles $\mathrm{K}_n(\sigma)$ of a random permutation σ, where “randomness” is defined using parameters $\theta_j = \zeta(2j)/2$, where $j = 1, 2, \dots$.

Keystone underlying results and further conjectures

Our results use the Delecroix–Goujard–Zograf–Zorich'19 conjecture proved in

Theorem (Aggarwal'21). *The Masur–Veech volume of the moduli space of holomorphic quadratic differentials has the following large genus asymptotics:*

$$\operatorname{Vol} \mathcal{Q}_g \sim \frac{4}{\pi} \cdot \left(\frac{8}{3}\right)^{4g-4} \quad \text{as } g \to +\infty\,.$$

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The similar conjecture of Eskin–Zorich'03 on the large genus asymptotics of Masur–Veech volumes of individual strata of *Abelian* differentials is recently proved by Aggarwal'19 and by Chen–Möller–Sauvaget–Zagier'20. The analogous conjecture for *quadratic* differentials still resists:

Conjecture (ADGZZ'20). *The Masur–Veech volume of any stratum of meromorphic quadratic differentials with at most simple poles has the following large genus asymptotics (with the error term uniformly small for all partitions $\boldsymbol{d}$):*

$$\operatorname{Vol}\mathcal{Q}(d_1,\dots,d_n) \overset{?}{\sim} \frac{4}{\pi}\cdot\prod_{i=1}^{n}\frac{2^{d_i+2}}{d_i+2} \quad \text{as } g \to +\infty\,,$$

under assumption that the number of simple poles is bounded or grows much slower than the genus.

Combinatorial formulation of Witten's conjecture

Initial data: $\langle\tau_0^3\rangle = 1, \qquad \langle\tau_1\rangle = \frac{1}{24}\,.$

String equation:

$$\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = \langle\tau_{d_1-1}\ldots\tau_{d_n}\rangle_{g,n} + \cdots + \langle\tau_{d_1}\ldots\tau_{d_n-1}\rangle_{g,n}\,.$$

Dilaton equation:

$$\langle\tau_1\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n+1} = (2g-2+n)\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{g,n}\,.$$

Virasoro constraints (in Dijkgraaf–Verlinde–Verlinde form; $k \geq 1$):

$$\langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\Bigg[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{r+s=k-1\\ r,s\geq 0}}(2r+1)!!(2s+1)!!\sum_{\{1,\ldots,n\}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg]\,.$$

Another Keystone result and one more conjecture

We also strongly use the uniform large genus asymptotics of ψ-classes which we conjectured in 2019 and which was proved by Aggarwal:

Theorem (Aggarwal'21). *The following* ***uniform*** *asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ ***uniformly*** *for all* $n = o(\sqrt{g})$ *and all partitions* $\boldsymbol{d}$*,* $d_1 + \dots + d_n = 3g-3+n$*, as* $g \to +\infty$*.*

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*where $\varepsilon(\boldsymbol{d}) = O\left(1+\frac{(n+\log g)^2}{g}\right)$ **uniformly** for all $n = o(\sqrt{g})$ and all partitions $\boldsymbol{d}$, $d_1+\cdots+d_n = 3g-3+n$, as $g \to +\infty$.*

Conjecture* (Delecroix–Goujard–Zograf–Zorich). *The distribution of the number of maximal horizontal cylinders in a random Abelian square-tiled surfaces of genus g gets very well approximated by the distribution of the number of disjoint cycles in a uniformly random permutation of $4g-3$ elements as $g \to \infty$.*

We already proved that a random square-tiled surface in a stratum $\mathcal{H}$ has a single cylinder with probability close to $\frac{1}{\dim \mathcal{H}}$.

* About 2 years of CPU-time of two independent computer experiments with strata of genera from 40 to 10 000.

Distribution of lengths of components of a random multicurve on a surface of large genus

Consider a random multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$ on a hyperbolic surface $X \in \mathcal{M}_g$ and rearrange the components of the vector of weighted lengths $(m_1\ell_X(\gamma_1), \ldots, m_k\ell_X(\gamma_k))$ in a decreasing order to produce a vector $\ell_X^\downarrow(\gamma)$. Normalize $\ell_X^\downarrow(\gamma)$ by $\ell_X(\gamma) = m_1\ell_X(\gamma_1) + \cdots + m_k\ell_X(\gamma_k))$.

Theorem (V. Delecroix, M. Liu, 2022). *For any X in $\mathcal{M}_g$ and for any $j \in \mathbb{N}$ the average of $\frac{(\ell_X^\downarrow)_j}{\ell_X}$ over multicurves of bounded length gives in the limit a well-defined random variable $L_j^{(g)\downarrow}$ which depends only on j and g.*

When $g \to +\infty$, the probability distribution of $L_j^{(g)\downarrow}$ weakly converges to a limiting probability distribution V_j. The distribution V_j coincides with the limiting distribution of the normalized length of the j-th longest cycle of a non-uniformly random permutation with respect to the Evens measure with parameter $\theta = \frac{1}{2}$ on S_n as $n \to +\infty$. It is the distribution of the Poisson–Dirichlet process with parameter $\theta = \frac{1}{2}$. In particular,

$$\mathbb{E}(V_1) \approx 0.758, \qquad \mathbb{E}(V_2) \approx 0.171, \qquad \mathbb{E}(V_3) \approx 0.049\,.$$

Poisson–Dirichlet process

Stick breaking process. Let $U_1, U_2, \ldots,$ be i.i.d. random variables supported on $[0,1]$ with density $\theta(1-x)^{\theta-1}$. Take a stick of length one and chop a piece of length U_1 out of it. Chop a piece of length proportional to U_2 out of the remaining part, etc. We get a random vector

$$V = (U_1, (1-U_1)U_2, (1-U_1)(1-U_2)U_3, \ldots)\,.$$

The law of V is the *Griffiths–Engen–McCloskey distribution with parameter* θ. The *Poisson–Dirichlet distribution with parameter* θ is the distribution of $V^{\downarrow}$, obtained from V by rearranging its components in the decreasing order. Both distributions are very well studied. In particular,

$$\mathbb{E}(V_j^{\downarrow}) = \int_0^{+\infty} \frac{\big(\theta E_1(x)\big)^{j-1}}{(j-1)!} e^{-x-\theta E_1(x)}\, dx\,,$$

where $E_1(x) = \int_x^{+\infty} \frac{e^{-y}}{y}\, dy$.

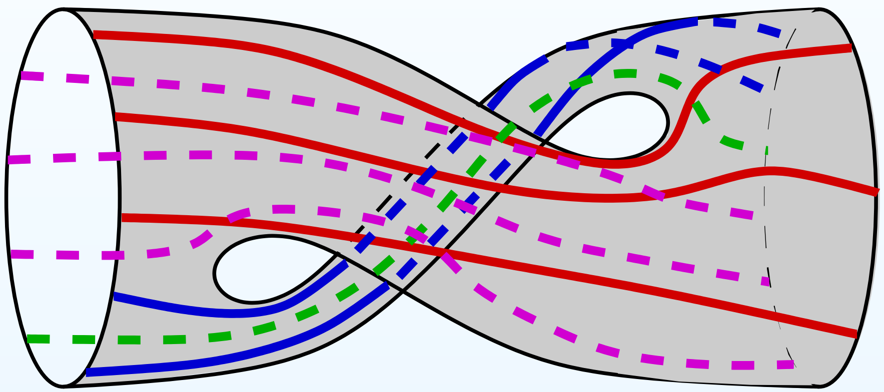
Equivalent statement for random square-tiled surfaces

Theorem (V. Delecroix, M. Liu, 2022). *Consider the decomposition of a random square-tiled surface of genus g into maximal horizontal cylinders. Consider the vector of normalized areas of these cylinders and rearrange its components in the decreasing order. The probability distribution of the resulting random vector weakly converges to the distribution of the Poisson–Dirichlet process with parameter $\theta = \frac{1}{2}$ as g tends to ∞.*

Restricting consideration to those random square-tiled surfaces, for which each cylinder contains at most m horizontal bands of squares, where $m = 1, 2, \dots,$ one gets in the limit the very same distribution of the Poisson–Dirichlet process with parameter $\theta = \frac{1}{2}$ as g tends to ∞.

Arnold’s problem (2002-8)

Glue randomly two boundary components of a braid with a large number N of strands on a surface of genus $g-1$ so that the endpoints fit.



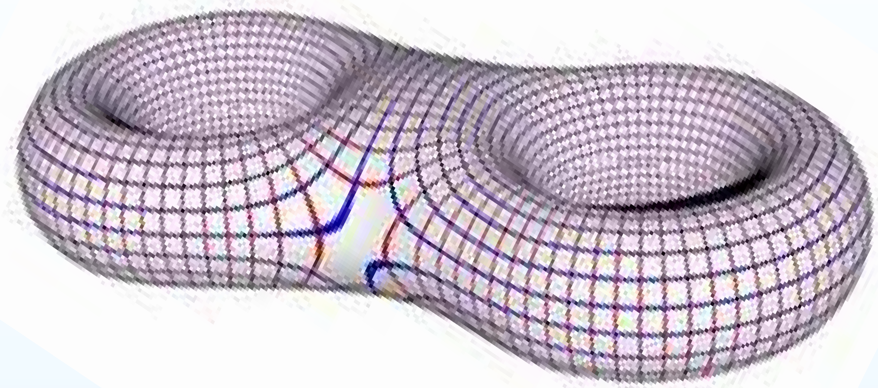
Theorem. *The probability p_g to get a single connected curve upon a random gluing of a random braid is*

$$p_g = \frac{1}{(4g-2)2^{2g-4}\,\mathrm{Vol}\,\mathcal{H}_g} \to \frac{1}{4g} + o\left(\frac{1}{g}\right) \text{ as } g \to +\infty\,.$$

Examples: $p_1 = \frac{6}{\pi^2}, \quad p_2 = \frac{45}{2\pi^4}, \quad p_3 = \frac{243}{2\pi^6}.$

Solution of homework problems

Problem: Which separatrix diagram?

