Dynamics and Geometry of Moduli Spaces

Lecture 2. Magic Wand Theorem

Anton Zorich
University Paris Cité

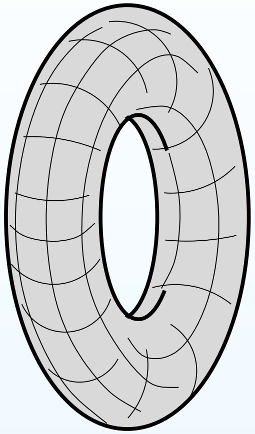
March 7, 2024

Diffeomorphisms of surfaces

Diffeomorphisms of surfaces

Observation 1. *Surfaces can wrap around themselves.*

Cut a torus along a horizontal circle.



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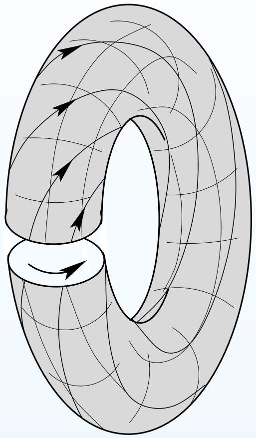

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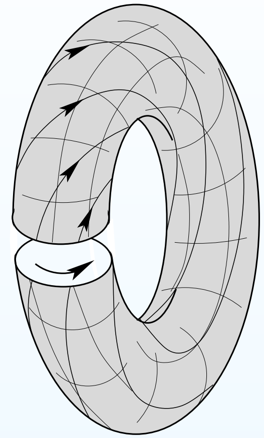

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Dehn twist corresponds to the linear map $\hat{f}_h : \mathbb{R}^2 \to \mathbb{R}^2$ with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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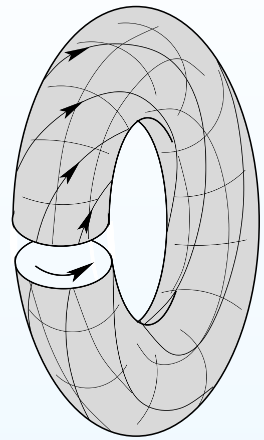

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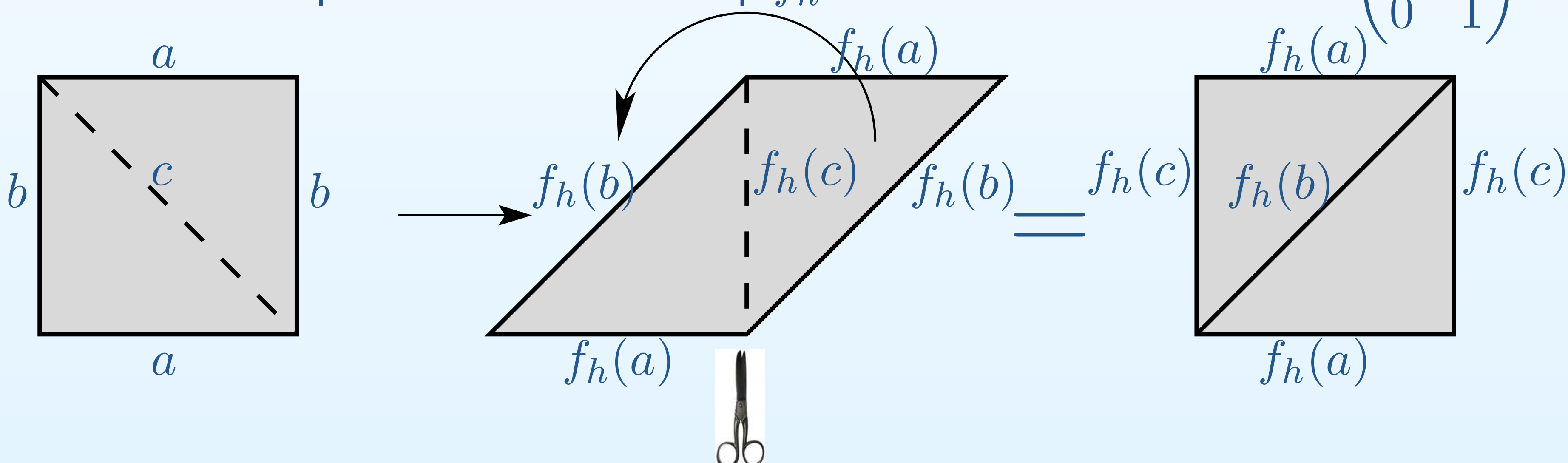

It maps the square pattern of the torus to a parallelogram pattern. Cutting and pasting appropriately we can transform the new pattern to the initial square.

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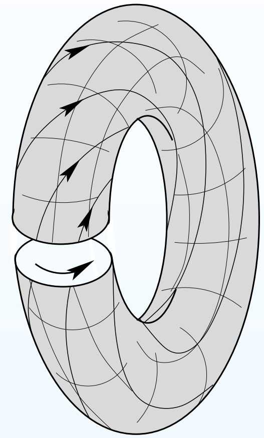

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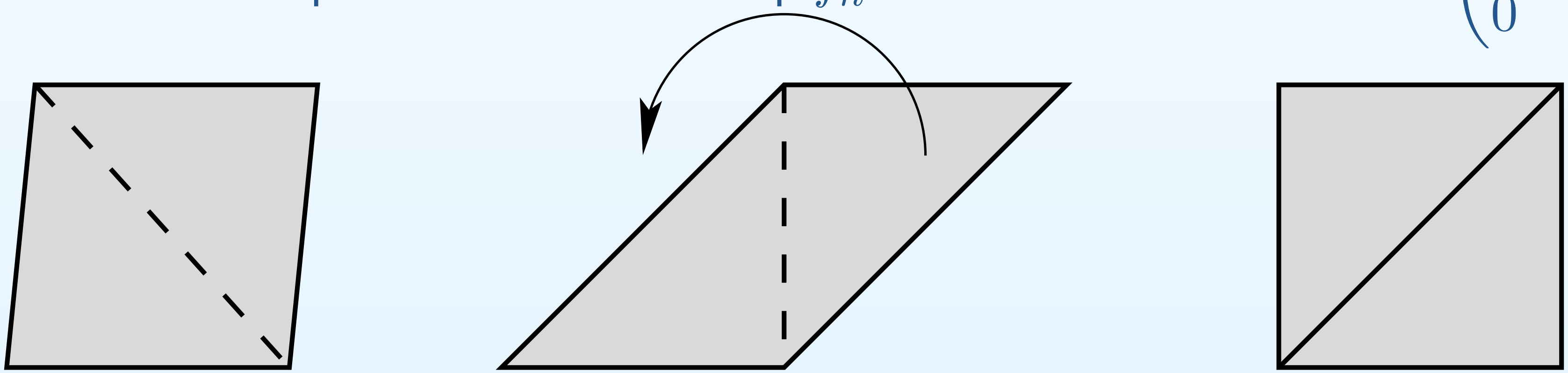

Changing the slope of the parallelogram pattern progressively we get a *closed path* in the space of flat tori.

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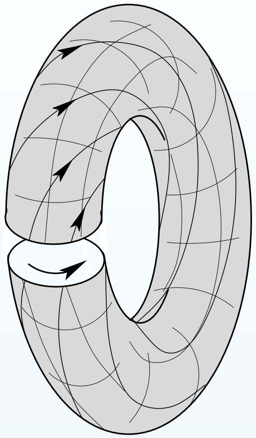

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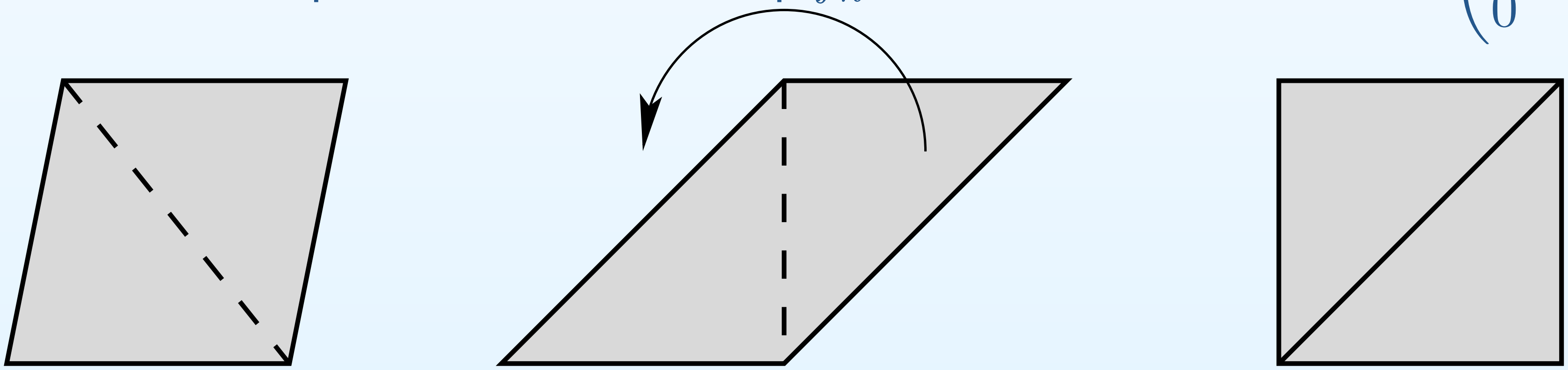

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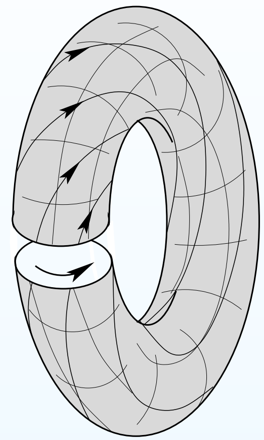

