Dynamics and Geometry of Moduli Spaces

Lecture 4. Idea of Renormalization

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Reminder: group action, ergodicity, Masur–Veech theorem, Magic Wand theorem

Volume element

Note that the vector space $H^1(S,\{P_1,\dots,P_n\}\,;\,\mathbb{C})$ contains a natural integer lattice $H^1(S,\{P_1,\dots,P_n\}\,;\,\mathbb{Z}\oplus\sqrt{-1}\,\mathbb{Z})$. Consider a linear volume element $d\nu$ normalized in such a way that the volume of the fundamental domain in this lattice equals one. Consider now the real hypersurface $\mathcal{H}_1(d_1,\dots,d_n)\subset\mathcal{H}(d_1,\dots,d_n)$ defined by the equation $area(S)=1$. The volume element $d\nu$ can be naturally restricted to the hypersurface defining the volume element $d\nu_1$ on $\mathcal{H}_1(d_1,\dots,d_n)$.

Theorem (H. Masur; W. A. Veech) *The total volume* $\mathrm{Vol}(\mathcal{H}_1(d_1,\dots,d_n))$ *of every stratum is finite.*

The Masur–Veech volumes of the first several low-dimensional strata were computed by M. Kontsevich and A. Zorich about 2000. The first efficient algorithm for evaluation of the Masur–Veech volume was found by A. Eskin and A. Okounkov. In particular, they proved that the Masur–Veech volume of any stratum always has the form $(p/q)\pi^{2g}$ where p/q is a rational number. By 2003 A. Eskin computed these rational numbers up for all strata to genus 10.

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By now there are further more efficient algorithms and high genus asymptotic results due to A. Eskin–A. Okounkov–R. Pandharipande (through quasimodularity and representation thery), J. Athreya–A. Eskin–A. Zorich (through dynamics), E. Goujard, D. Chen–M. Möller–A. Sauvaget–D. Zagier (mostly — algebraic geometry), A. Aggarwal (combinatorics), V. Delecroix–E. Goujard–P. Zograf–A. Zorich (combinatorics and Witten–Kontsevich correlators).

Ergodic transformations

Let μ be a finite measure on a topological space M (for example, a volume element on a manifold M, with a finite total volume). A map $T : M \to M$ *preserves measure* μ (corresp. is *volume preserving*) if for any measurable subset $A \subset M$ one has $\mu(T^{-1}(A)) = \mu(A)$.

A subset $A \subset M$ is called T-*invariant* if $T^{-1}(A) = A$.

The map T is called *ergodic* with respect to the measure μ if any invariant set has measure 0, or the full measure $\mu(M)$.

Examples.

● Rotations of a circle are measure preserving. Irrational rotations are ergodic; rational ones are not.

● The map $z \to z^2$ where $z \in \mathbb{C},\ |z| = 1$ is not invertible but it preserves the measure and is ergodic.

● A (pseudo)Anosov diffeomorphism preserves the area form and and is ergodic.

Ergodic Theorem

Consider the orbit $x, T(x), T(T(x)), \ldots, T^{(n-1)}(x)$ of a point $x \in M$. By *time average* of a μ-measurable function f on M we call the average

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^{(i)}(x))\,,$$

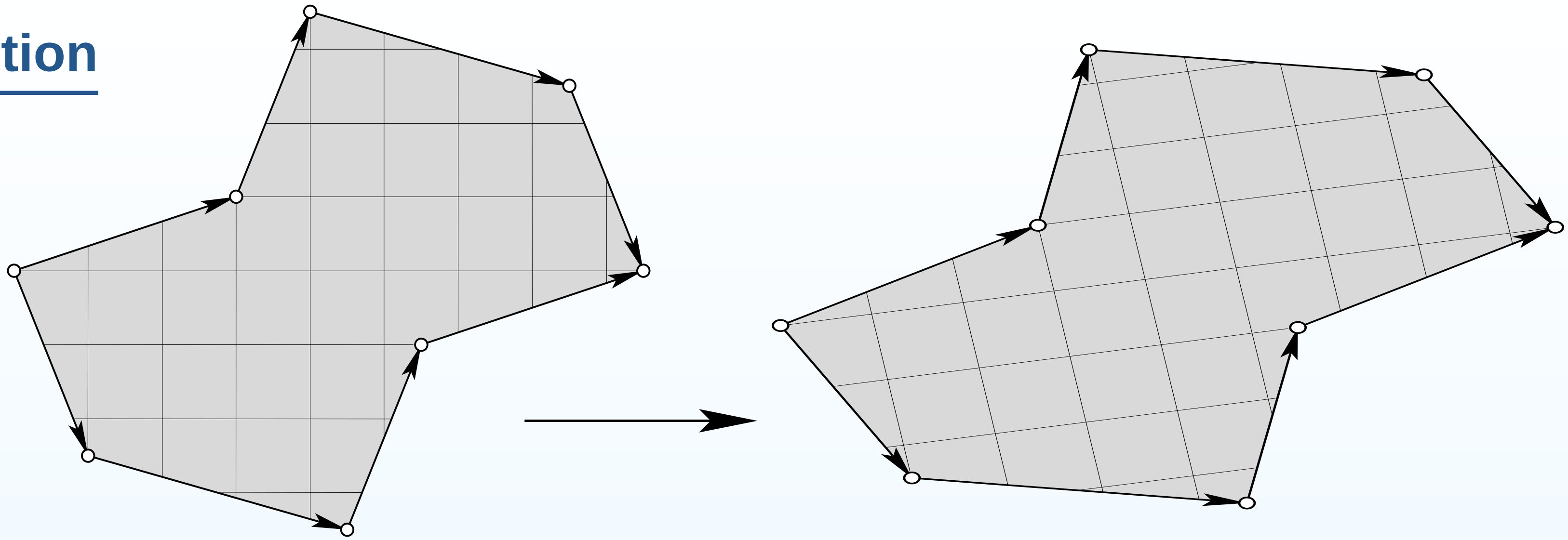
i.e. the mean value of f along first n points of the orbit of x. By *space average* we call

$$\frac{1}{\mu(M)}\int_M f(x)\,d\mu\,.$$

Ergodic Theorem. *Let T be an ergodic map preserving finite measure μ. For μ-almost any point x of M the time averages converge to space averages:*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^{(i)}(x)) = \frac{1}{\mu(M)}\int_M f(x)\,d\mu\,.$$

Group action

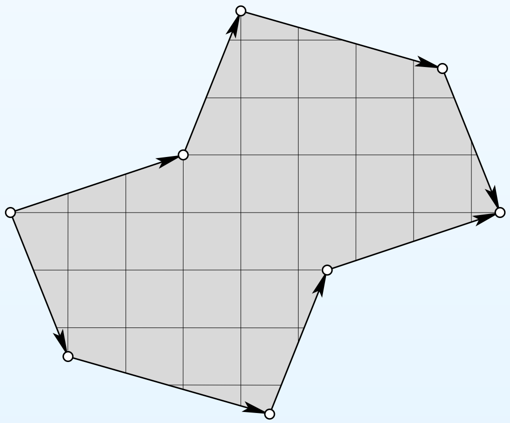


The subgroup $SL(2,\mathbb{R})$ of area preserving linear transformations acts on the "unit hyperboloid" $\mathcal{H}_1(d_1,\dots,d_n)$. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2,\mathbb{R})$ induces a natural flow on the stratum, which is called the *Teichmüller geodesic flow*.

Key Theorem (H. Masur; W. A. Veech) *The action of the groups* $SL(2,\mathbb{R})$ *and* $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ *preserves the measure* $d\nu_1$*. Both actions are ergodic with respect to this measure on each connected component of every stratum* $\mathcal{H}_1(d_1,\dots,d_n)$*.*

Masur—Veech Theorem

Theorem of Masur and Veech claims that taking almost any octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon. Moreover, the corresponding trajectory would spend the time in a neighborhood $\mathcal{U}$ of the regular octagon proportional to the measure of $\mathcal{U}$ (on a long scale of time).



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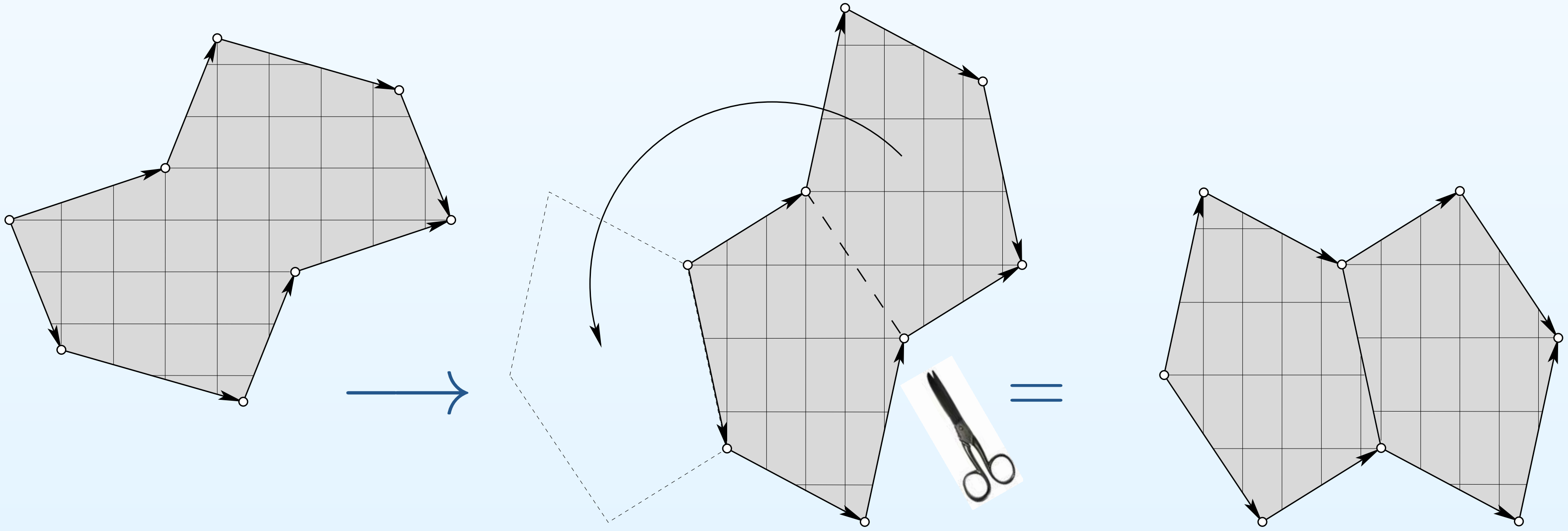
There is no paradox since we are allowed to cut-and-paste!



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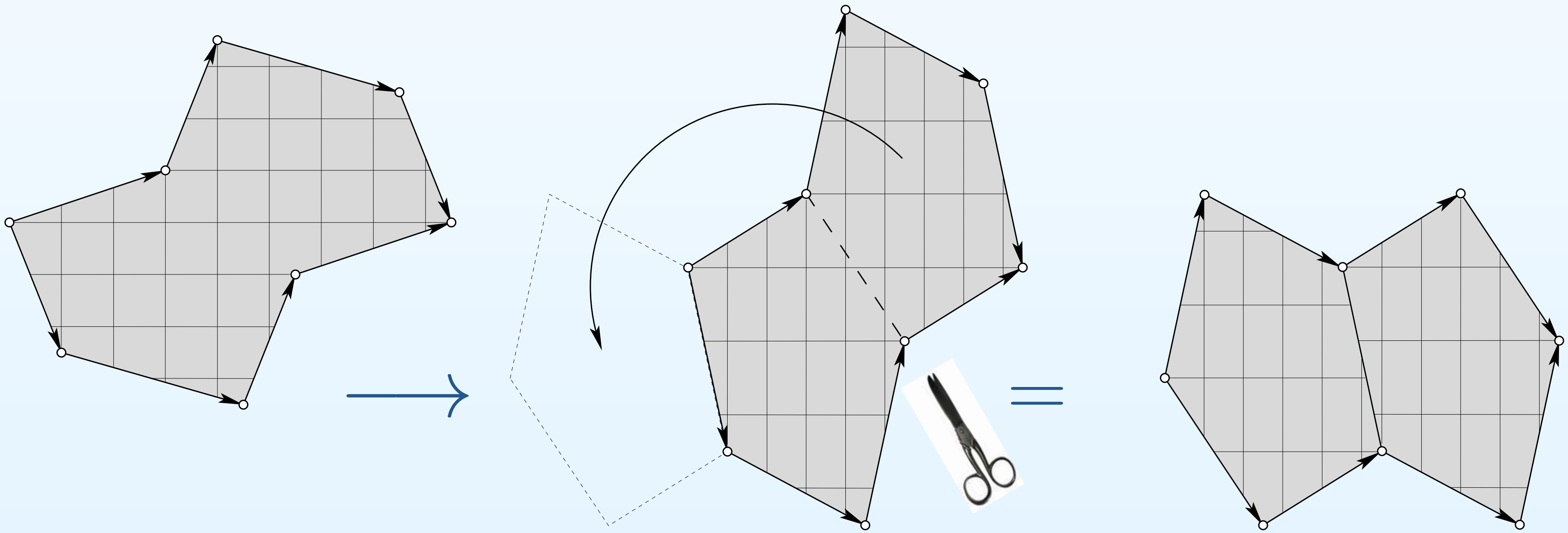

The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface

Moduli spaces of Abelian differentials

We have seen that any stratum $\mathcal{H}(m_1, \ldots, m_n)$ of all pairs *(Riemann surface S, holomorphic 1-form with n zeroes of degrees $m_1, \ldots, m_n$)* is locally modeled on $H^1(S, \{n \text{ points}\}; \mathbb{C})$. The action of the group $\mathrm{GL}(2, \mathbb{R})$ can be seen as the action on the second term in the product

$$H^1(S, \{n \text{ points}\}; \mathbb{R} \oplus i\mathbb{R}) \simeq H^1(S, \{n \text{ points}\}; \mathbb{R}) \otimes \mathbb{R}^2 \,.$$

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A projectivized stratum $\mathrm{P}\mathcal{H}(m_1, \dots, m_n) \simeq \mathcal{H}_1(m_1, \dots, m_n)/\,\mathrm{SO}(2, \mathbb{R}) \simeq \mathcal{H}(m_1, \dots, m_n)/\mathbb{C}^*$ is foliated by hyperbolic planes $\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\,\mathrm{SO}(2, \mathbb{R})$ called *Teichmüller discs*. A natural projection of such a disc to $\mathcal{M}_g$ is an isometric immersion with respect to Teichüller metric on $\mathcal{M}_g$, so Teichmüller discs can be seen as *complex geodesics* in the Teichmüller metric on $\mathcal{M}_g$.

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Similarly, any stratum of meromorphic quadratic differentials with at most simple poles is locally modeled on the anti-invariant subspace of $H^1(\hat S,\{n \text{ points}\};\mathbb{C})$, where $p:\hat S\to S$ is the canonical double cover such that $p^*q=\omega^2$ becomes a global square of a holomorhic form ω.

Moduli spaces of Abelian differentials

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Excercise*. Verify by an explicit computation in coordiantes that disintegrating the Masur–Veech volume element on $\mathcal{H}(0)$ one gets the standard hyperbolic volume element on $\mathrm{P}\mathcal{H}(0)=\mathbb{H}^2$.

Invariant measures and orbit closures

Magic Wand Theorem (A. Eskin–M. Mirzakhani–A. Mohammadi, 2014). *The closure of any $\mathrm{SL}(2,\mathbb{R})$-orbit is a suborbifold. In period coordinates any $\mathrm{GL}(2,\mathbb{R})$-orbit closure is represented by a complexification of an $\mathbb{R}$-linear subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant measure is supported on a suborbifold. In period coordinates this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.

Theorem (S. Filip, 2014) *Any $\mathrm{GL}(2,\mathbb{R})$-invariant orbifold is, actually, an algebraic variety characterized by special arithmetic conditions.*

Theorem (A. Avila, A. Eskin. M. Möller, 2017) *Let L be a linear subspace representing a $\mathrm{GL}(2,\mathbb{R})$-orbit closure in period coordinates. The restriction of the natural symplectic form in $H^1(C,\mathbb{C})$ to the image of L under the projection $H^1(C,\{\text{zeroes}\};\mathbb{C}) \to H^1(C,\mathbb{C})$ is non-degenerate.*

“But still, my homeward way has proved too long.
While we were wasting time there, old Poseidon,
it almost seems, stretched and extended space.”

J. Brodsky

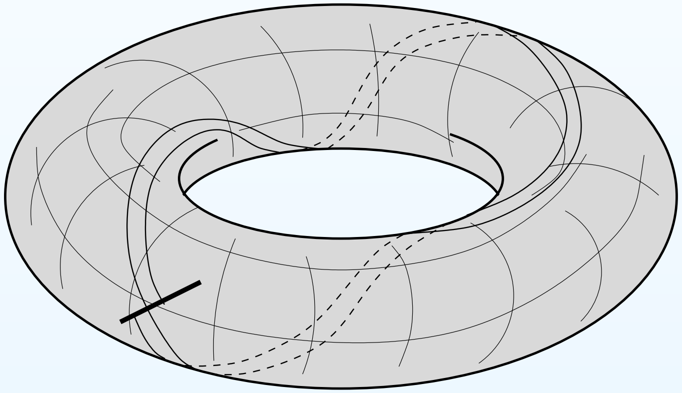
И все-таки ведущая домой
дорога оказалась слишком длинной,
как будто Посейдон, пока мы там
теряли время, растянул пространство.

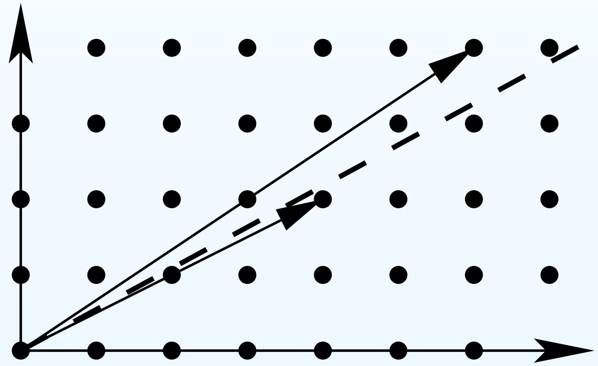
И. Бродский

Idea of Renormalization

Asymptotic cycle for a torus

Consider a leaf of a measured foliation on a surface. Choose a short transversal segment X. Each time when the leaf crosses X we join the crossing point with the point x_0 along X obtaining a closed loop. Consecutive return points $x_1, x_2, \ldots$ define a sequence of cycles $c_1, c_2, \ldots$.





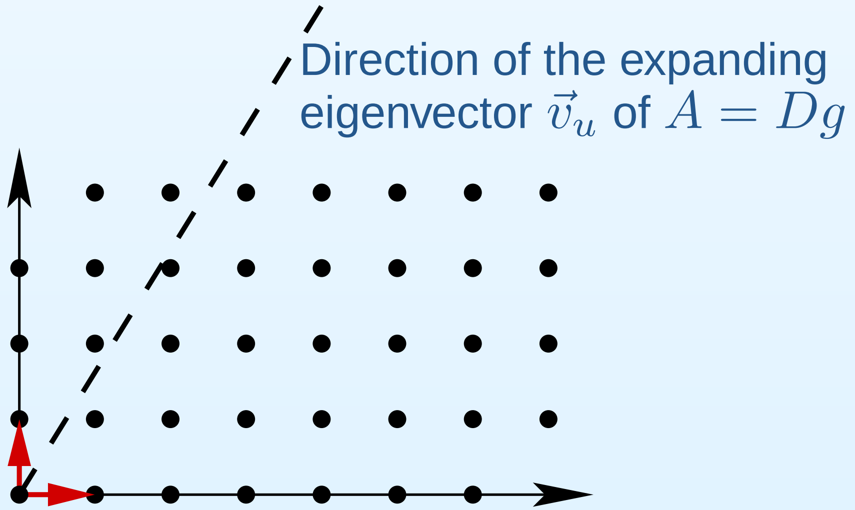
The *asymptotic cycle* is defined as $\lim_{n\to\infty} \frac{c_n}{n} = c \in H_1(\mathbb{T}^2; \mathbb{R})$.

Theorem (S. Kerckhoff, H. Masur, J. Smillie, 1986.) *For any flat surface directional flow in almost any direction is uniquely ergodic.*

This implies that for almost any direction the asymptotic cycle exists and is the same for all points of the surface.

Asymptotic cycle in the pseudo-Anosov case

Consider a model case of the foliation in direction of the expanding eigenvector $\vec{v}_u$ of the Anosov map $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $Dg = A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.



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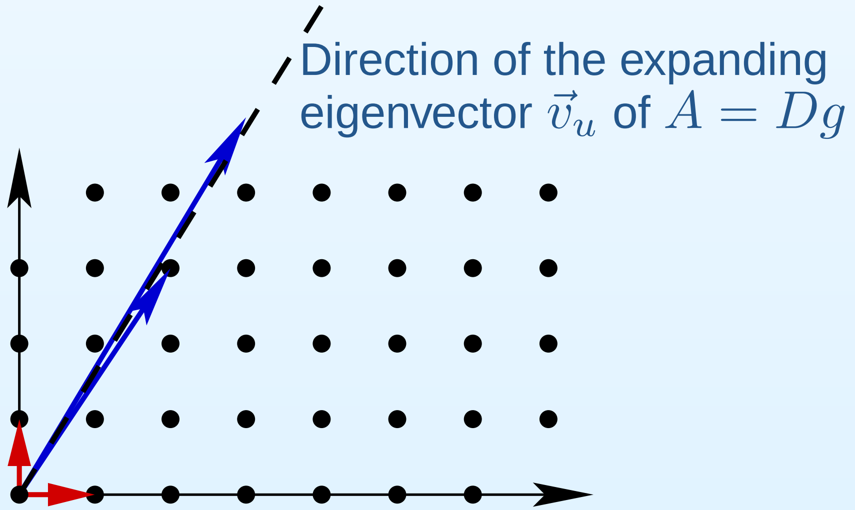

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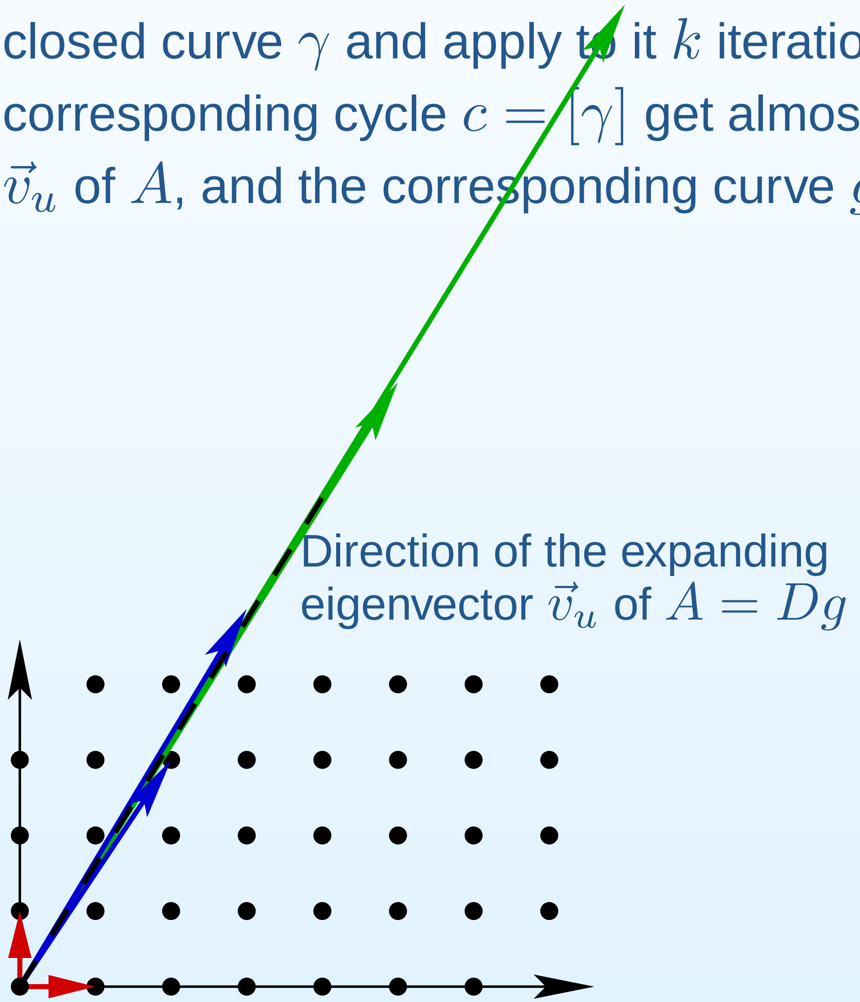

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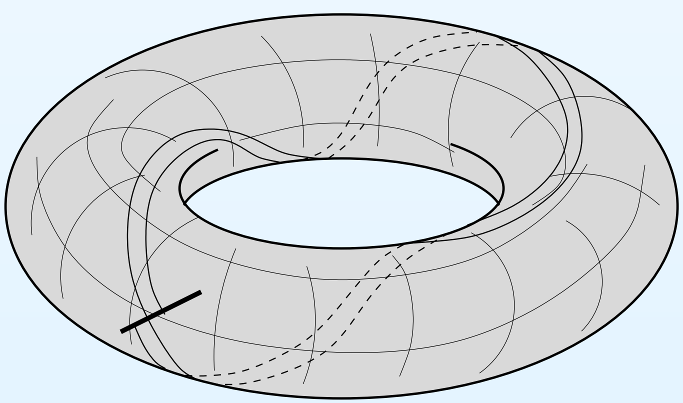
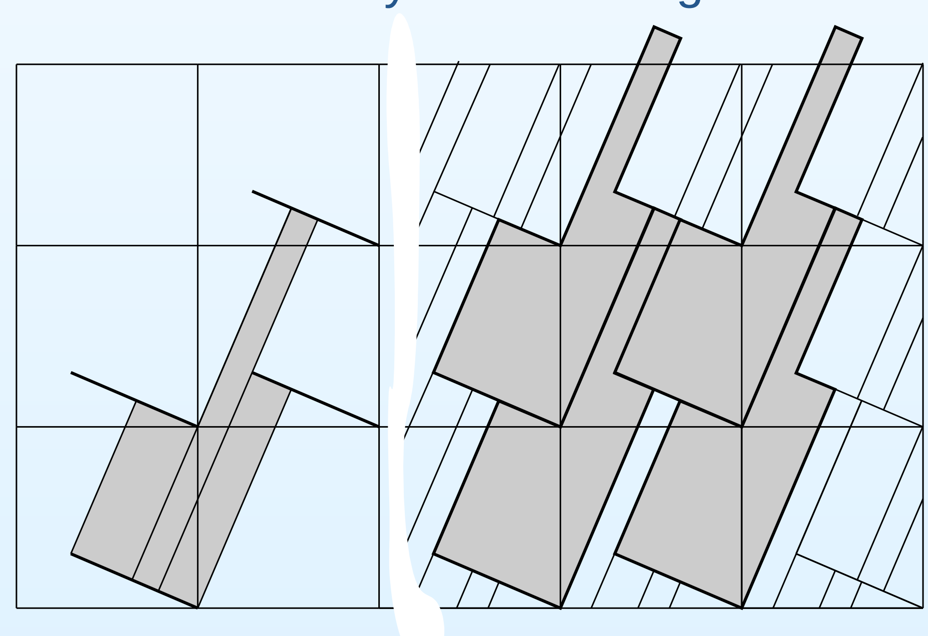
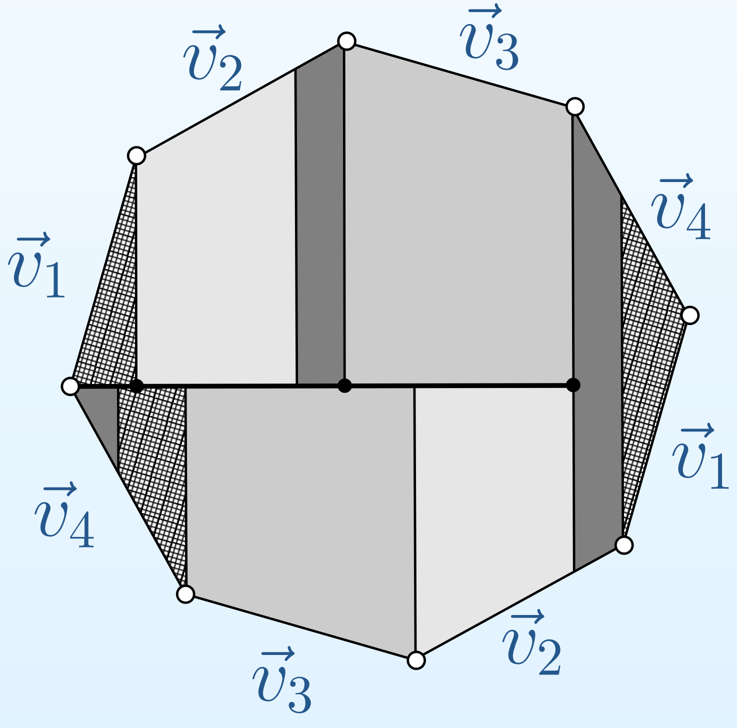
The first return cycles to a short subinterval exhibit exactly the same behavior by a simple reason that they are images of the first return cycles to a longer subinterval under a high iteration of g.