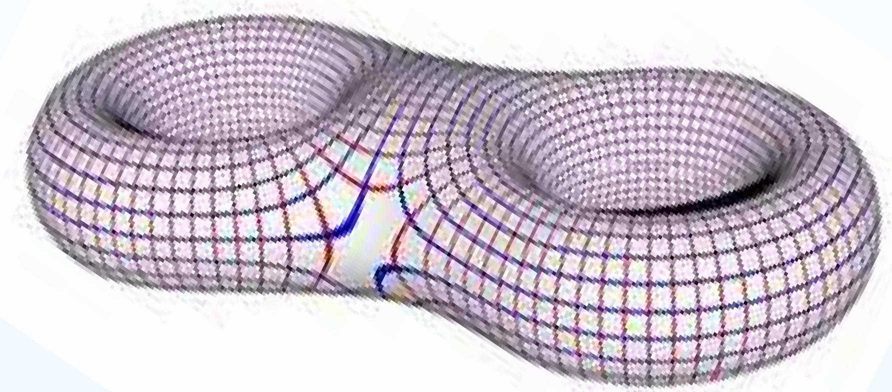


Picture created by Jian Jiang

Questions.

- *To what stratum belongs this square-tiled surface?*
- *Find all realizabe separatrix diagrams for this stratum.*
- *To which of the found diagrams corresponds the square-tiled surface from the picture?*

Which stratum?



Picture created by Jian Jiang

Question.

- *To what stratum belongs this square-tiled surface?*

Answer.

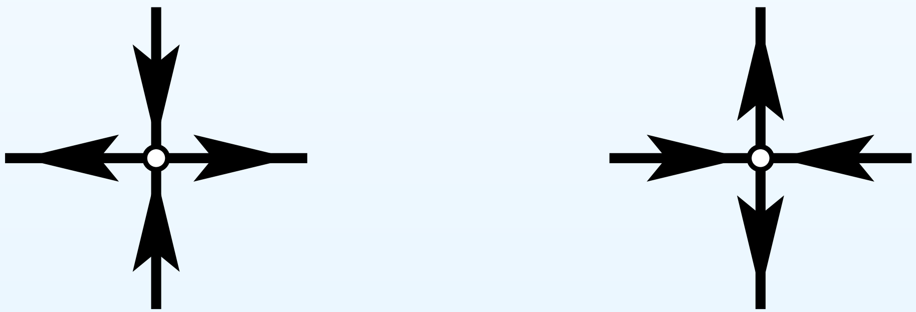
There are two strata in genus two: $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$. The surface in the picture has two symmetric conical singularities, so the ambient stratum is $\mathcal{H}(1,1)$.

One can also honestly count the cone angle at the visible conical singularity. The neighborhood is an octagon composed of four horizontal (blue) sides of the squares and of four vertical (red) sides. Thus, the cone angle is 4π, which excludes stratum $\mathcal{H}(2)$.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*



We have two zeroes. Each has two outgoing and two incoming horizontal separatrices.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

Let us start with critical graphs (separatrix diagram) having no closed loops. Let us draw one saddle connection and discuss how we can complete it.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

On the left there is a single outgoing separatrix and on the right — only one incoming. We are forced to join them.

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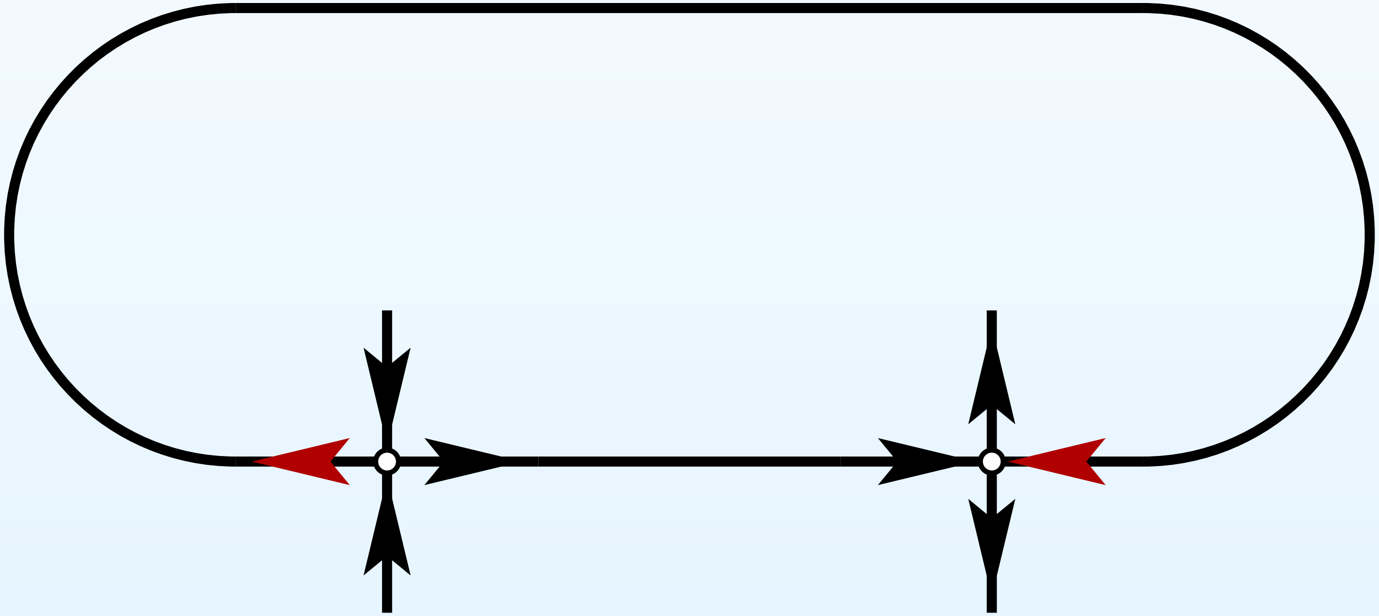

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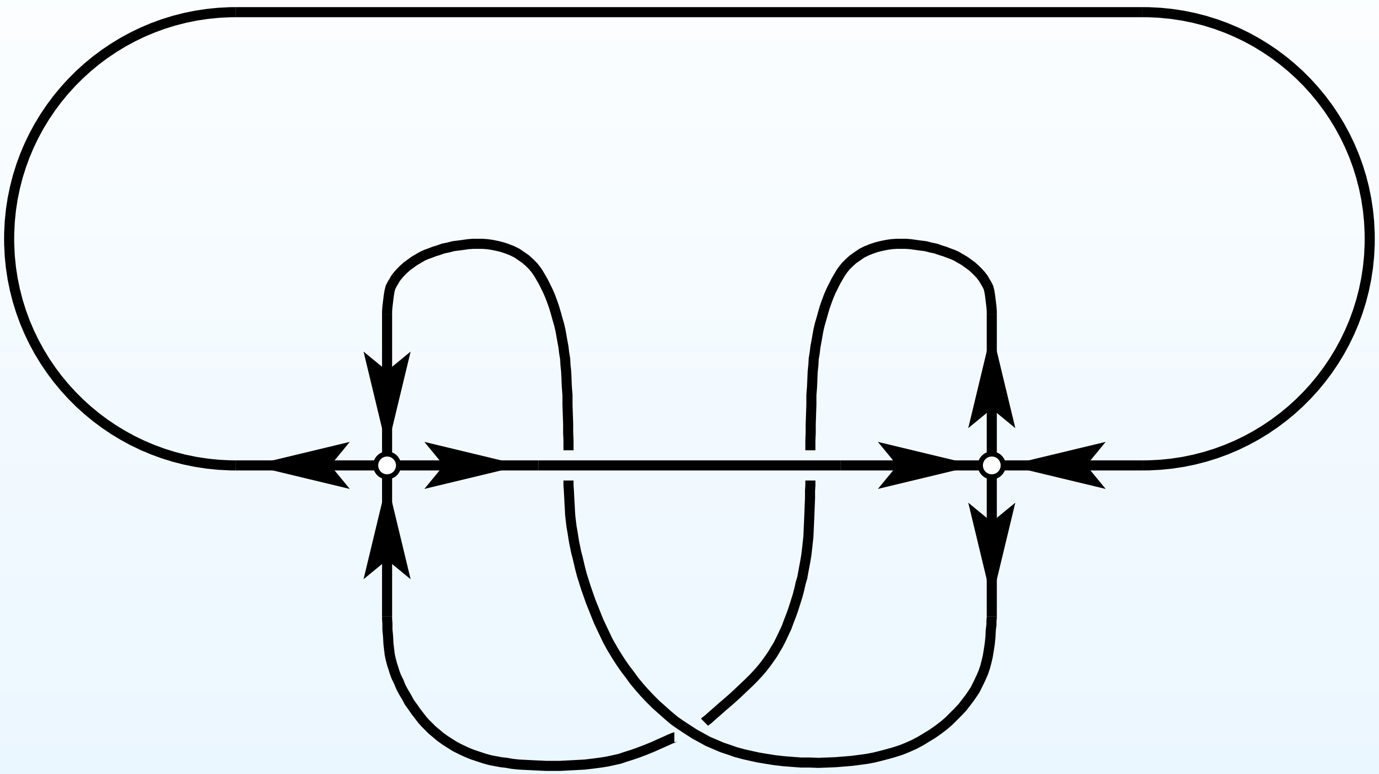
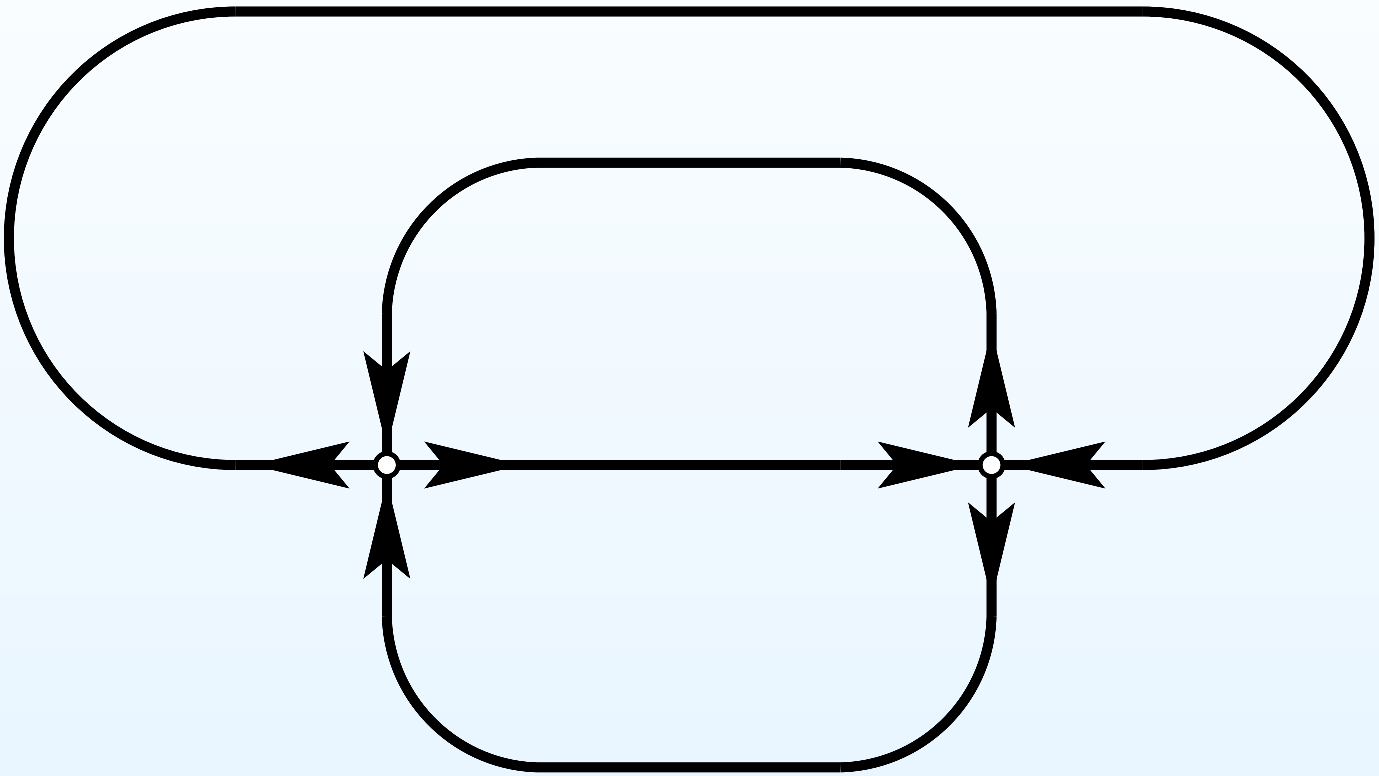
1-cylinder diagram in $\mathcal{H}(1,1)$