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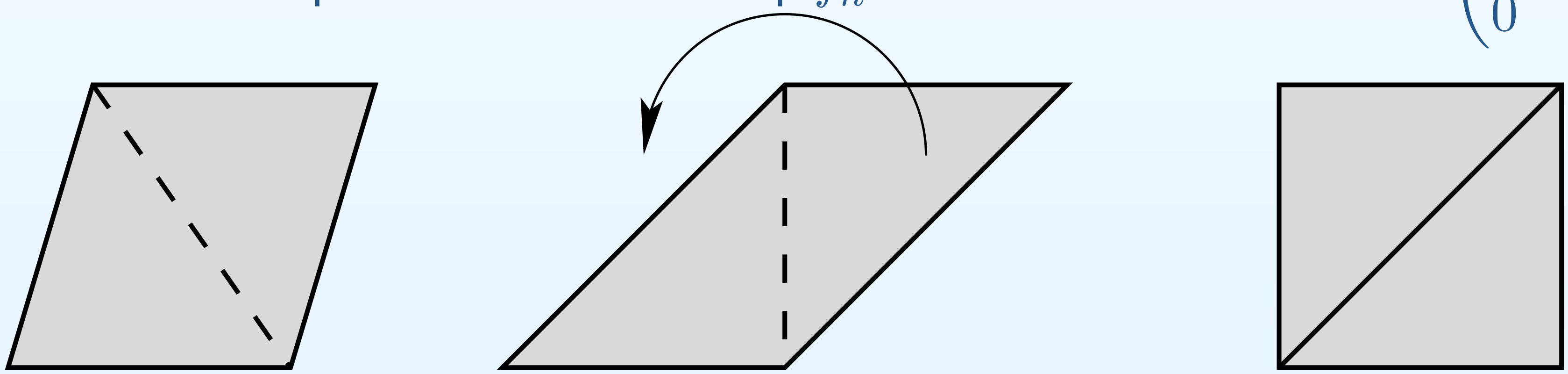

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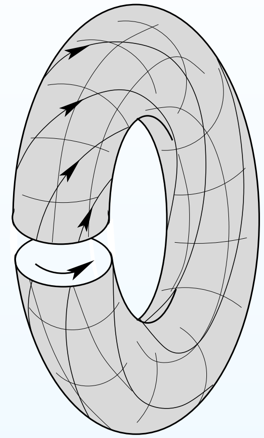

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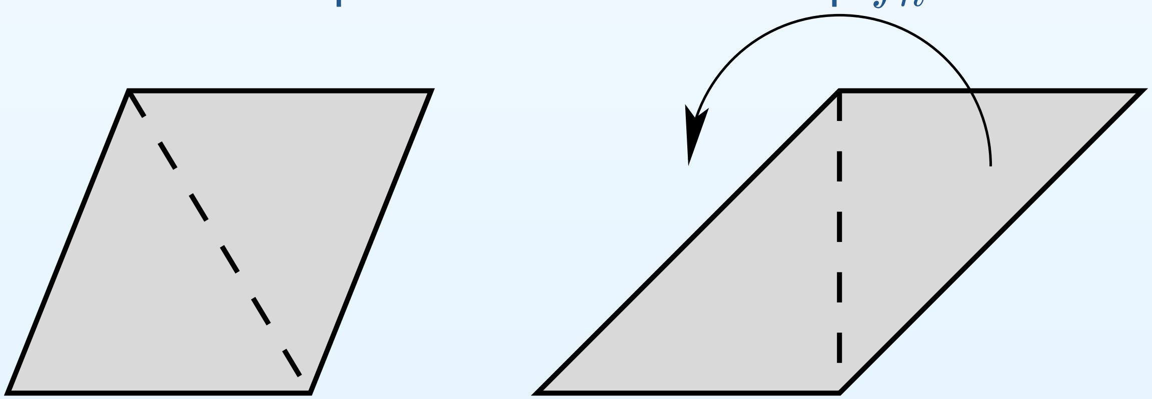

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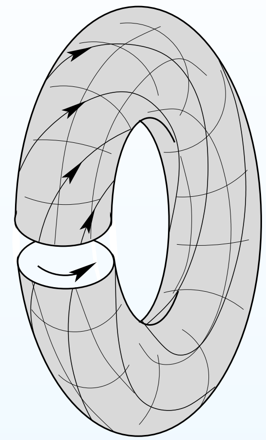

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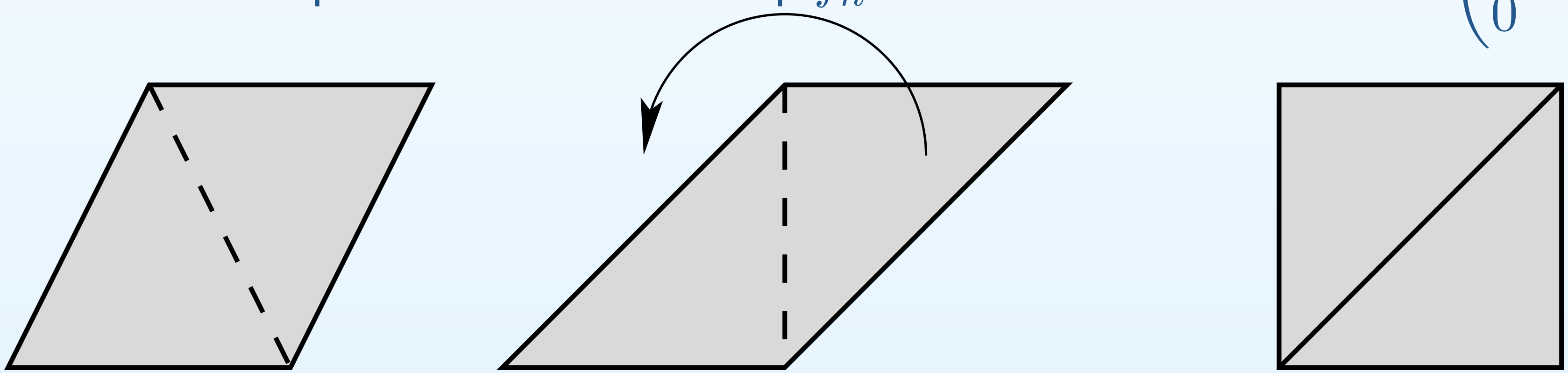

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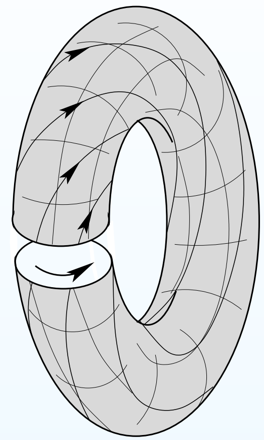

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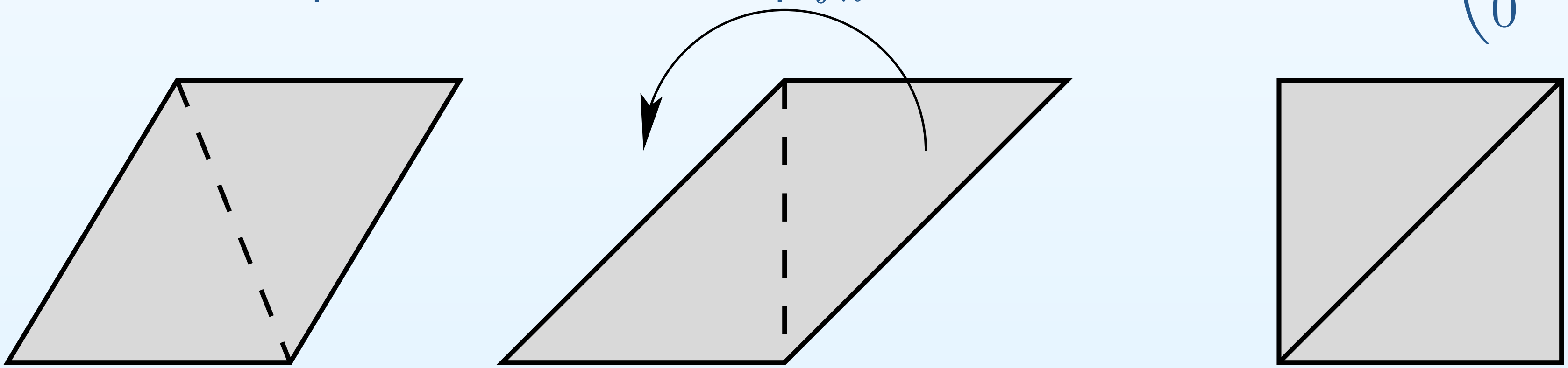

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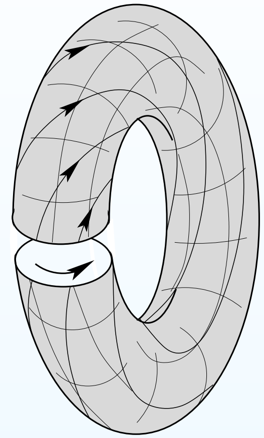

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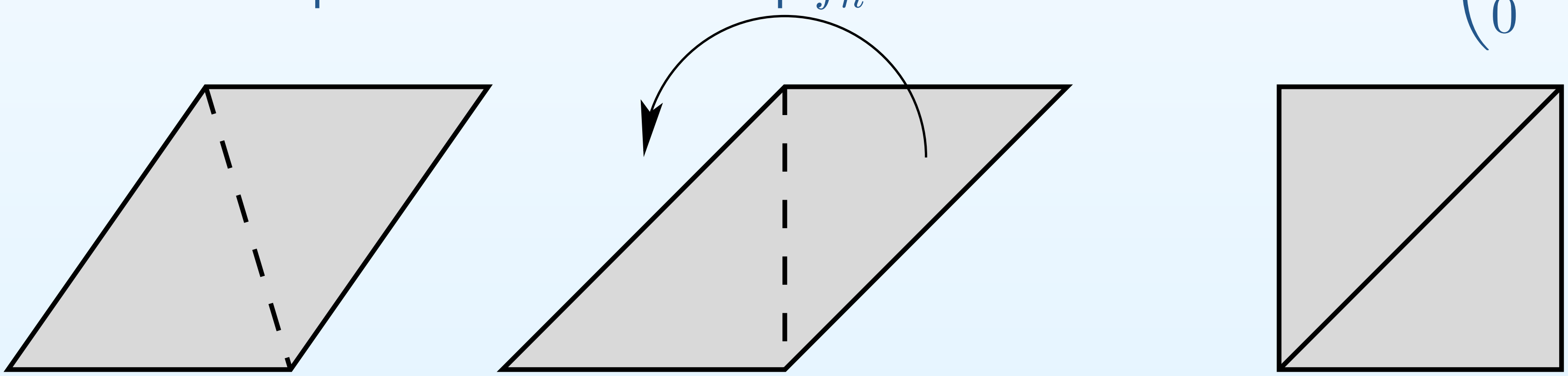

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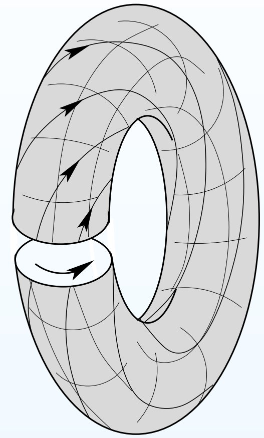

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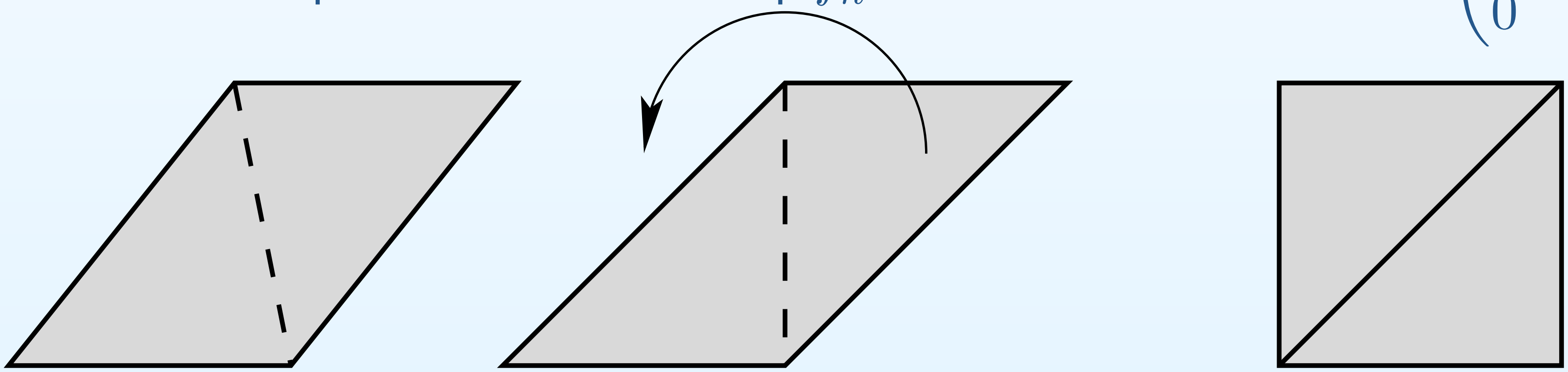

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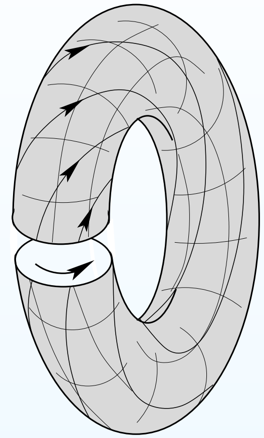

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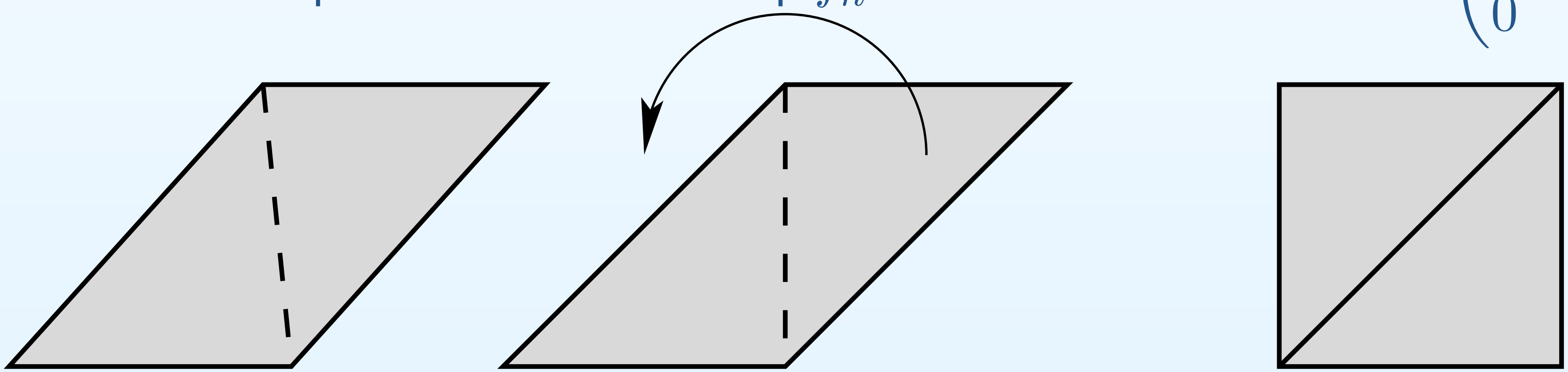

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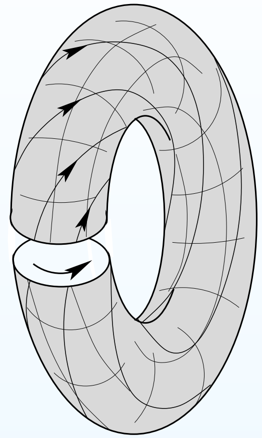

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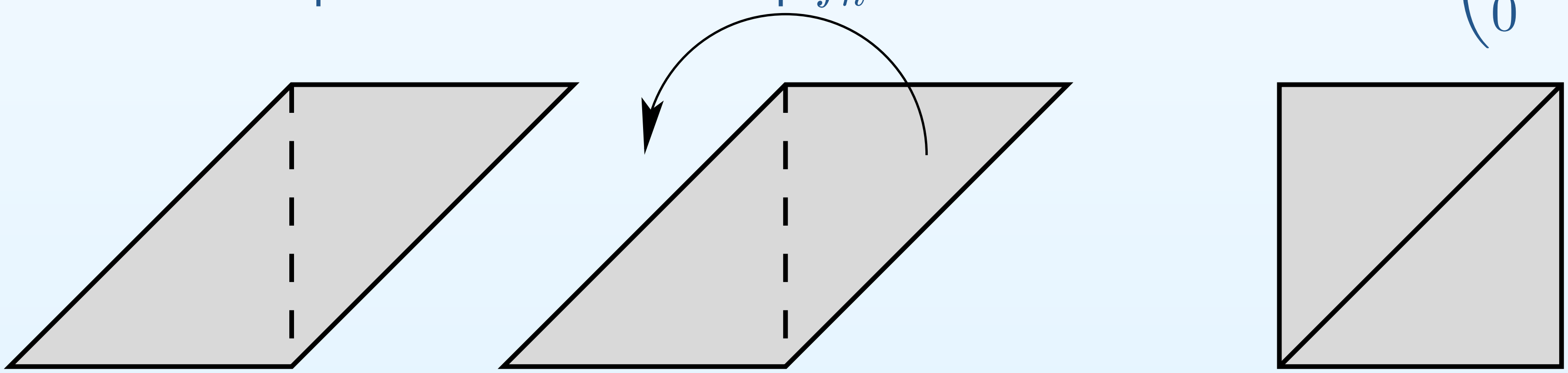

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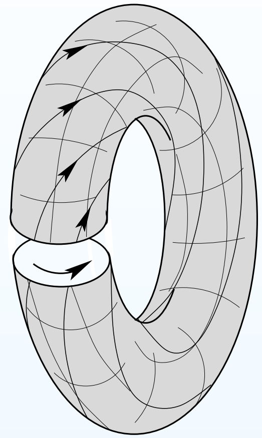

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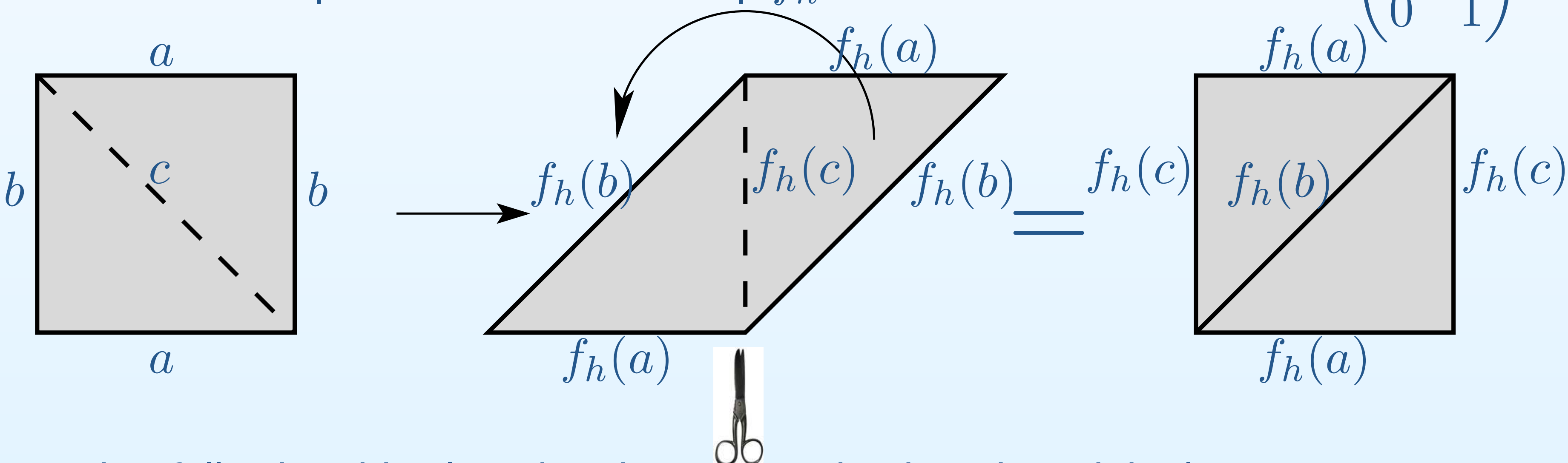

Note that following this closed path we come back to the original square torus having twisted the homology!

Closed horocycle in the moduli space of tori

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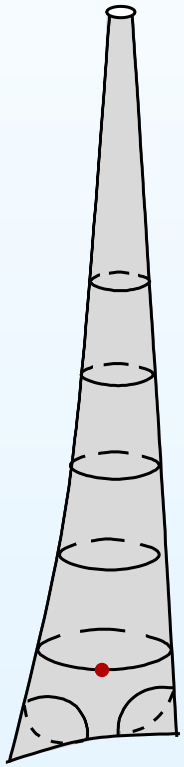

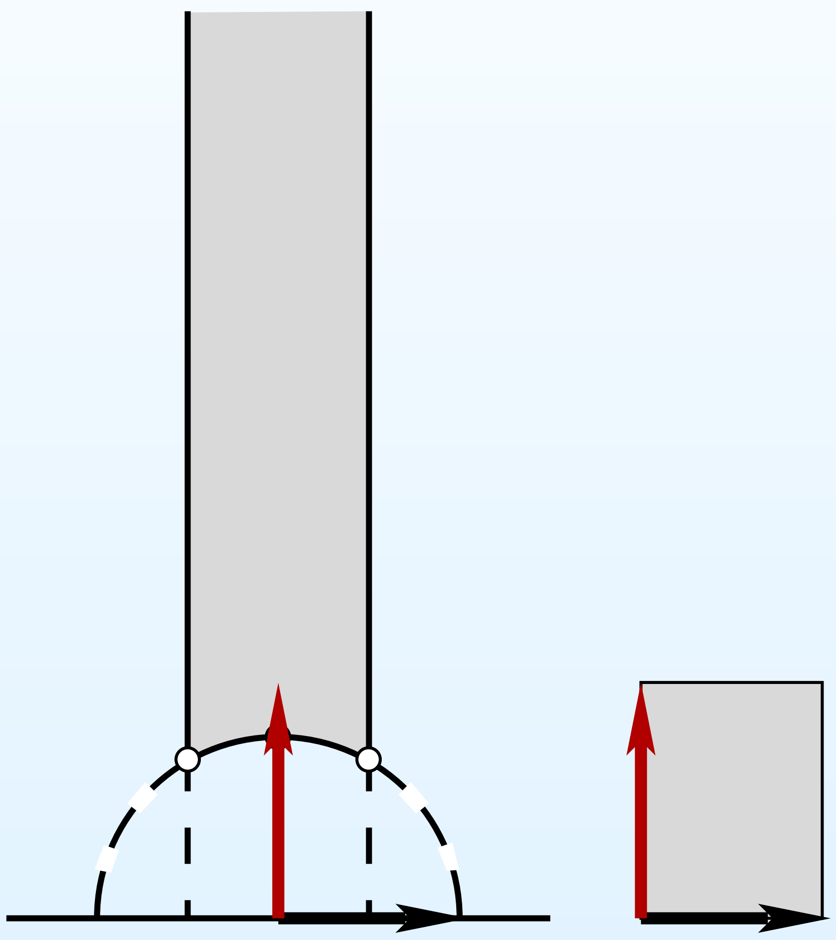

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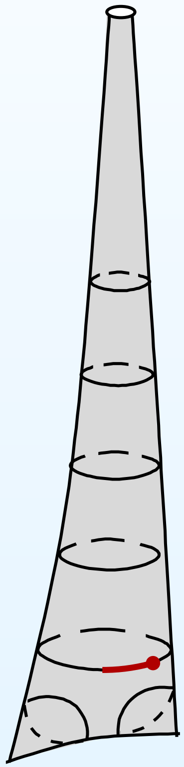

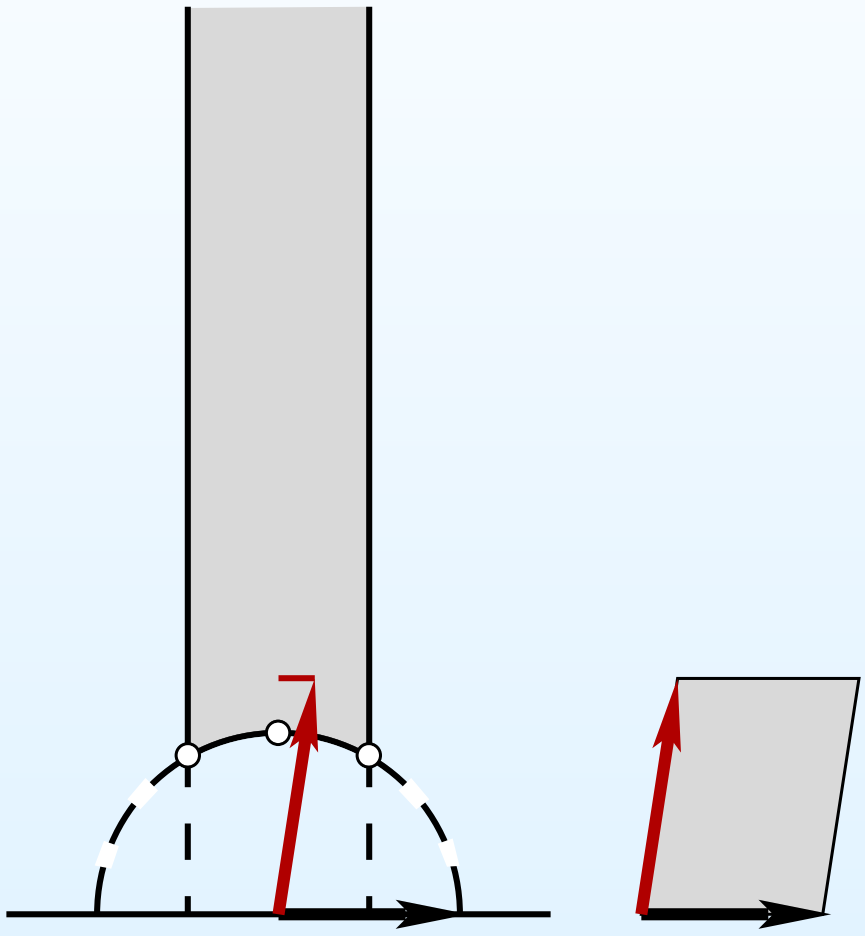

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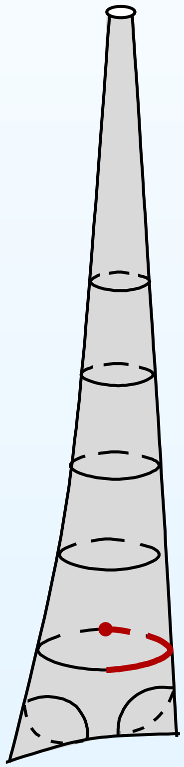

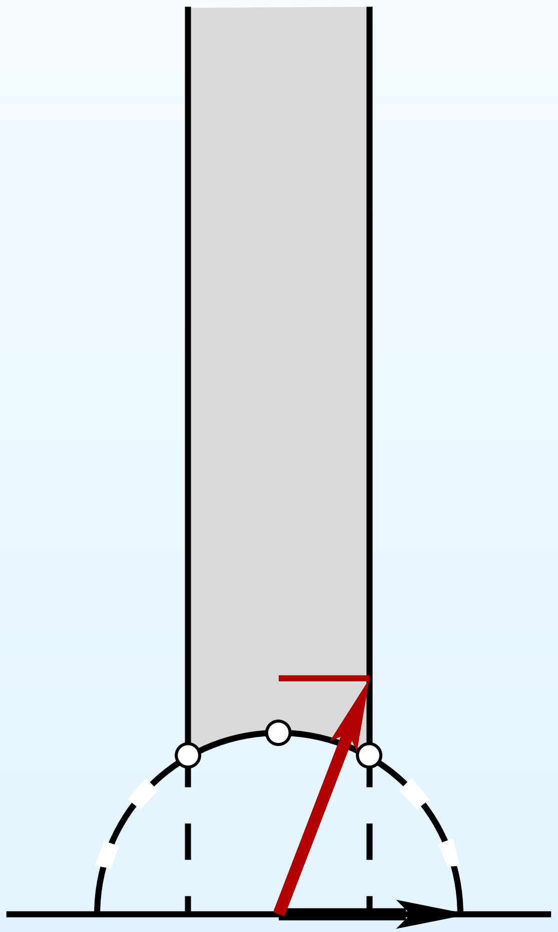

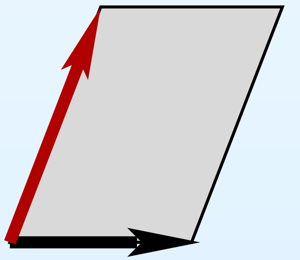

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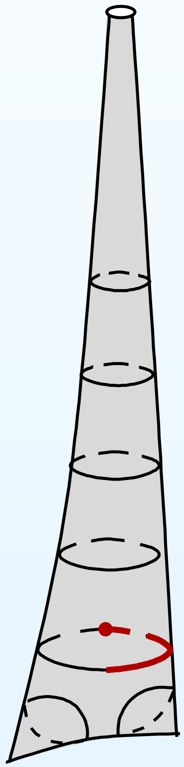

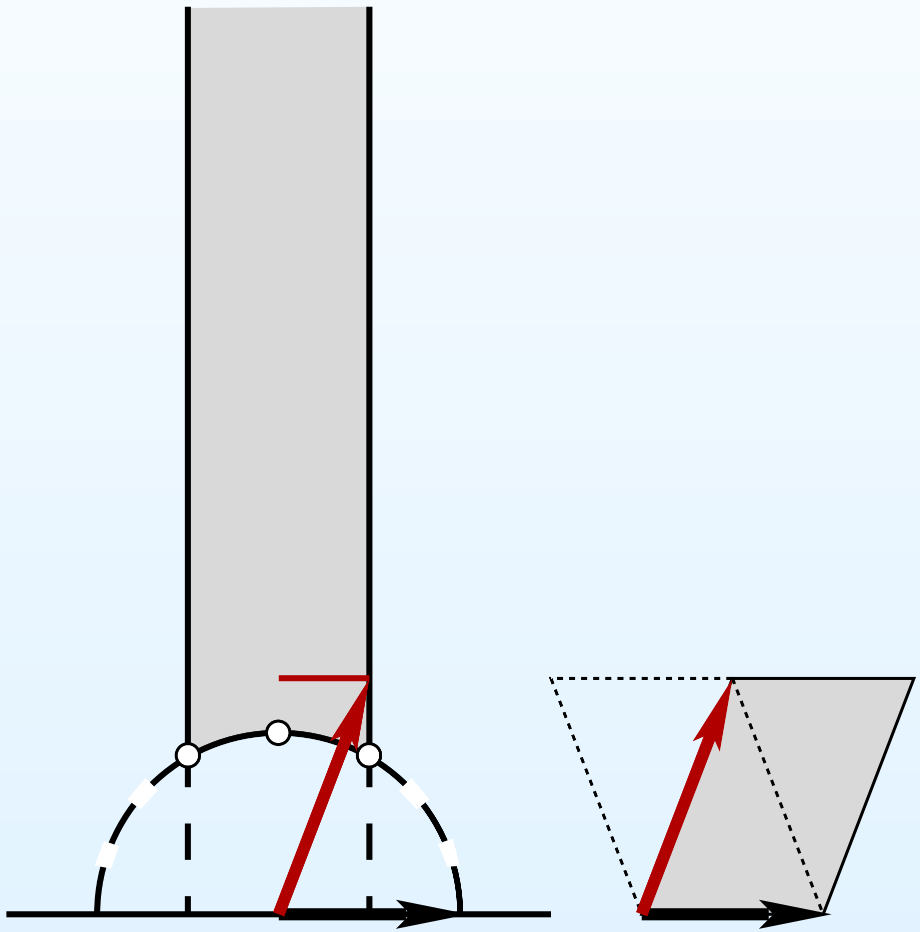

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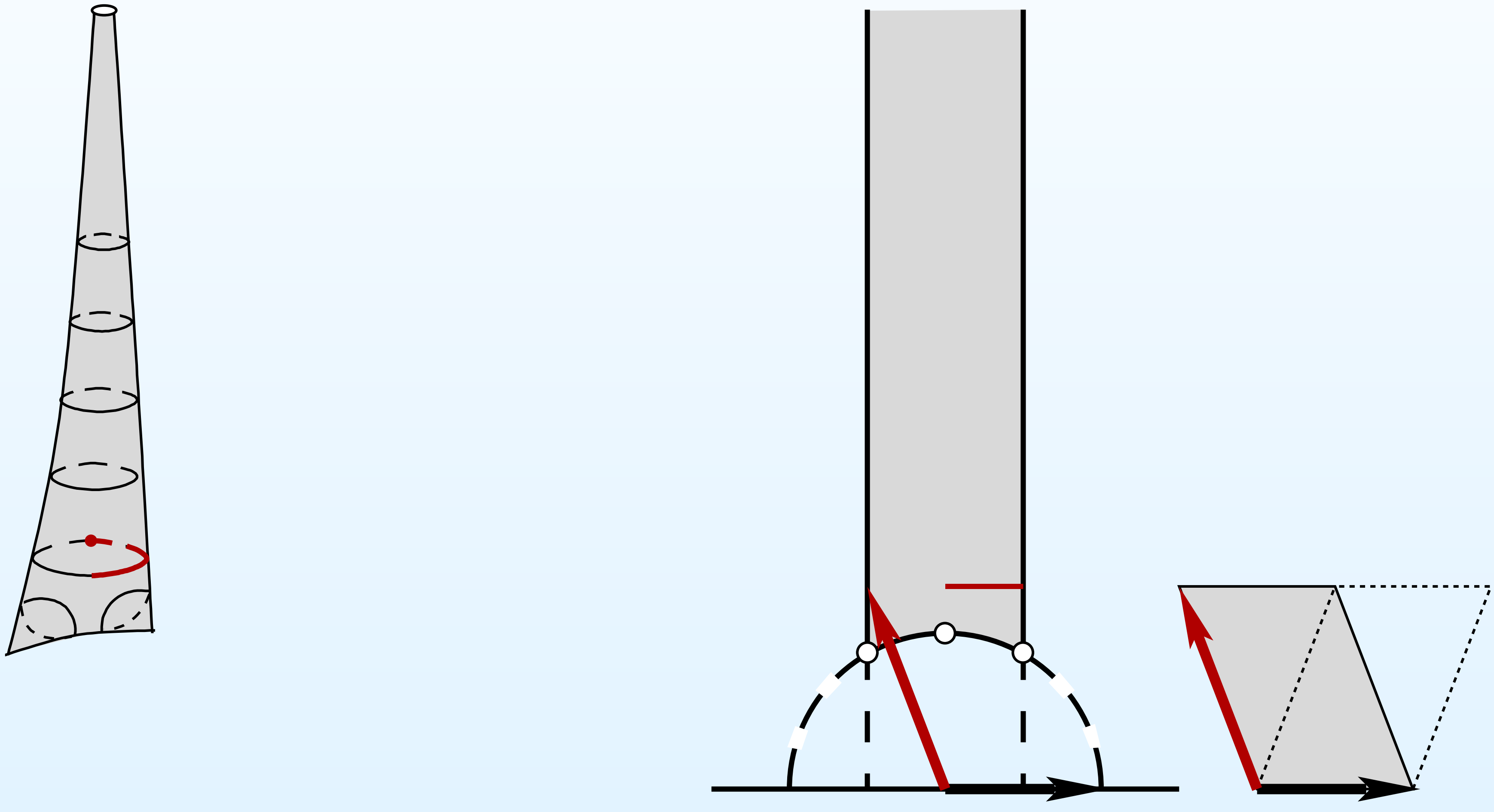

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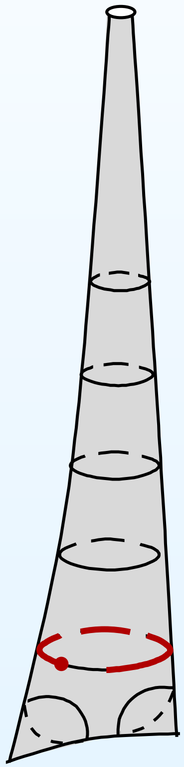

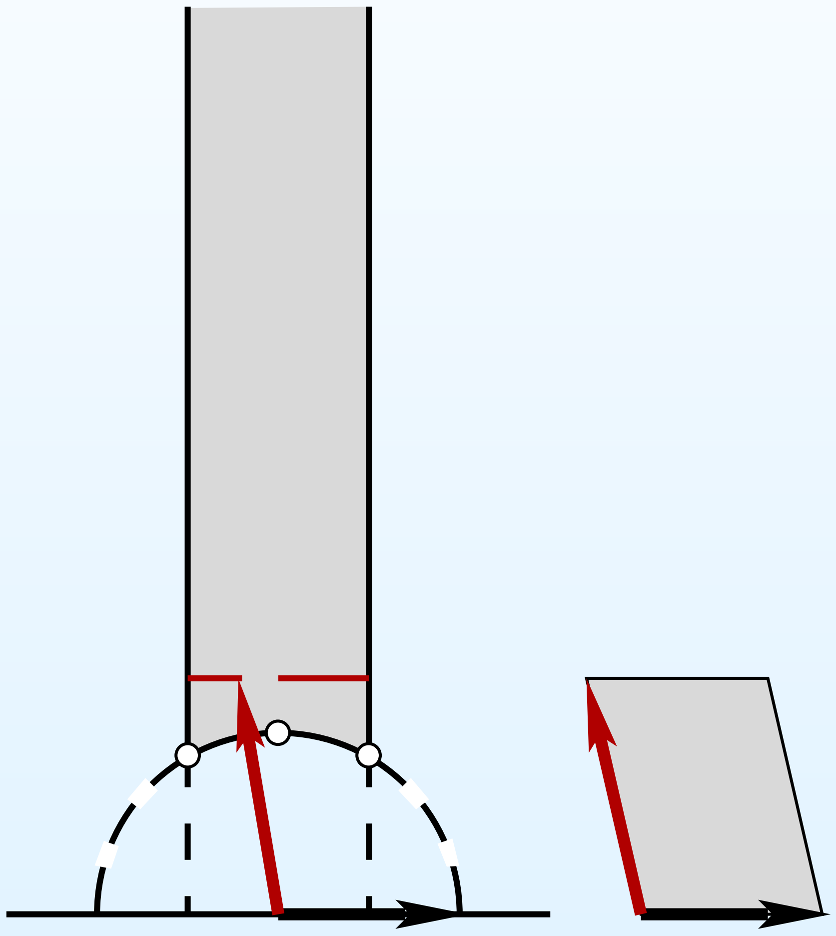

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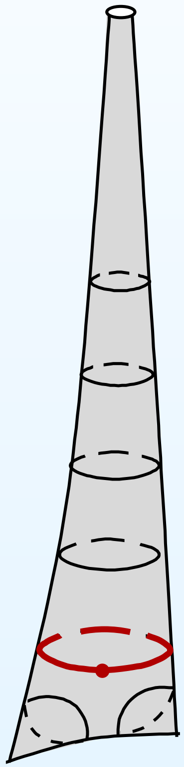

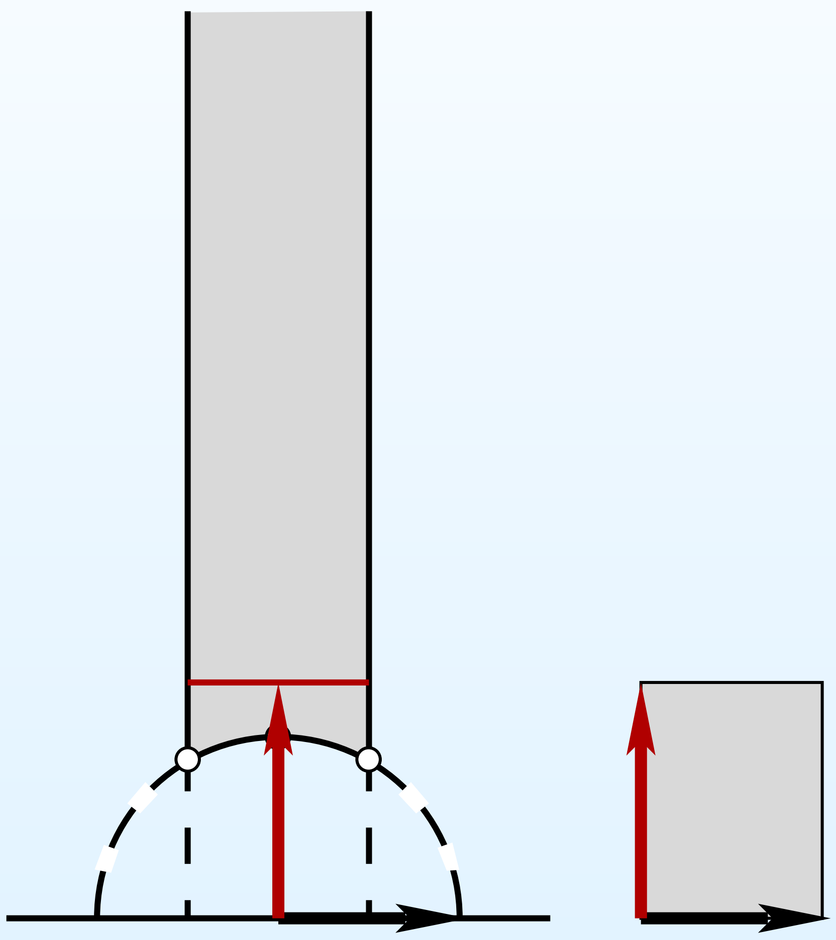

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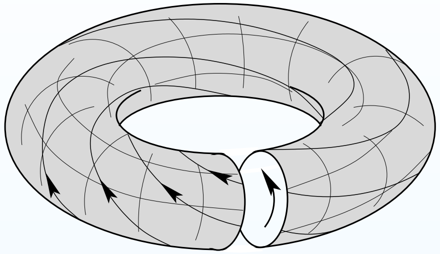
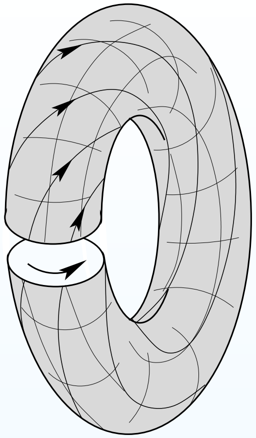
$g = f_v \circ f_h =$  $\circ$ 

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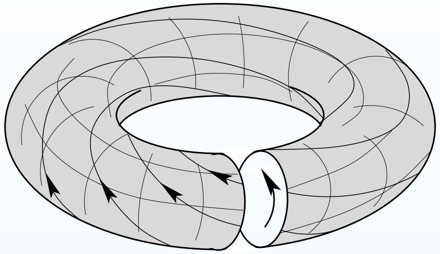
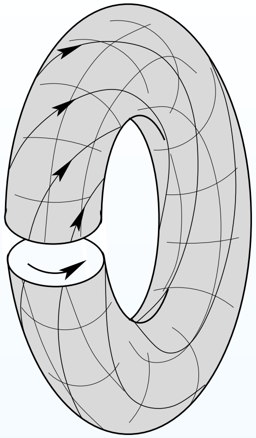
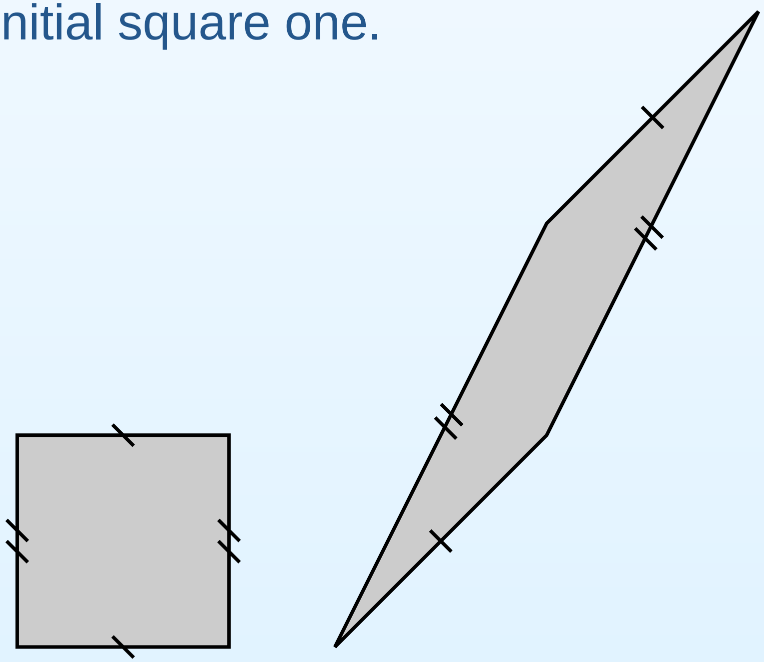
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

It corresponds to the integer linear map $\hat{g} : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Cutting and pasting appropriately the image parallelogram pattern we can check by hands that we can transform the new pattern to the initial square one.



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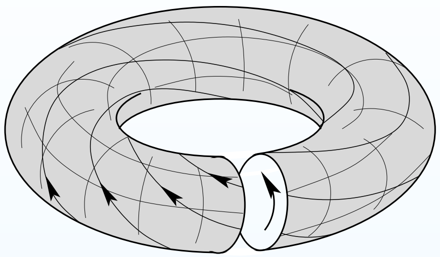
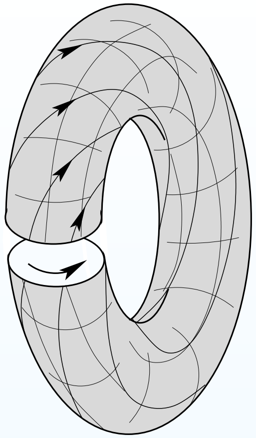
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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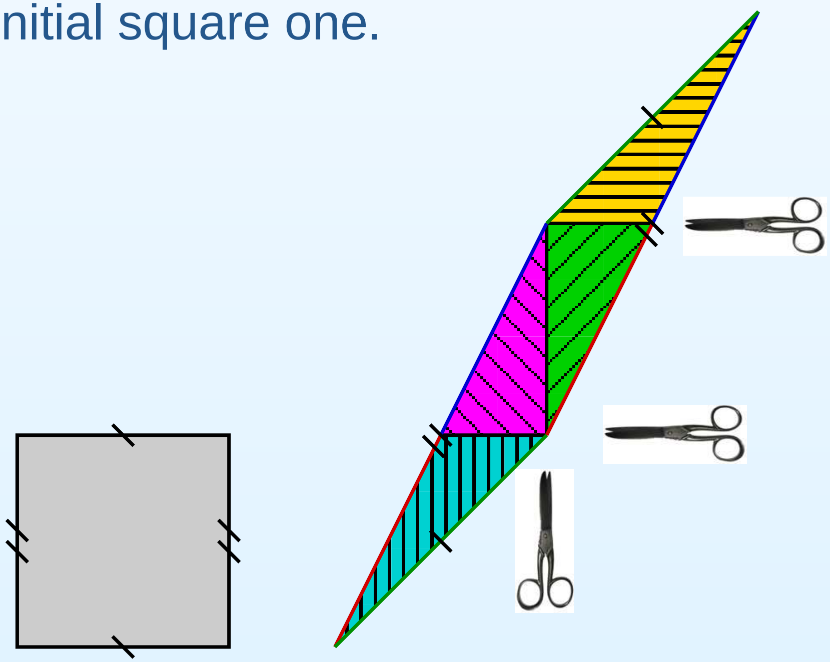

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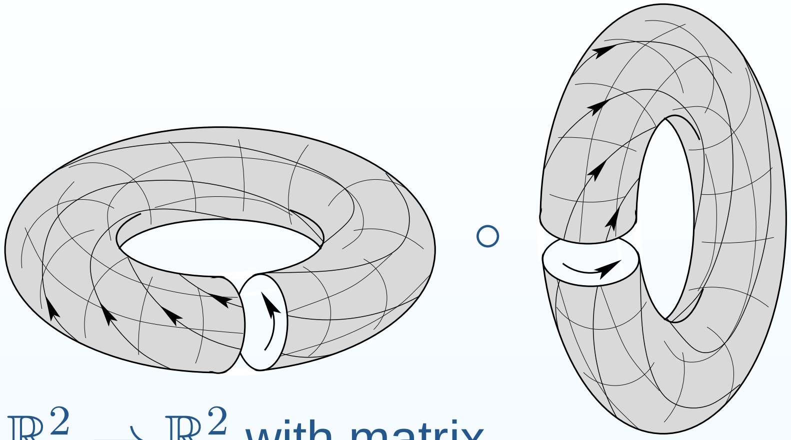

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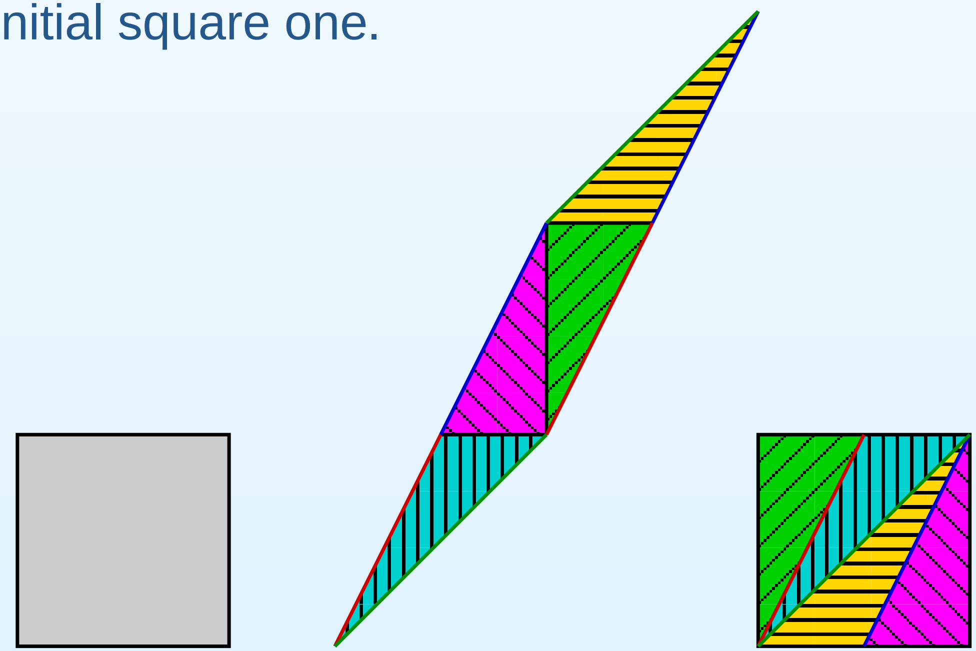

Closed geodesics in the space of tori

Consider eigenvectors $\vec{v}_{exp}$ and $\vec{v}_{contr}$ of the linear transformation $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ corresponding to the eigenvalues $\lambda > 1$ and to $1/\lambda < 1$ respectively. Consider two transversal foliations on the original torus in directions of $\vec{v}_{exp}$ and of $\vec{v}_{contr}$. We have just proved that expanding our torus $\mathbb{T}^2$ by factor λ in direction $\vec{v}_{exp}$ and contracting it by the factor λ in direction $\vec{v}_{contr}$ we get the original torus.

Consider a one-parameter family of flat tori obtained from the initial square torus by a continuous deformation expanding with a factor e^t in directions $\vec{v}_{exp}$ and contracting with a factor e^{-t} in direction $\vec{v}_{contr}$. By construction such one-parameter family defines a closed curve in the space of flat tori: after the time $t_0 = \log \lambda$ it closes up and follows itself.

One can check that this closed curve is, actually, a closed geodesics in the moduli spaces of tori.

Compute asymptotic intersection number in this particular case

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Dynamics in the moduli spaces

Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane, where lives the polygon. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points inherits the complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

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Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is the **degree** of zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \dots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \dots + d_n = 2g - 2$.
- Any holomorphic 1-forms corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly n zeroes $P_1, \dots, P_n$ of degrees $d_1, \dots, d_n$.
- The vectors defining the polygon from the previous picture considered as complex numbers are the relative periods $\int_{P_i}^{P_j} \omega$ of ω, so each stratum $\mathcal{H}(d_1, \dots, d_n)$ is modelled on the relative cohomology $H^1(S, \{P_1, \dots, P_n\}\,;\, \mathbb{C})$ serving as *period coordinates*.

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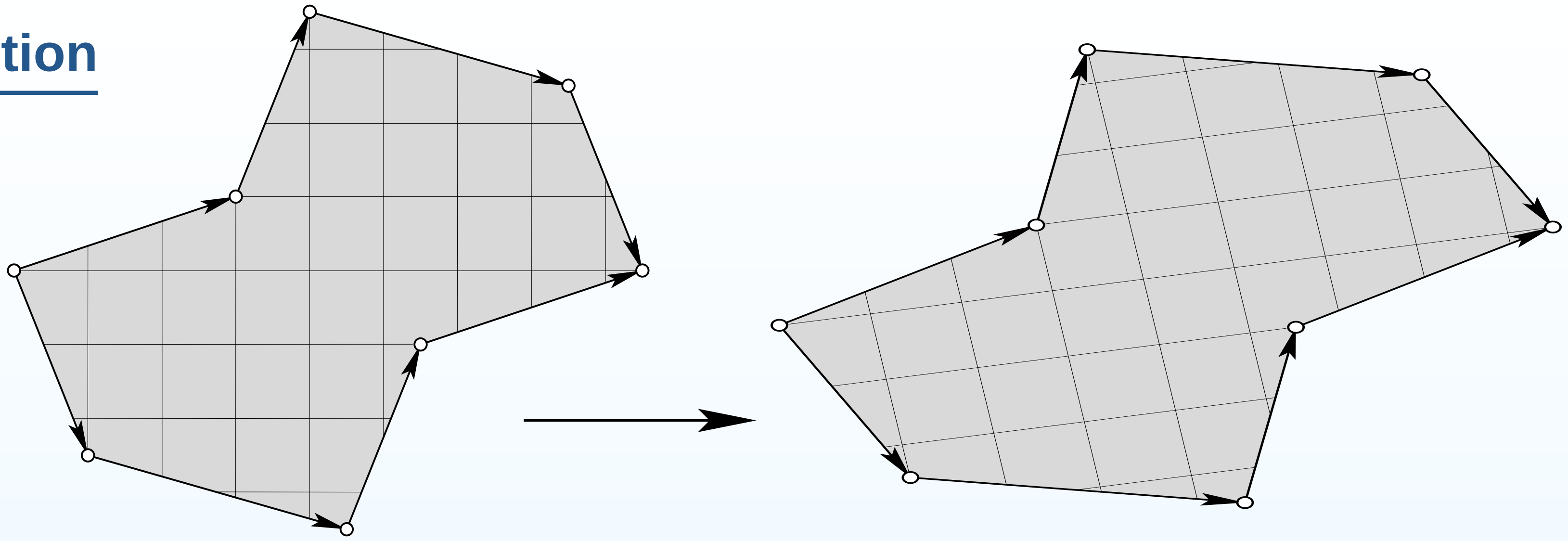
Volume element

Note that the vector space $H^1(S,\{P_1,\dots,P_n\}\,;\mathbb{C})$ contains a natural integer lattice $H^1(S,\{P_1,\dots,P_n\}\,;\mathbb{Z}\oplus\sqrt{-1}\,\mathbb{Z})$. Consider a linear volume element $d\nu$ normalized in such a way that the volume of the fundamental domain in this lattice equals one. Consider now the real hypersurface $\mathcal{H}_1(d_1,\dots,d_n)\subset\mathcal{H}(d_1,\dots,d_n)$ defined by the equation $area(S)=1$. The volume element $d\nu$ can be naturally restricted to the hypersurface defining the volume element $d\nu_1$ on $\mathcal{H}_1(d_1,\dots,d_n)$.

Theorem (H. Masur; W. A. Veech) *The total volume* $\operatorname{Vol}(\mathcal{H}_1(d_1,\dots,d_n))$ *of every stratum is finite.*

The Masur–Veech volumes of the first several low-dimensional strata were computed by M. Kontsevich and A. Zorich about 2000. The first efficient algorithm for evaluation of the Masur–Veech volume was found by A. Eskin and A. Okounkov. In particular, they proved that the Masur–Veech volume of any stratum always has the form $(p/q)\pi^{2g}$ where p/q is a rational number. By 2003 A. Eskin computed these rational numbers up for all strata to genus 10. By now we have much better knowledge of Masur–Veech volumes; we will discuss them in more details later in these lectures.

Group action

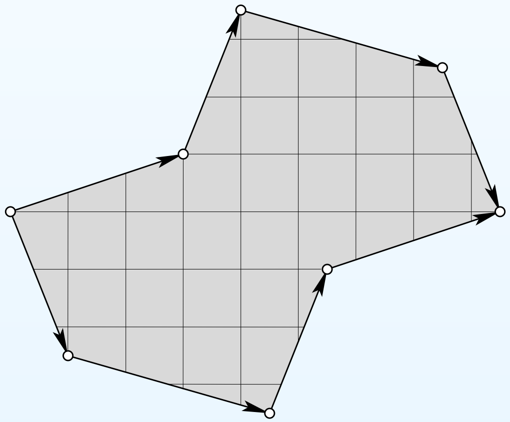


The subgroup $SL(2,\mathbb{R})$ of area preserving linear transformations acts on the "unit hyperboloid" $\mathcal{H}_1(d_1,\dots,d_n)$. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2,\mathbb{R})$ induces a natural flow on the stratum, which is called the *Teichmüller geodesic flow*.

Key Theorem (H. Masur; W. A. Veech) *The action of the groups $SL(2,\mathbb{R})$ and $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ preserves the measure $d\nu_1$. Both actions are ergodic with respect to this measure on each connected component of every stratum $\mathcal{H}_1(d_1,\dots,d_n)$.*

Masur—Veech Theorem

Theorem of Masur and Veech claims that taking an arbitrary octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon.



Compute asymptotic intersection number again

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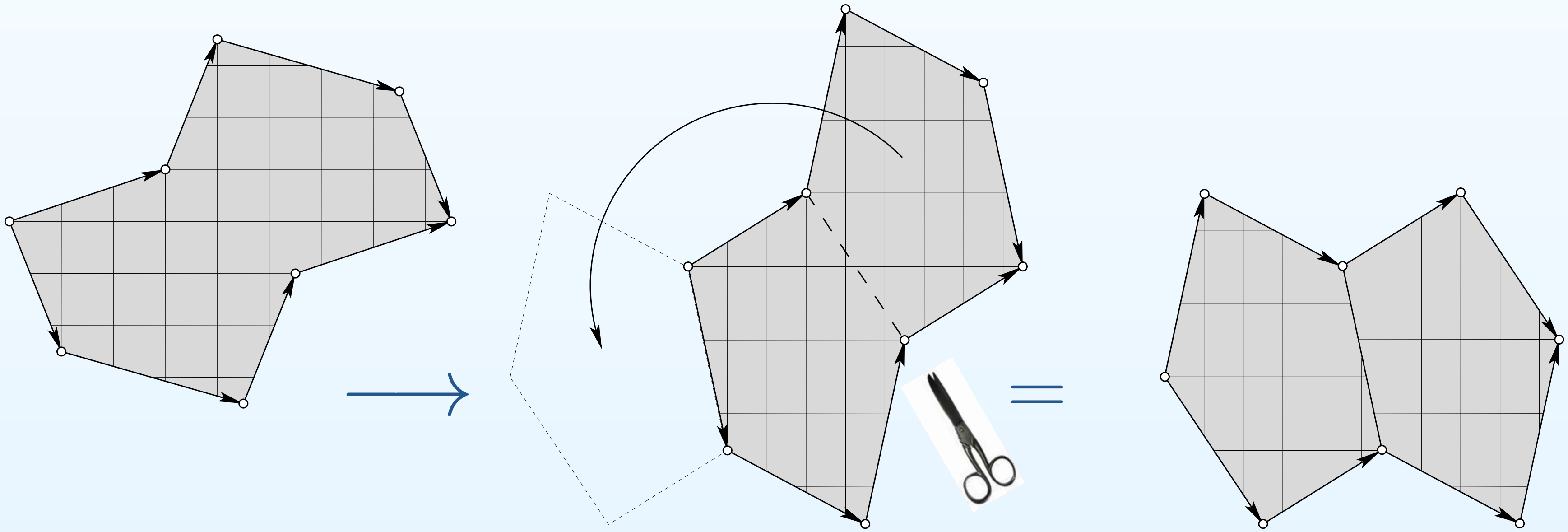
There is no paradox since we are allowed to cut-and-paste!



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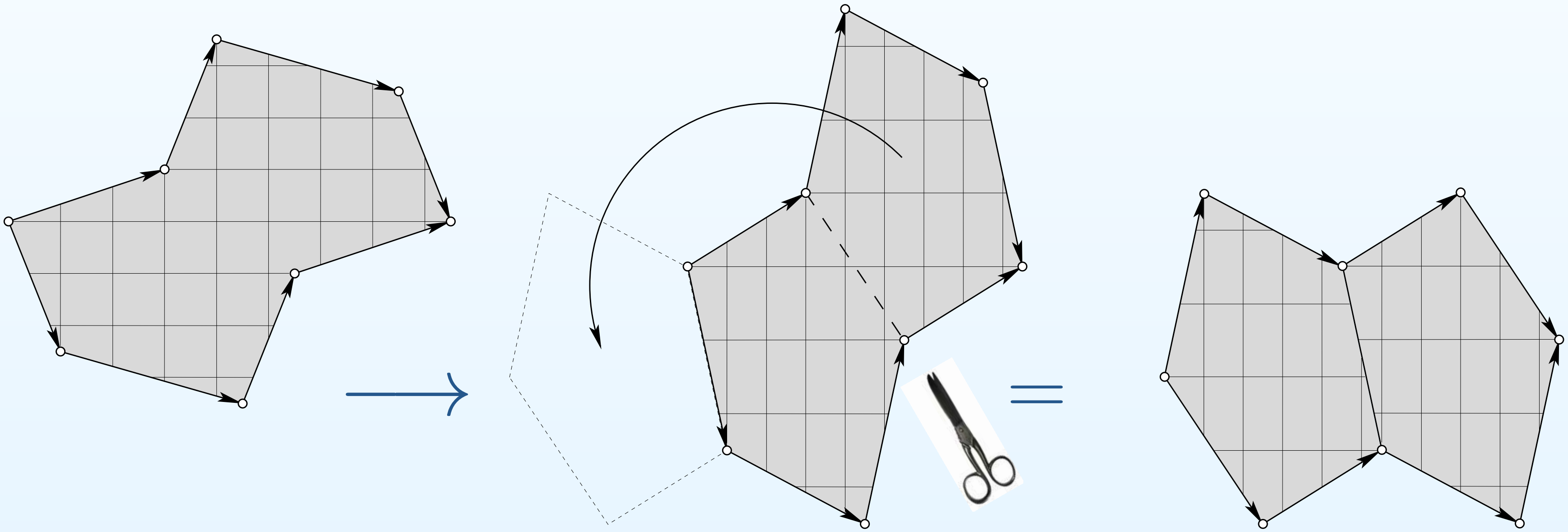

The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface

Compute asymptotic intersection number again

Magic Wand Theorem

Invariant measures and orbit closures

Magic Wand Theorem (A. Eskin–M. Mirzakhani–A. Mohammadi, 2014). *The closure of any* $\mathrm{SL}(2,\mathbb{R})$*-orbit is a suborbifold. In period coordinates any* $\mathrm{GL}(2,\mathbb{R})$*-orbit closure is represented by a complexification of an* $\mathbb{R}$*-linear subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$*-invariant measure is supported on a suborbifold. In period coordinates this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.*

Theorem (S. Filip, 2014) *Any* $\mathrm{GL}(2,\mathbb{R})$*-invariant orbifold is, actually, an algebraic variety characterized by special arithmetic conditions.*

Further developements (A. Eskin–C. McMullen–R. Mukamel–A. Wright, 2017). *New examples of nontrivial* $\mathrm{SL}(2,\mathbb{R})$*–invariant orbifolds coming from families of "optimal billiards in quadrilaterals".*

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Fields Medal

At the International Congress of Mathematics in 2014 Maryam Mirzakhani has received a Fields Medal for *“for her exceptional contributions to dynamics and geometry of Riemann surfaces and their moduli spaces”* becoming the first woman to receive the Fields Medal.

Breakthrough Prize

Alex Eskin got 2020 Breakthrough Prize in Mathematics *“for revolutionary discoveries in the dynamics and geometry of moduli spaces of Abelian differentials, including the proof of the “Magic Wand Theorem” with Maryam Mirzakhani.”*

Why the Magic Wand Theorem is astonishing

For most of dynamical systems (including very nice and gentle ones) certain individual trajectories are disastrously complicated. In particular, after many iterations they might fill wired fractal sets.

For example, the map $f : x \mapsto \{2x\}$ homogeneously winding the circle $S^1 = \mathbb{R}/\mathbb{Z}$ twice around itself has orbits with orbit closures of (basically) any Hausdorff dimension between 0 and 1. The same map has infinite orbits avoiding certain arcs of the circle, etc. Even such elementary maps have certain (rare) orbits with a very bizarre behavior.

Bernoulli shift. In the binary representation of a real number $x \in [0; 1[$

$$x = \frac{n_1}{2} + \cdots + \frac{n_k}{2^k} + \cdots ,$$

all the binary digits n_k are zeroes or ones. The map f acts on a sequence $(n_1, n_2, \ldots, n_k, \ldots)$ by erasing the first digit. This coding shows that we have, basically, a complete freedom in constructing orbits of f with peculiar behavior.

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For example, the map $f : x \mapsto \{2x\}$ homogeneously winding the circle $S^1 = \mathbb{R}/\mathbb{Z}$ twice around itself has orbits with orbit closures of (basically) any Hausdorff dimension between 0 and 1. The same map has infinite orbits avoiding certain arcs of the circle, etc. Even such elementary maps have certain (rare) orbits with a very bizarre behavior.

Bernoulli shift. In the binary representation of a real number $x \in [0; 1[$

$$x = \frac{n_1}{2} + \cdots + \frac{n_k}{2^k} + \cdots,$$

all the binary digits n_k are zeroes or ones. The map f acts on a sequence $(n_1, n_2, \ldots, n_k, \ldots)$ by erasing the first digit. This coding shows that we have, basically, a complete freedom in constructing orbits of f with peculiar behavior.

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Folklore Theorem (H. Furstenberg versus B. Weiss). *For any Riemann surface C of constant negative curvature and any real number d, such that $1 \leq d \leq 3$, there is a trajectory of the geodesic flow on the unit tangent bundle to C such that its closure has Hausdorff dimension d.*

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Situation with "geodesics" of higher dimensions is completely different.

Theorem (N. Shah). *In a compact manifold of constant negative curvature, the closure of a totally geodesic, complete (immersed) submanifold of dimension at least 2 is a totally geodesic immersed submanifold.*

The moduli space is **not** a homogeneous space, so a priori there were no reasons to hope for a rigidity theorem like the Magic Wand Theorem of Eskin, Mirzakhani, and Mohammadi!