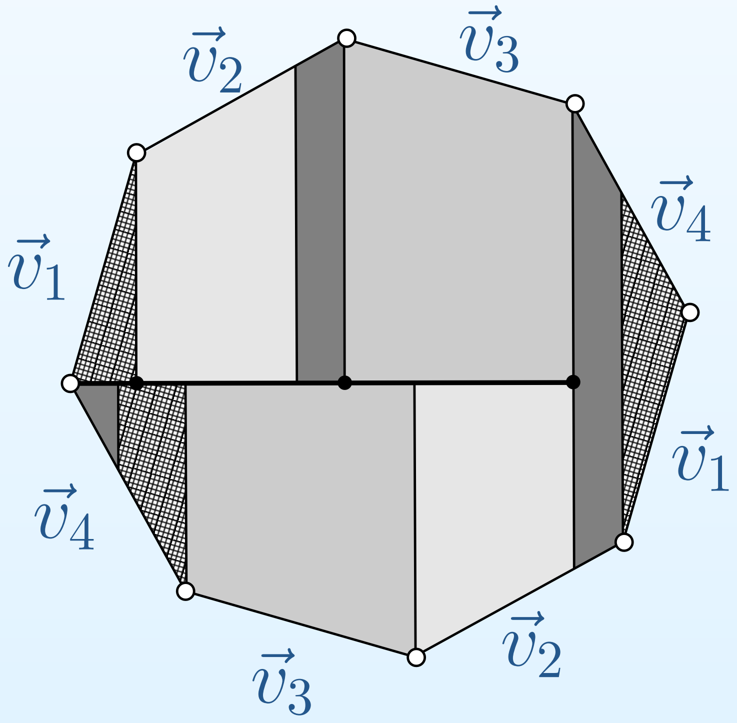
Zippered rectangles

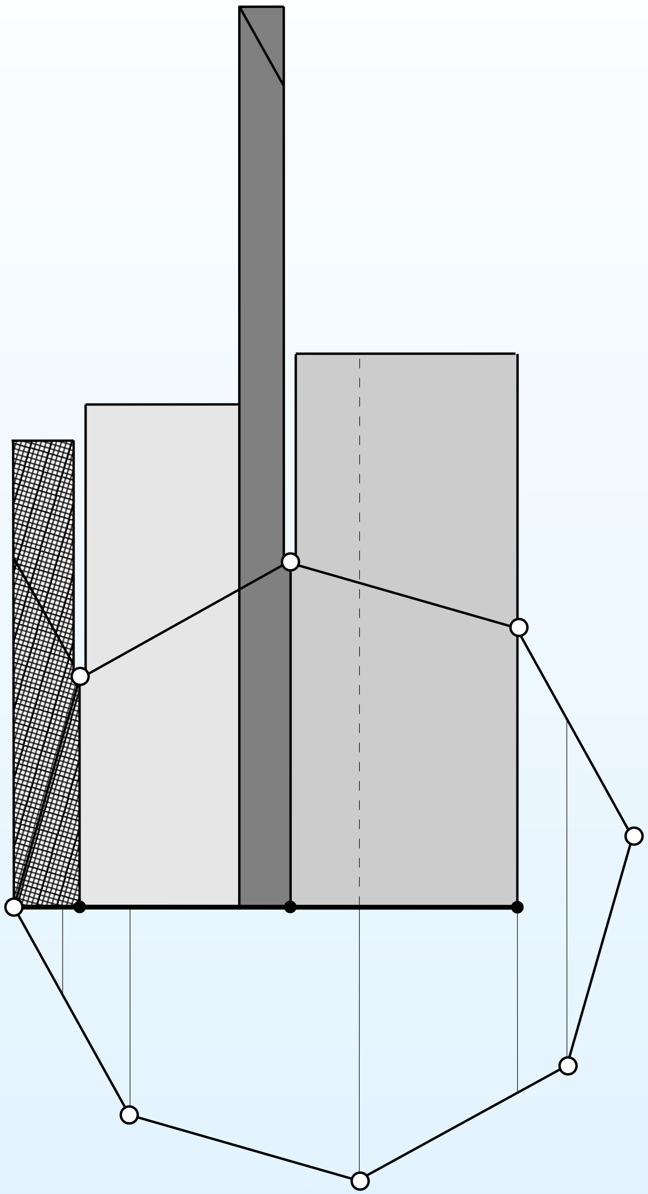
For a general flat surface S the first return map of the vertical flow to a horizontal segment X also induces an interval exchange transformation $T : X \to X$.



Zippered rectangles

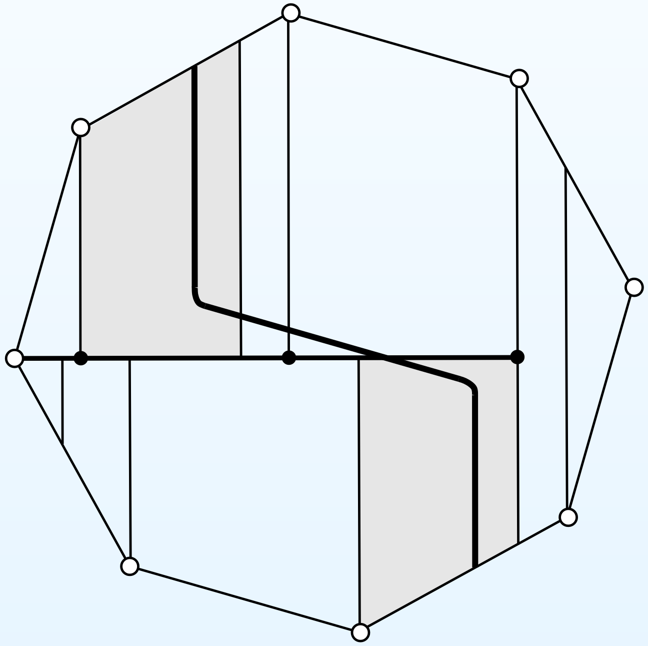
We get a decomposition of S into *zippered rectangles*.





First return cycles

Launch the vertical trajectory from a point $x \in X$. When the trajectory intersects X for the first time join the corresponding point $T(x)$ to the original point x along X to obtain a closed loop $c(x)$. (In the picture this “first return cycle” is smoothed.)



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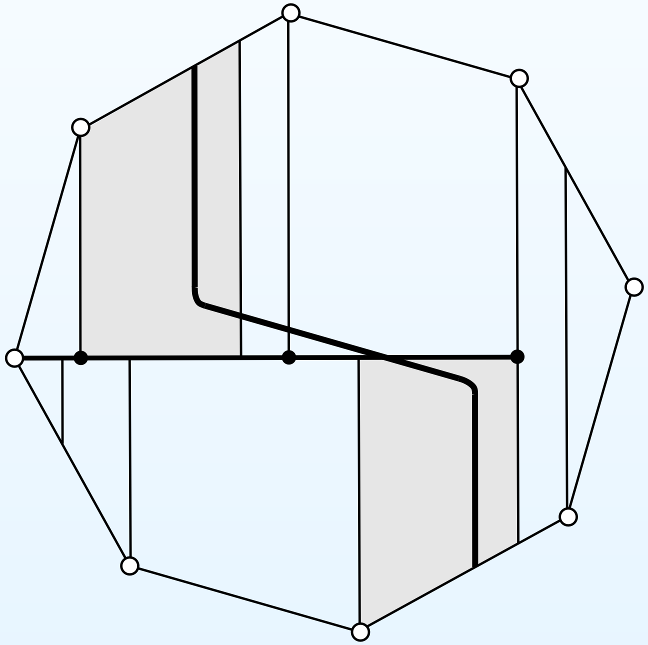
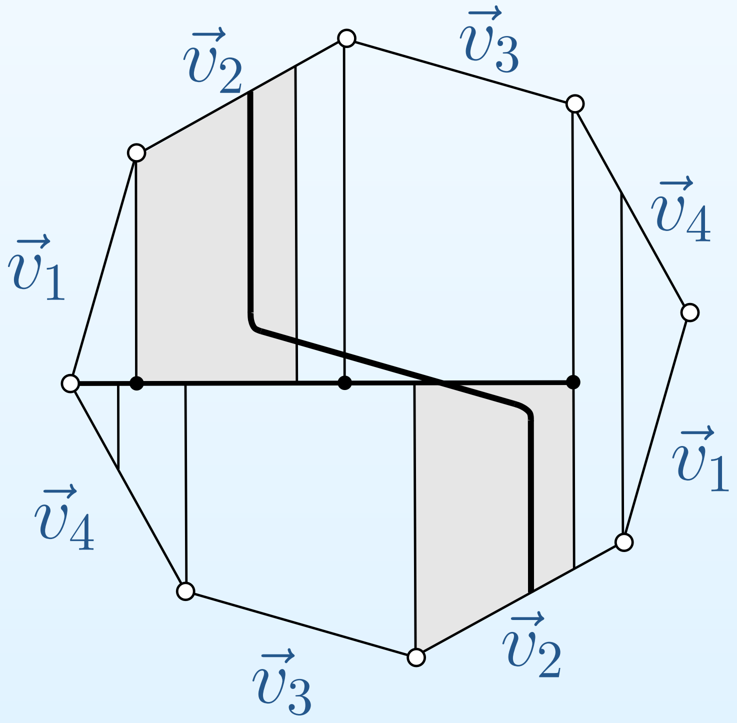


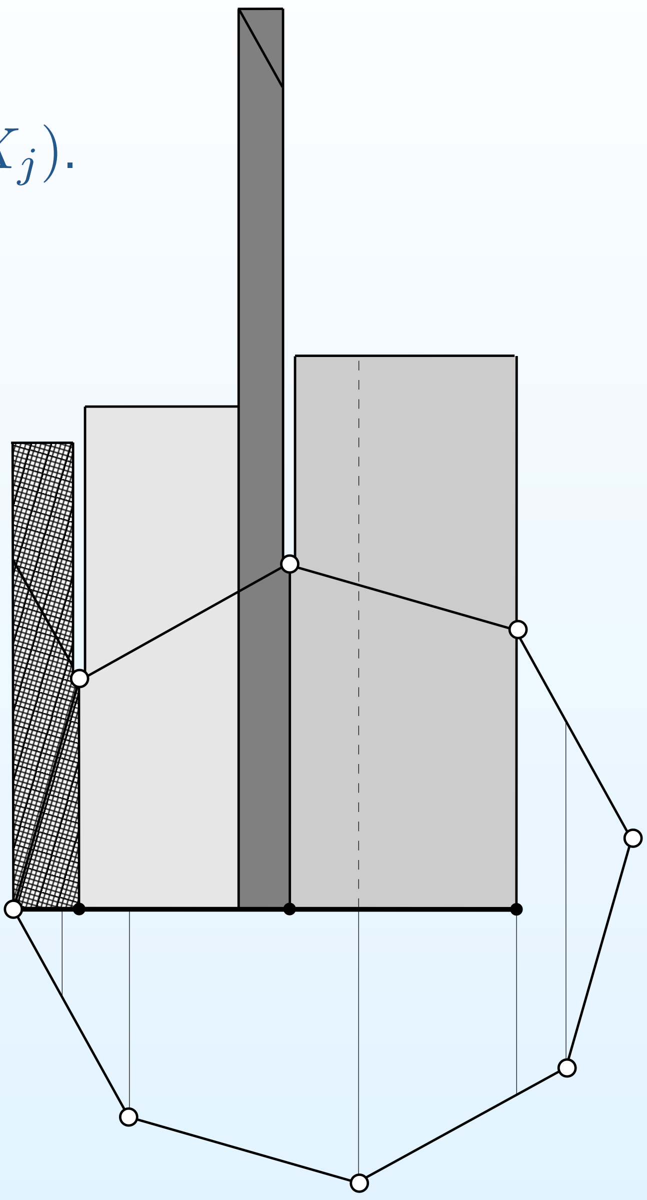
The cycle $c_N(x)$ obtained after N returns of the vertical trajectory to X can be computed as:

$$c_N(x) = c(x) + c(T(x)) + \cdots + c(T^{N-1}(x))$$

First return cycles

The "first return cycle" $c(x)$ is constant on every subinterval X_j; denote it by $c(X_j)$.





One step of renormalization

Consider a subinterval $X' \subset X$. Choose it in such way that that the first return map to X' induces an interval exchange transformation $T' : X' \to X'$ of the same number n of subintervals.

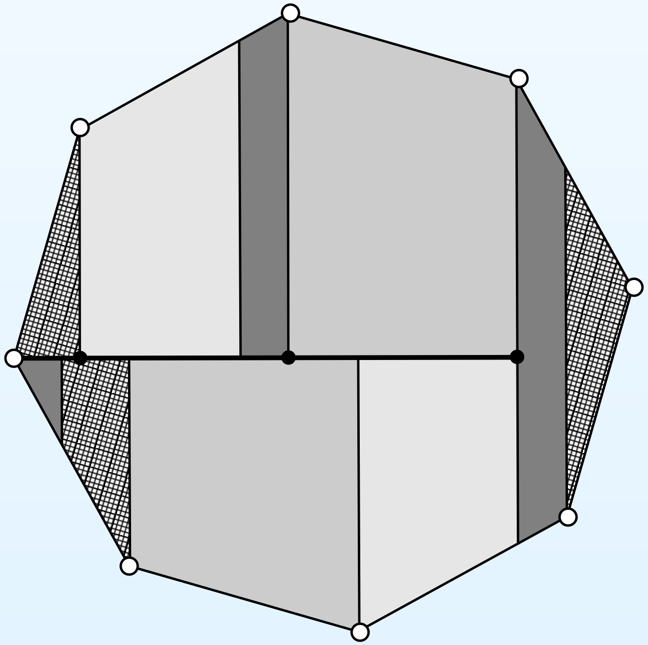
New first return cycles $c'(X'_k)$ to the interval X' are expressed in terms of the initial first return cycles $c(X_j)$ by linear relations; the lengths $|X'_k|$ of subintervals of the new partition $X' = X'_1 \sqcup \cdots \sqcup X'_n$ are expressed in terms of the lengths $|X_j|$ of subintervals of the initial partition by dual linear relations:

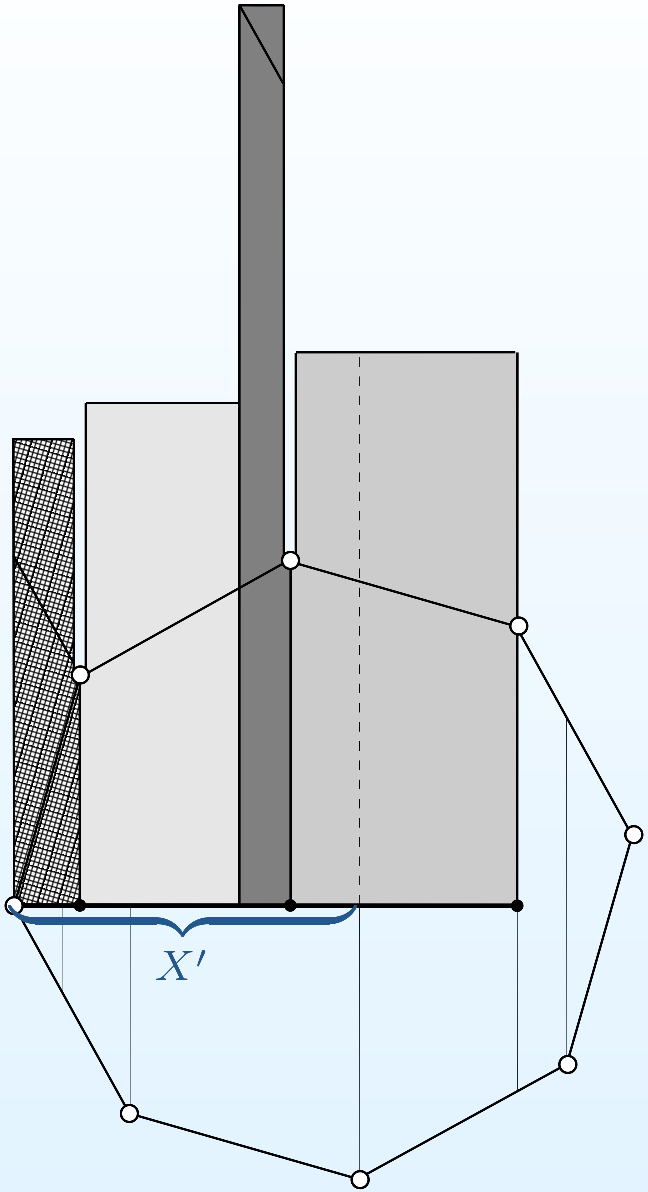
$$c'(X'_k) = \sum_{j=1}^{n} A_{jk} \cdot c(X_j) \qquad\qquad |X_j| = \sum_{k=1}^{n} A_{jk} \cdot |X'_k|\,,$$

Here a nonnegative integer matrix A_{jk} is completely determined by the initial interval exchange transformation $T : X \to X$ and by the choice of $X' \subset X$.

Idea of renormalization

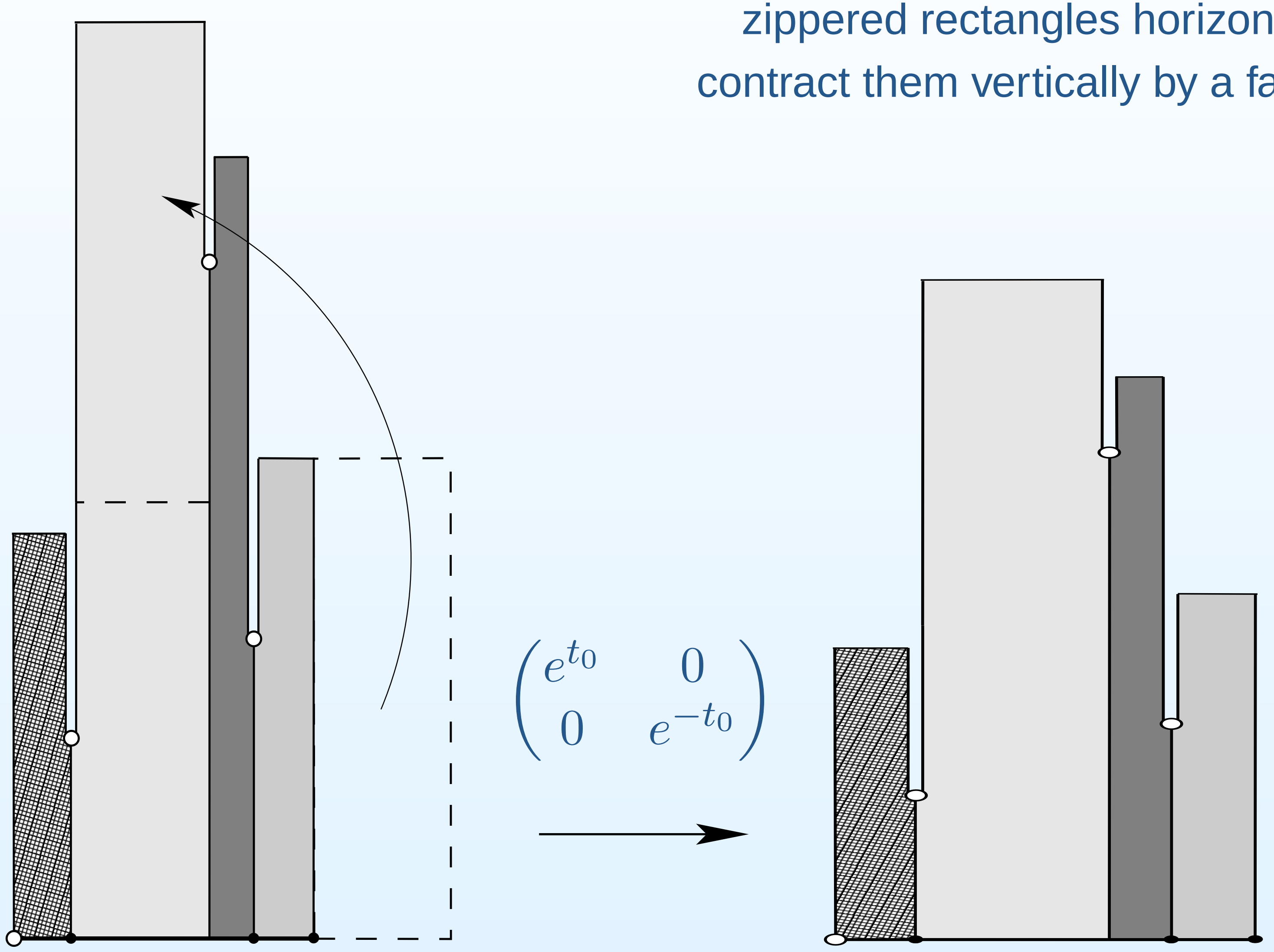
Unwrap the flat surface into "zippered rectangles". Shorten the base.





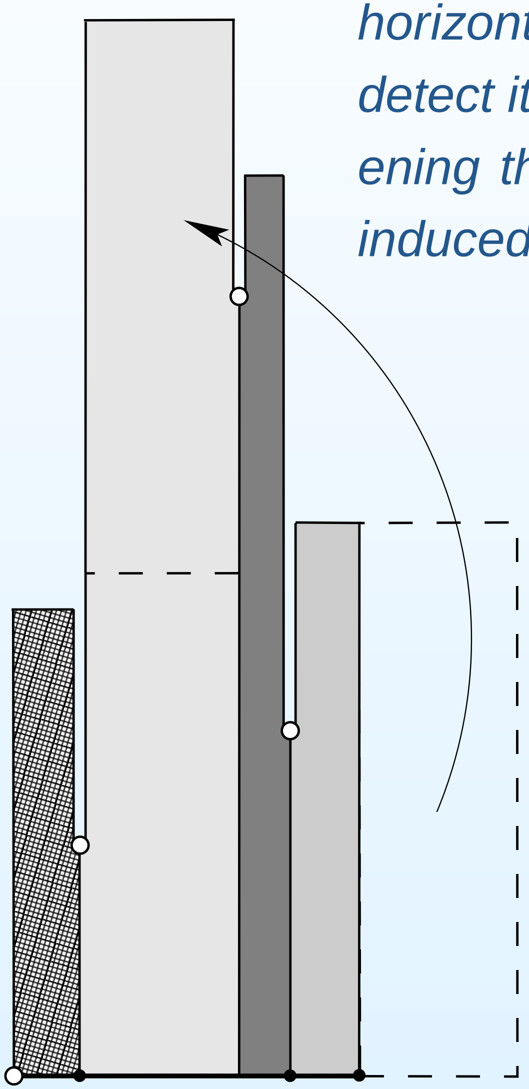
Idea of a renormalization

Expand the resulting tall and narrow zippered rectangles horizontally and contract them vertically by a factor e^{t_0}.