This is the first of the two ways of joining the remaining two pairs of separatrix rays.

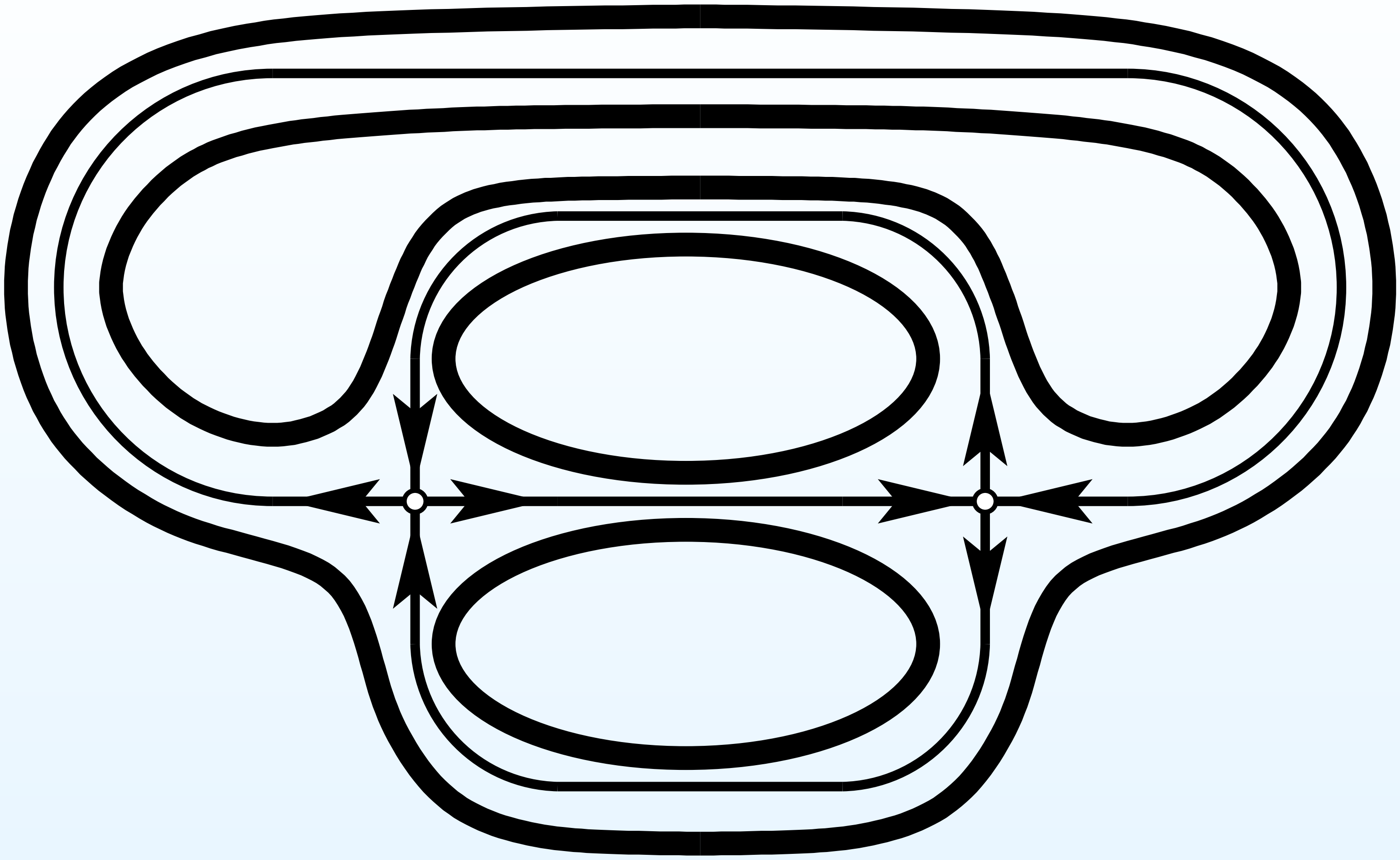
Mandatory Exercise. Check all of the following: The corresponding ribbon graph has two boundary components. Each component follows once each of the four saddle connection, so that the length of each of the two saddle connections is $\ell_1 + \ell_2 + \ell_3 + \ell_4$. There are no realtions on ℓ_i: this diagram is realizable for any choice of the lengths ℓ_i, where $i = 1, \ldots, 4$.

2-cylinder diagram in $\mathcal{H}(1,1)$



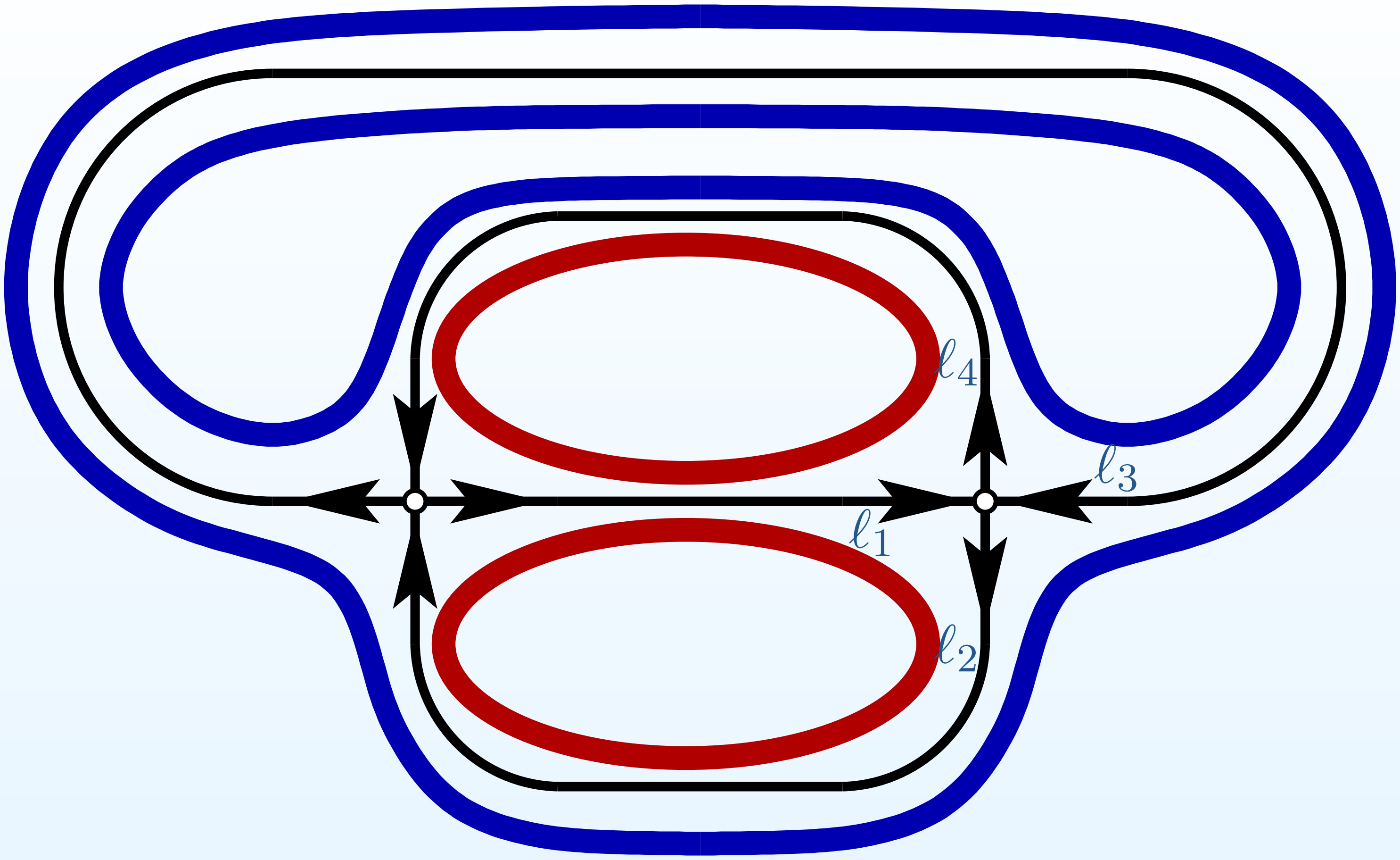
This is the other way to join the remaining two pairs of separatrix rays. Note that every maximal horizontal cylinder has one top and one boundary component. Thus, for every pair of boundary components to which we glue a cylinder, one component has the critical graph on the left and the other component has it on the right.

2-cylinder diagram in $\mathcal{H}(1,1)$



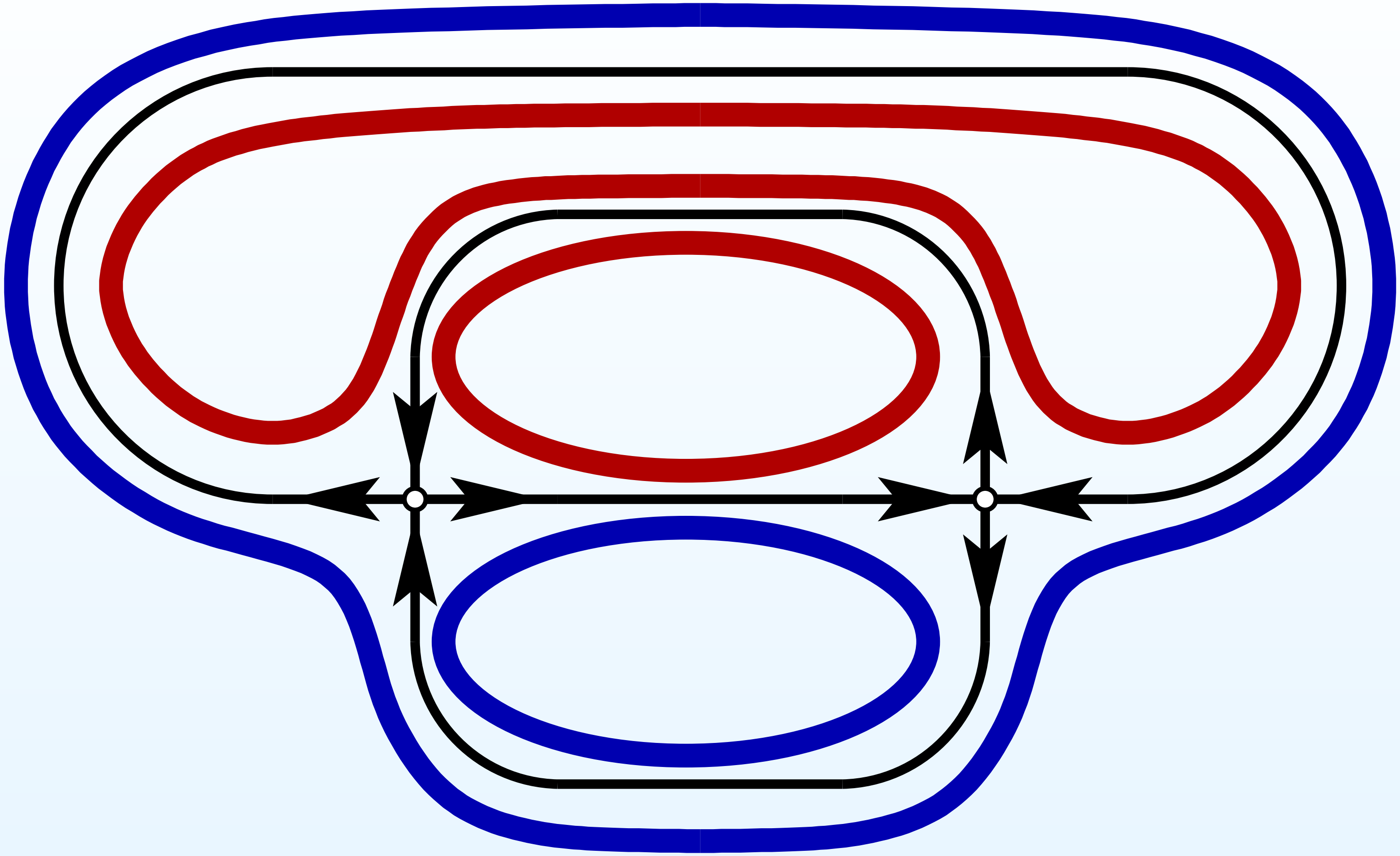
It gives us two ways in which we can organize the four boundary components into two pairs.

2-cylinder diagram in $\mathcal{H}(1,1)$



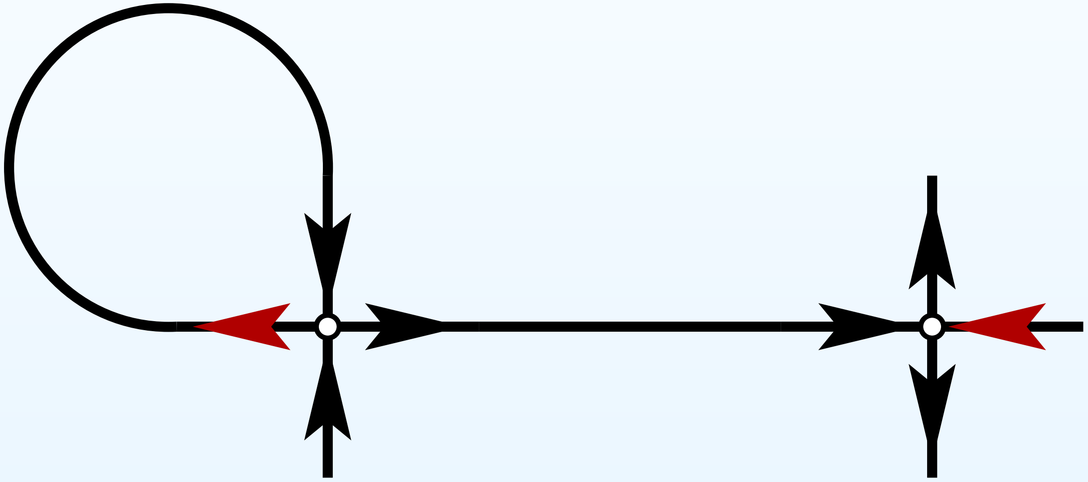
If we choose this way, we see that we have to impose the following conditions on the lengths of saddle connections: $\ell_2 = \ell_4$. Then the red cylinder has the waist curve of length $\ell_1 + \ell_2$ and the blue cylinder has the waist curve of length $\ell_3 + \ell_2$. We get an admissible diagram.

2-cylinder diagram in $\mathcal{H}(1,1)$



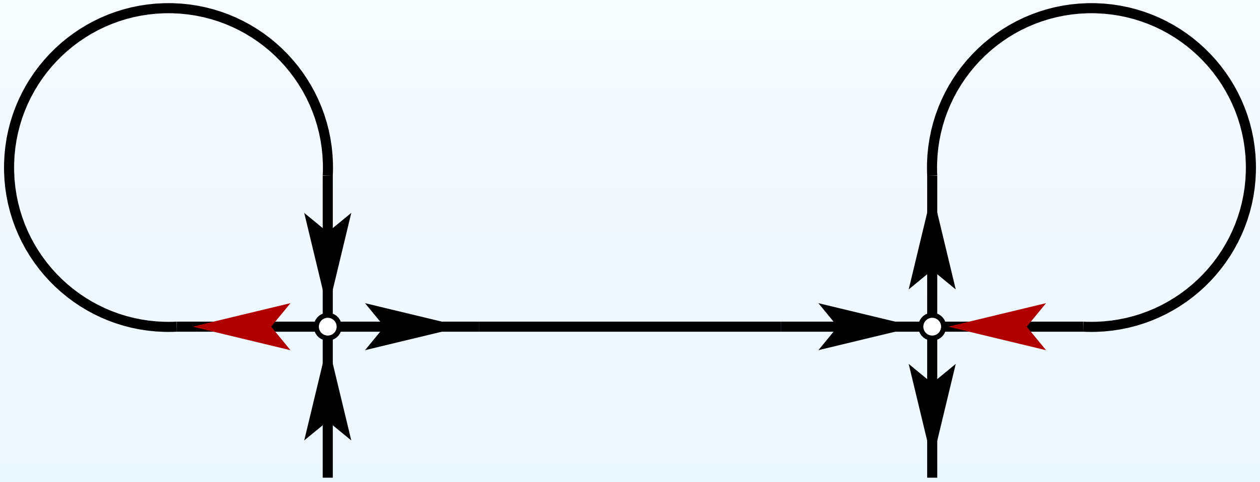
Exercise. Verify that the second way to arrange boundary components into pairs (as in the picture) is symmetric to the first one under interchanging the labels of the two singularities.