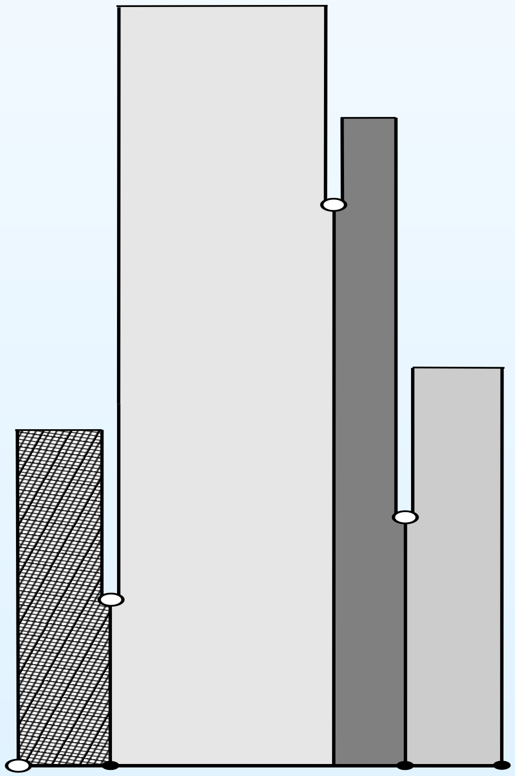


Idea of a renormalization

Lemma (Veech). *If a translation surface S admits a pseudo-Anosov diffeomorphism $f = g_{t_0}$ contracting the vertical and expanding the horizontal directions, (i.e. if $g_{t_0}(S) = S$ for some t_0), then we will detect it (possibly as a power f^k) in our discrete procedure of shortening the base X. The corresponding matrix A_{jk} represents the induced map $f_*^k : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$.*

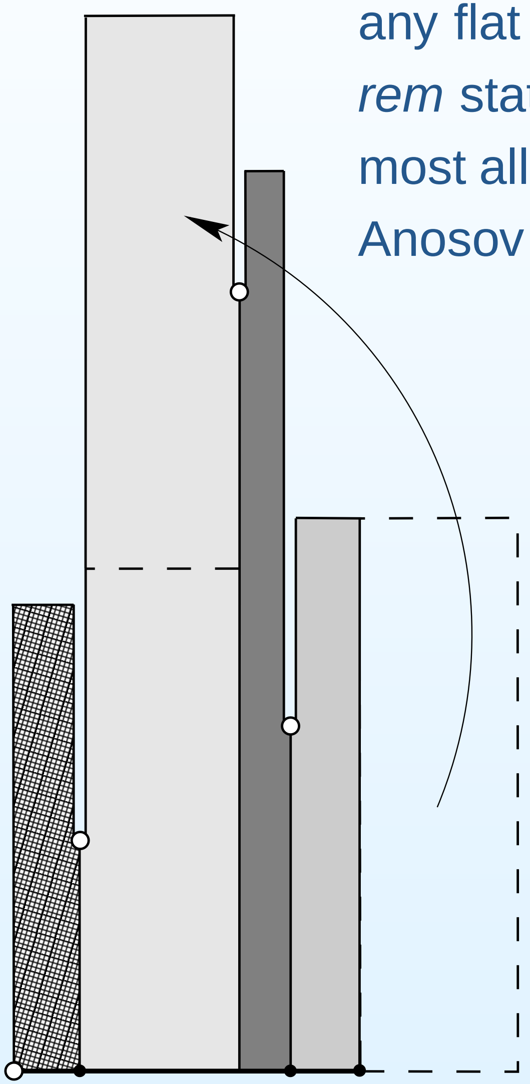


$$\begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix}$$



Idea of a renormalization

By the theorem of Masur and Veech, the homogeneous expansion-contraction in vertical-horizontal directions regularly brings almost any flat surface, basically, back to itself. *Multiplicative ergodic theorem* states that, in a sense, there a matrix (one and the same for almost all flat surfaces) which mimics a matrix of a "universal" pseudo-Anosov diffeomorphism as if the Teichmüller flow would be periodic.



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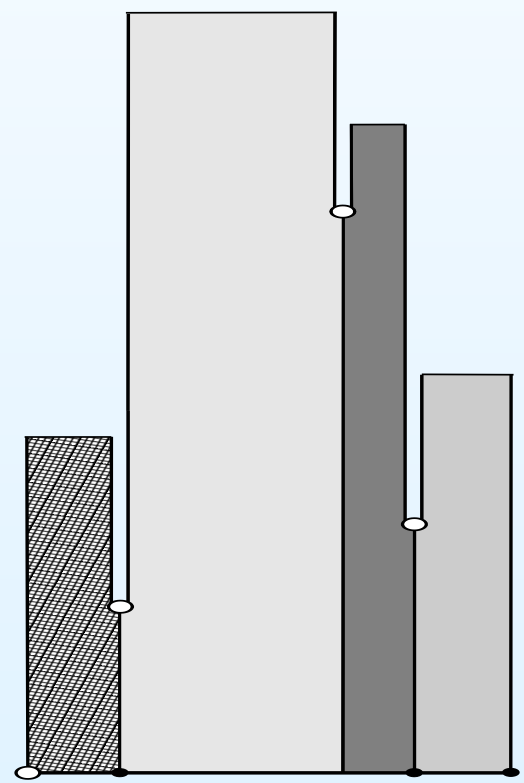

Time acceleration machine

To construct the cycle c_N representing a long piece of trajectory of the vertical flow we follow the trajectory $x, T(x), \dots, T^{N-1}(x)$ of the corresponding interval exchange transformation and compute the corresponding ergodic sum $c_N(x) = c(x) + \cdots + c(T^{N-1}(x))$.

Passing to a subinterval $X' \subset X$ we can follow the trajectory $x, T'(x), \dots, (T')^{N'-1}(x)$ of the new interval exchange transformation $T' : X' \to X'$. Since X' is shorter than X we cover the initial piece of trajectory of the vertical flow in a smaller number N' of steps.

Passing from T to T' we accelerate the time: that the trajectory $x, T'(x), \dots, (T')^{N'-1}(x)$ follows the trajectory $x, T(x), \dots, T^{N-1}(x)$ but jumps over several iterations of T at a time.

Our renormalization consists in considering first return cycles to a special shorter subinterval. Formally, it can be seen as a map on the space of interval exchange transformations, combined with rescaling the interval to keep unit length. Applying several iterations of the renormalization map we obtain exponentially long trajectory of the initial first return map.

Spectrum of “mean monodromy”

Consider a vector bundle endowed with a flat connection over a manifold X^n. Having a flow on the base we can take a fiber of the vector bundle and transport it along a trajectory of the flow. When the trajectory comes close to the starting point we identify the fibers using the connection and we get a linear transformation $\mathcal{A}(x,1)$ of the fiber; the next time we get a matrix $\mathcal{A}(x,2)$, etc.

The multiplicative ergodic theorem says that when the flow is ergodic a *“matrix of mean monodromy”* along the flow

$$A_{mean} := \lim_{N\to\infty} \left(\mathcal{A}^*(x,N)\cdot\mathcal{A}(x,N)\right)^{\frac{1}{2N}}$$

is well-defined and constant for almost every starting point.

Lyapunov exponents correspond to logarithms of eigenvalues of this “matrix of mean monodromy”. They measure the average growth rate of the norm of vectors of the bundle when we pull them along the flow using the connection. Lyapunov exponents are dynamical analogs of characteristic numbers of the bundle. It is known that they are responsible for the diffusion rate.

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Hodge bundle and Gauss–Manin connection

Consider a natural vector bundle over the stratum with a fiber $H^1(S;\mathbb{R})$ over a "point" (S,ω), called the *Hodge bundle*. It carries a canonical flat connection called *Gauss—Manin connection*: we have a lattice $H^1(S;\mathbb{Z})$ in each fiber, which tells us how we can locally identify the fibers. Thus, Teichmüller flow on $\mathcal{H}_1(d_1,\dots,d_n)$ defines Lyapunov exponents.

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Theorem (A. Eskin, M. Kontsevich, A. Z., 2014). *The Lyapunov exponents λ_i of the Hodge bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L}\subseteq\mathcal{H}_1(d_1,\dots,d_n)$ satisfy:*

$$\lambda_1+\lambda_2+\cdots+\lambda_g \;=\; \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+2)}{d_i+1}+$$

$$\sum_{\substack{\textit{Combinatorial types}\\ \textit{of flat analogs}\\ \textit{of stable curves}}}(\text{explicit combinatorial factor})\cdot\frac{\prod_{j=1}^{k}\operatorname{Vol}\mathcal{H}_1(\text{adjacent simpler strata})}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}.$$

Renormalization applied to the wind-tree problem

We have reformulated the model problem of windtree billiard in terms of intersection indices $c(T) \circ h$ and $c(T) \circ v$ of a cycle $c(T)$ obtained by closing up a very long piece of vertical trajectory with two given cycles h and v on a given translation surface S.

Idea: apply the Teichmüller geodesic flow to S for an appropriate time t to get a flat surface $g_t S$ located very close to the original surface S. Close up the corresponding segment of the Teichmüller geodesic to get an associated pseudo-Anosov diffeomorphism $f : S \to S$.

Note that g_t exponentially contracts the vertical direction. Choosing $t \simeq \log T$ we can transform the very long cycle $c(T)$ to an ordinary integer cycle $f_* c(T)$ of length comparable to 1.

Conclusion: to compute $c(T) \circ h = f_* c(T) \circ f_* h$ we have to figure out how the pseudo-Anosov diffeomorphism f corresponding to a very long piece of a Teichmüller geodesic twists the distinguished cycles h and v. In other words, we have to compute the *Lyapunov exponents* for the cycles cycles h and v.

Solution of the windtree problem

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Theorem (J. Chaika–A. Eskin, 2014). *For any flat surface S almost all vertical directions define a Lyapunov-generic point in the orbit closure $\overline{\mathrm{SL}(2,\mathbb{R})\cdot S}$.*

Schematic solution of a generalized windtree problem

1. Find the family of flat surfaces $\mathcal{B}$ associated to the original family of rational billiards;
2. Find the orbit closure $\mathcal{L}=\overline{\mathrm{SL}(2,\mathbb{R})\cdot\mathcal{B}}$ of $\mathcal{B}$ inside the ambient moduli space (stratum).
3. Compute or estimate the relevant Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow on $\mathcal{L}$.

Currently we do not have a slightest idea on how to approach the problem when the periodic obstacles are irrational.

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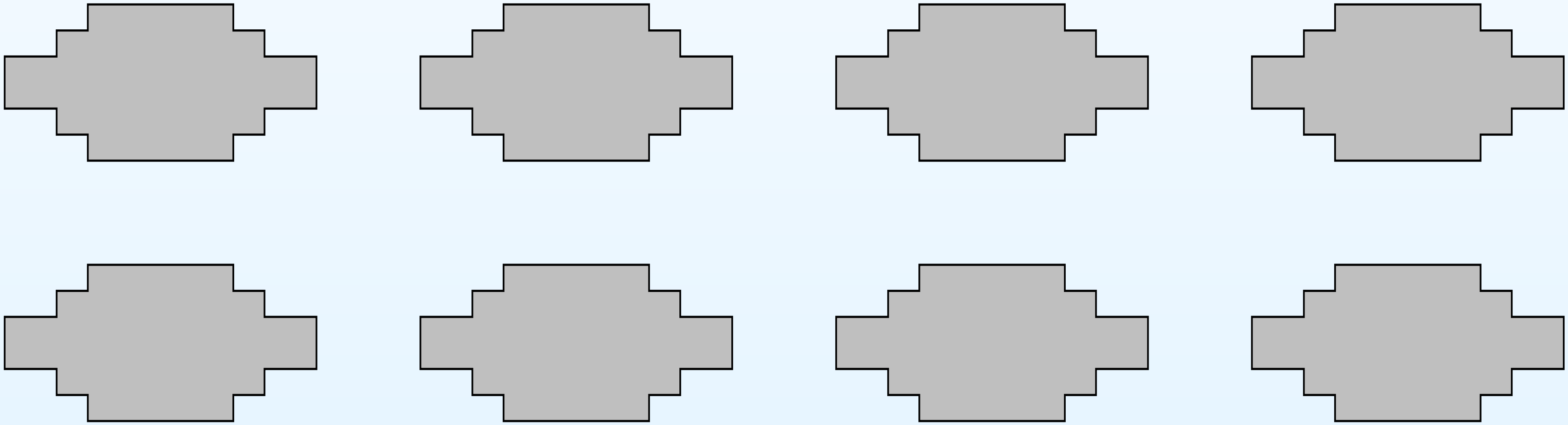
Question. *What diffusion rate has a windtree billiard with "generic" (in any reasonable sense) irrational polygonal obstacles? Is it, by any chance, $\frac{1}{2}$?*

Changing the shape of the obstacle

Theorem (V. Delecroix, A. Z., 2015). *Changing the shape of the obstacle we get a different diffusion rate. Say, for a symmetric obstacle with* $4m-4$ *angles* $3\pi/2$ *and* $4m$ *angles* $\pi/2$ *the diffusion rate is*

$$\frac{(2m)!!}{(2m+1)!!} \sim \frac{\sqrt{\pi}}{2\sqrt{m}} \quad \text{as } m \to \infty\,.$$



Note that once again the diffusion rate depends only on the number of the corners, but not on the (almost all) lengths of the sides, or other details of the shape of the obstacle.

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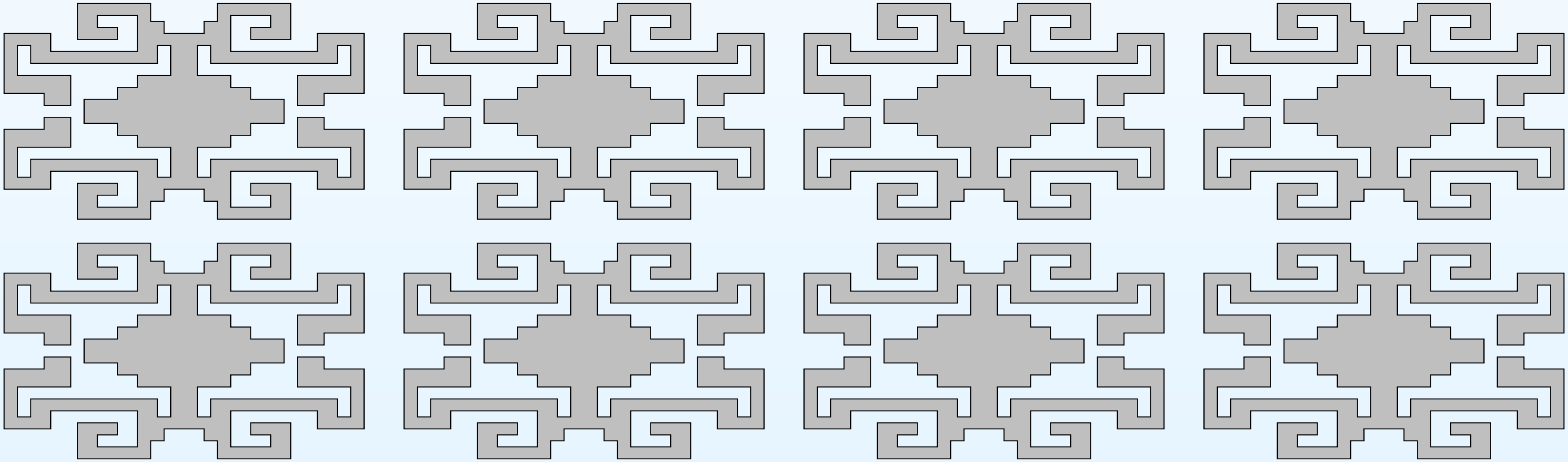

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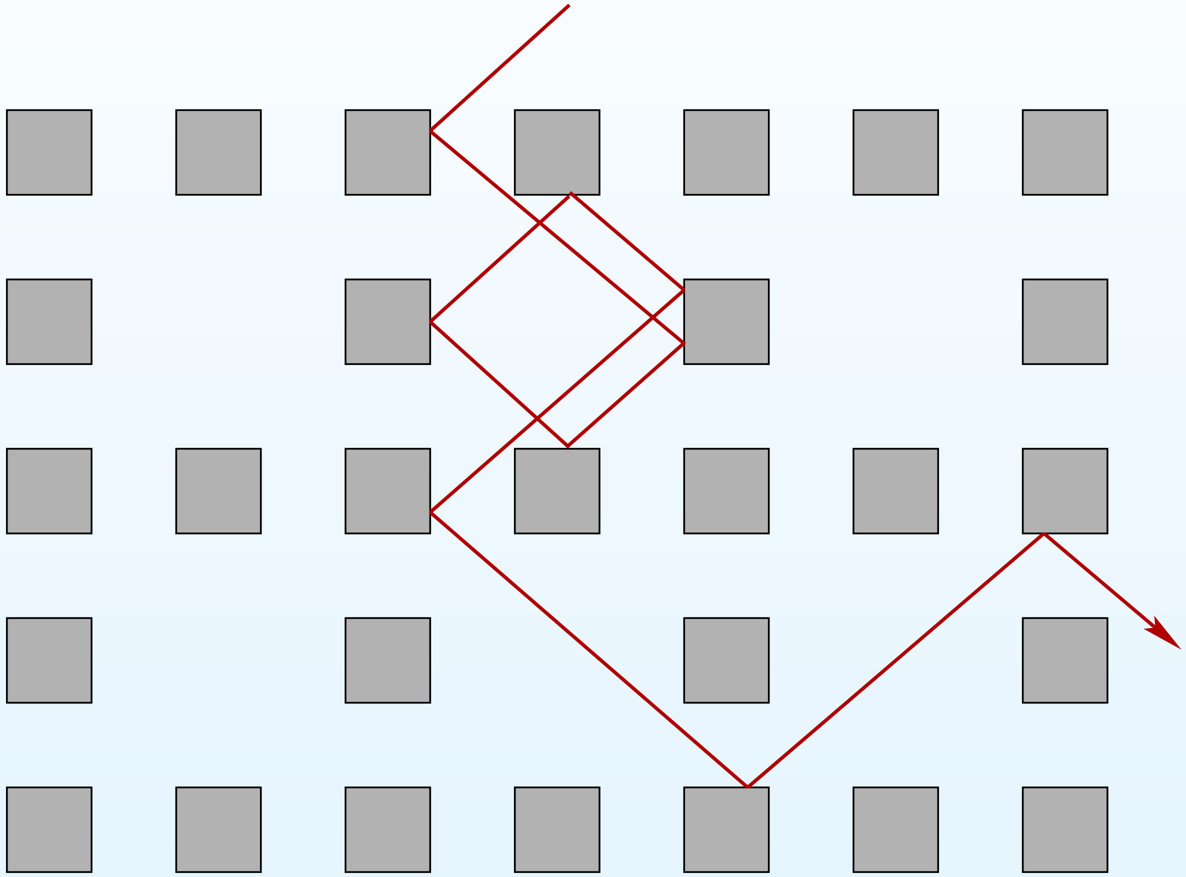
Removing part of the obstacles

How would change the diffusion rate if we remove periodically one out of four obstacles in every 2×2 group of squares?



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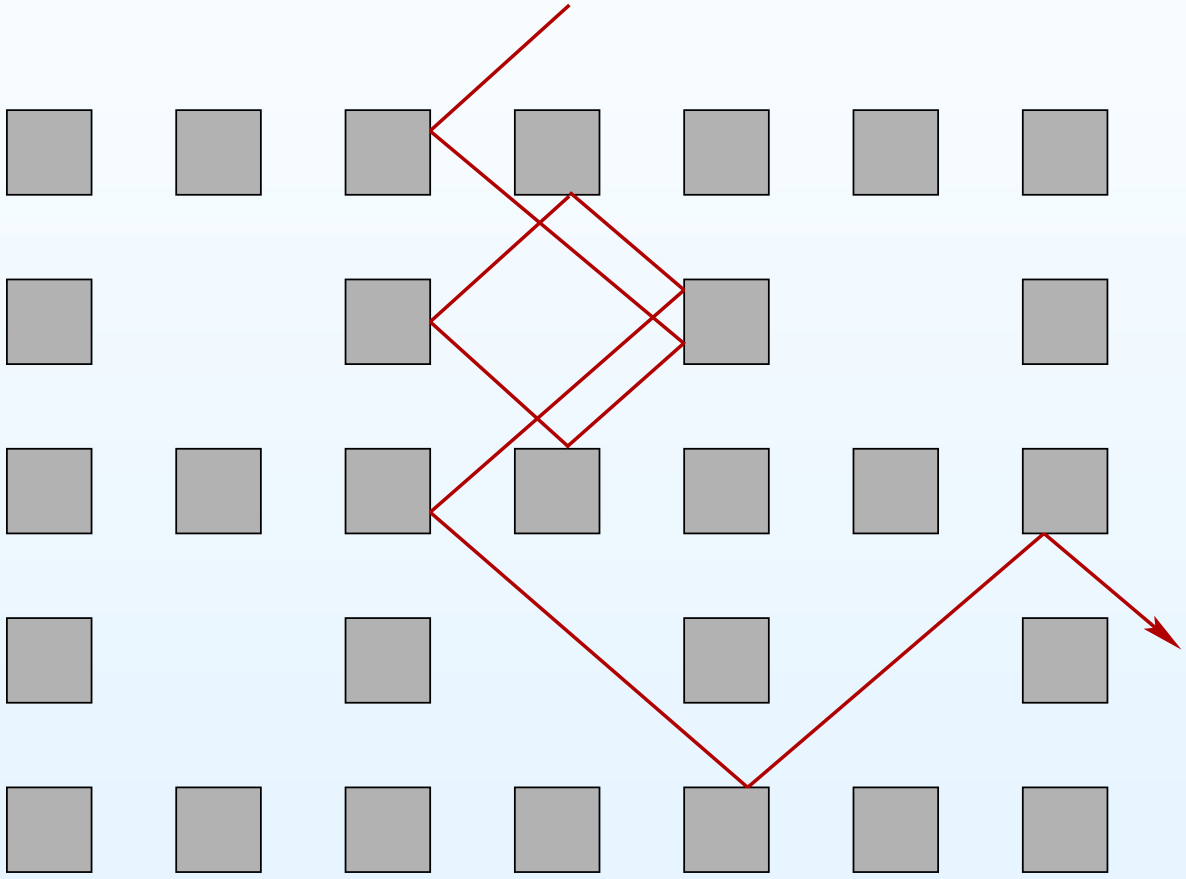
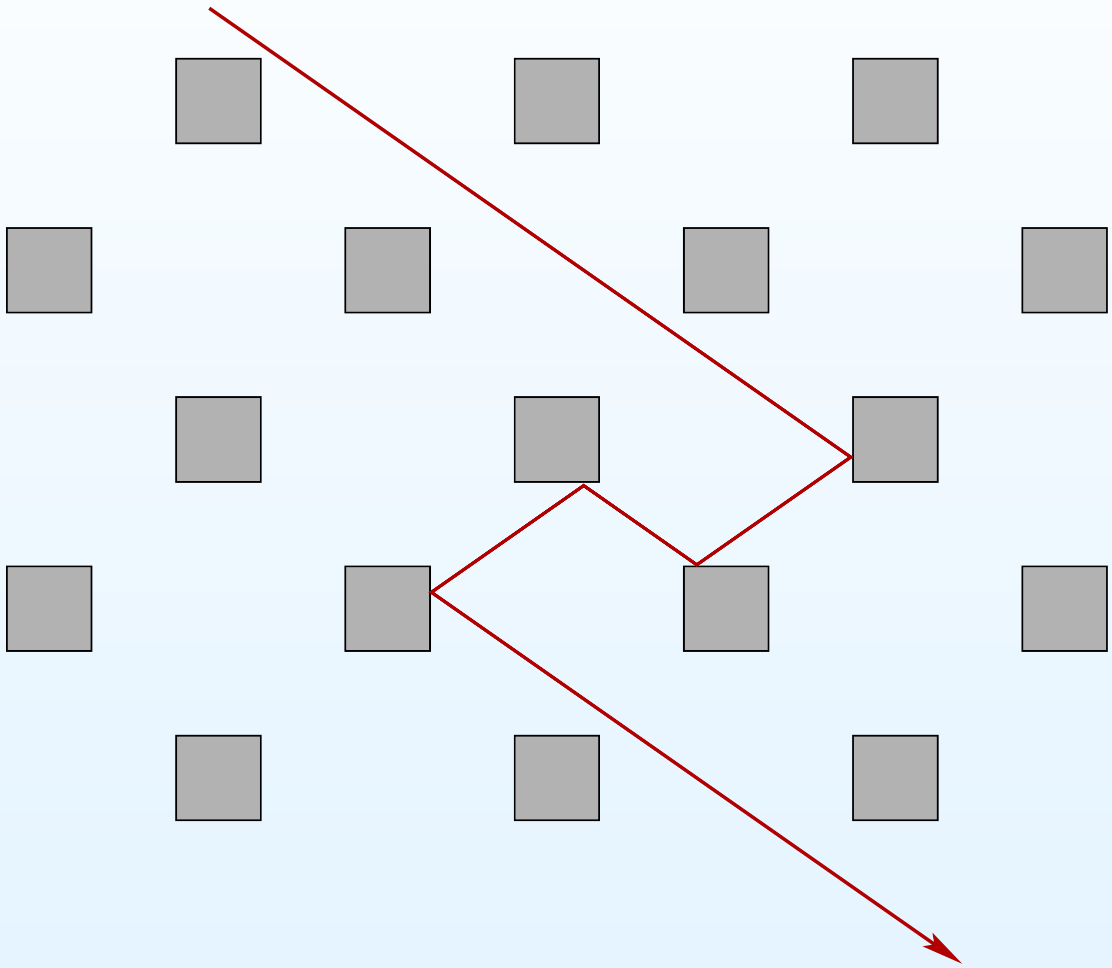


Lemma (V. Delecroix, A. Z., 2015). *Diffusion rate* $= \dfrac{491}{1053}$.

Removing part of the obstacles

And what about removing periodically two obstacles in every 2×2 group?