Diagrams with two loops in $\mathcal{H}(1,1)$



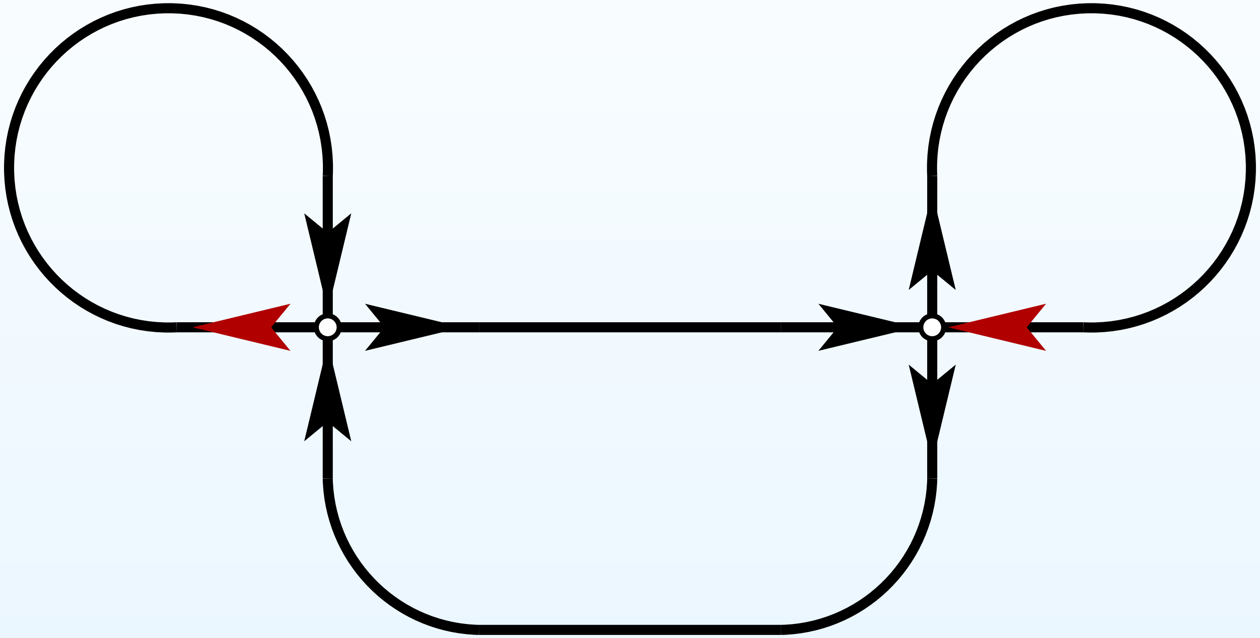
Now we have to consider diagrams having at least one loop. It is clear, that if a diagram has a loop and a saddle connection joining the two zeroes, it has to have another loop at the other zero.

Diagrams with two loops in $\mathcal{H}(1,1)$



There are two choices for the second loop. This is the first possible choice.

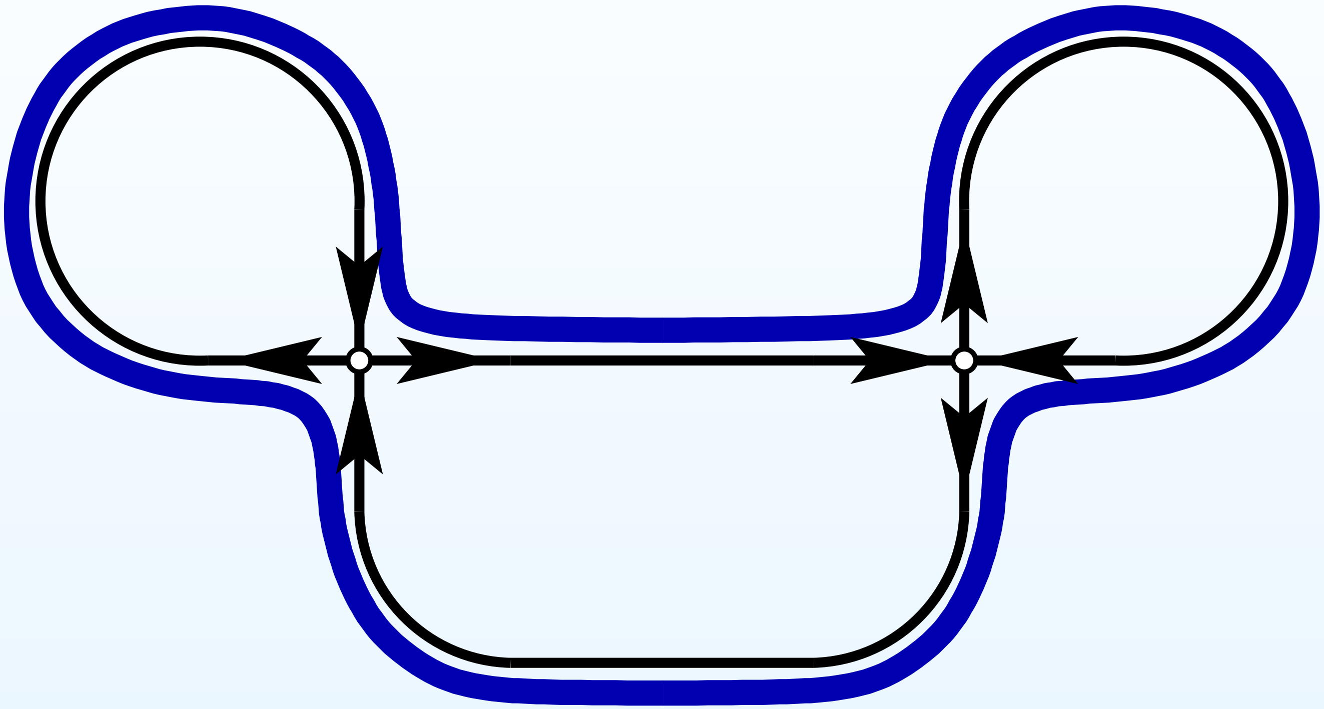
Diagrams with two loops in $\mathcal{H}(1,1)$



There are two choices for the second loop. This is the first possible choice.

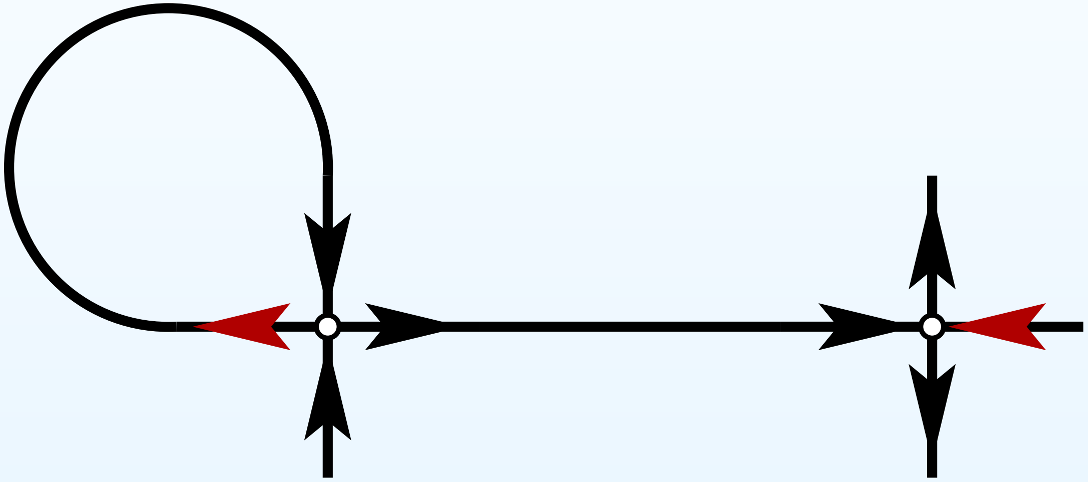
This is the unique way to join the remaining pair of separatrix rays.

Diagrams with two loops in $\mathcal{H}(1,1)$



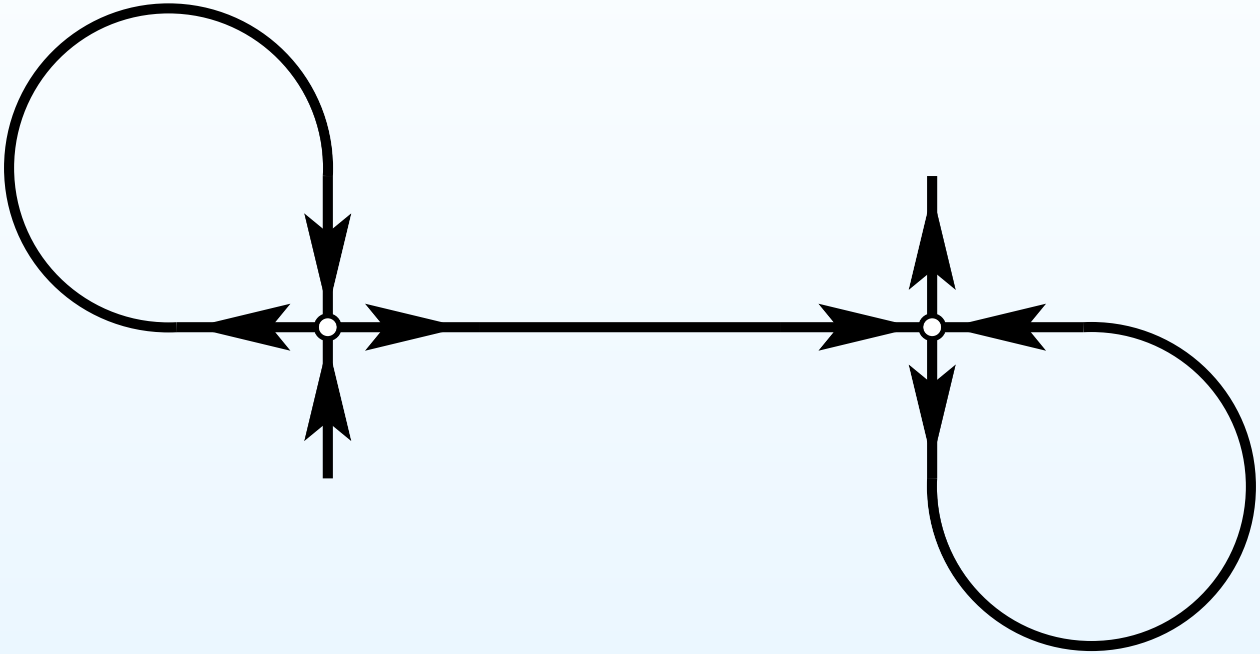
The boundary component of the resulting ribbon graph is longer than any other component for any choice of lengths of saddle connections (edges of the graph). This diagram is not realizable.

Diagrams with two loops in $\mathcal{H}(1,1)$



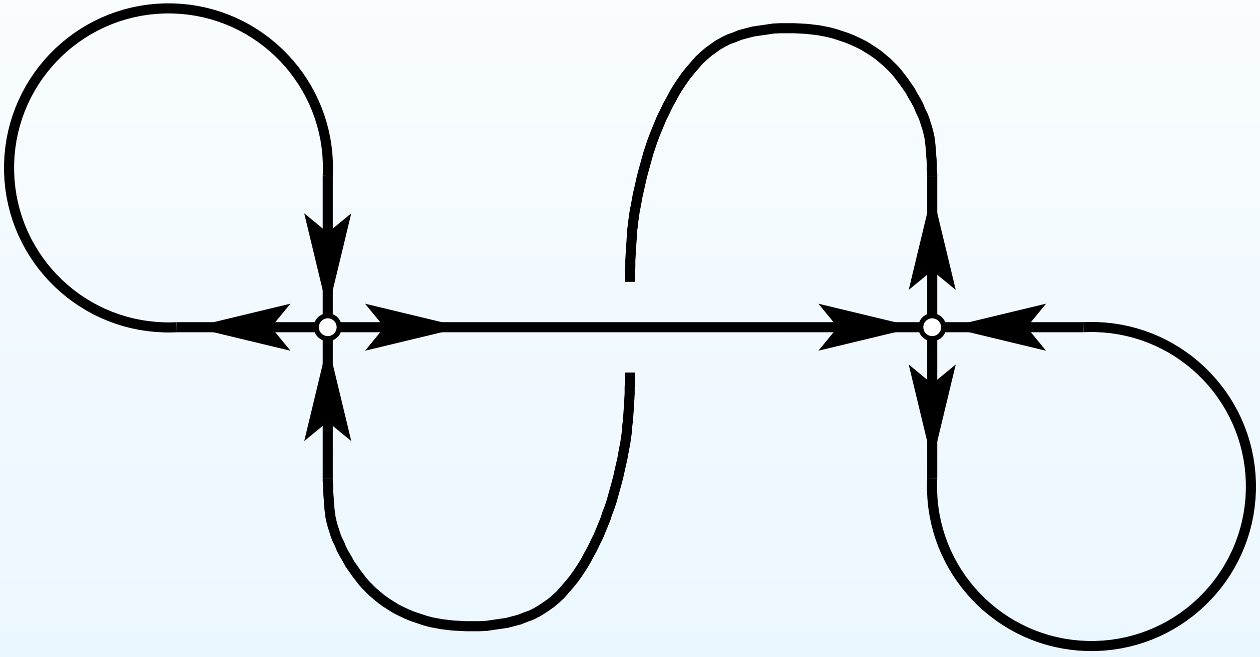
Recall that we are considering diagrams having at least one loop and a saddle connection joining the two zeroes.

Diagrams with two loops in $\mathcal{H}(1,1)$



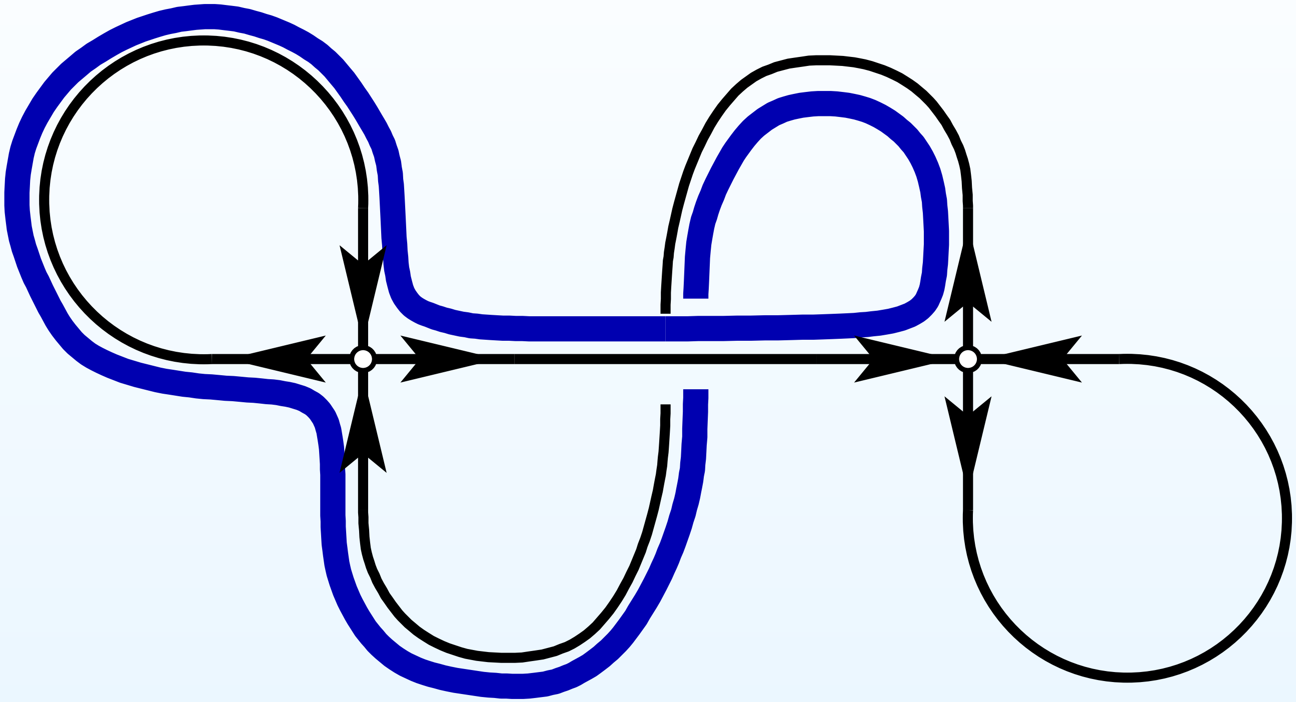
The second choice for the second loop as in the picture.

Diagrams with two loops in $\mathcal{H}(1,1)$



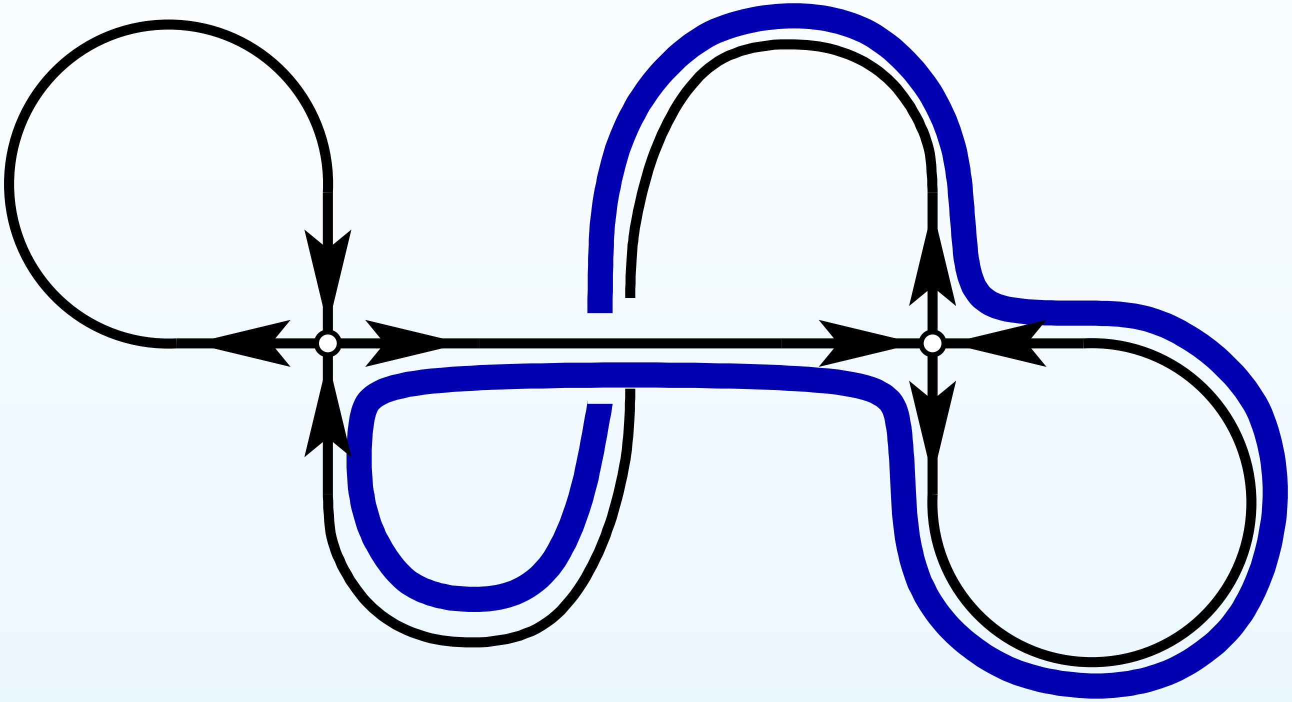
The second choice for the second loop as in the picture. This is the unique way to join the remaining pair of separatrix rays.