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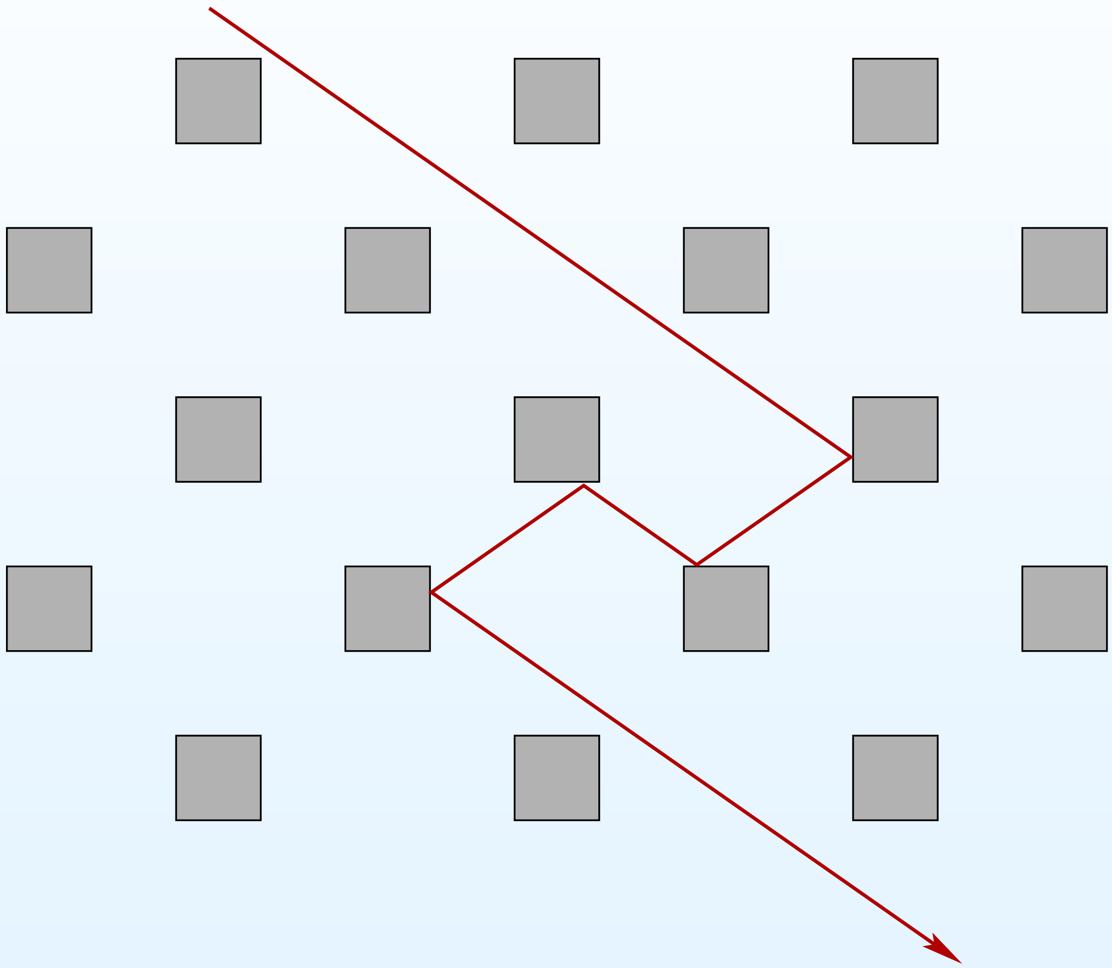
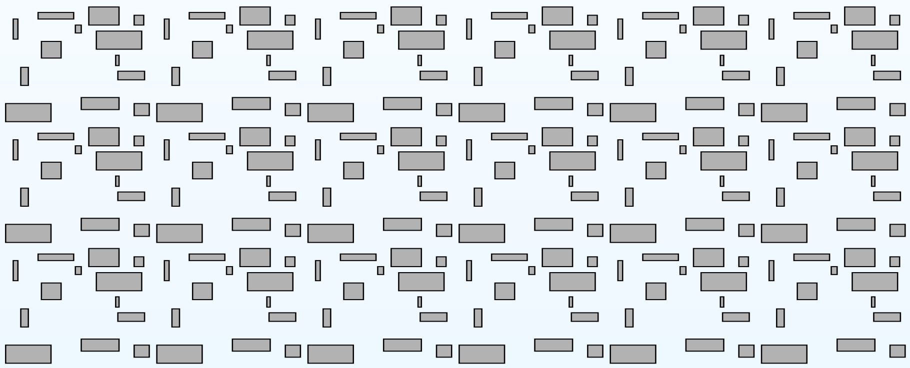


Lemma (V. Delecroix, A. Z., 2015). *Diffusion rate* $= \dfrac{2}{3}$.

Generic windtree model of high complexity

Theorem (Fougeron'20). *The diffusion rate of a periodic billiard with* $n \geq 2$ *random rectangular obstacles placed as in the picture equals the top Lyapunov exponent* $\lambda_1^+(\mathcal{Q}_{n+1})$ *of the Kontsevich–Zorich cocycle over the moduli space of holomorphic quadratic differentials of genus* $g = n + 1$.



Conjectures (Zorich'98, Delecroix'15, Fougeron'19).

$$\lambda_2(\mathcal{H}(m_1, \dots, m_n)) \to \frac{1}{2} \qquad \lambda_1^+(\mathcal{Q}(d_1, \dots, d_n)) \to \frac{1}{2} \quad \text{as } g \to +\infty$$

uniformly for all $m_1 + \cdots + m_n = 2g - 2$ *and* $d_1 + \cdots + d_n = 4g - 4$.

The conjecture is confirmed by extensive computer experiments. Conceptually, it indicates that *parabolic* dynamical systems of large complexity in certain aspects mimic *hyperbolic* dynamical systems. For hyperelliptic strata we have $\lambda_2(\mathcal{H}_g^{hyp}) \to 1$ (Eskin–Kontsevich–Möller–Zorich + Fei Yu'18).

Computation of diffusion rate

1. Find a surface S endowed with a flat metric with conical singularities associated to the original *rational* periodic polygonal billiard. (Straightforward).

2. The surface S represents a point in the moduli space. Find an orbit closure $\mathcal{L} = \overline{\mathrm{GL}(2,\mathbb{R}) \cdot S}$ of S inside an ambient stratum in the moduli space. Uses very recent highly elaborated technology based on Eskin–Mirzakhani Magic Wand rigidity theorem. (Difficult, but in many cases doable due to works of P. Apisa, J. Chaika, C. McMullen, M. Mirzakhani, R. Mukamel, A. Wright, . . . , and due to a computer assisted tool currently developped by V. Delecroix–A. Eskin–J. Rüth–A. Wright incorporating all known tools.)

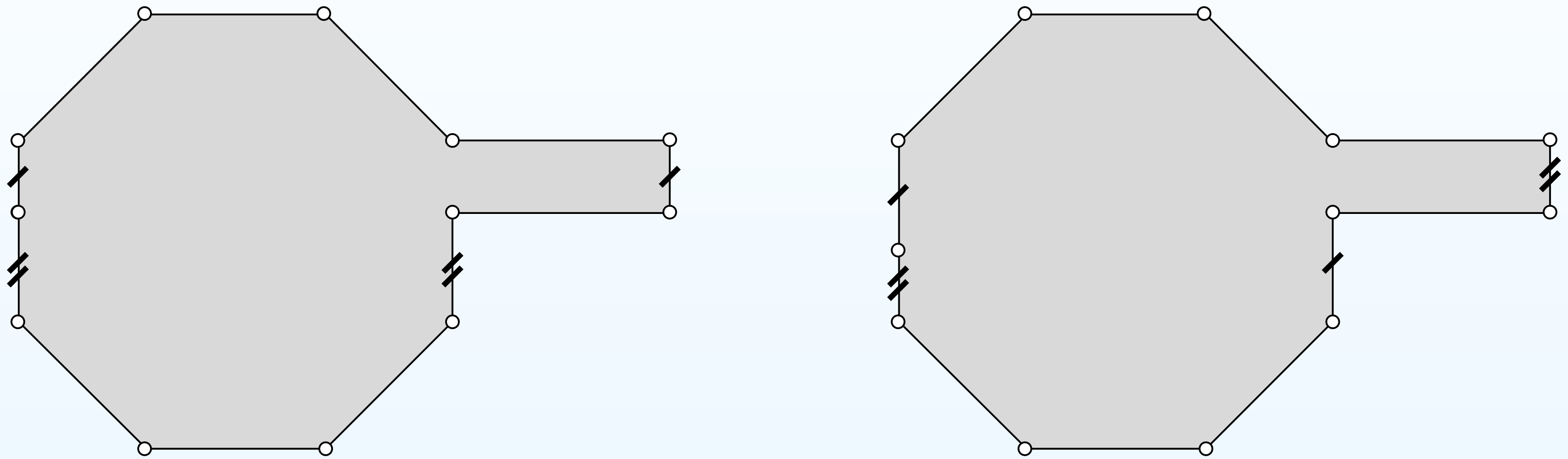
3. Compute or estimate the relevant Lyapunov exponent of the Hodge bundle along the Teichmüller geodesic flow on $\mathcal{L}$ (i.e. compute mean monodromy of the Hodge bundle along Teichmüller geodesics). (Currently can be done only in very special cases admitting extra symmetries leading to an equivariant splitting of the Hodge bundle. In these cases Eskin–Kontsevich–Zorich formula for the sum of the Lyapunov exponents is applicable to subbundles.)

We aim to advance in the last point in large genus *beyond symmetric cases*.

Exercise with representatives of $\mathcal{H}(4)$

Exercise

• Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.

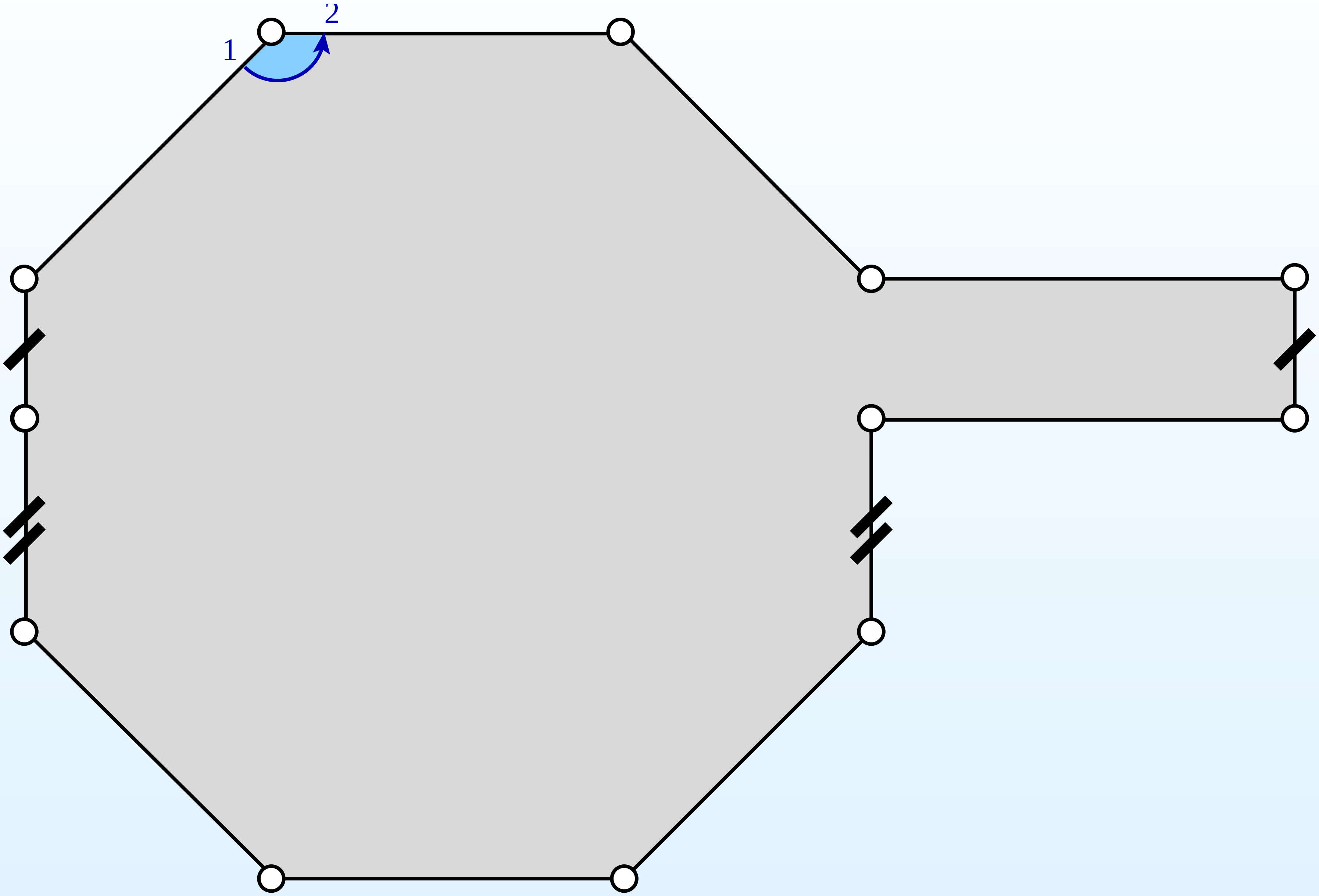


• Compute a matrix of intersection numbers between cycles representing the sides of the left polygon. Prove that these cycles form a basis in homology.

• Determine which of the two surfaces is hyperelliptic.

• Find the hyperelliptic involution of this surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.

- Proof that the following flat surface belongs to the stratum $\mathcal{H}(4)$.



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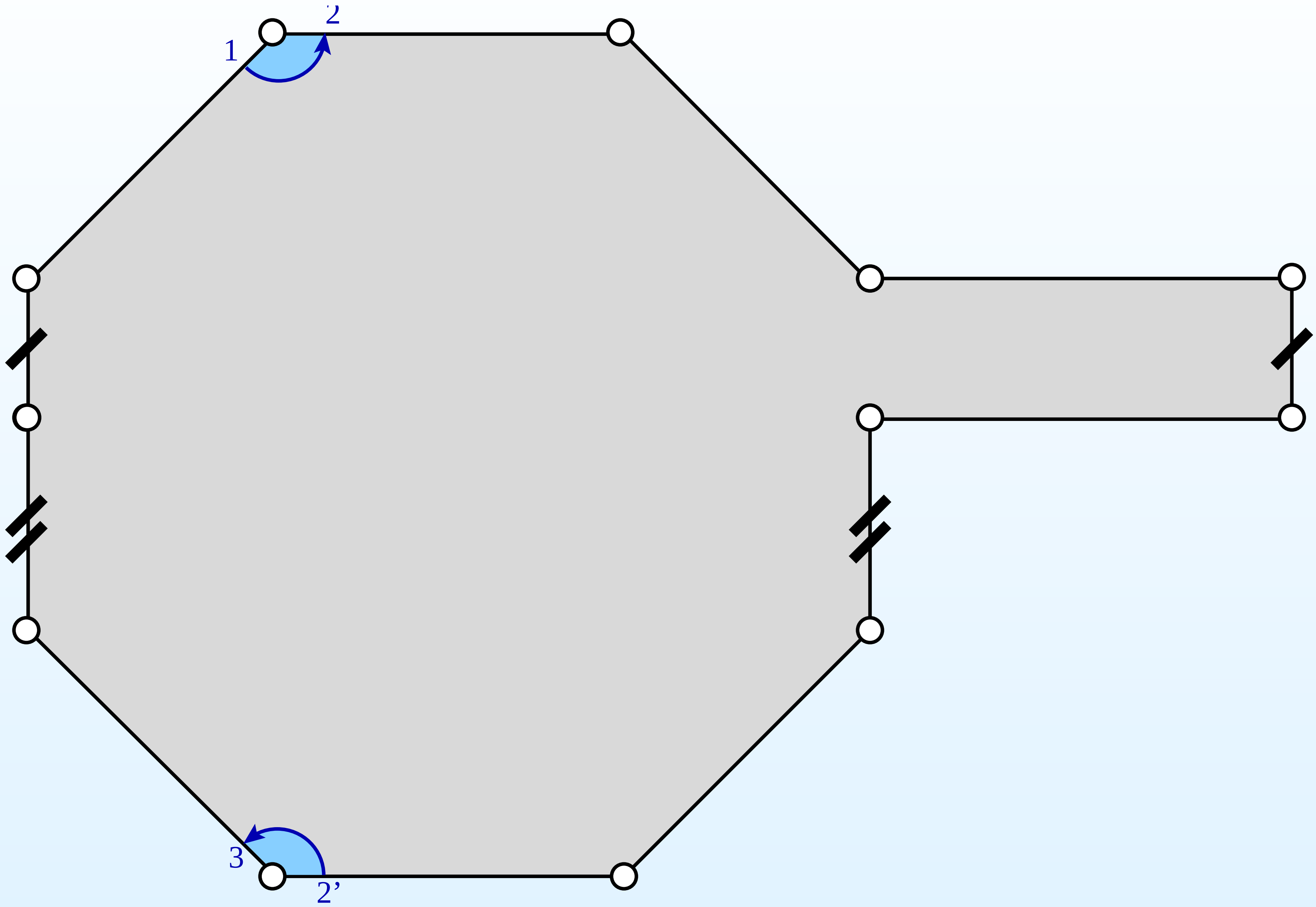

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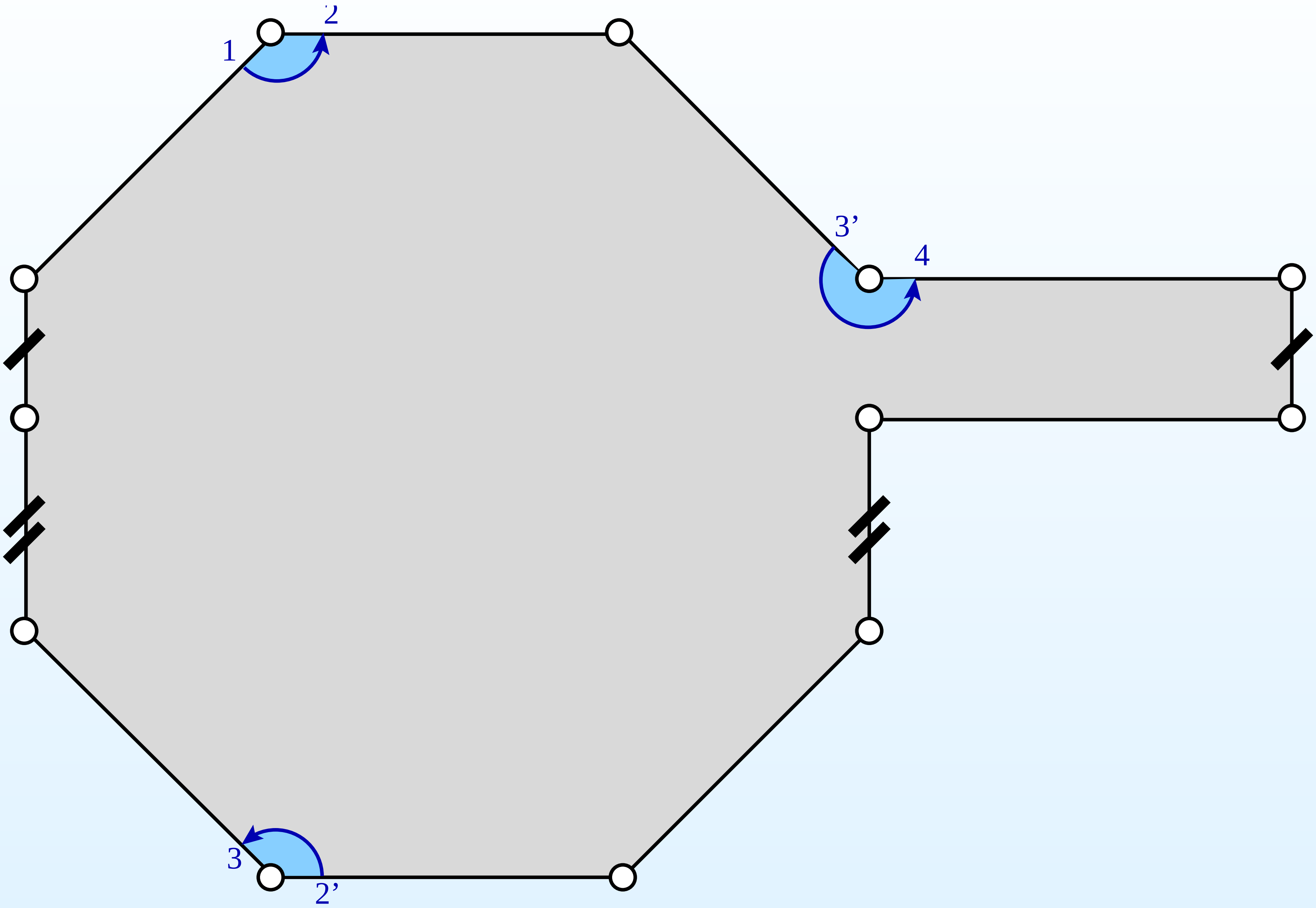

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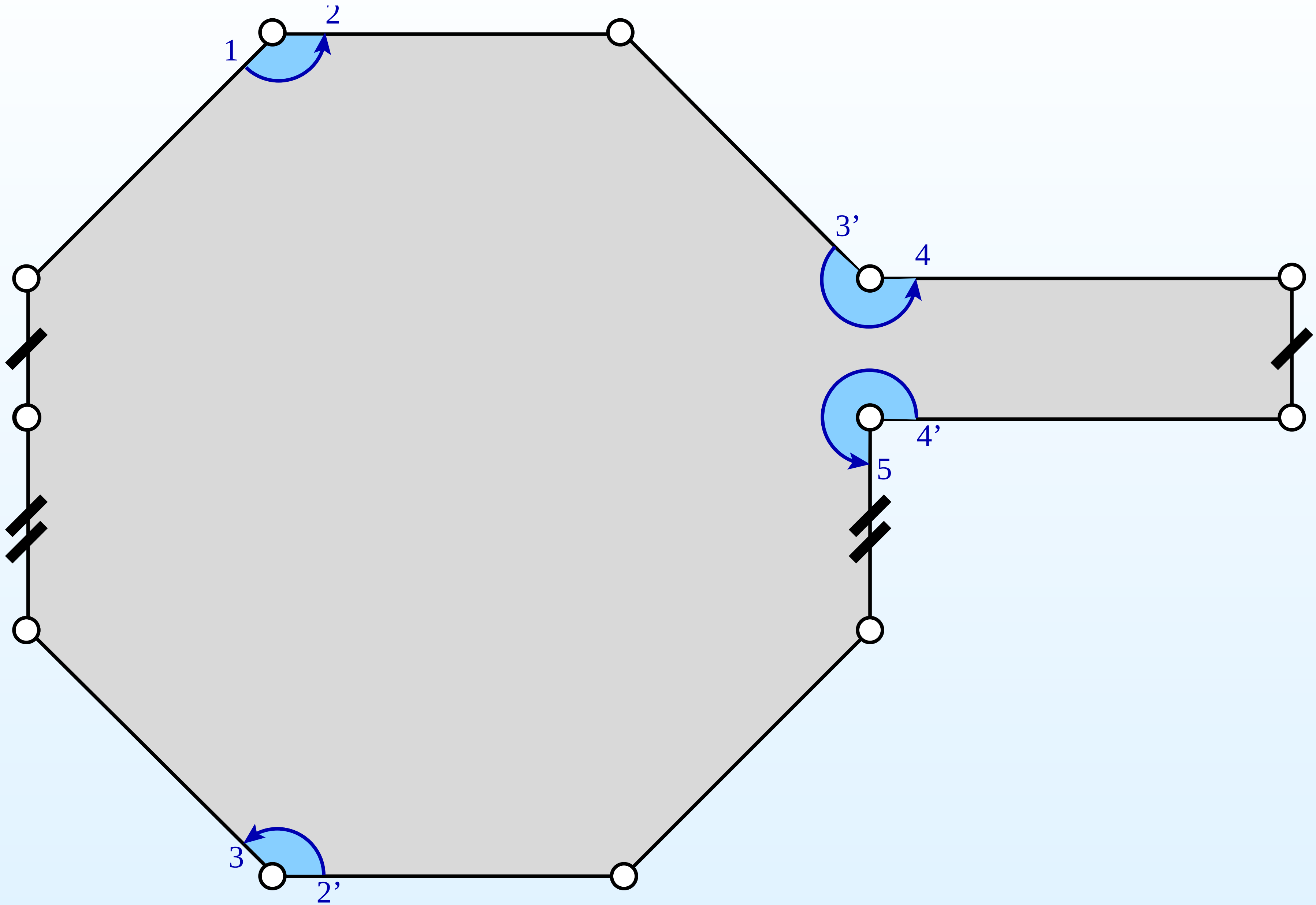

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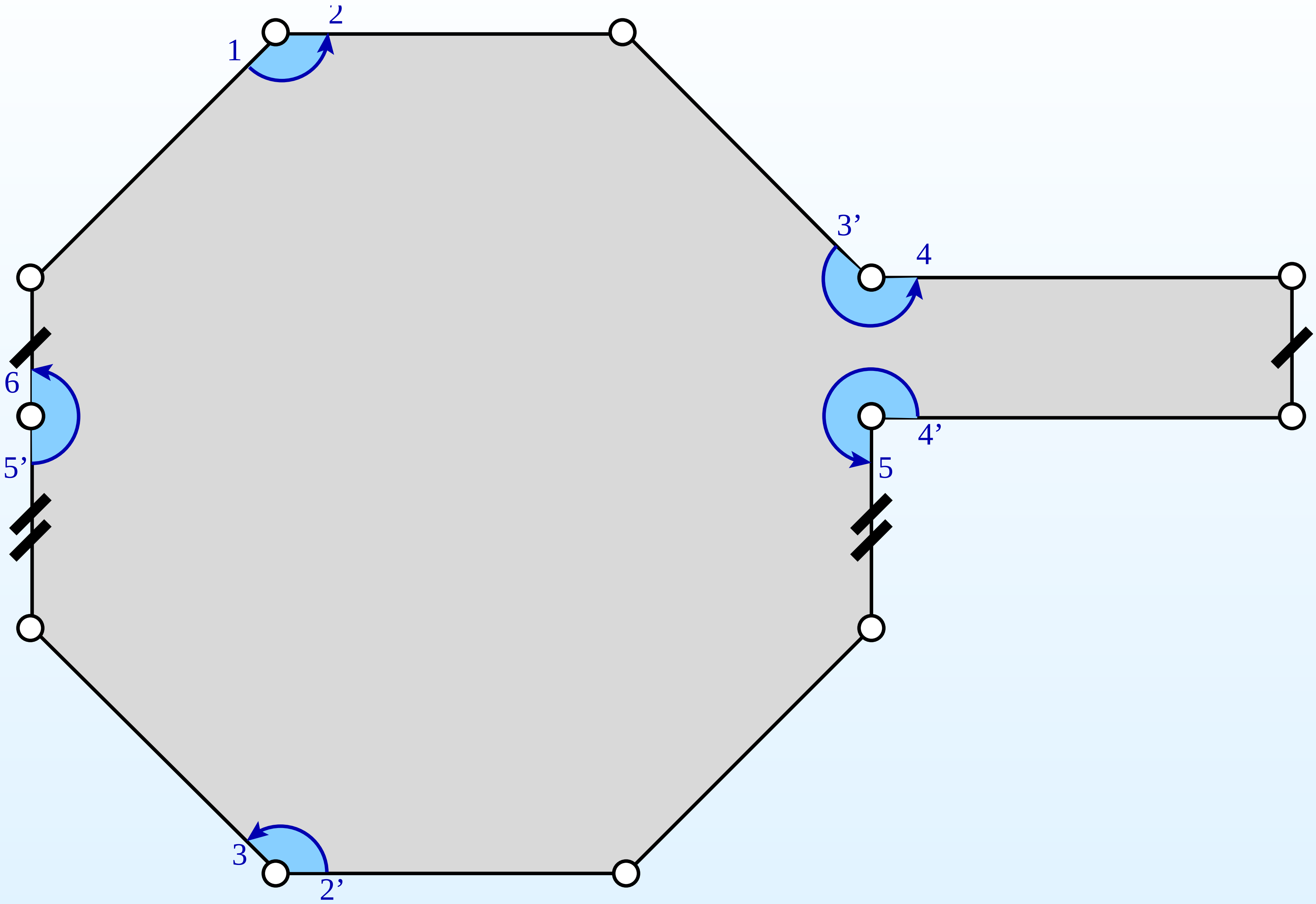

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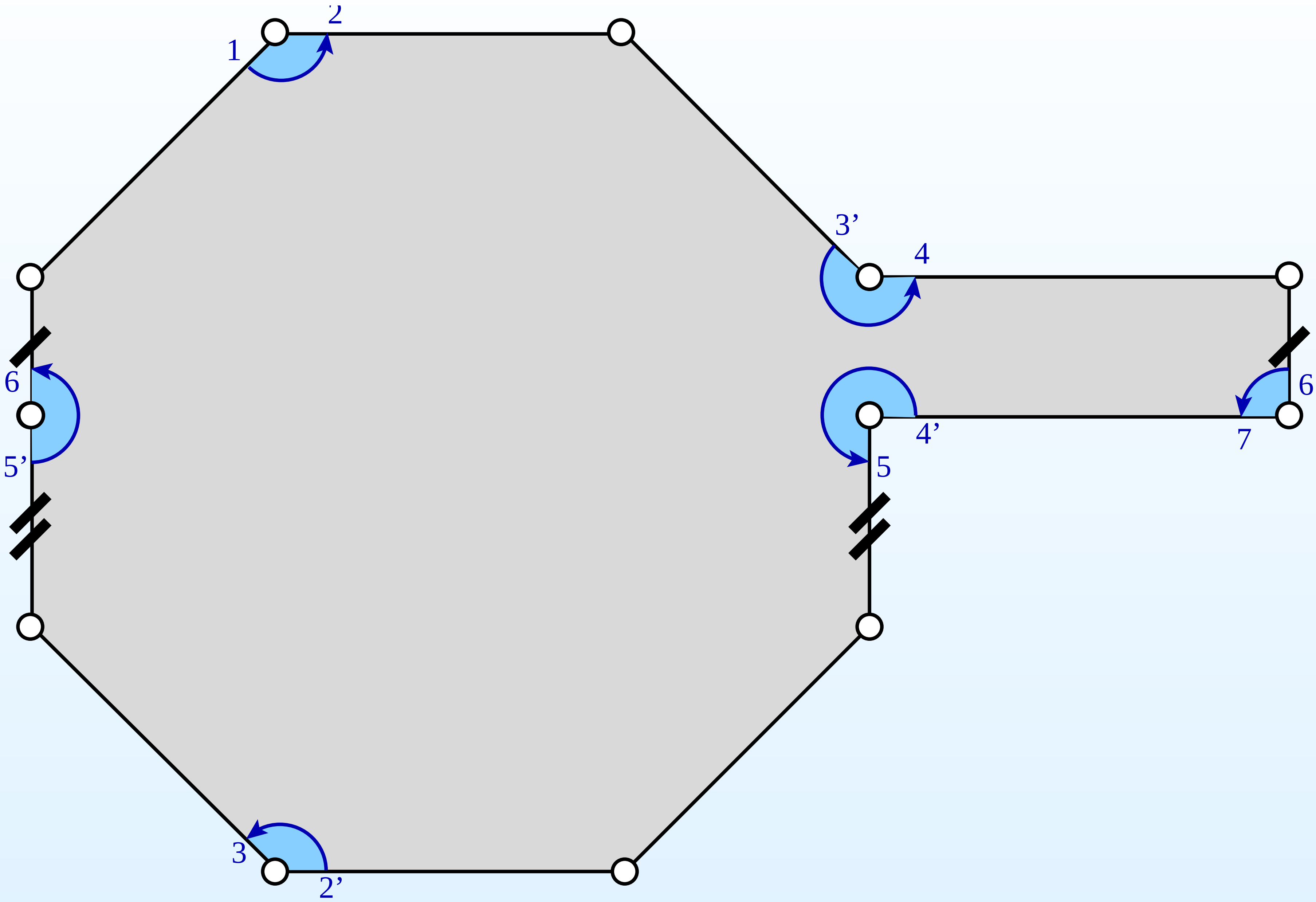

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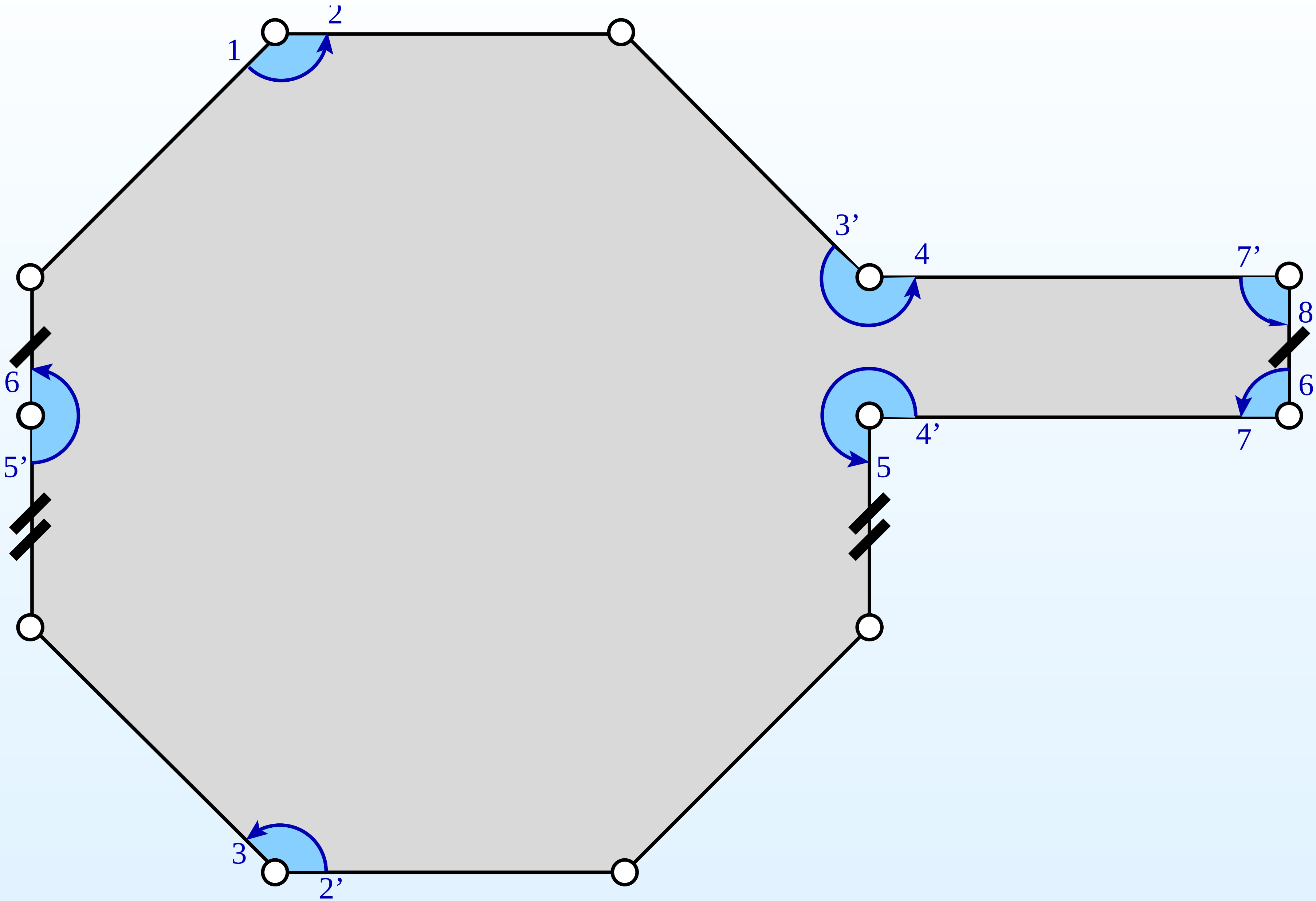

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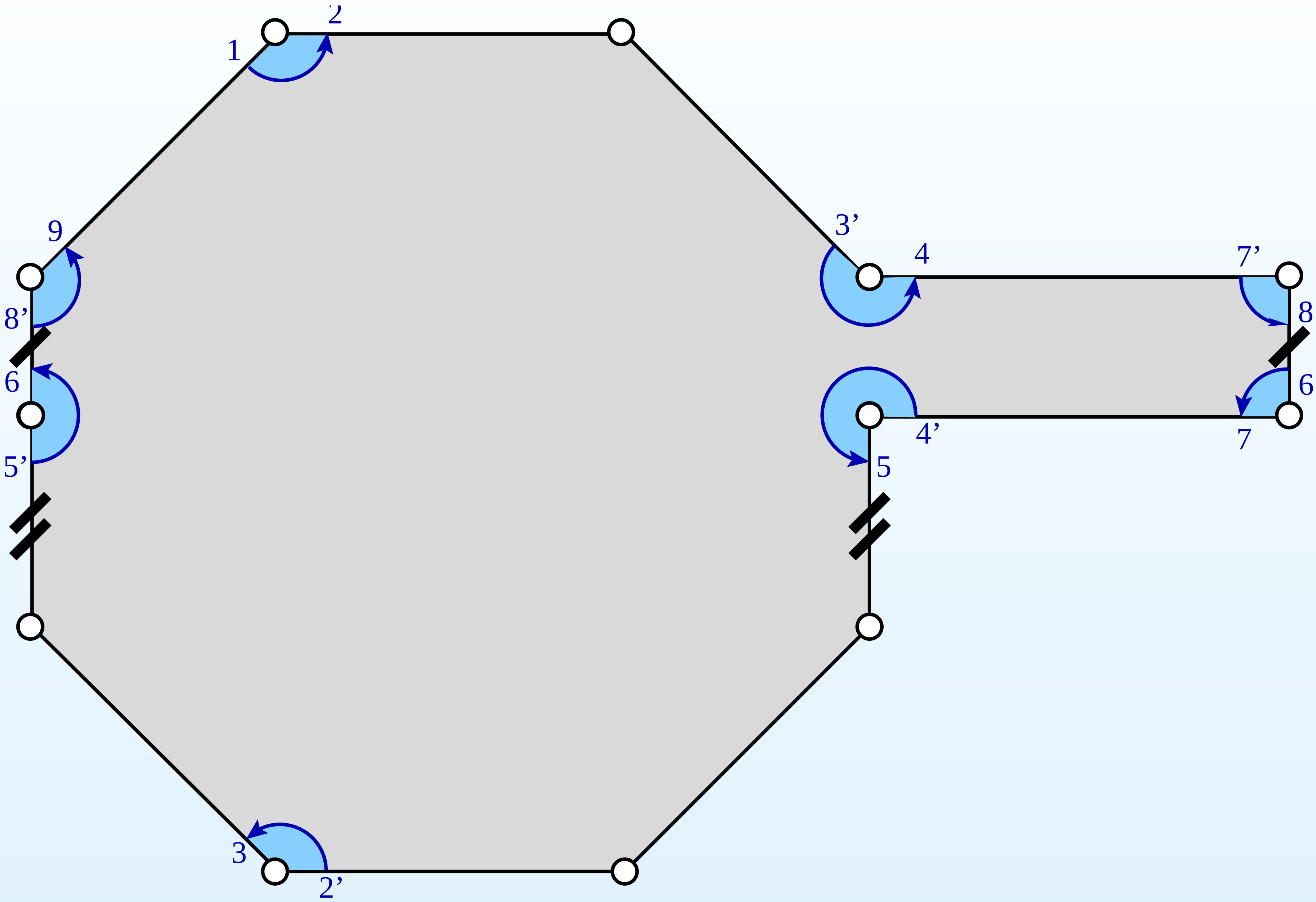

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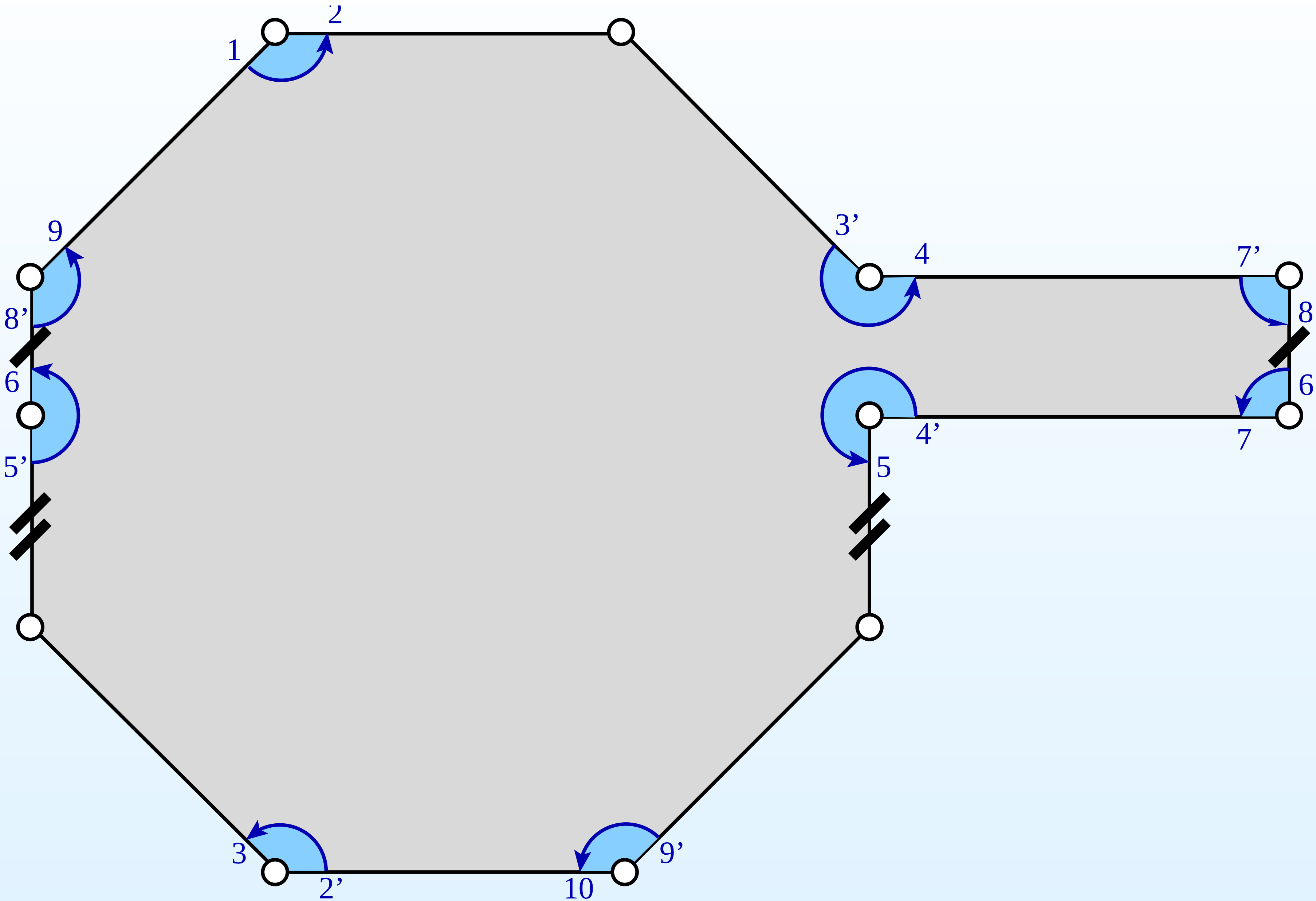

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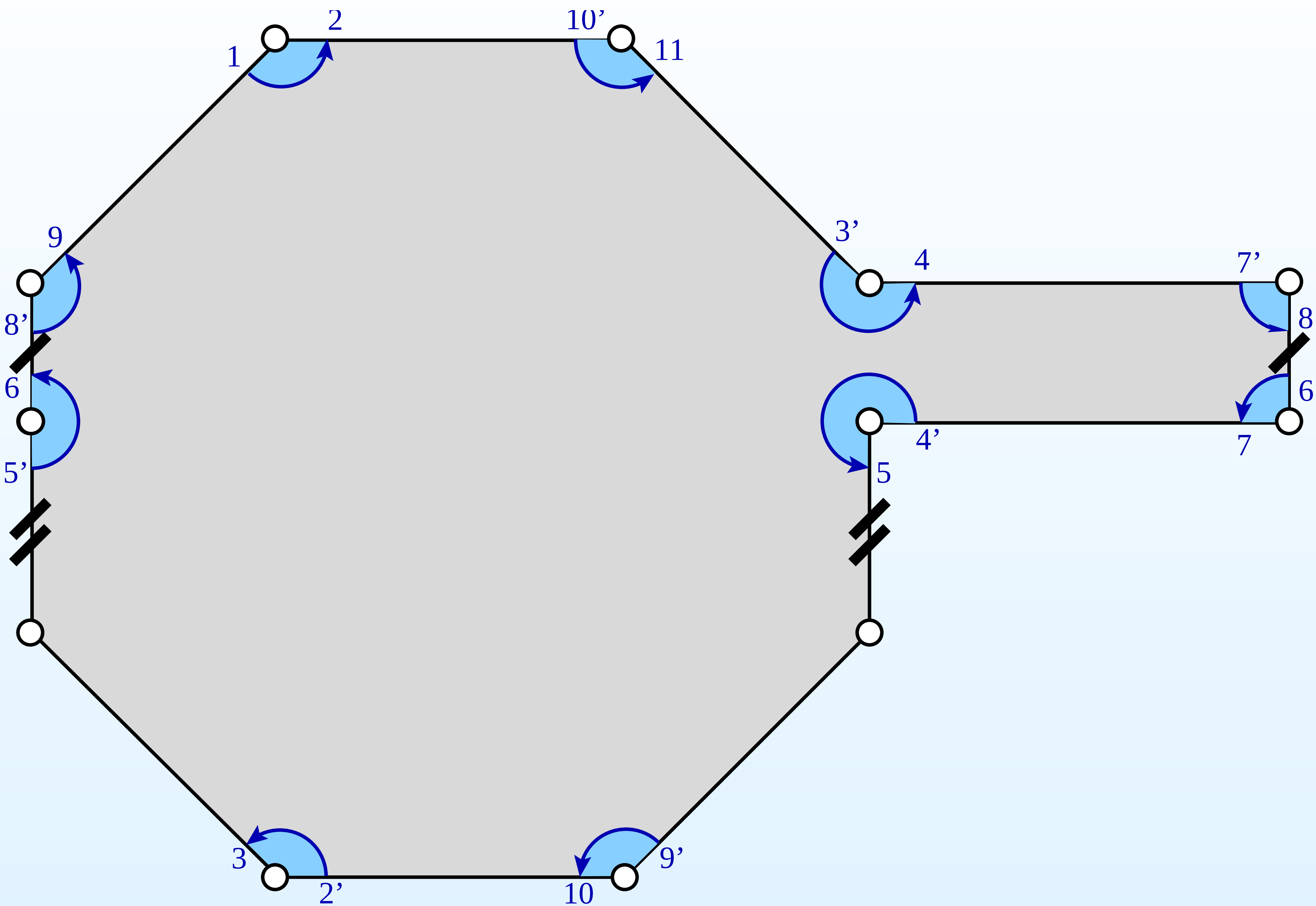

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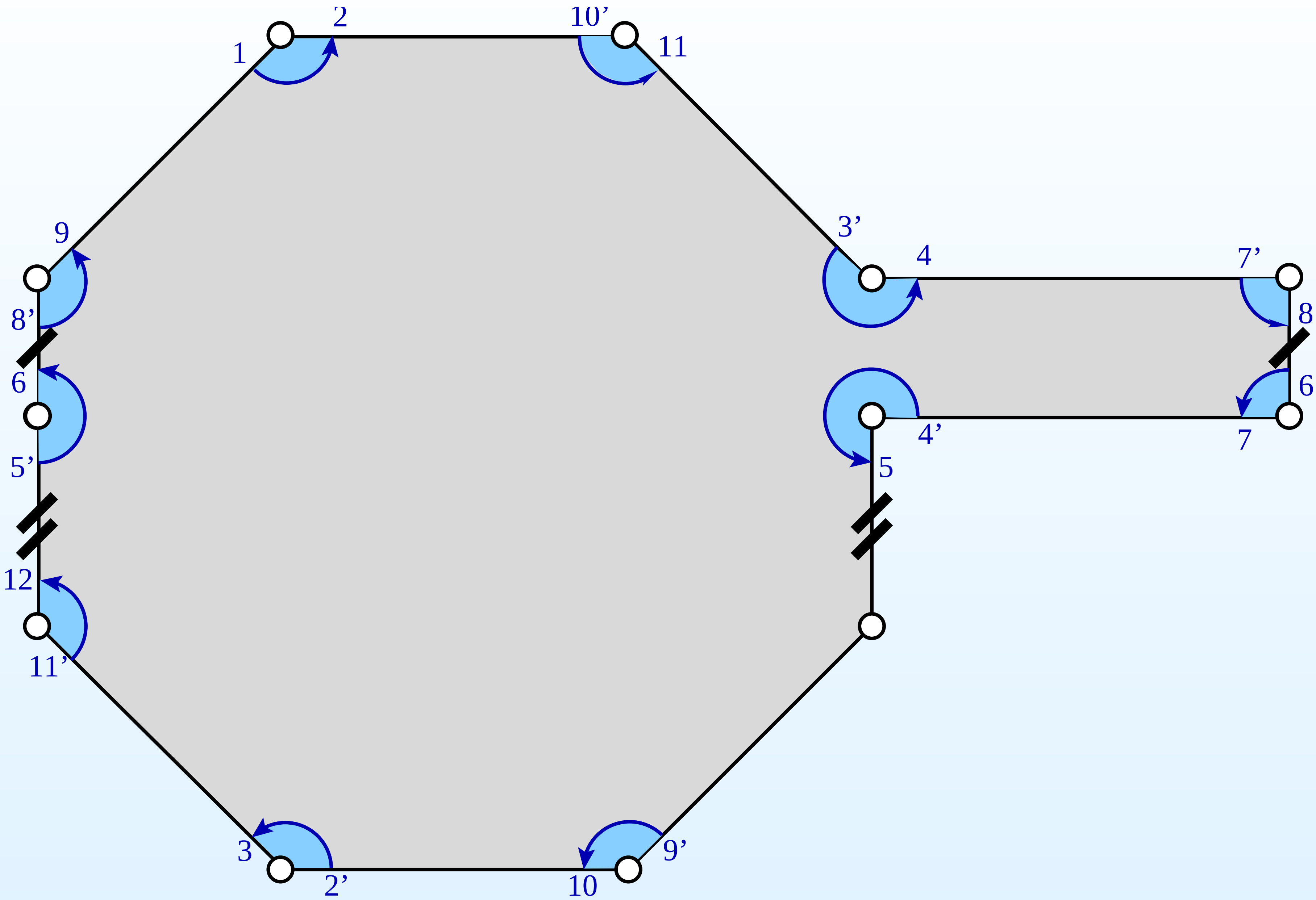

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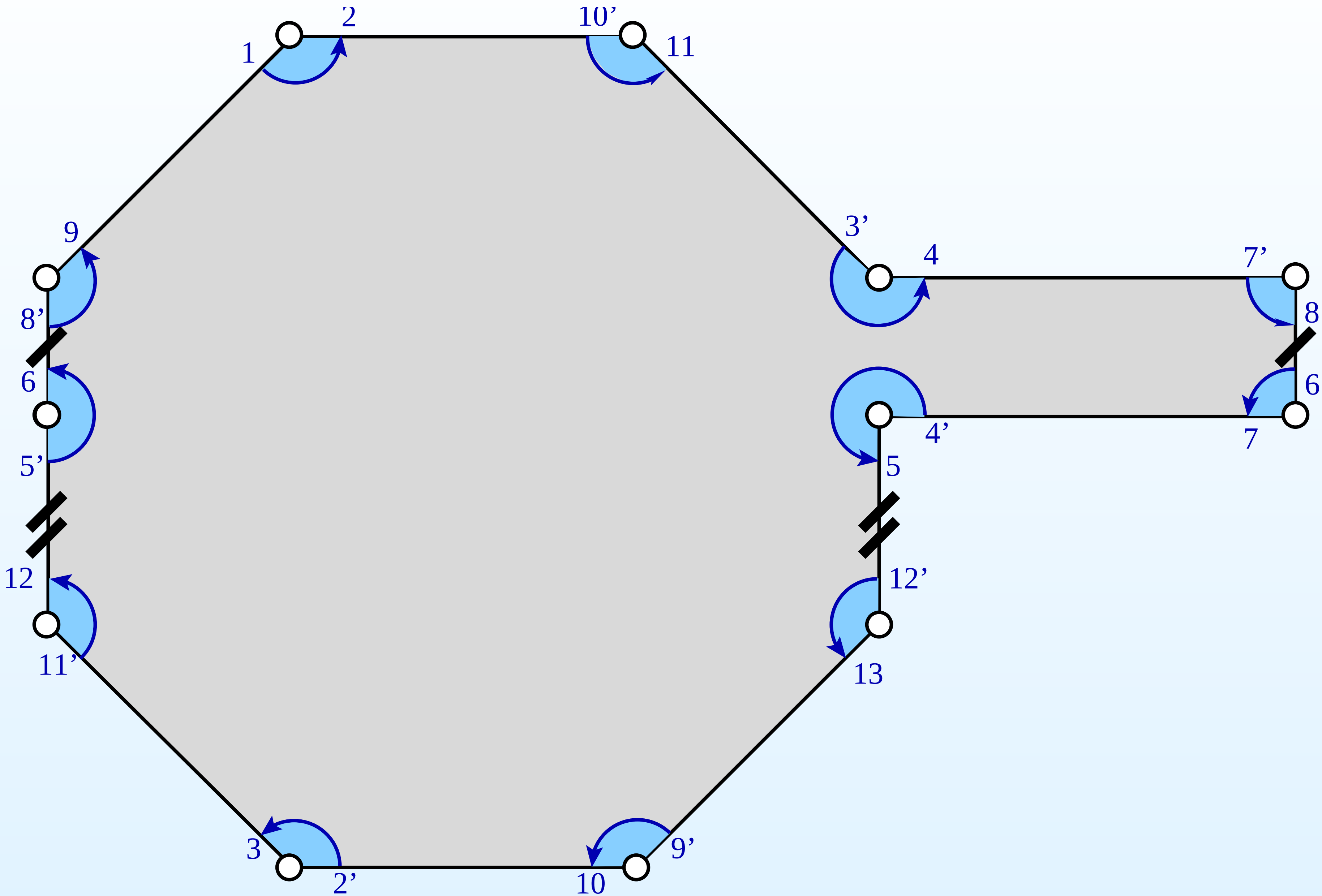

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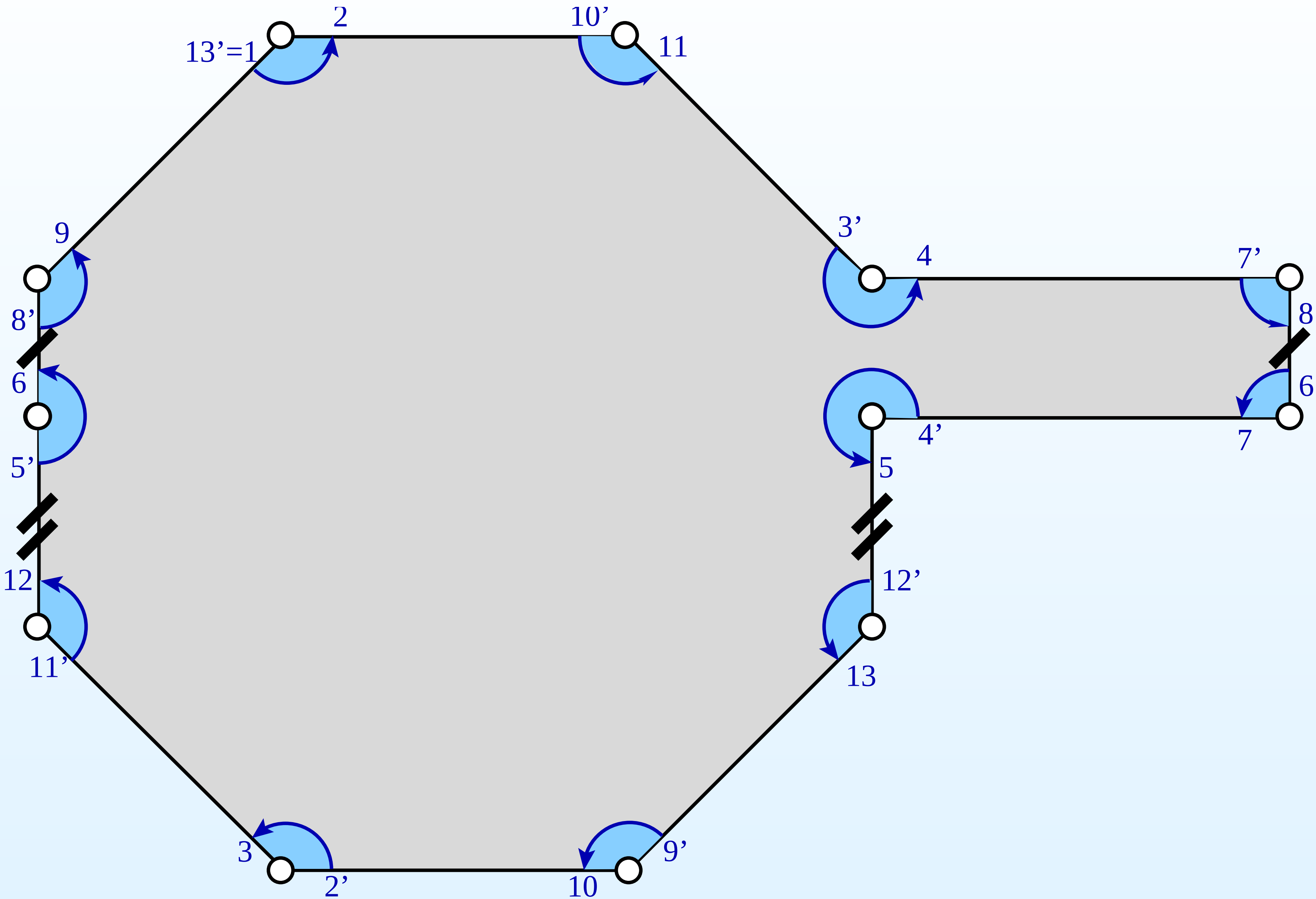
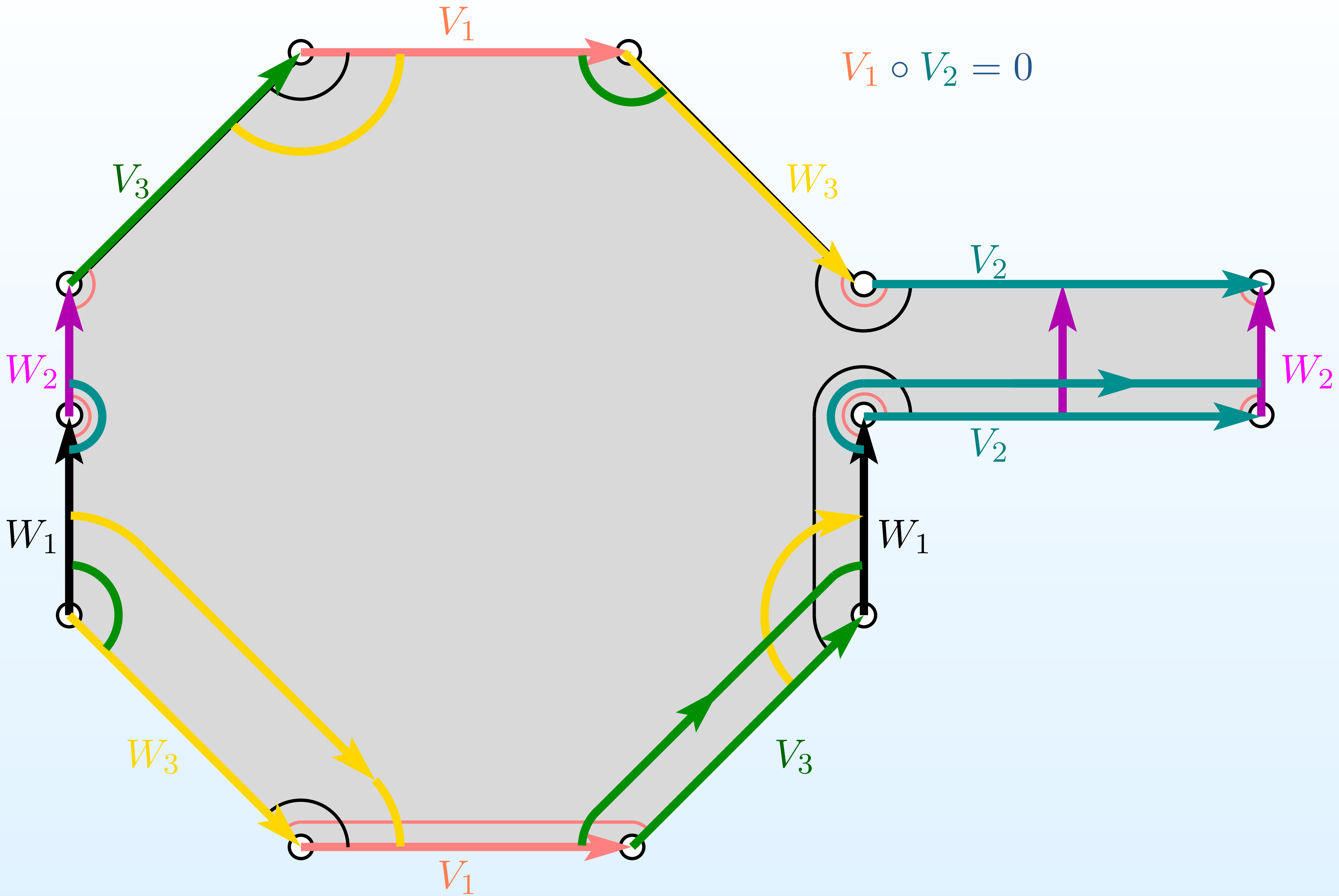


Computation of intersection numbers



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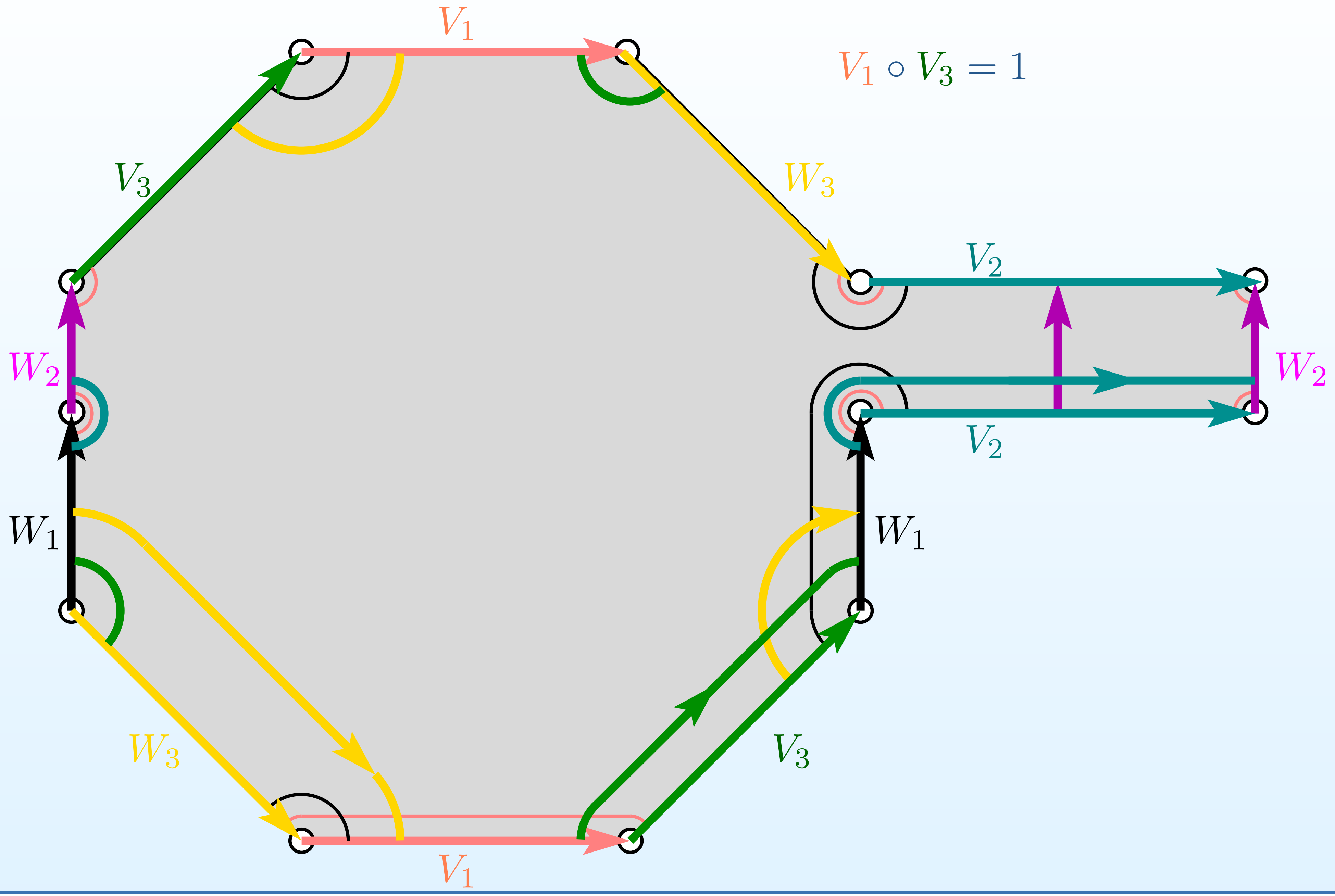

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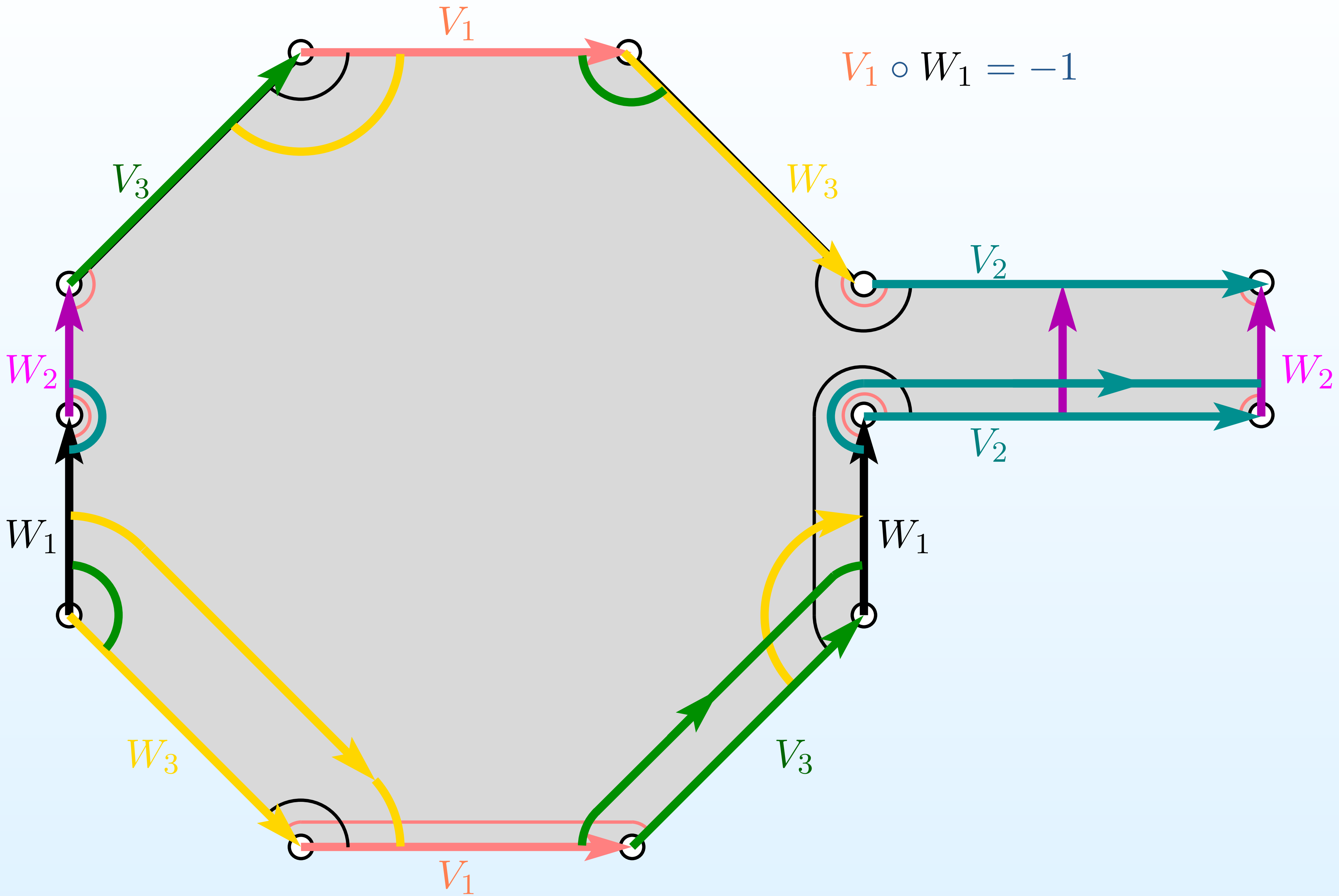

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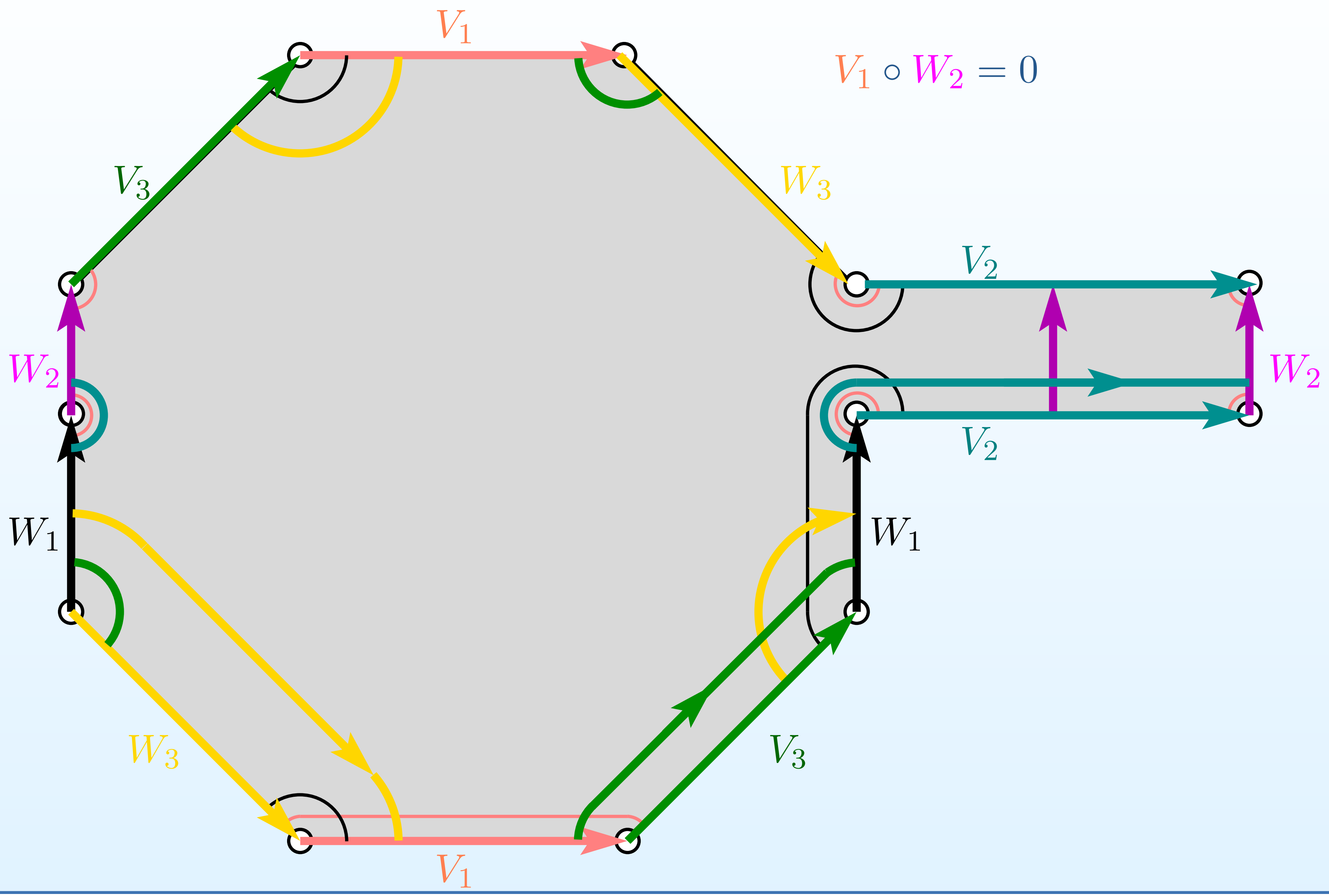
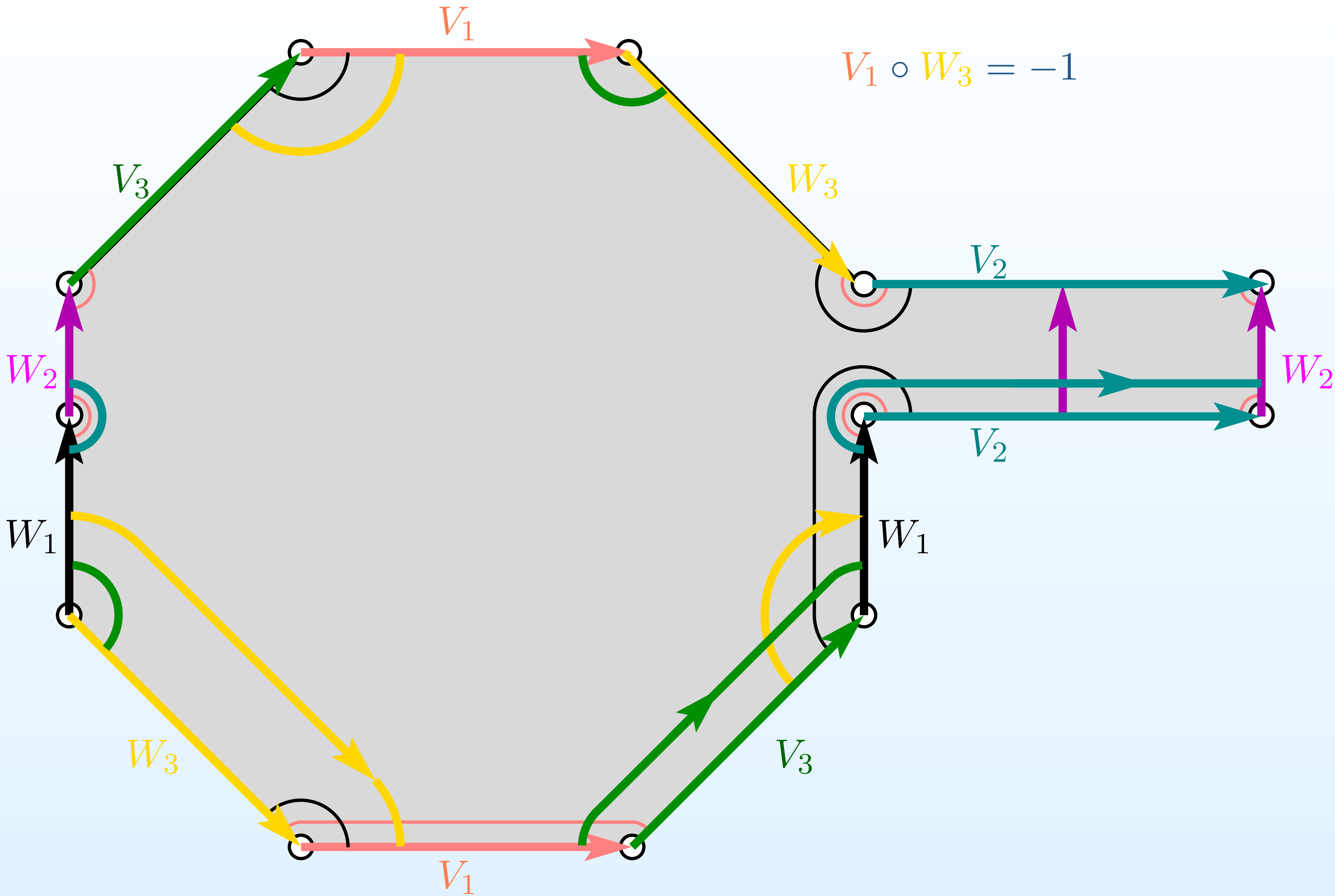


Computation of intersection numbers

$V_1 \circ W_3 = -1$



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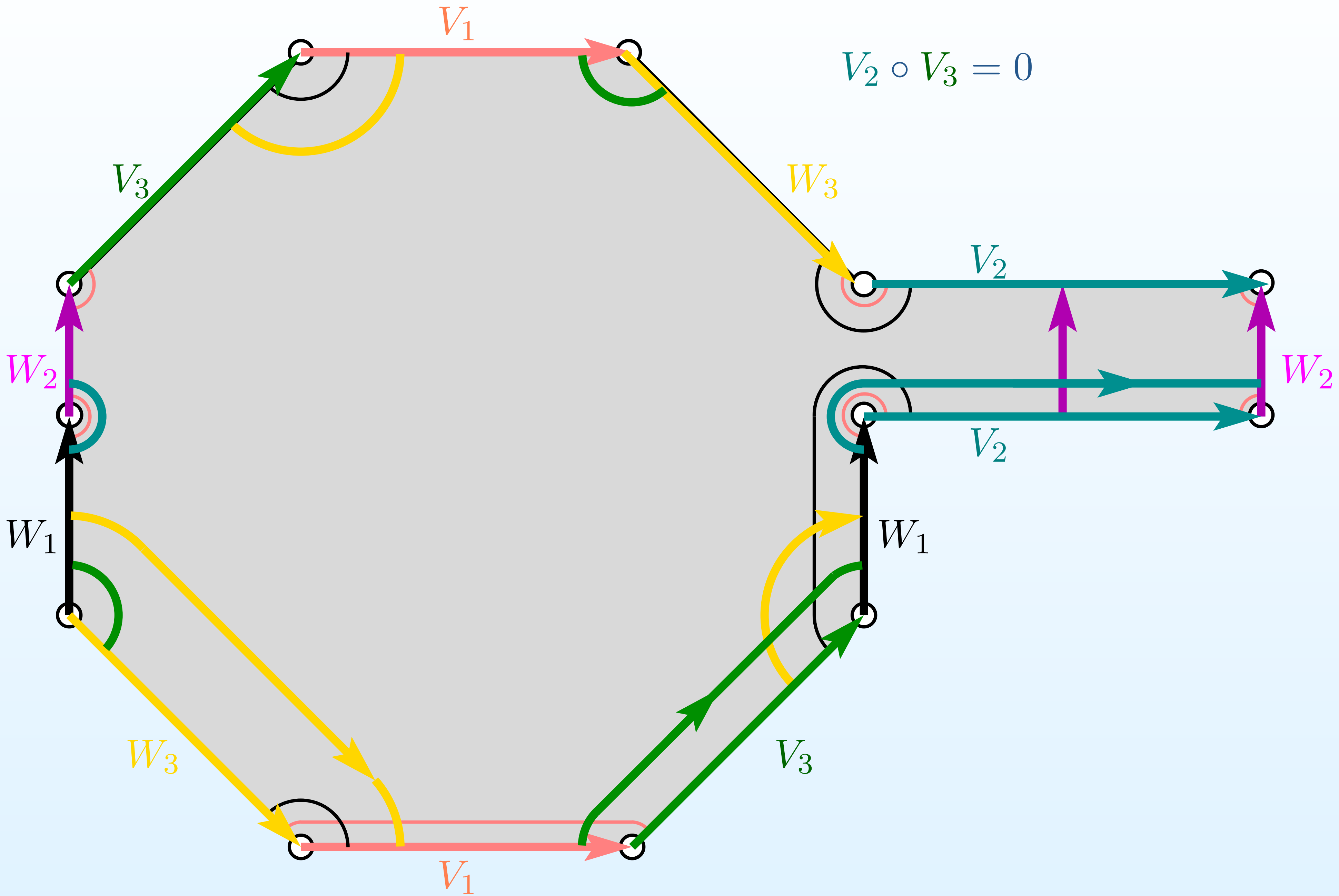

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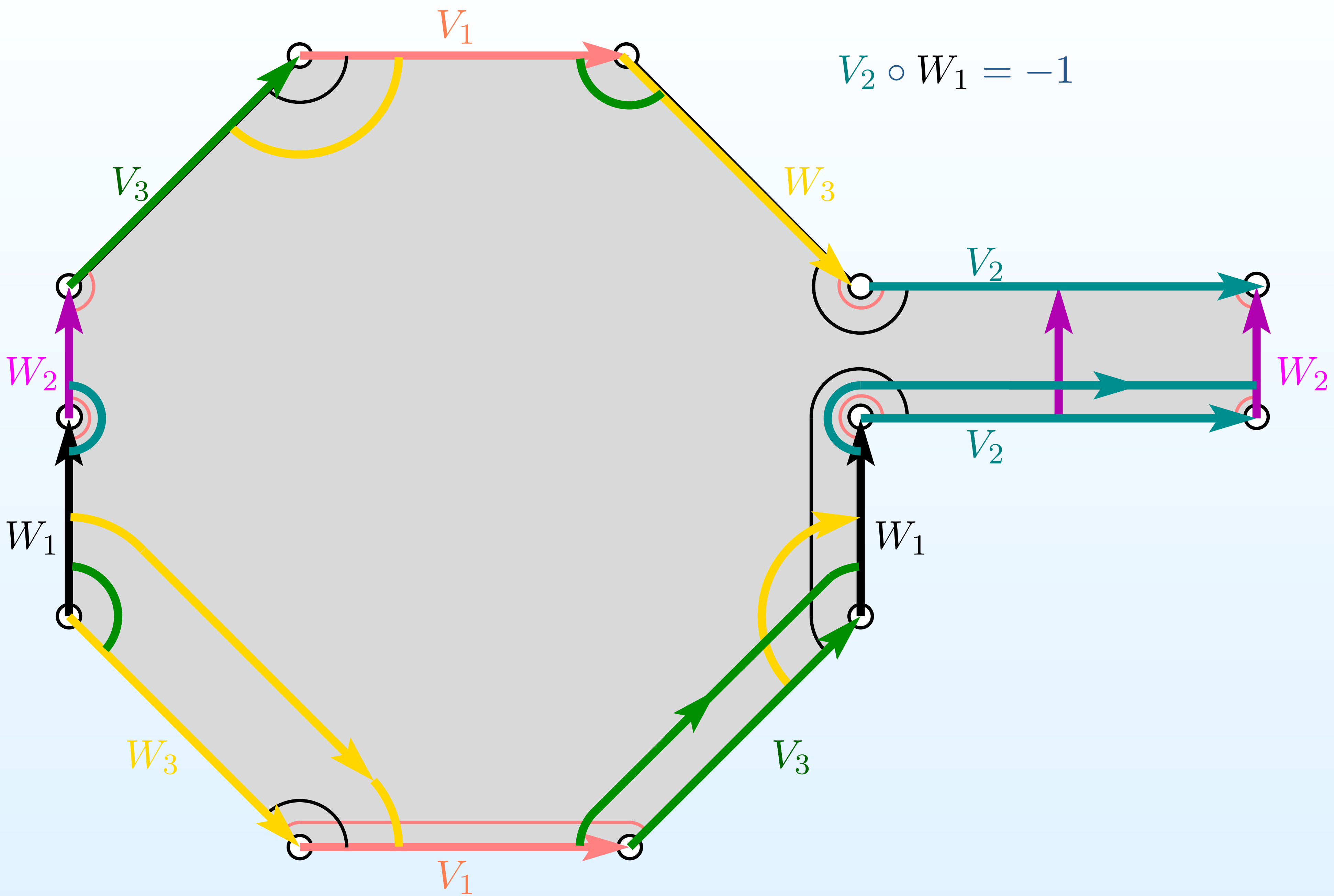

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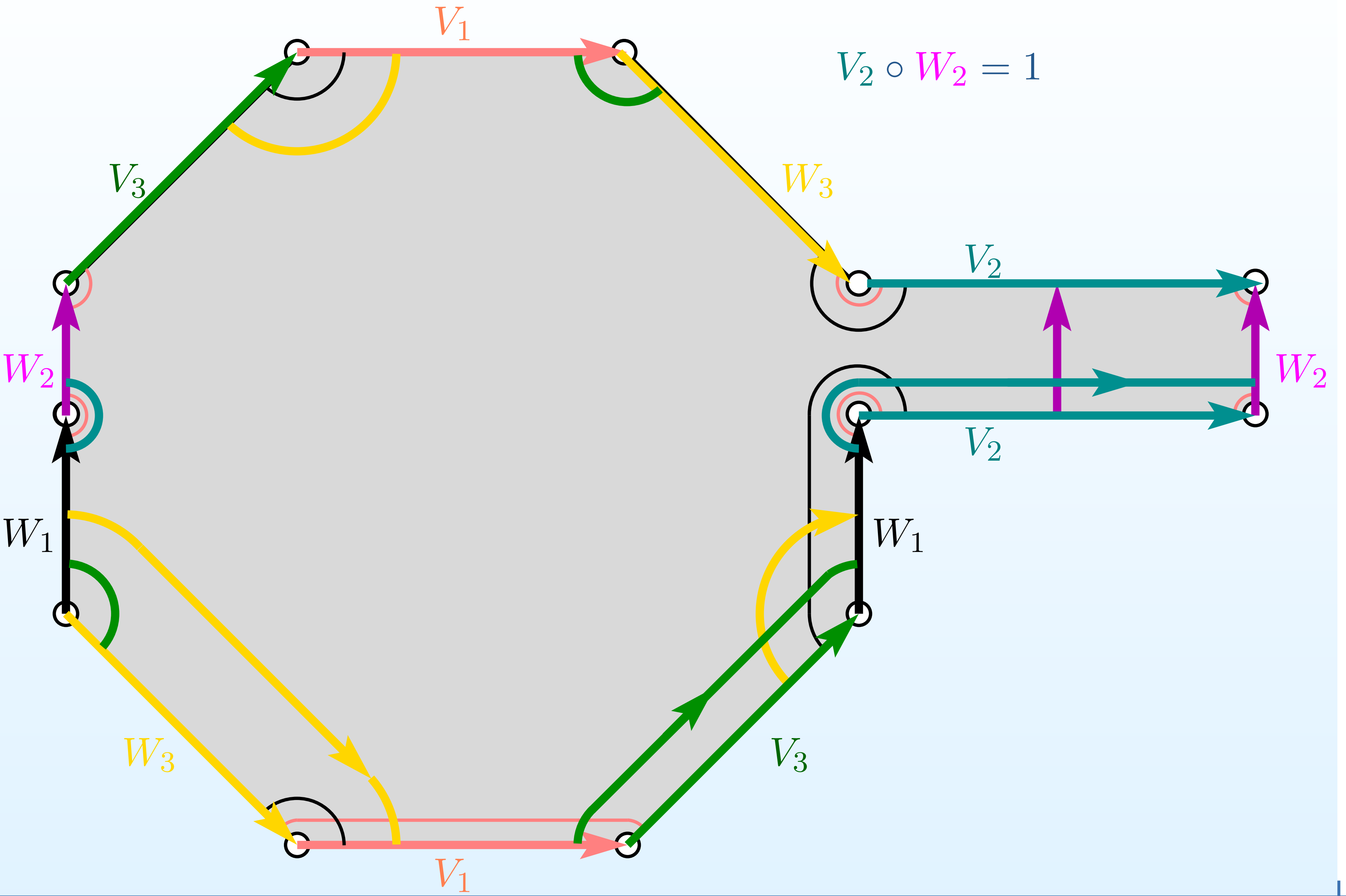

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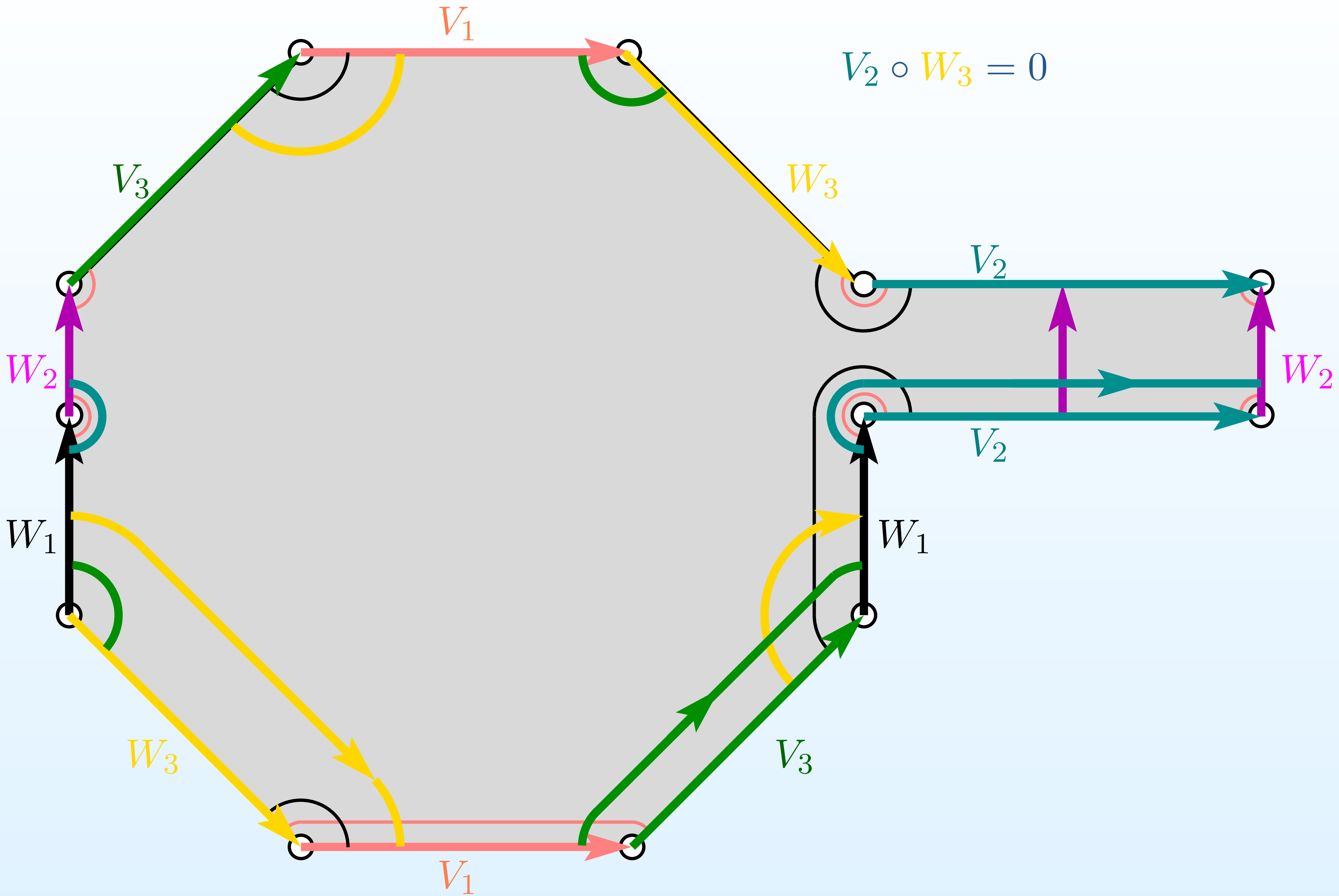

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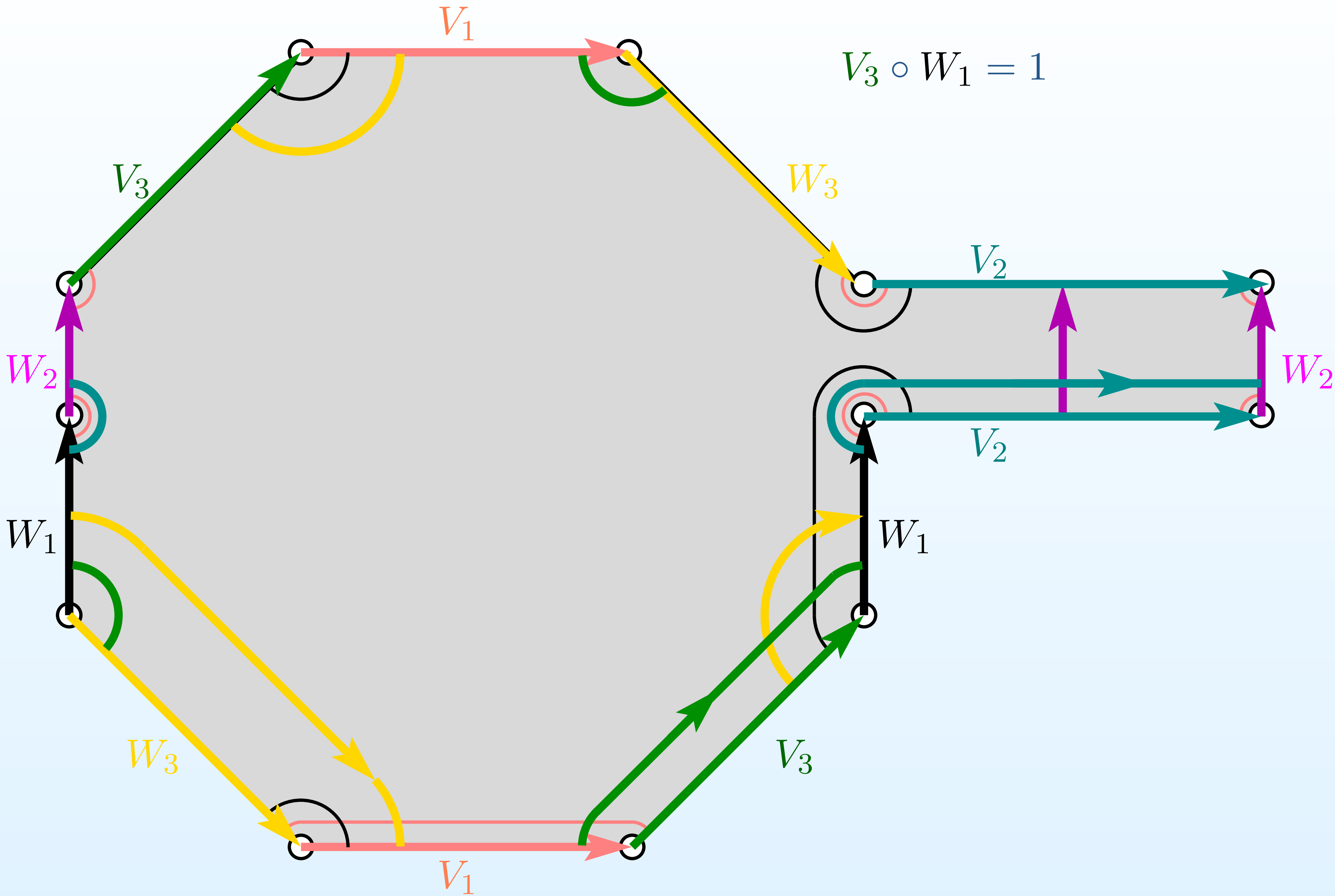

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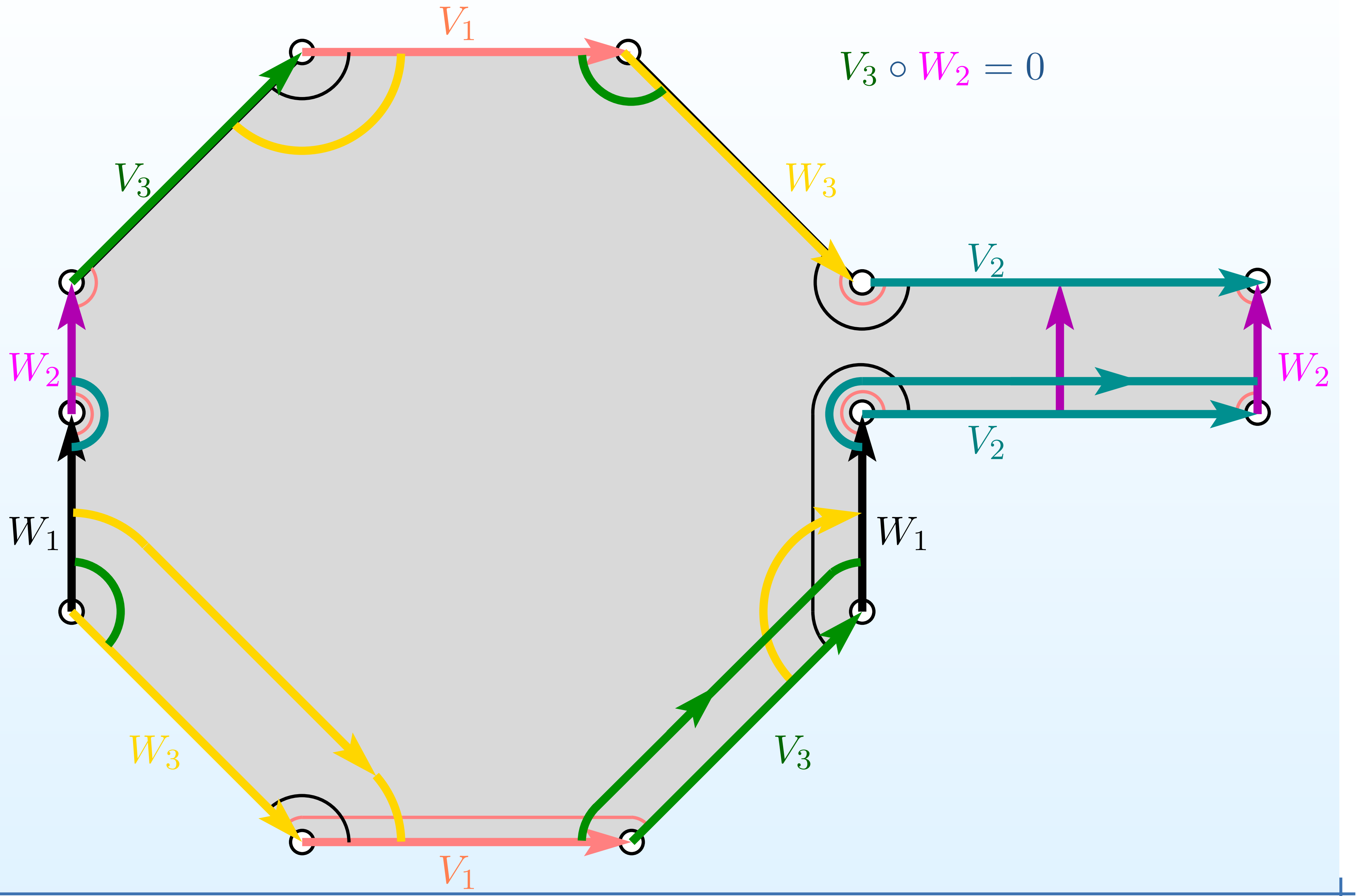

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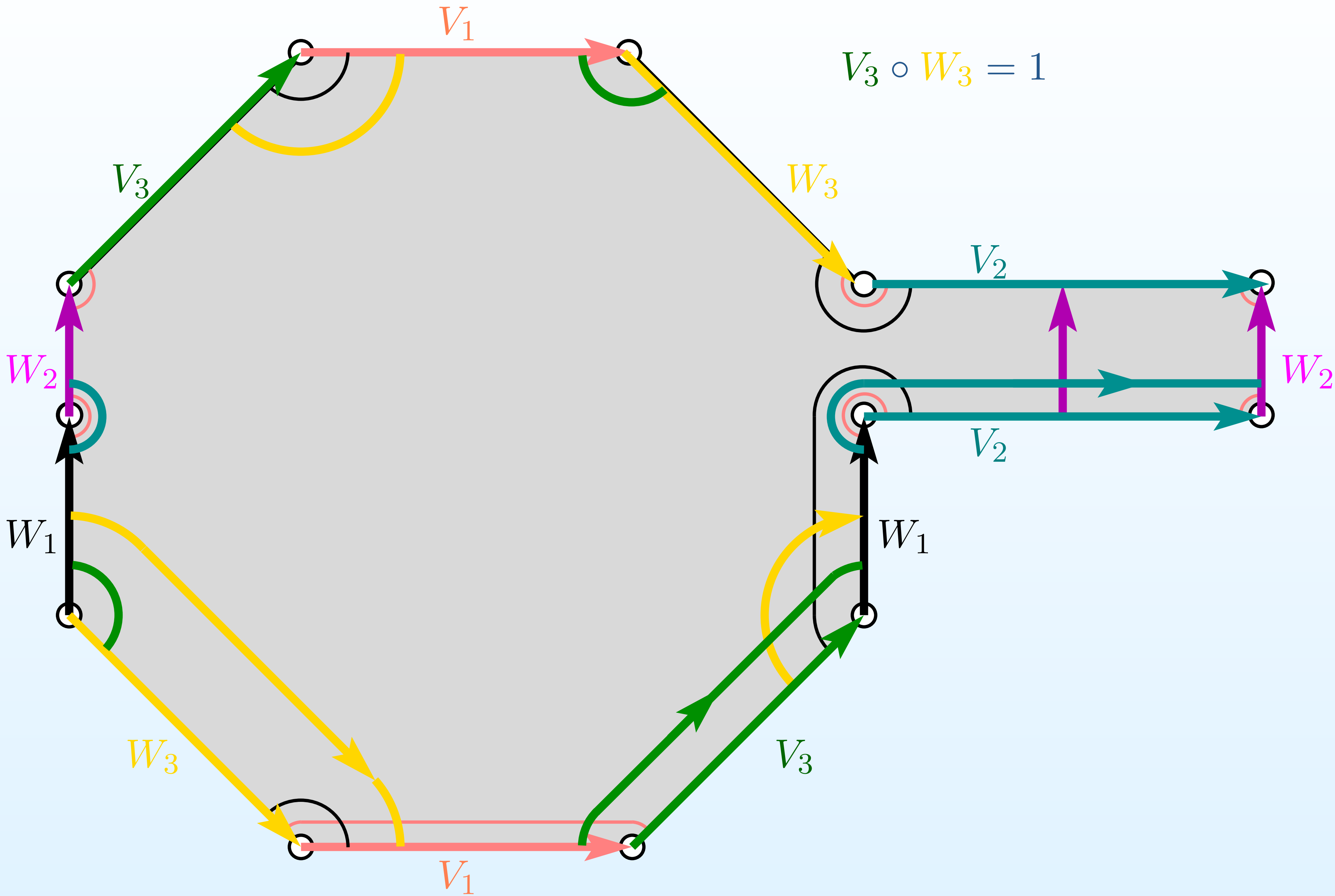

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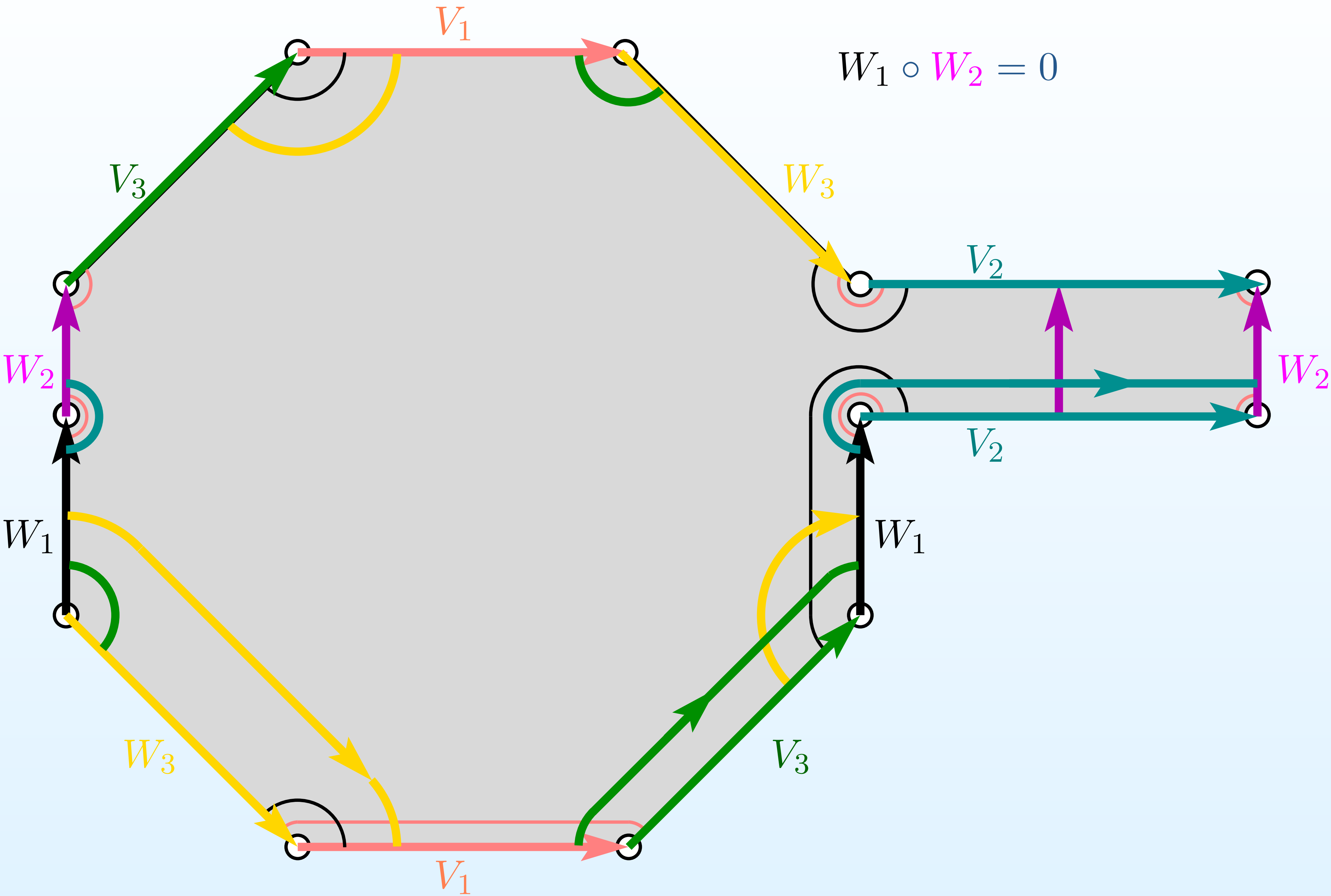

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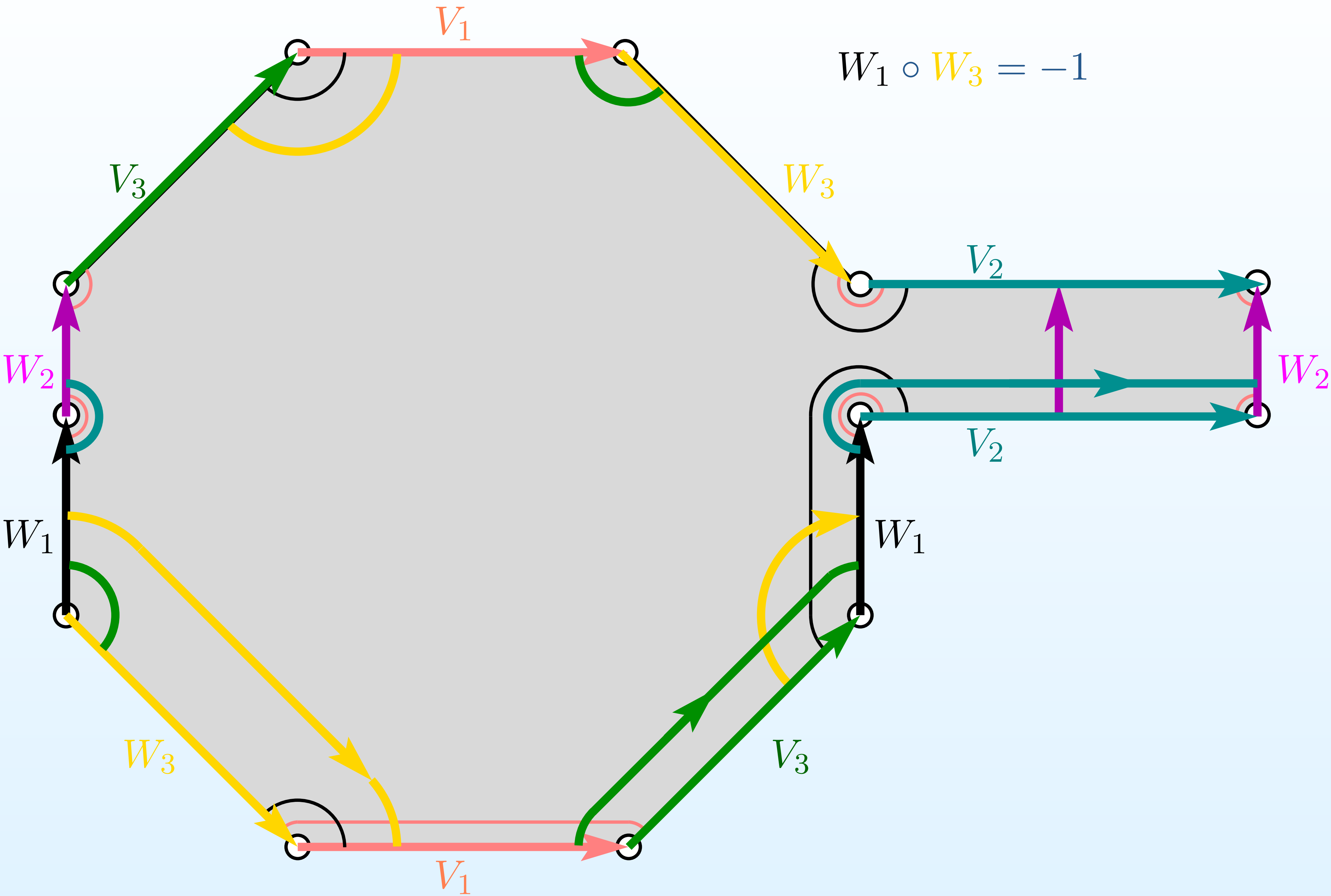
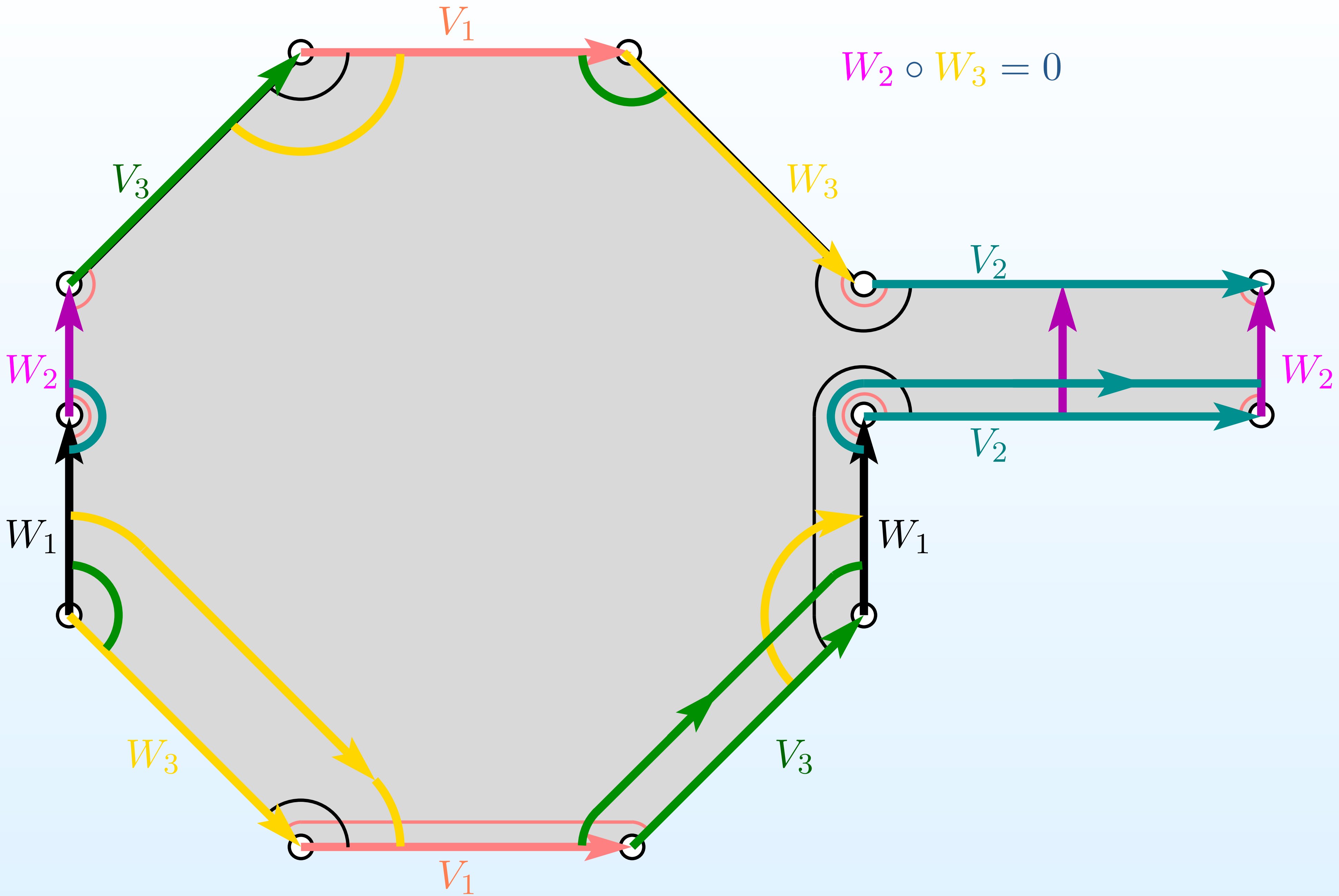


Computation of intersection numbers

$W_2 \circ W_3 = 0$



Computation of intersection numbers

Collecting our calculations we get the following matrix of intersection numbers:

	V_1	V_2	V_3	W_1	W_2	W_3
V_1	$*$	0	1	-1	0	-1
V_2	$*$	$*$	0	-1	1	0
V_3	$*$	$*$	$*$	1	0	1
W_1	$*$	$*$	$*$	$*$	0	-1
W_2	$*$	$*$	$*$	$*$	$*$	0
W_3	$*$	$*$	$*$	$*$	$*$	$*$

Since it is skew-symmetric, we can reconstruct the missing values.

Excercise. It is easy to check that this matrix is nondegenerate. Why this implies that the cycles form a basis? Verify that the cycles $a_1 = V_1 + V_3 + W_3$, $b_1 = V_3 + W_1 + W_2 - W_3$, $a_2 = V_2$, $b_2 = W_2$, $a_3 = V_3$, $b_3 = W_3$ form a *canonical basis of cycles*, that is

$$a_i \circ a_j = b_i \circ b_j = 0; \qquad a_i \circ b_j = \delta_{ij} \, .$$

Computation of intersection numbers

Collecting our calculations we get the following matrix of intersection numbers:

	V_1	V_2	V_3	W_1	W_2	W_3
V_1	0	0	1	−1	0	−1
V_2	0	0	0	−1	1	0
V_3	1	0	0	1	0	1
W_1	1	1	−1	0	0	−1
W_2	0	−1	0	0	0	0
W_3	1	0	−1	1	0	0

Since it is skew-symmetric, we can reconstruct the missing values.

Excercise. It is easy to check that this matrix is nondegenerate. Why this implies that the cycles form a basis? Verify that the cycles $a_1 = V_1 + V_3 + W_3$, $b_1 = V_3 + W_1 + W_2 - W_3$, $a_2 = V_2$, $b_2 = W_2$, $a_3 = V_3$, $b_3 = W_3$ form a *canonical basis of cycles*, that is

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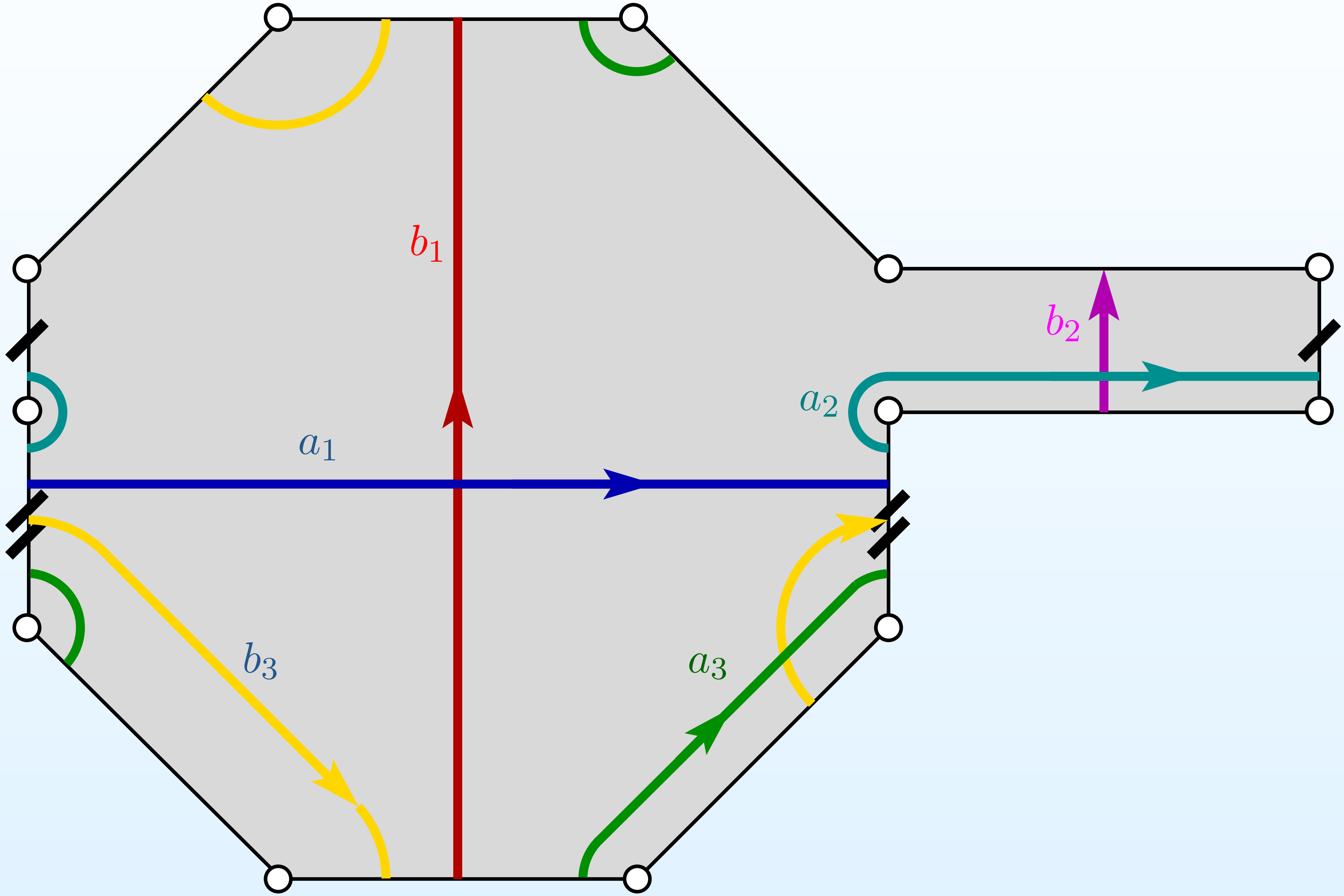
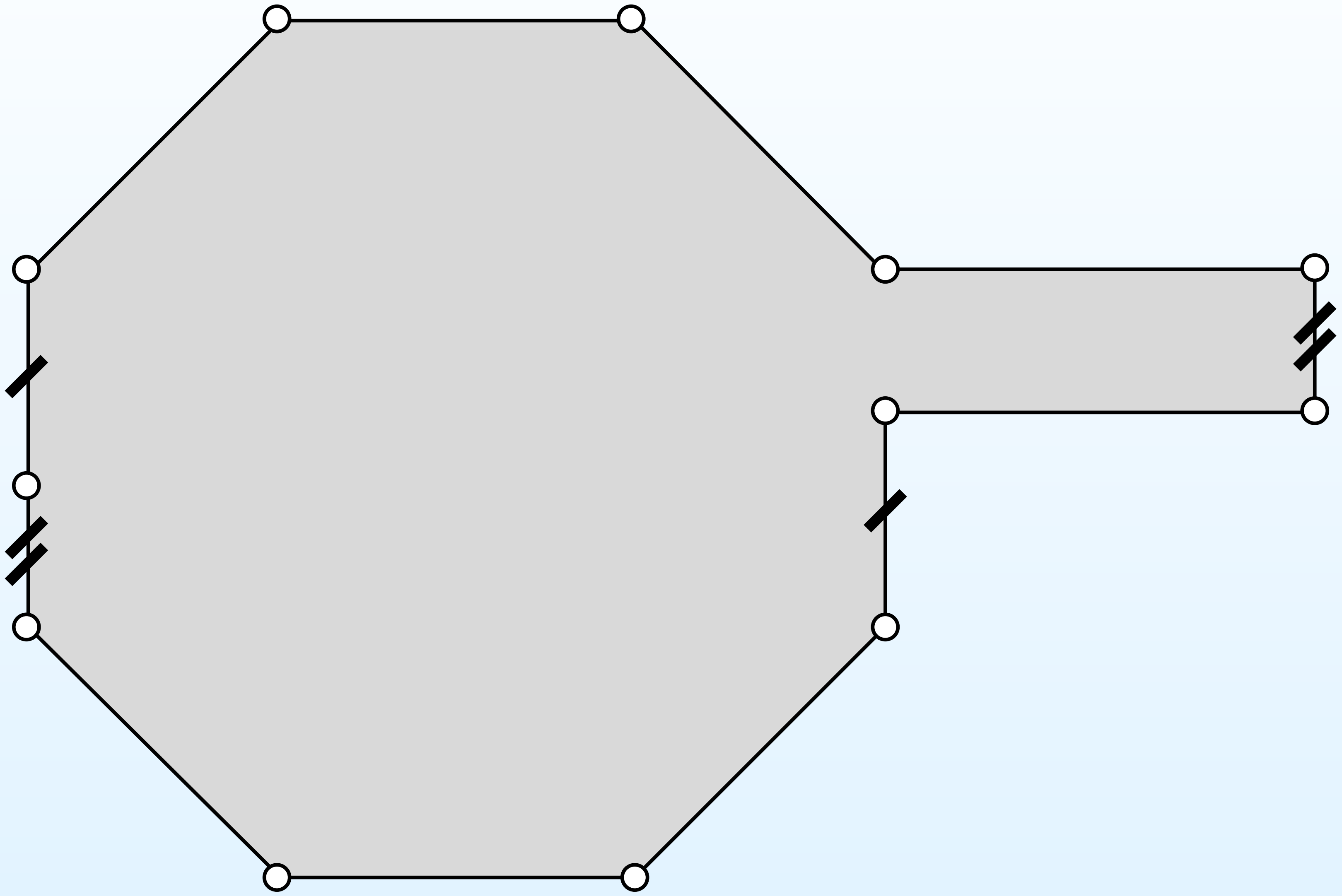
Canonical basis of cycles



Excercise. Find an involution of the surface below in geometric terms. Check that there are $2g + 2 = 2 \cdot 3 + 2 = 8$ such points. Determine the genus of the quotient. Recognize a hyperelliptic involution. Find the Weierstrass points.



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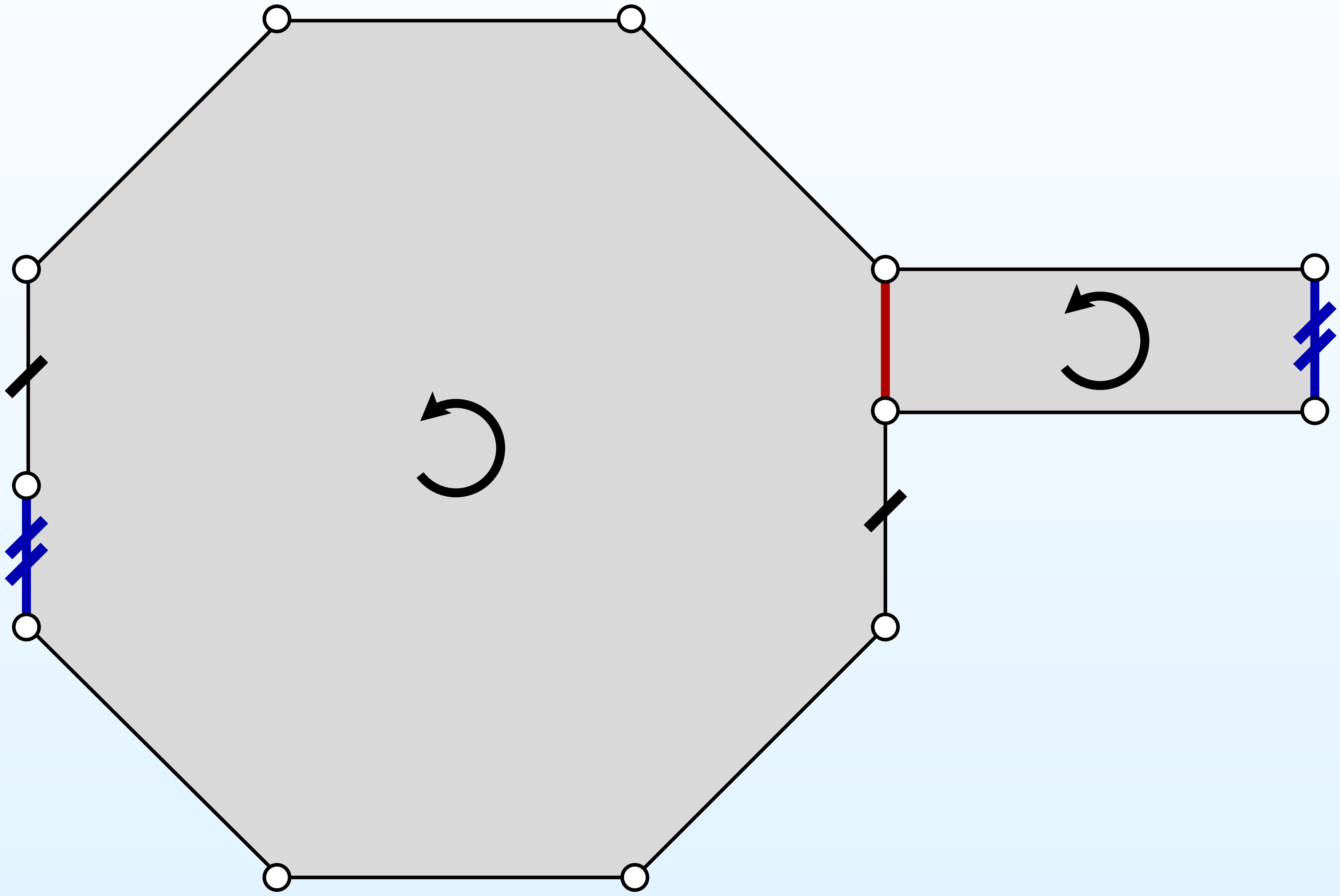

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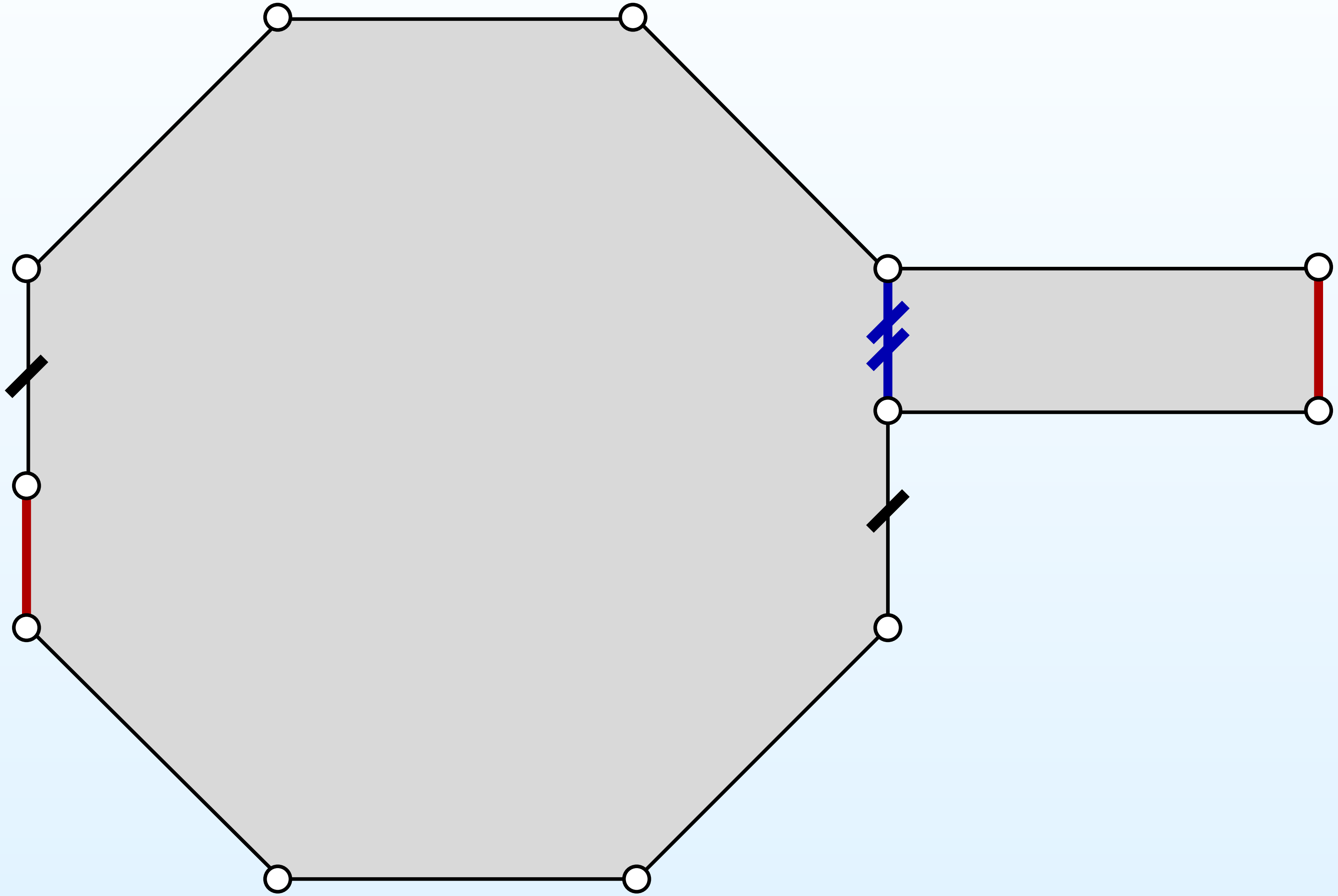

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