Diagrams with two loops in $\mathcal{H}(1,1)$



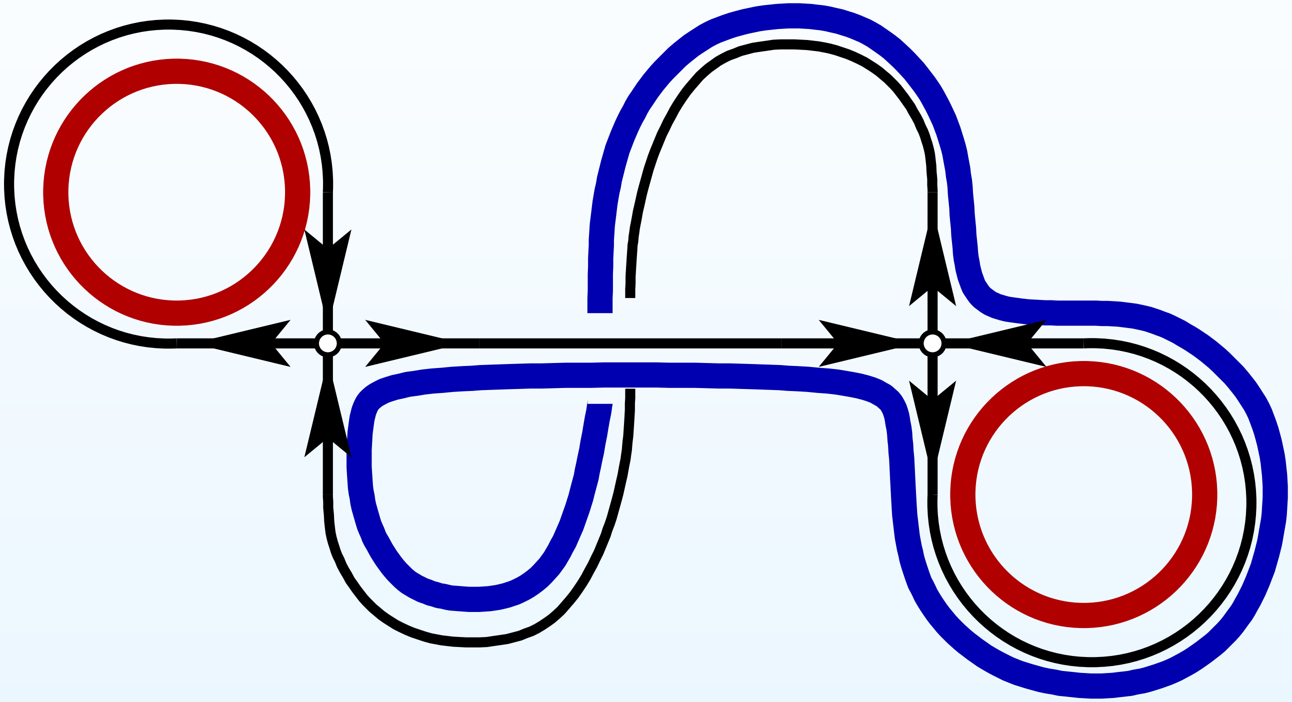
This is one of the four boundary components of the resulting ribbon graph.

Diagrams with two loops in $\mathcal{H}(1,1)$



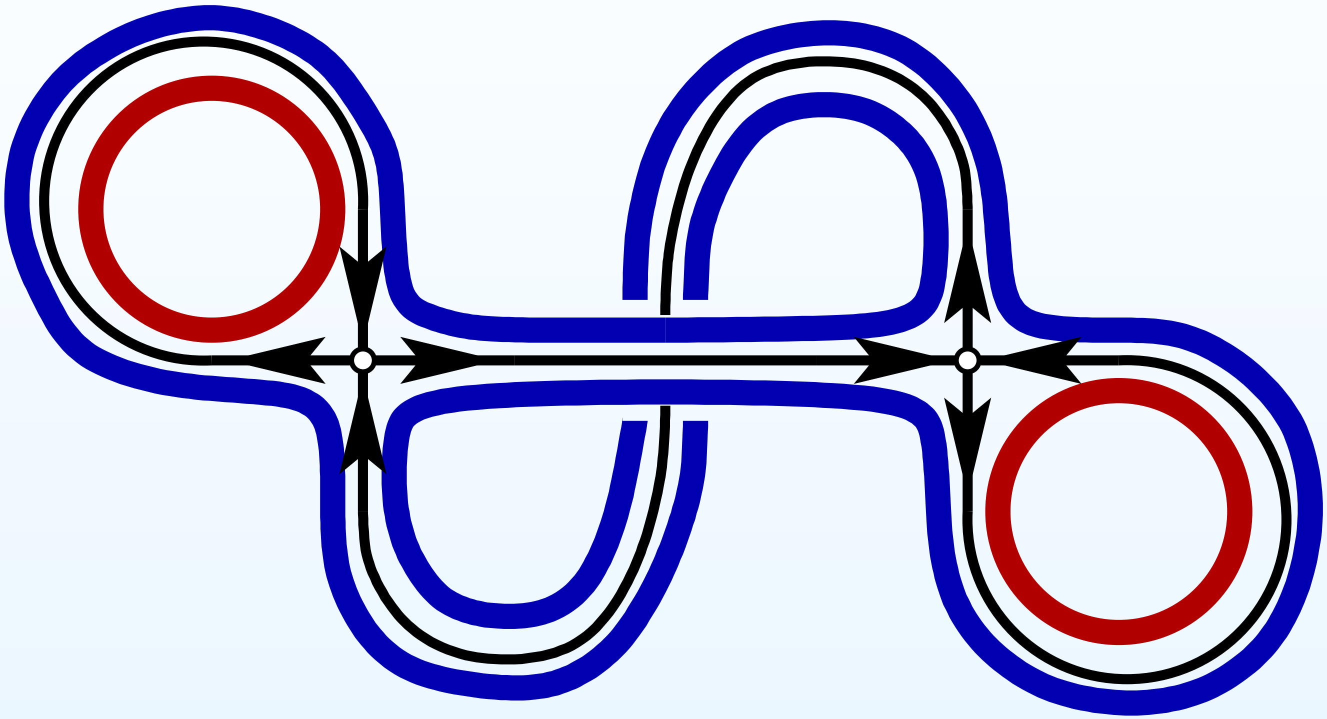
This is one more boundary component.

Diagrams with two loops in $\mathcal{H}(1,1)$



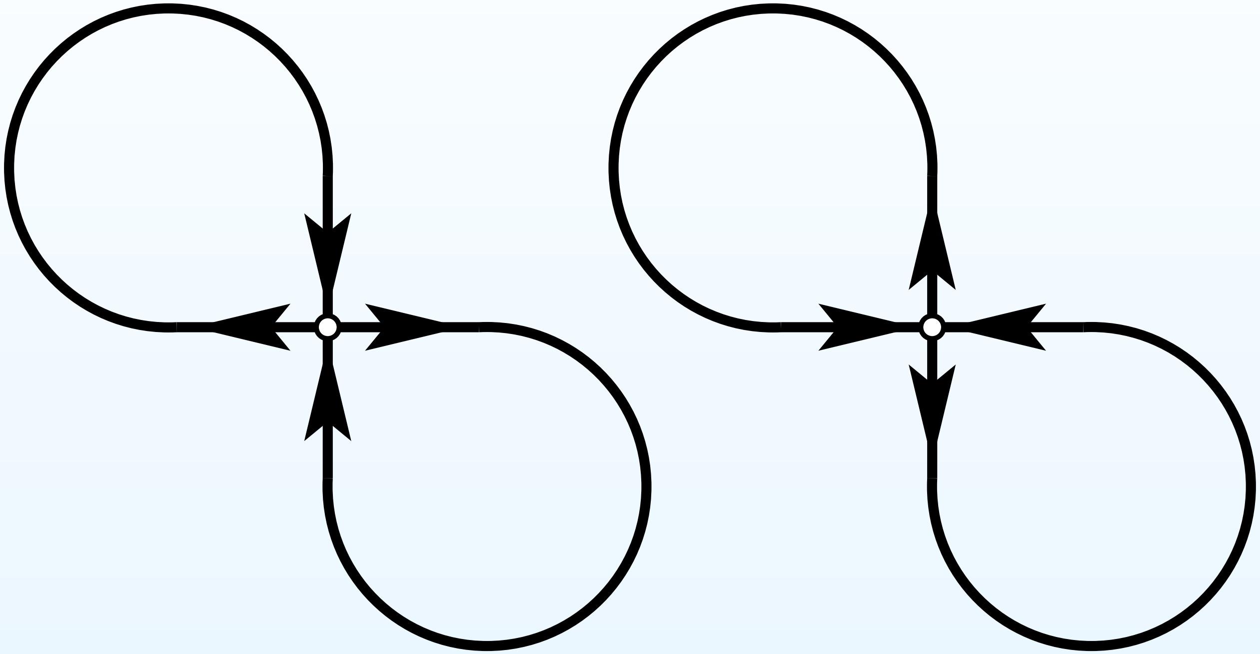
It is really easy to check that the only choice is to paste a cylinder to the pair of red boundary components. This implies a condition that the lengths of the corresponding loops are the same.

Diagrams with two loops in $\mathcal{H}(1,1)$



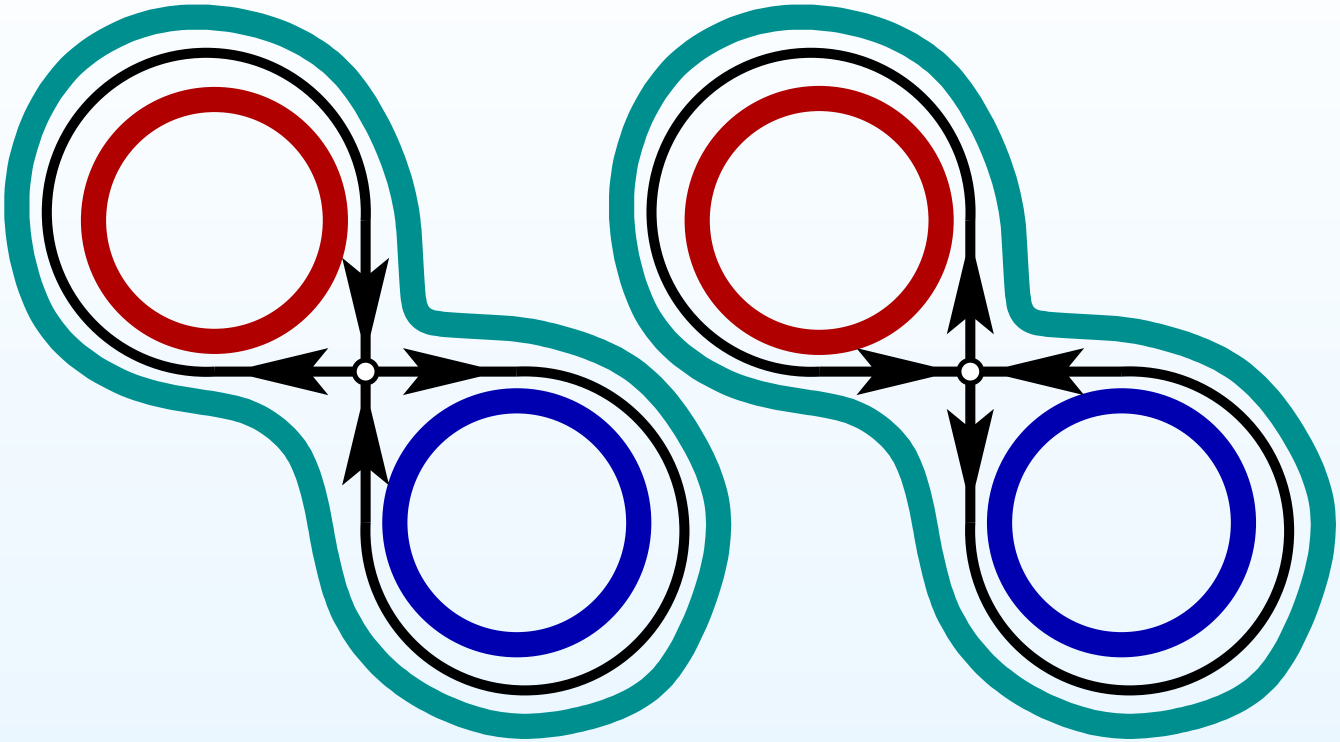
It is really easy to check that the only choice is to paste a cylinder to the pair of red boundary components. This implies a condition that the lengths of the corresponding loops are the same. This automatically implies that the lengths of the blue boundary components are the same. We get one more realizable diagram with two cylinders.

Diagrams with four loops in $\mathcal{H}(1,1)$



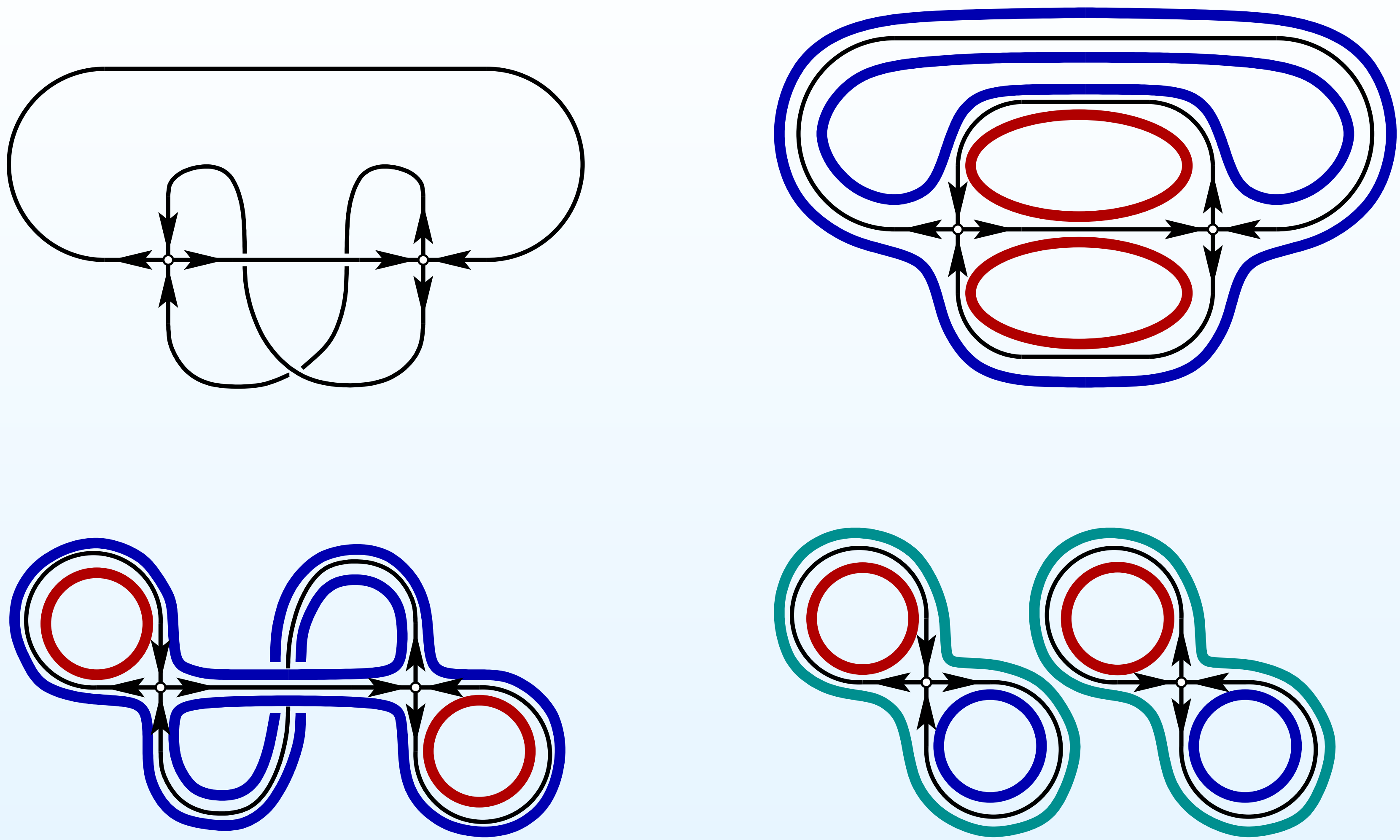
In the remaining case all the edges are loops.

Diagrams with four loops in $\mathcal{H}(1,1)$



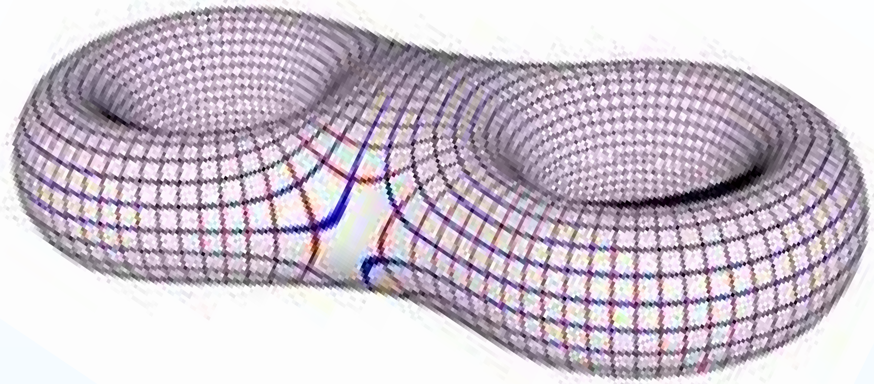
In the remaining case all the edges are loops. There is, clearly only one way to arrange boundary components into pairs. We get the last admissible (realizable) diagram in the stratum $\mathcal{H}(1,1)$.

Admissible diagrams in $\mathcal{H}(1,1)$



These four separatrix diagrams are admissible (realizable) diagrams in the stratum $\mathcal{H}(1,1)$ and there are no other ones (up to interchange of the labelling of the two zeroes).

Which diagram?



Picture created by Jian Jiang

Question.

● *To which of the found diagrams corresponds the red foliation of the square-tiled surface from the picture?*

Answer.

There are, clearly, three distinct cylinders. There only one 3-cylinder diagram in the stratum $\mathcal{H}(1,1)$: