Dynamics and Geometry of Moduli Spaces

Lecture 7. Train tracks. Integral measured laminations. Idea of the proof of Mirzakhani's count of simple closed geodesics

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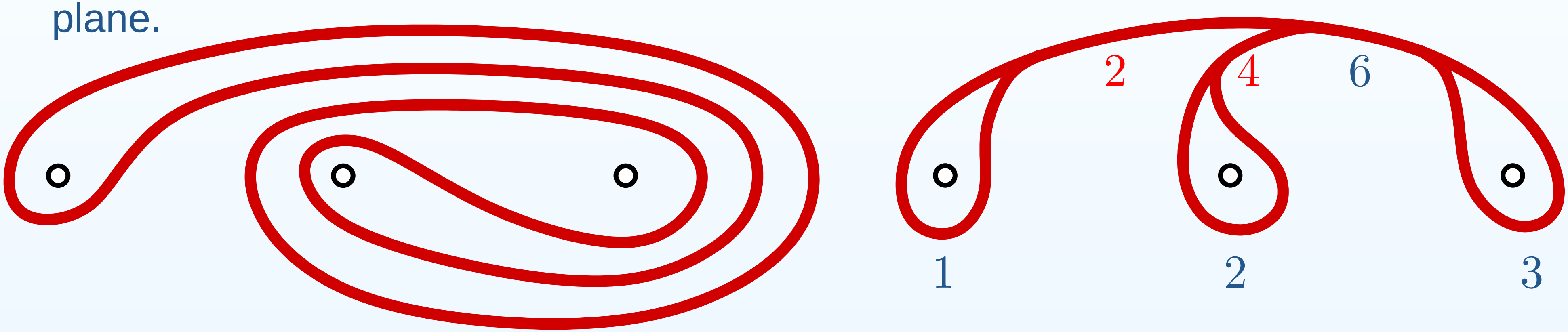
University Paris Cité

March 26, 2024

Space of multicurves

Train tracks carrying simple closed curves

Working with simple closed curves it is convenient to encode them (following Thurston) by *train tracks*. Following Farb and Margalit we consider the model case of four-punctured sphere $S_{0,4}$ which we represent as a three-punctured plane.



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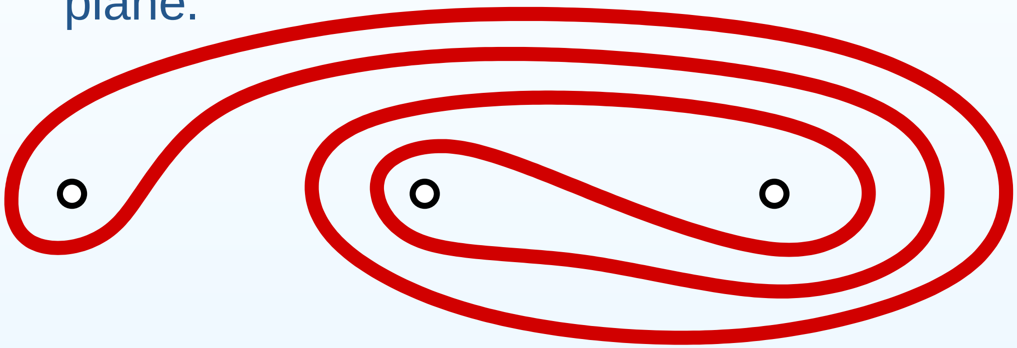

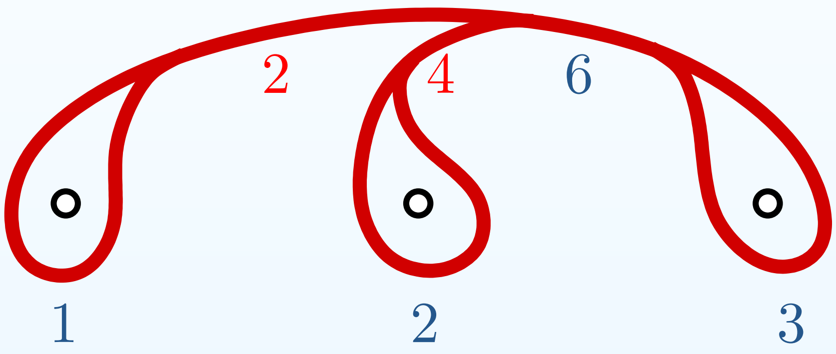

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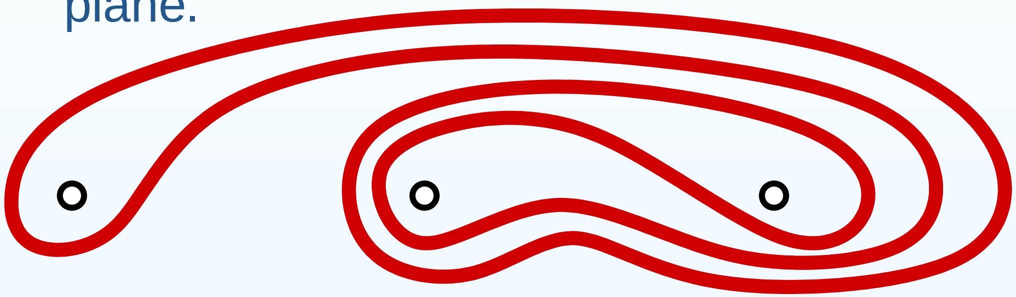

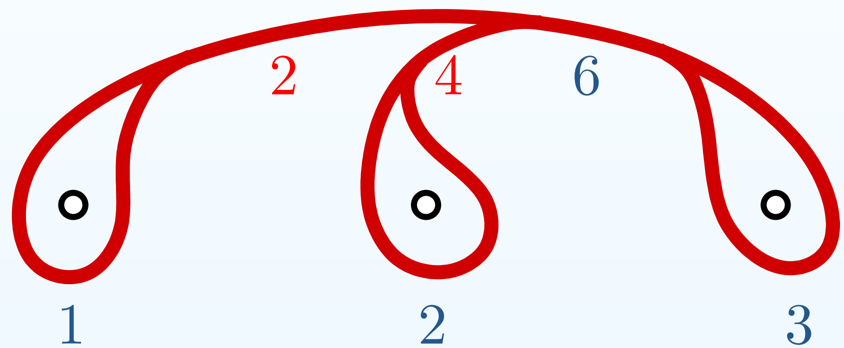

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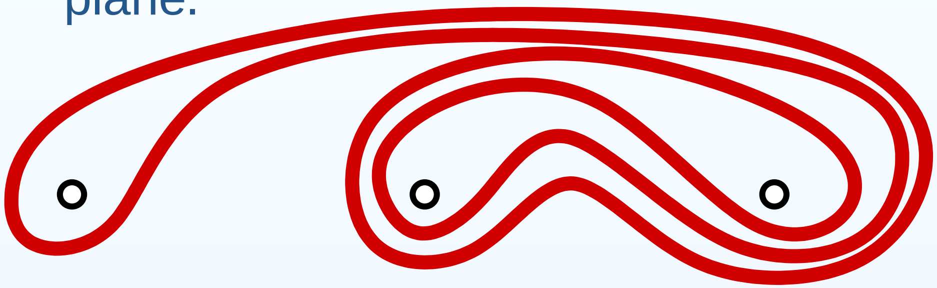

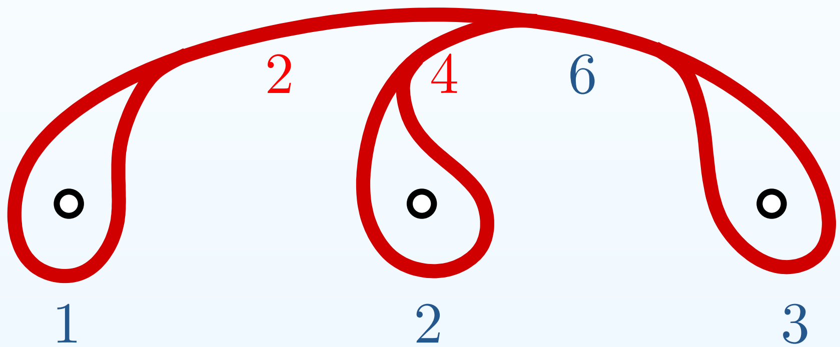

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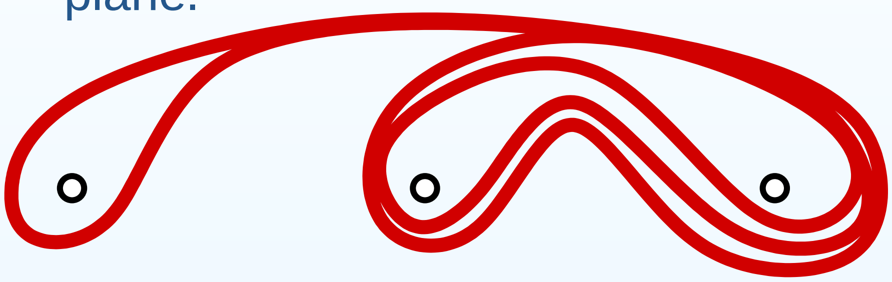

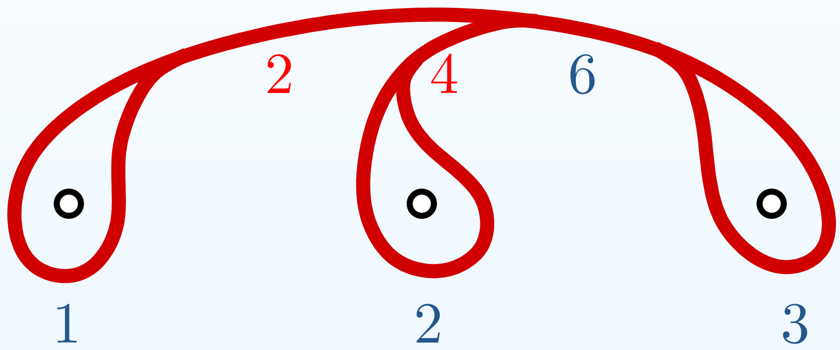

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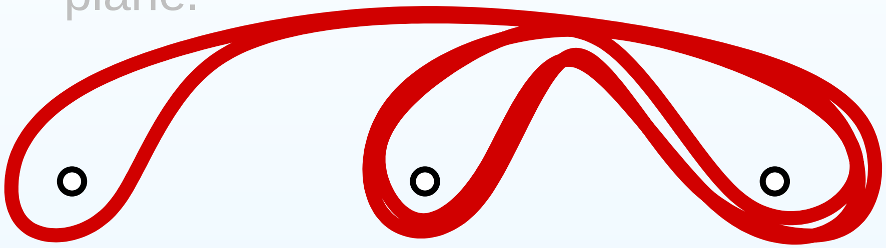

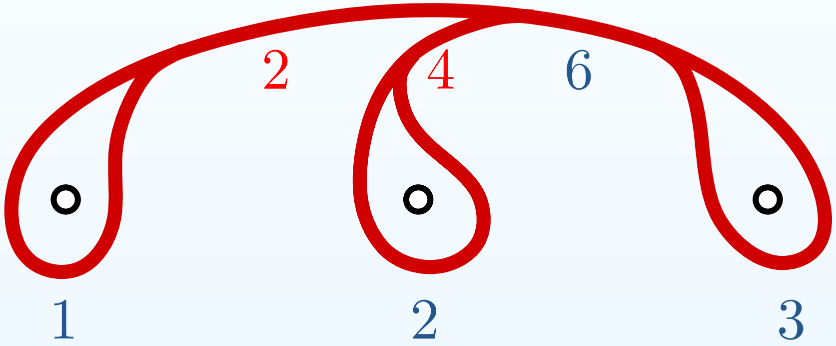

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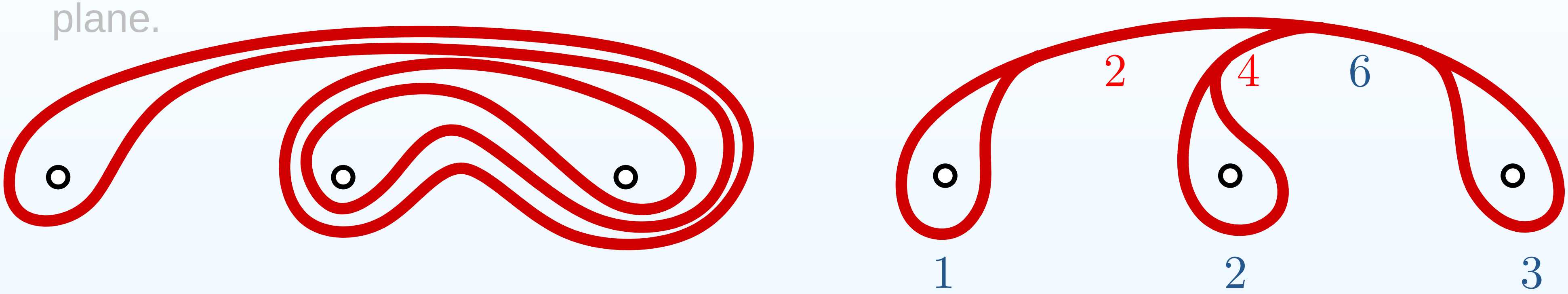

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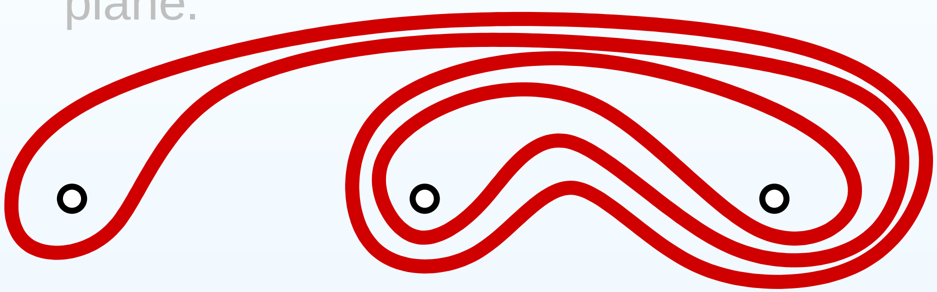

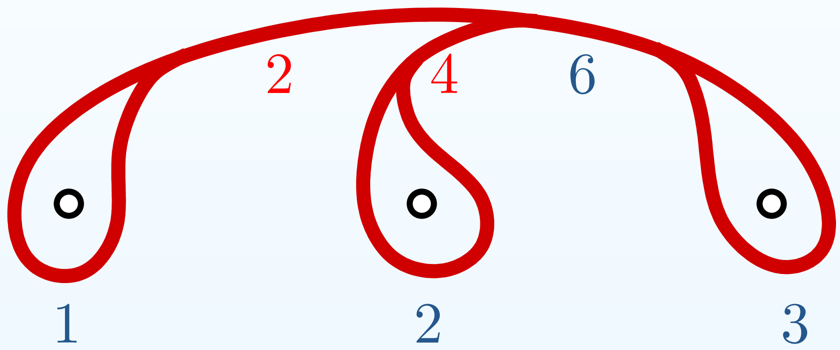

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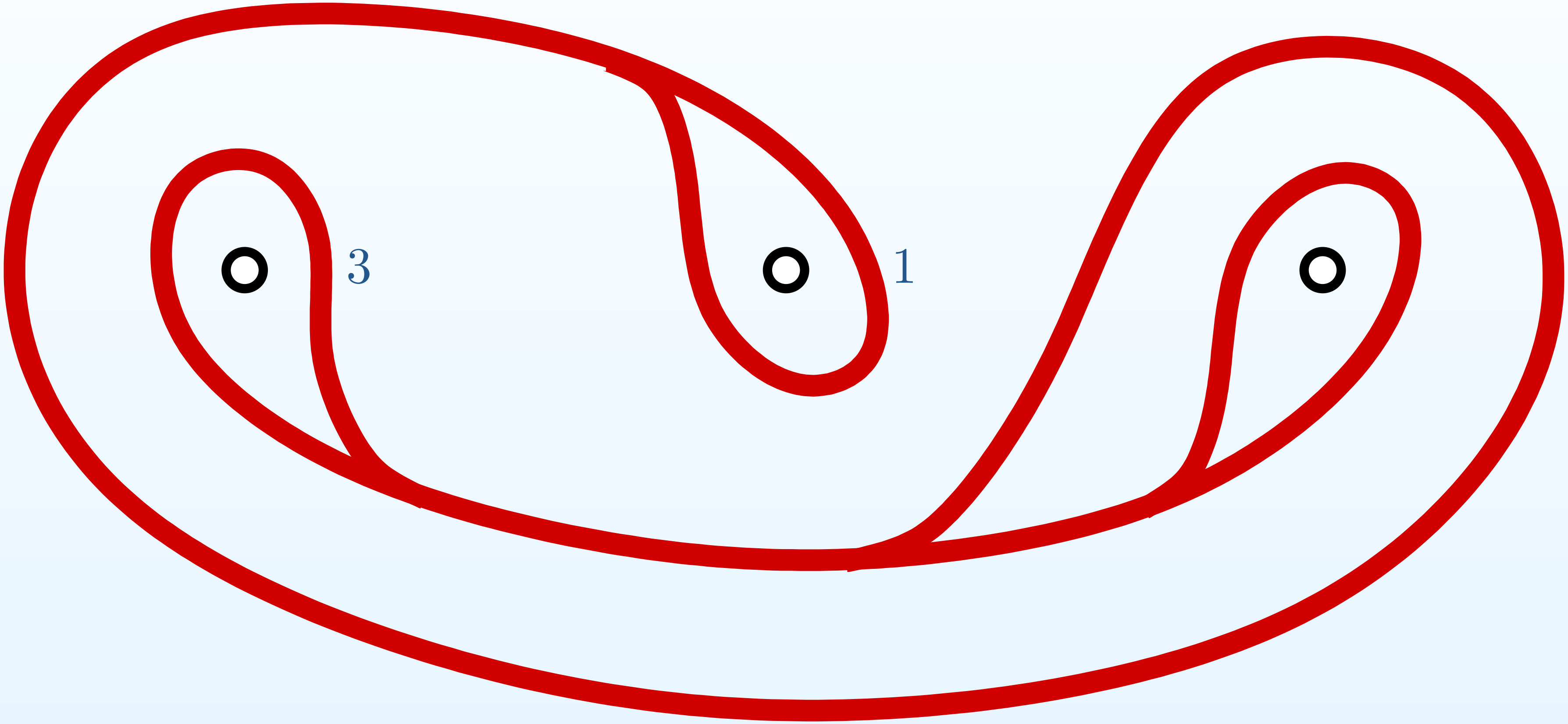
Zipping and unzipping train tracks



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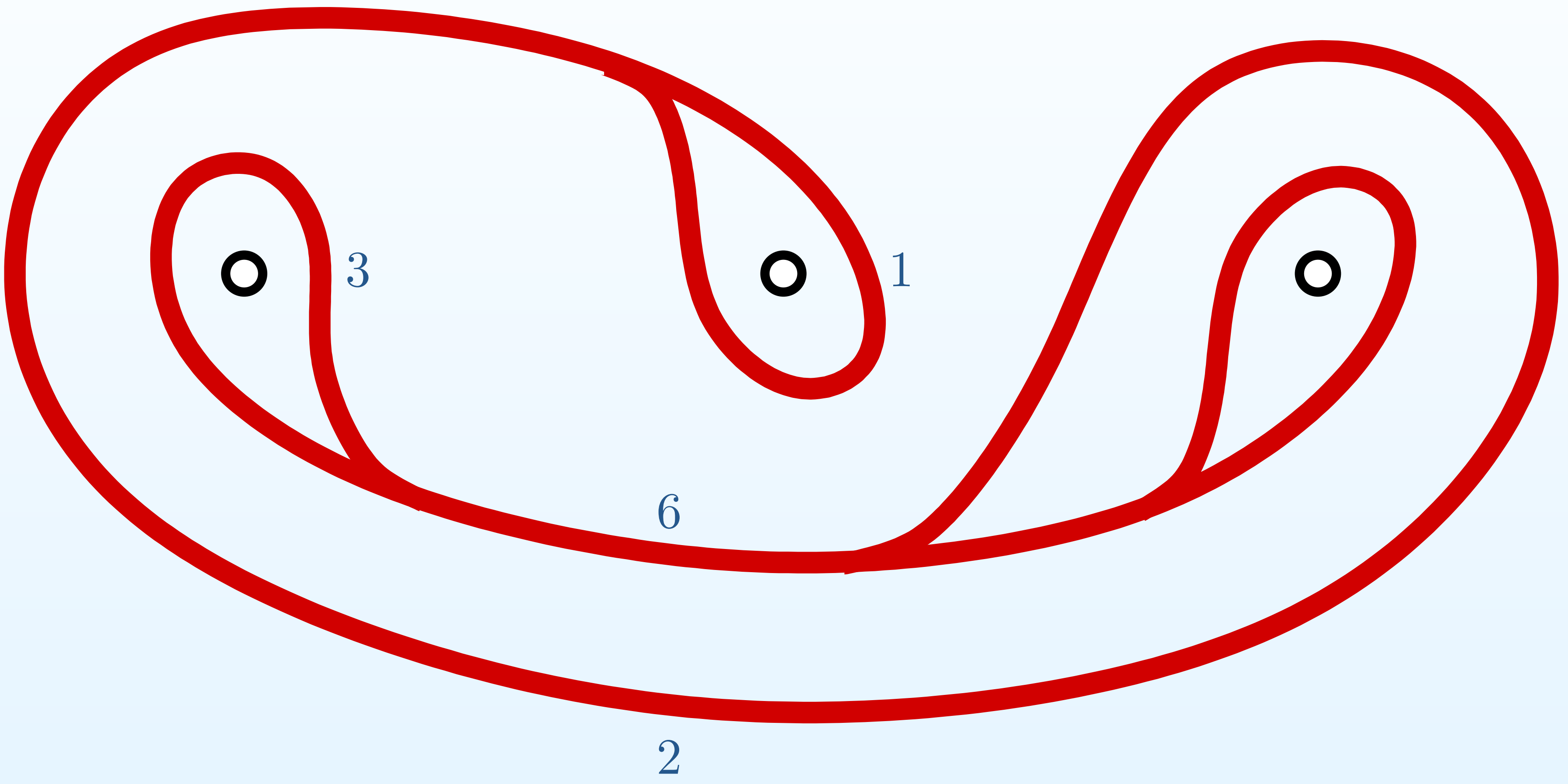

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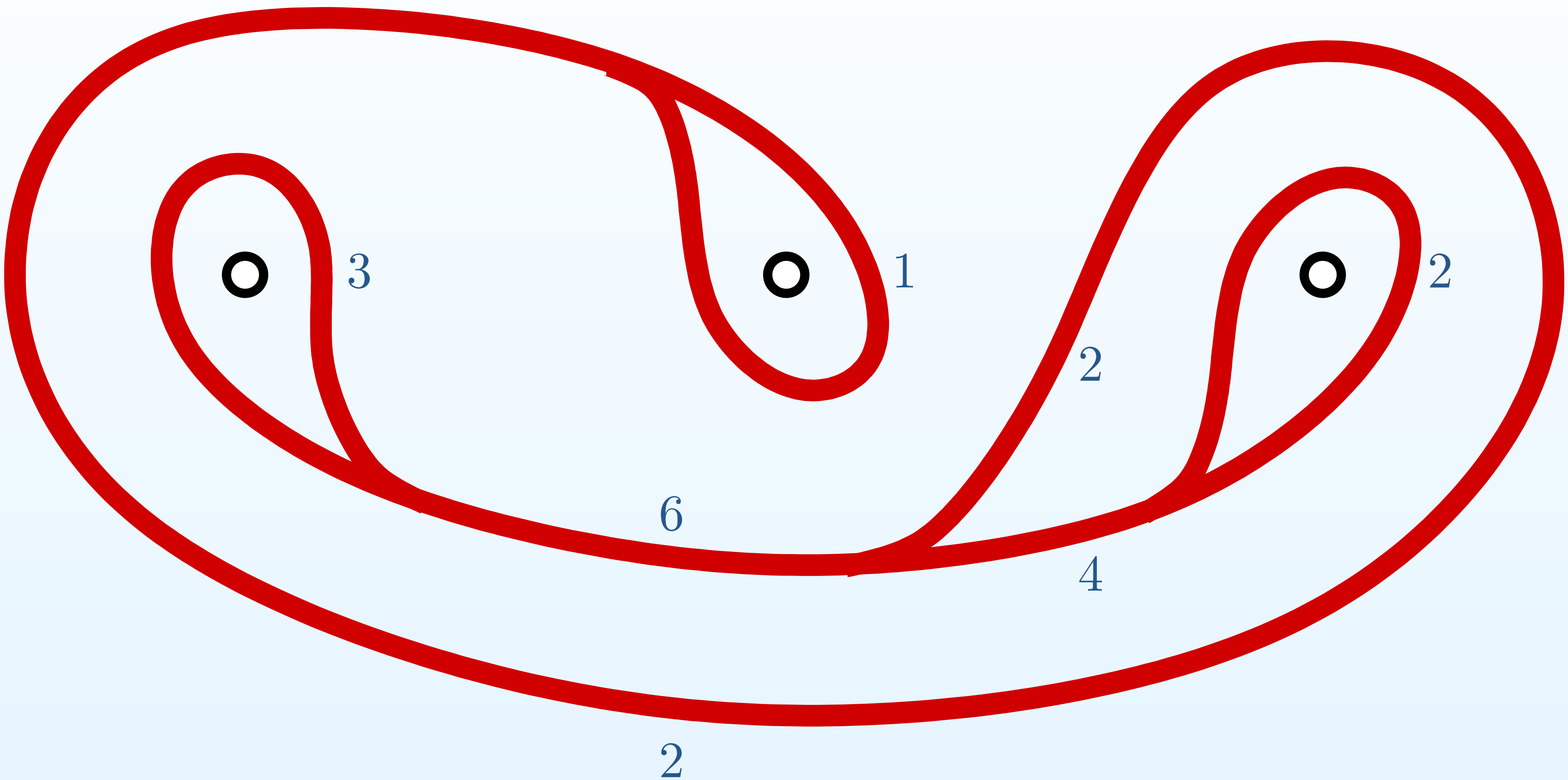

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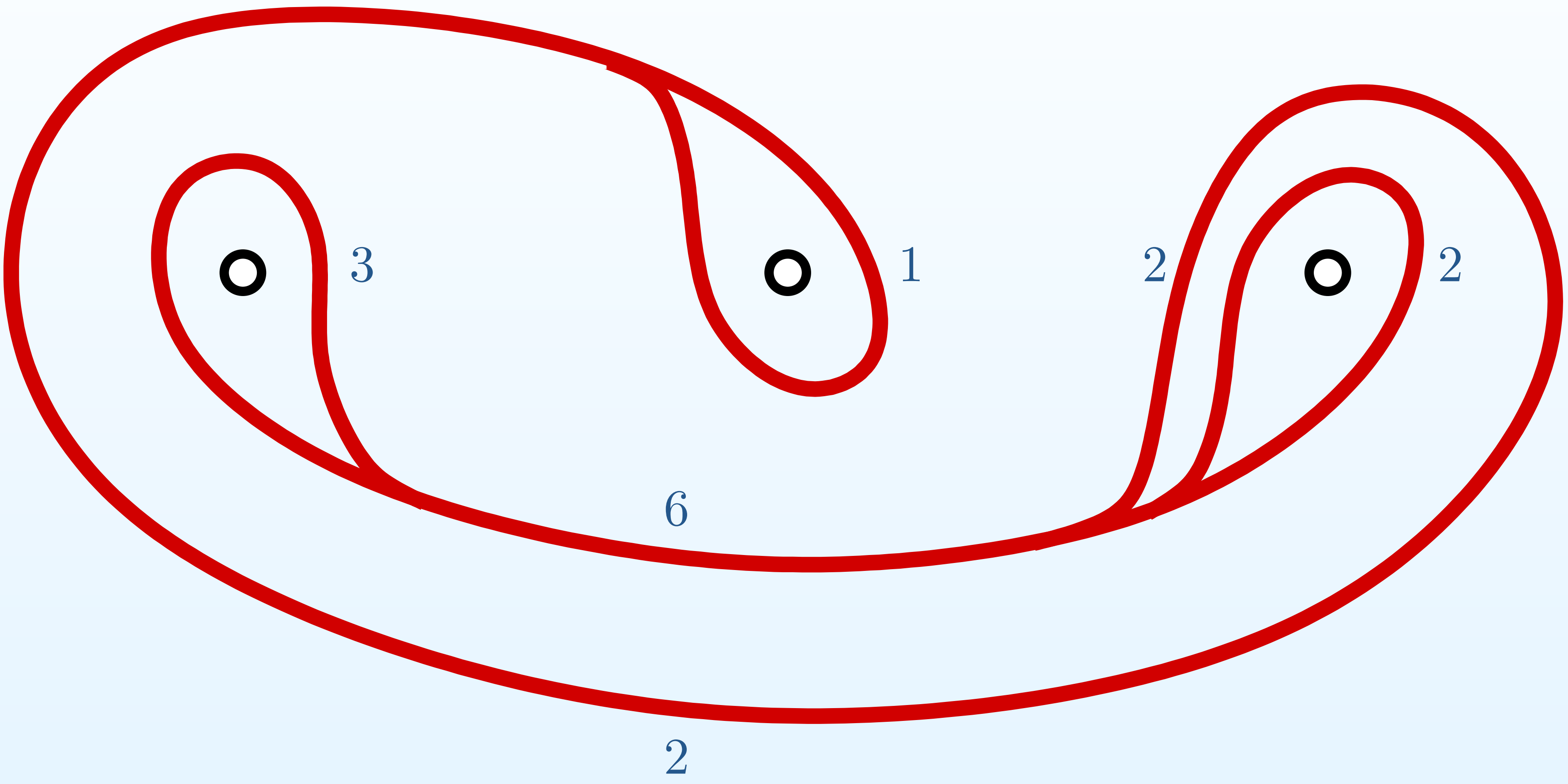

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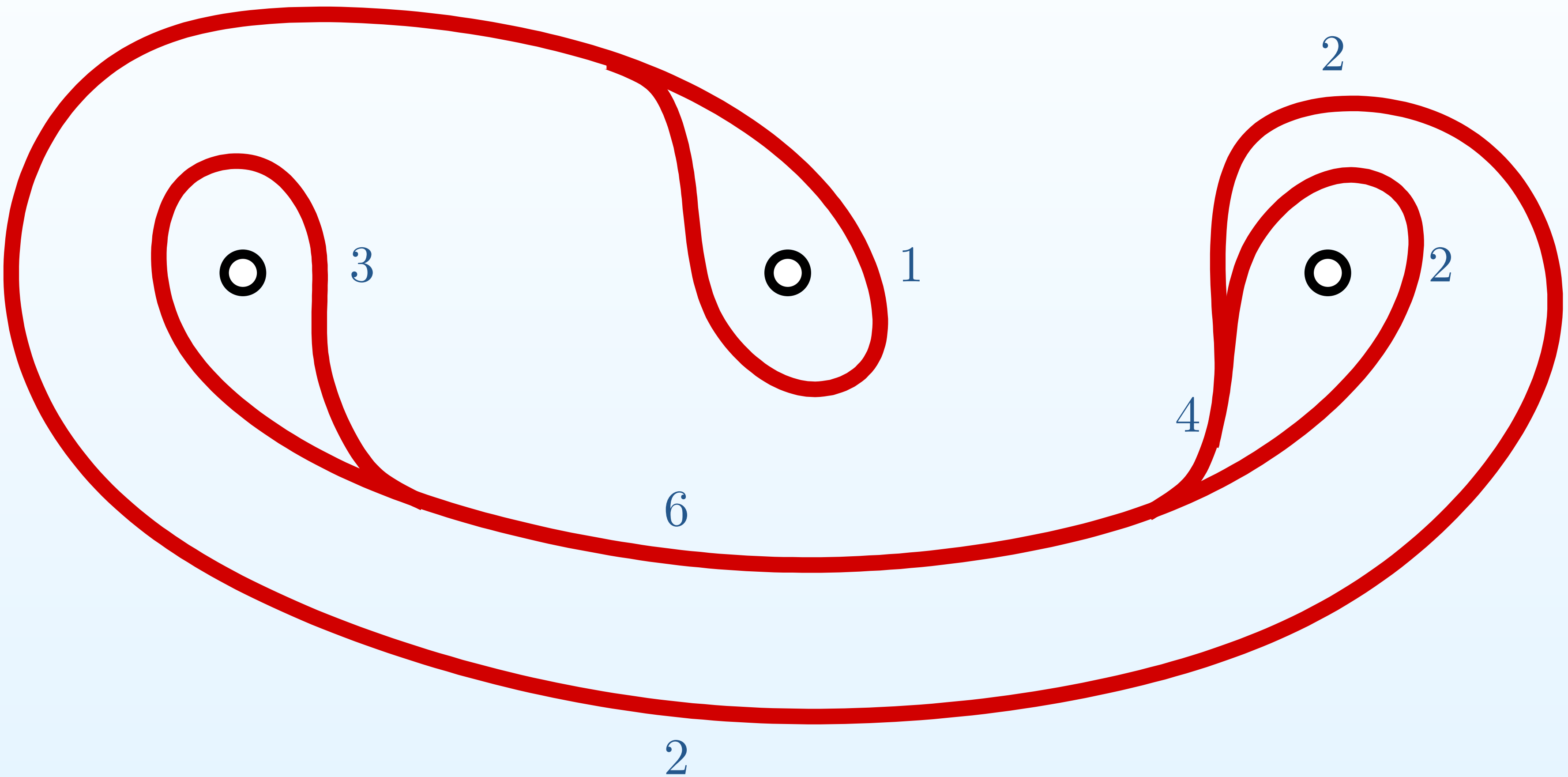

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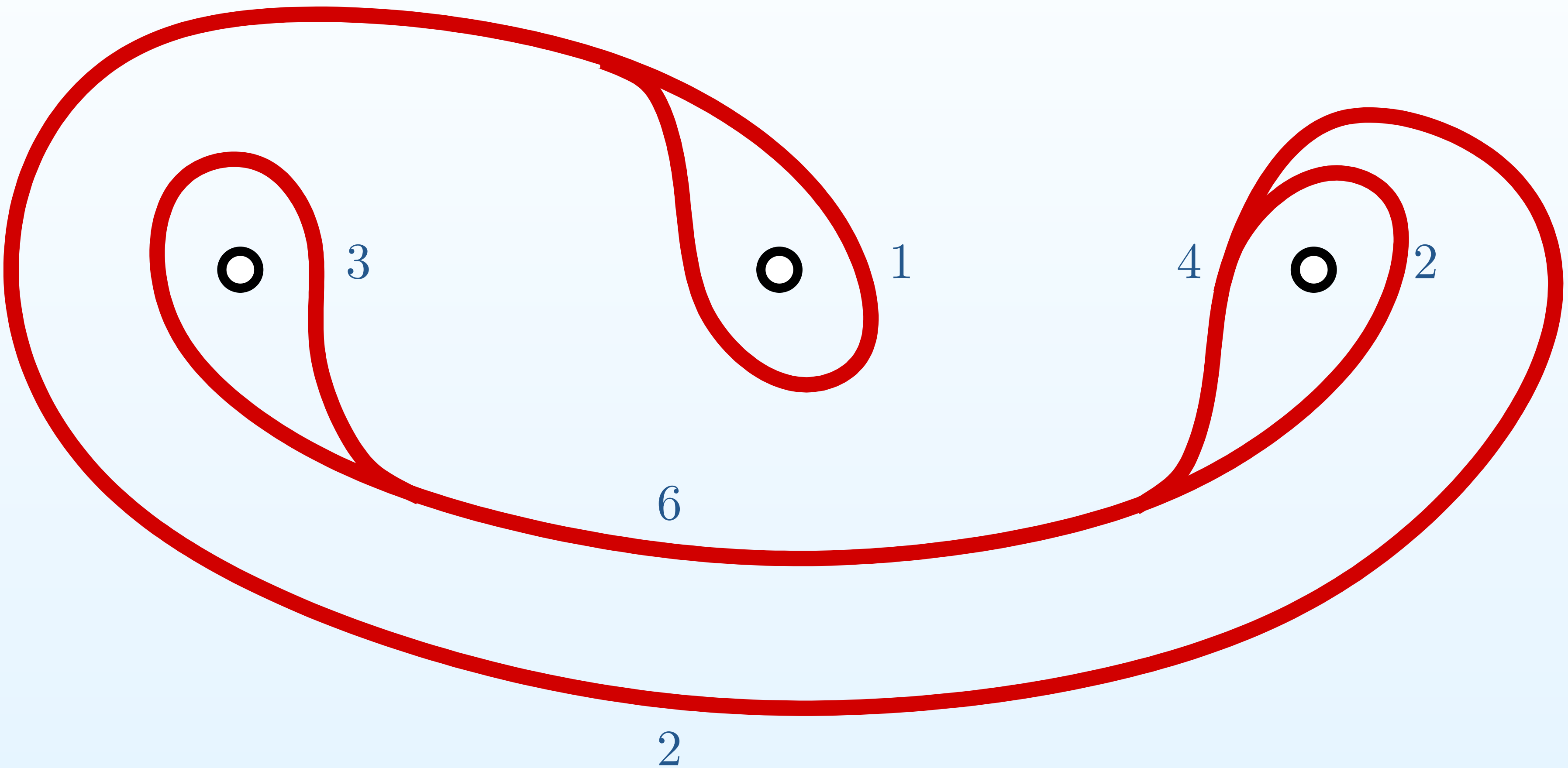

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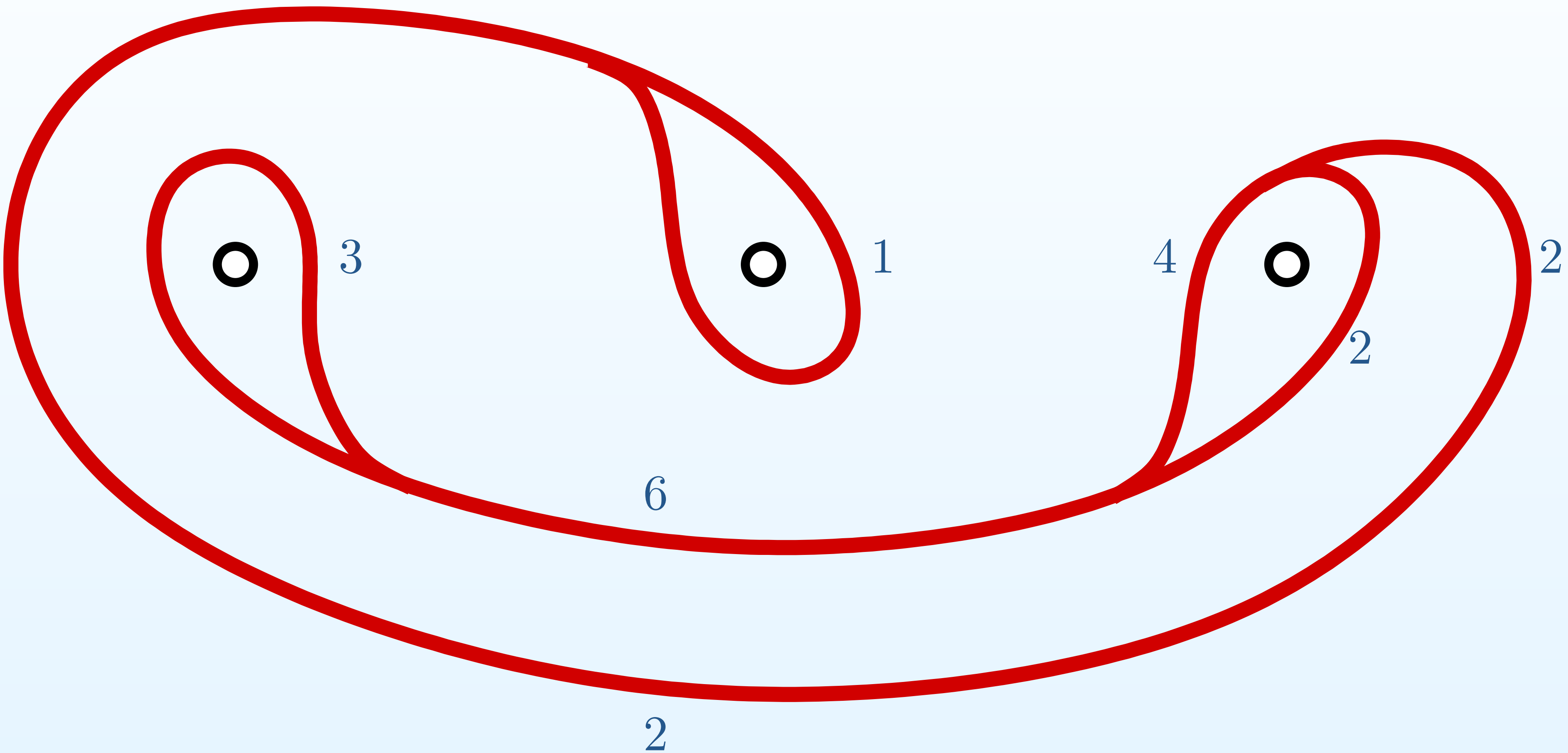

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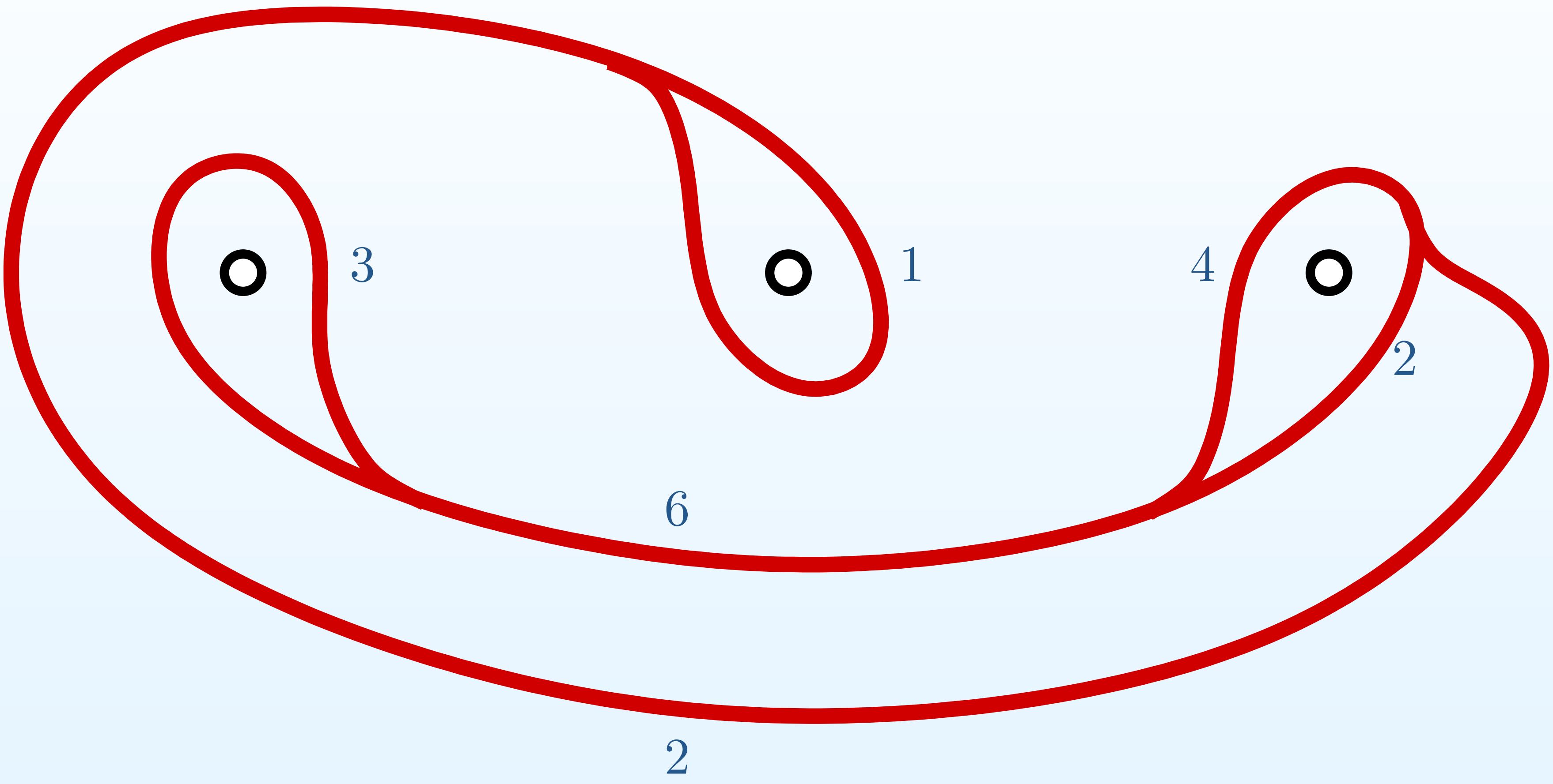

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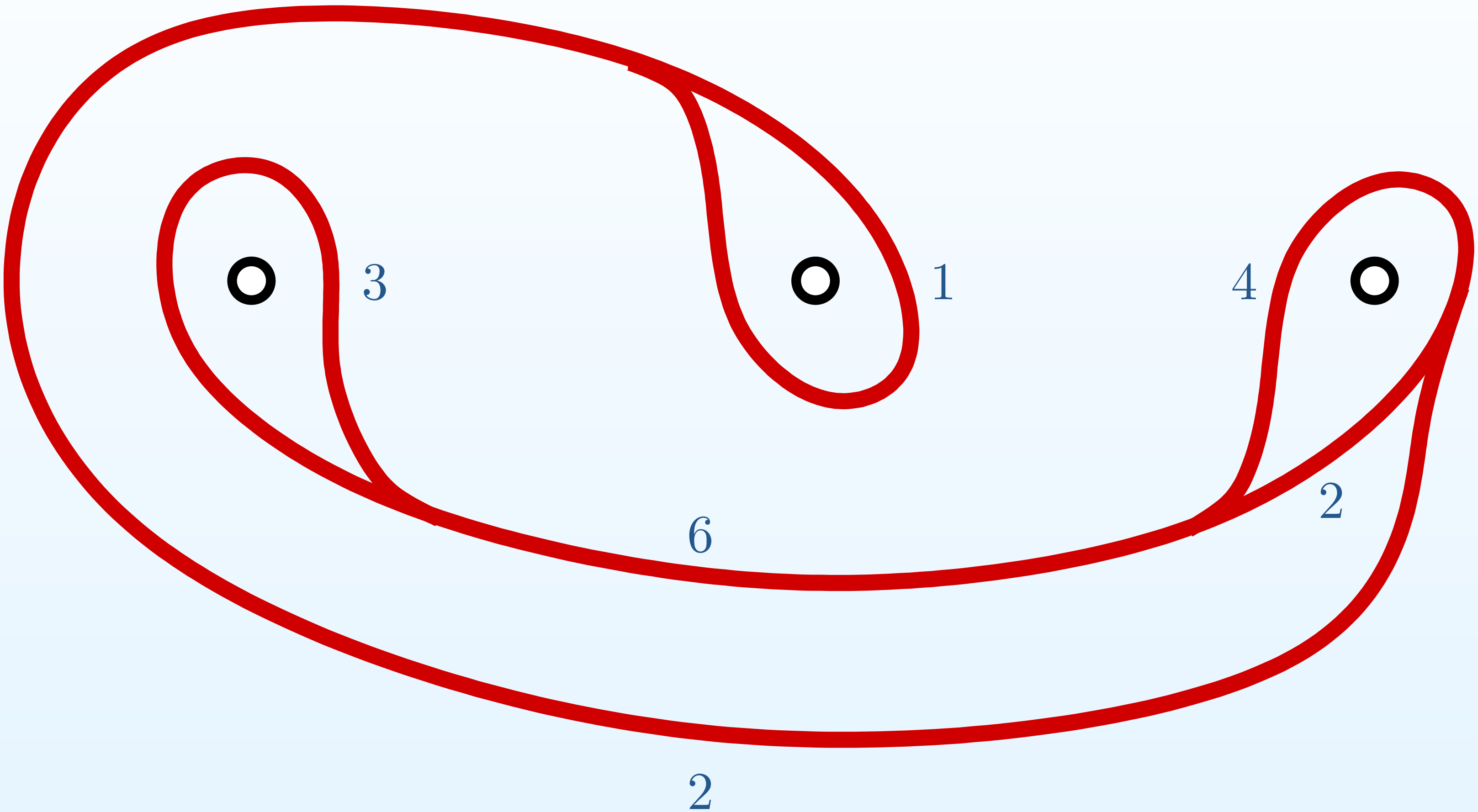

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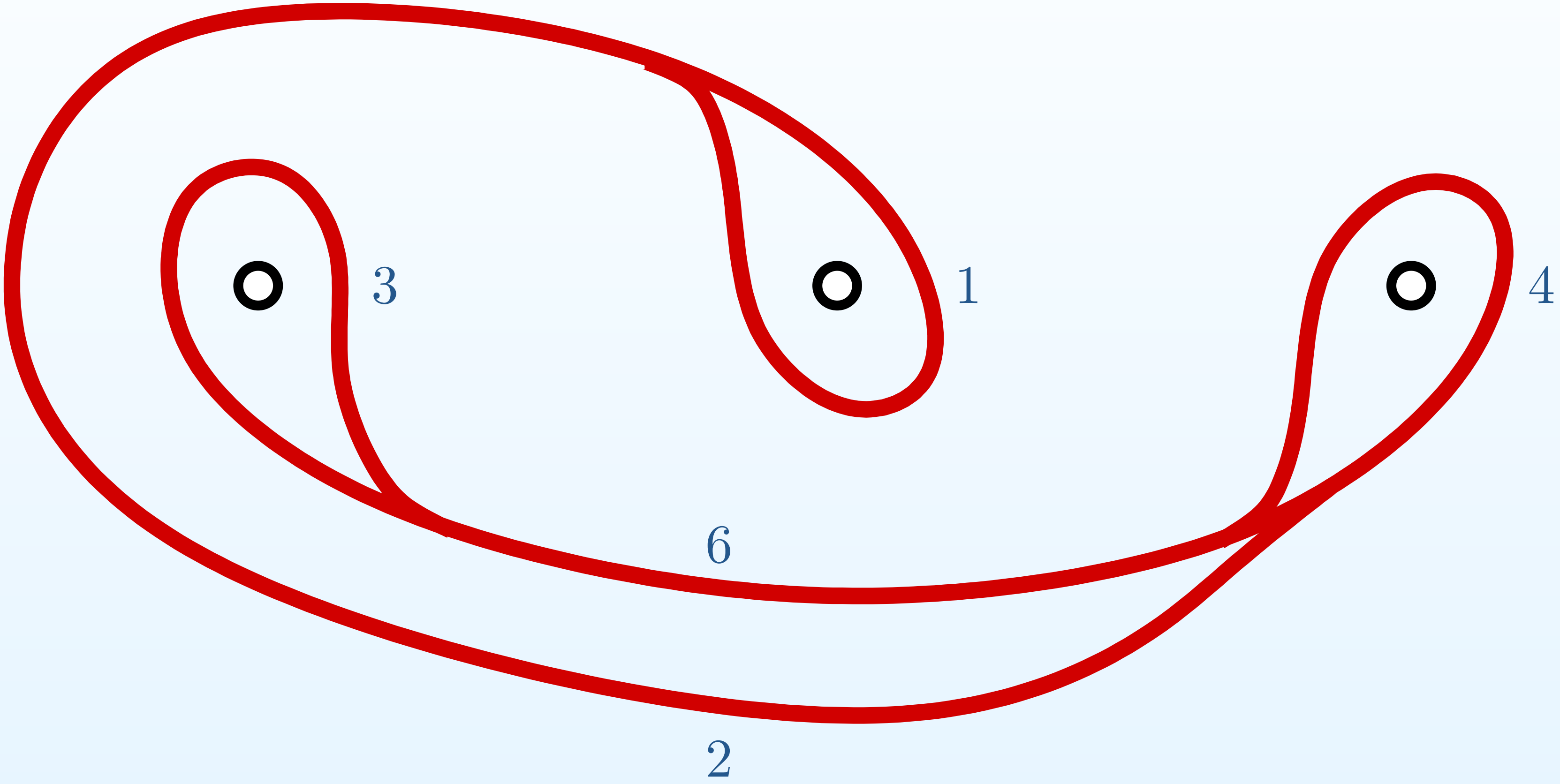

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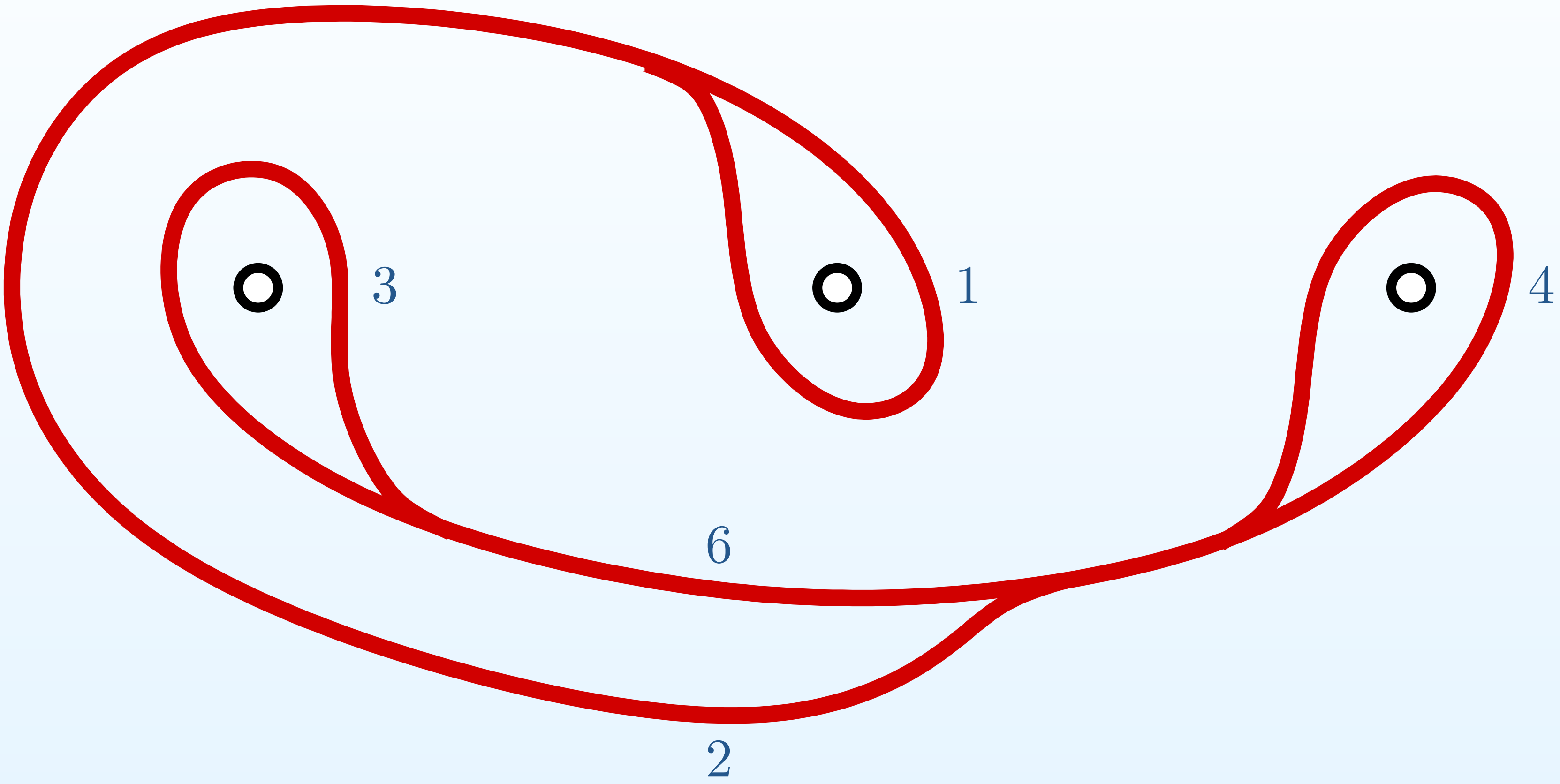

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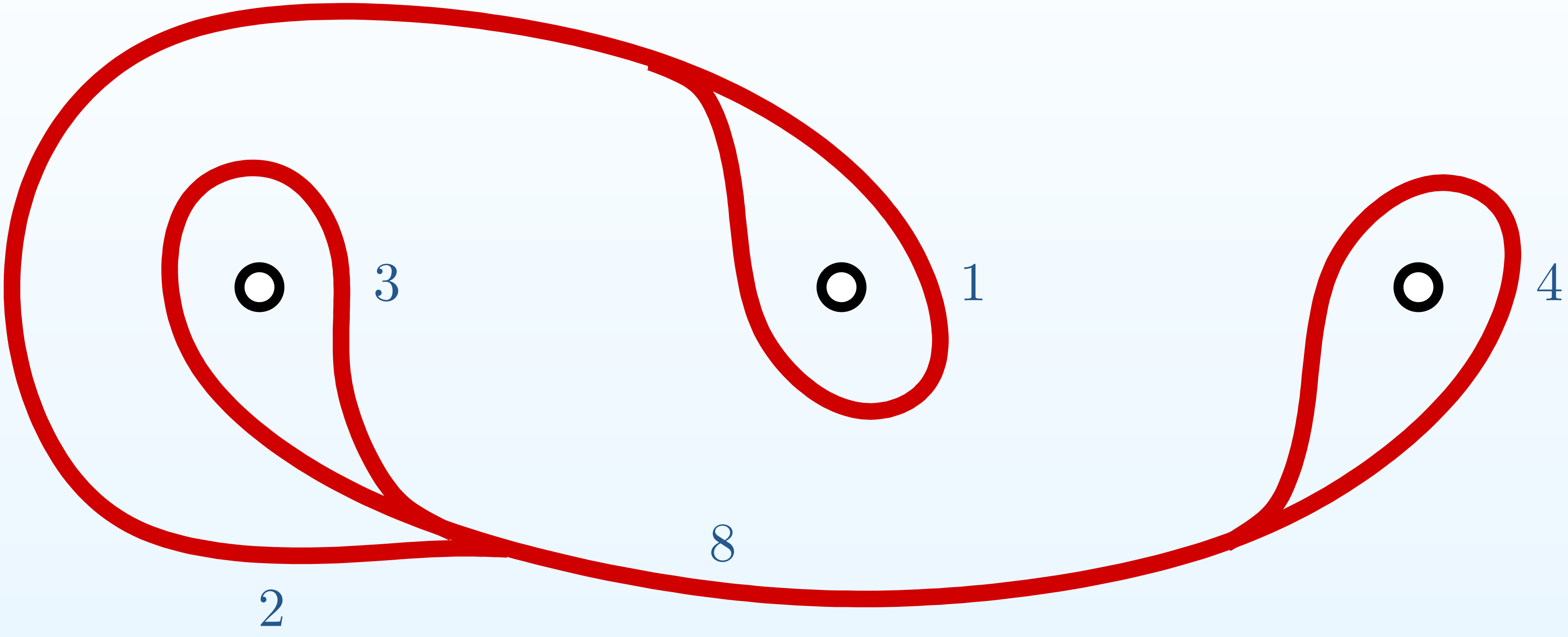

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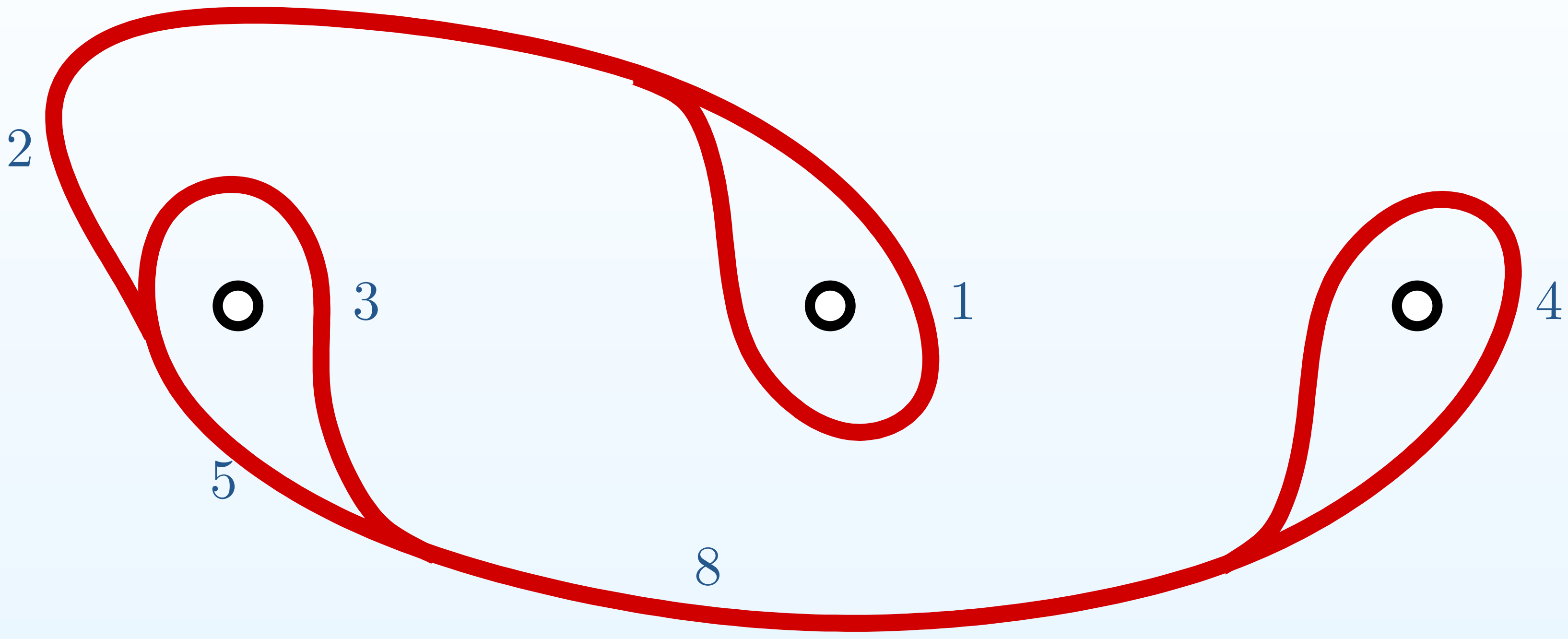

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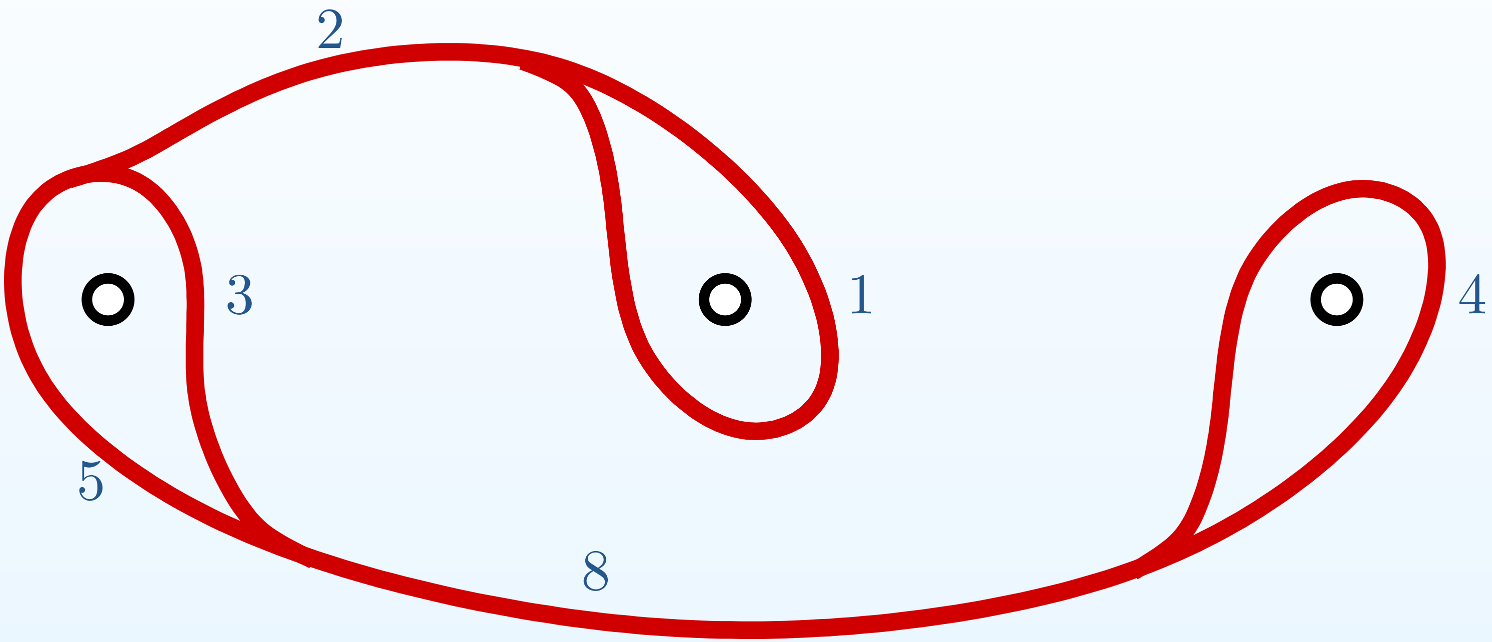

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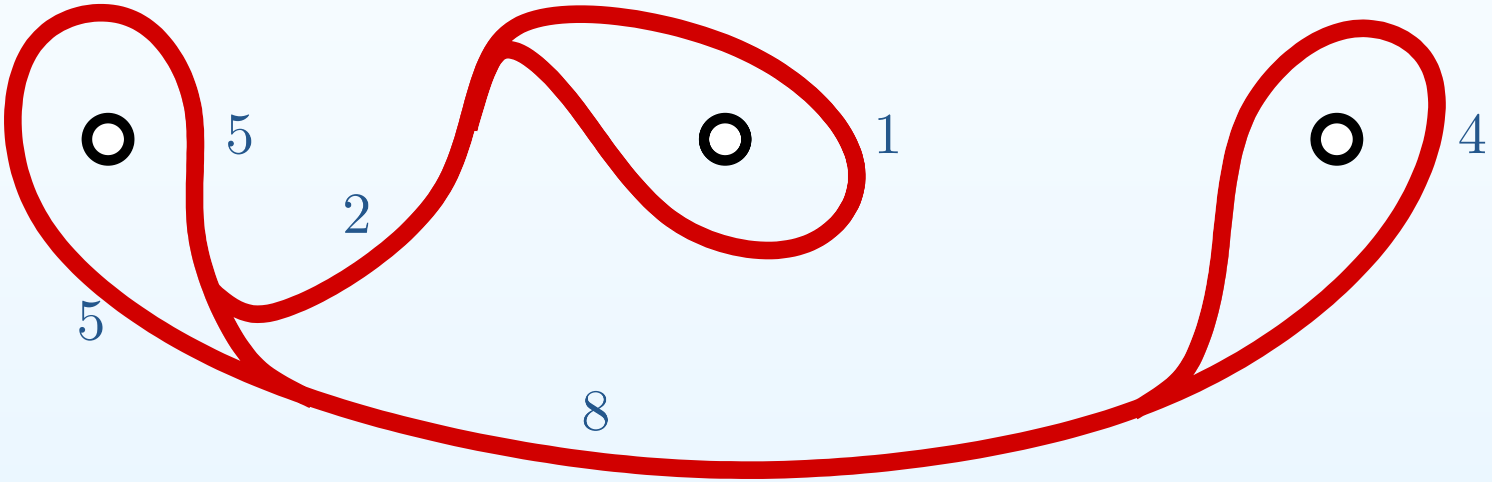

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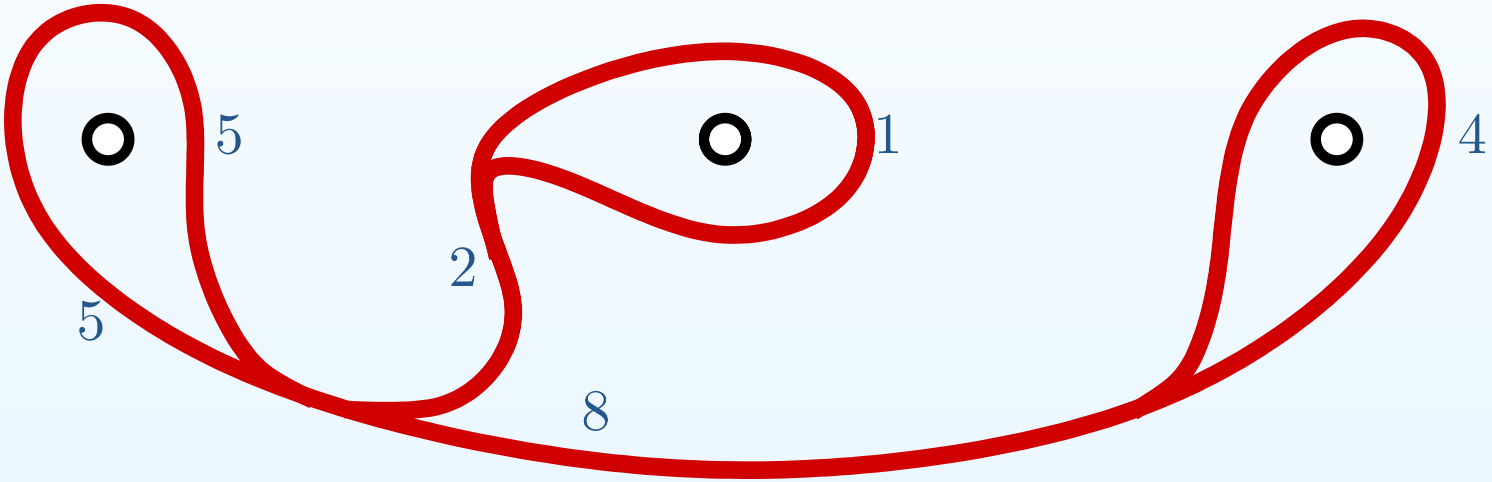

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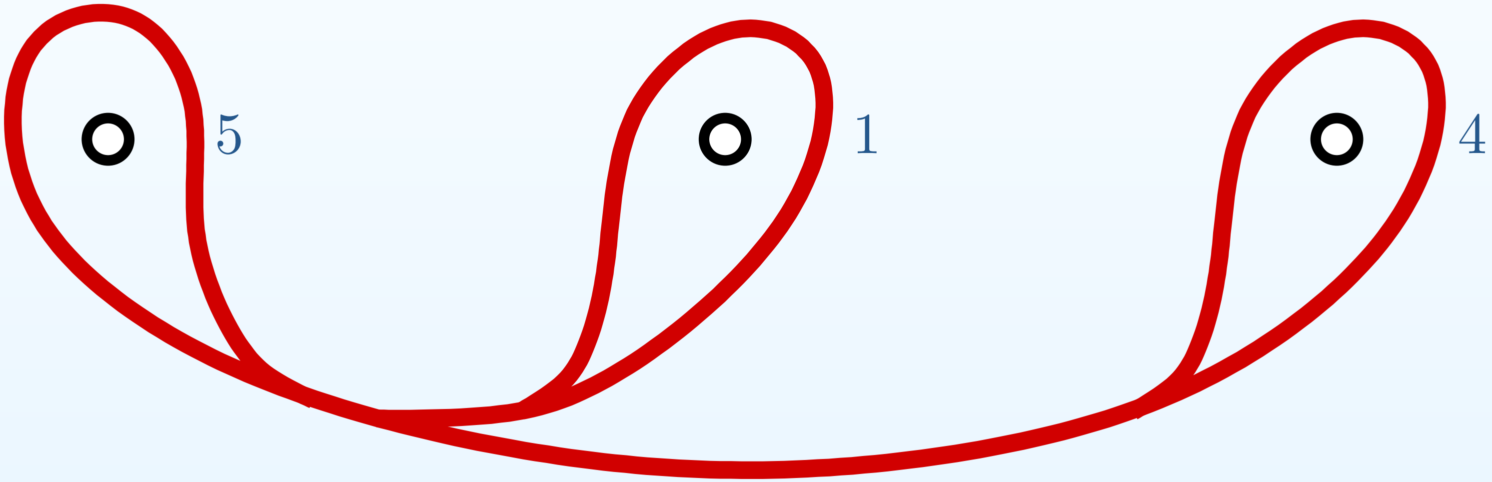
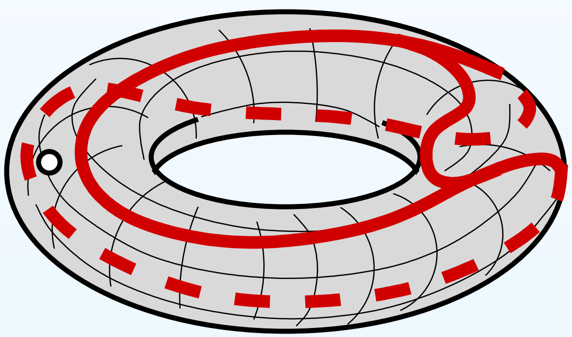
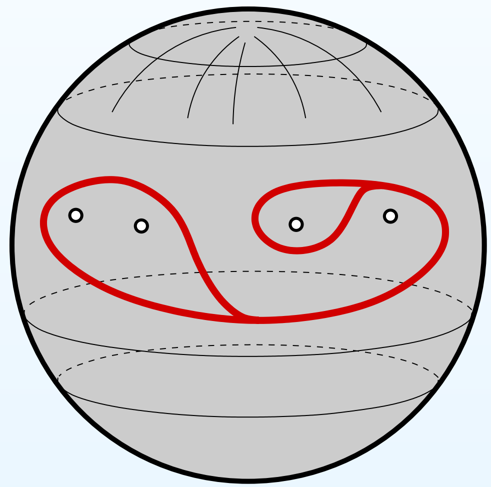


Exercise on train-tracks

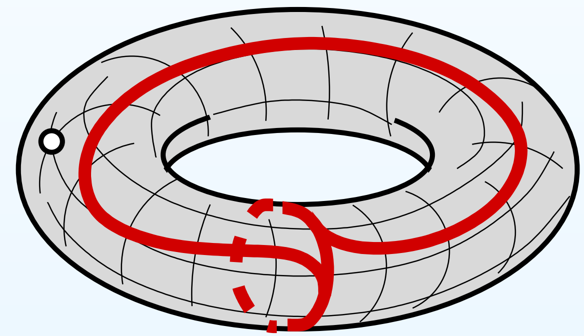
Which of the given train-tracks τ_1, τ_2, τ_3 might carry a simple closed hyperbolic geodesic? Indicate some legitimate weights if you claim that the train track carries a simple closed hyperbolic geodesic.



τ_1



τ_2



τ_3

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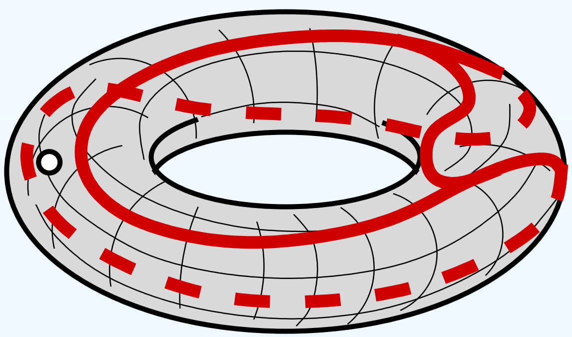

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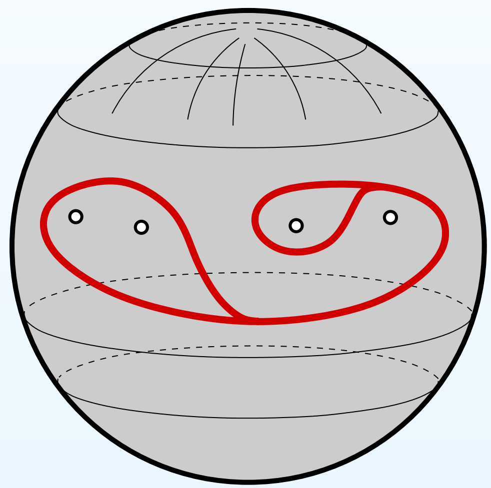

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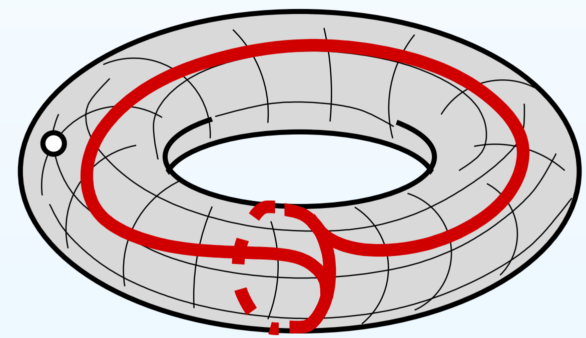

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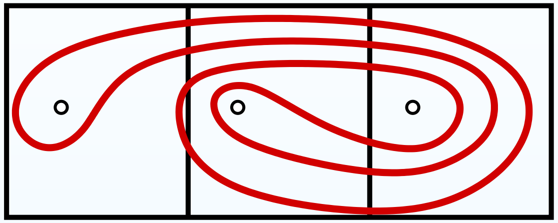
Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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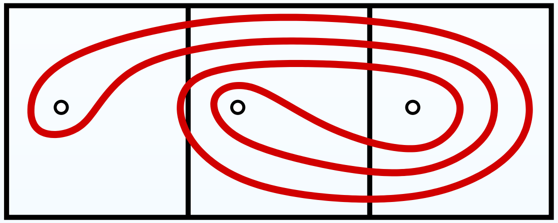

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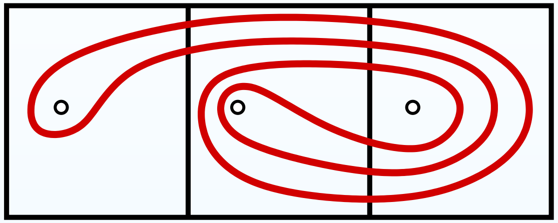

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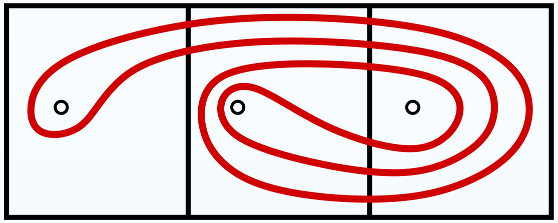

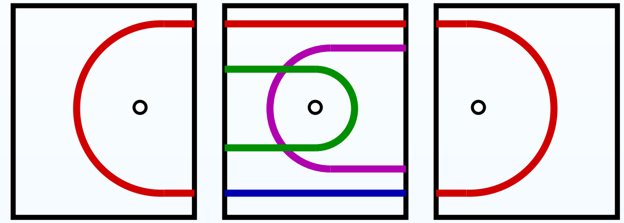

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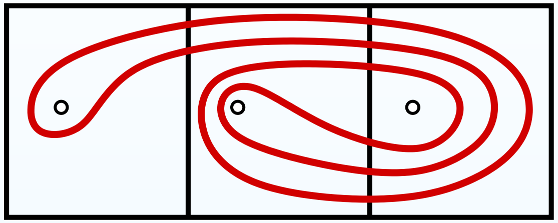

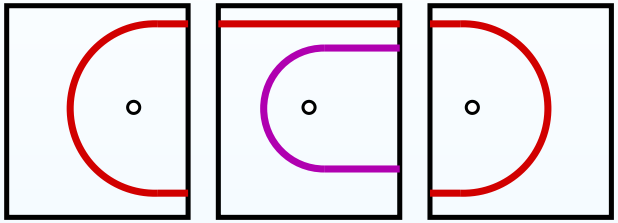

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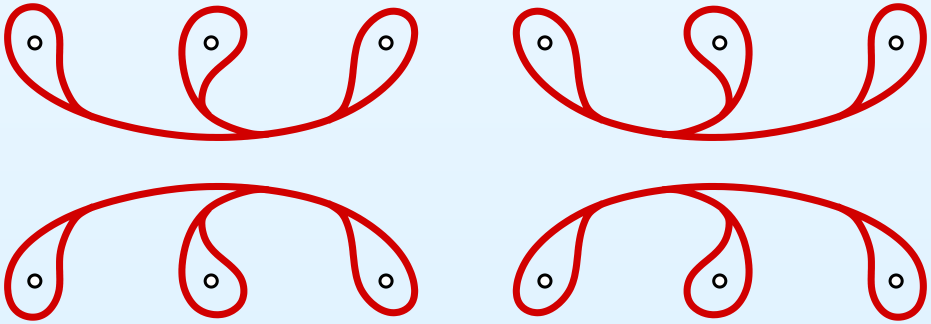
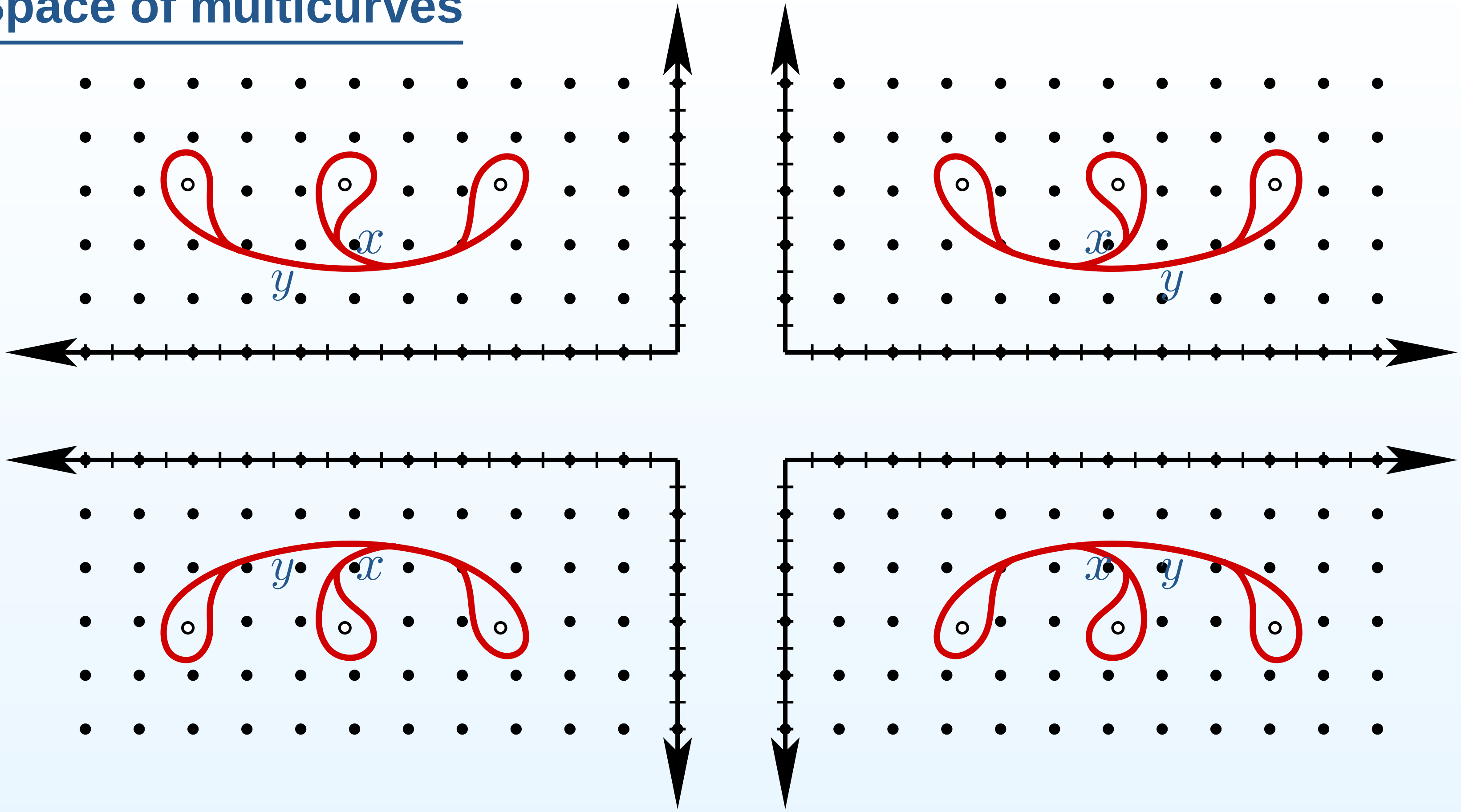
Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve is carried by one of the following four train tracks:



Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

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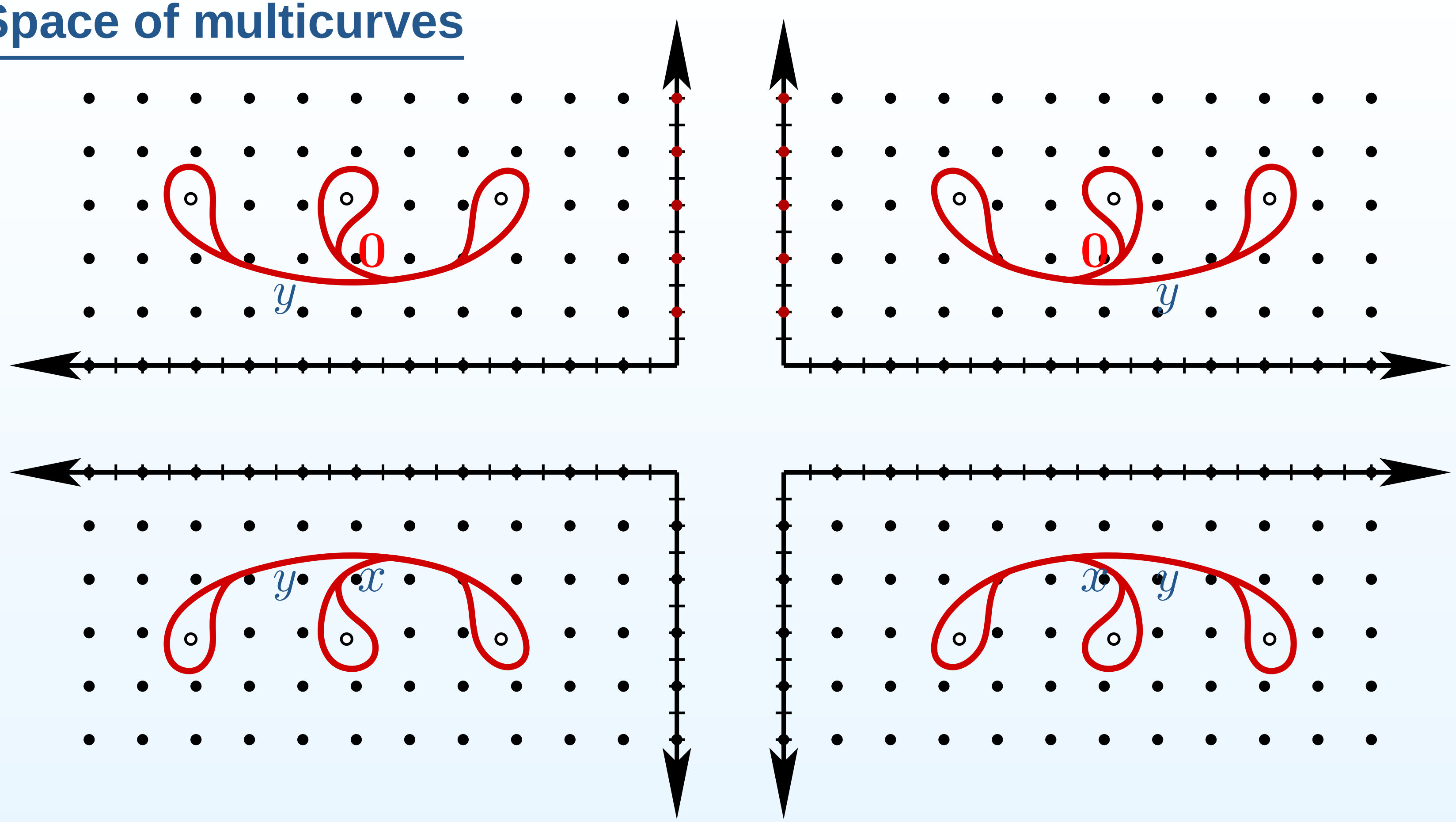

Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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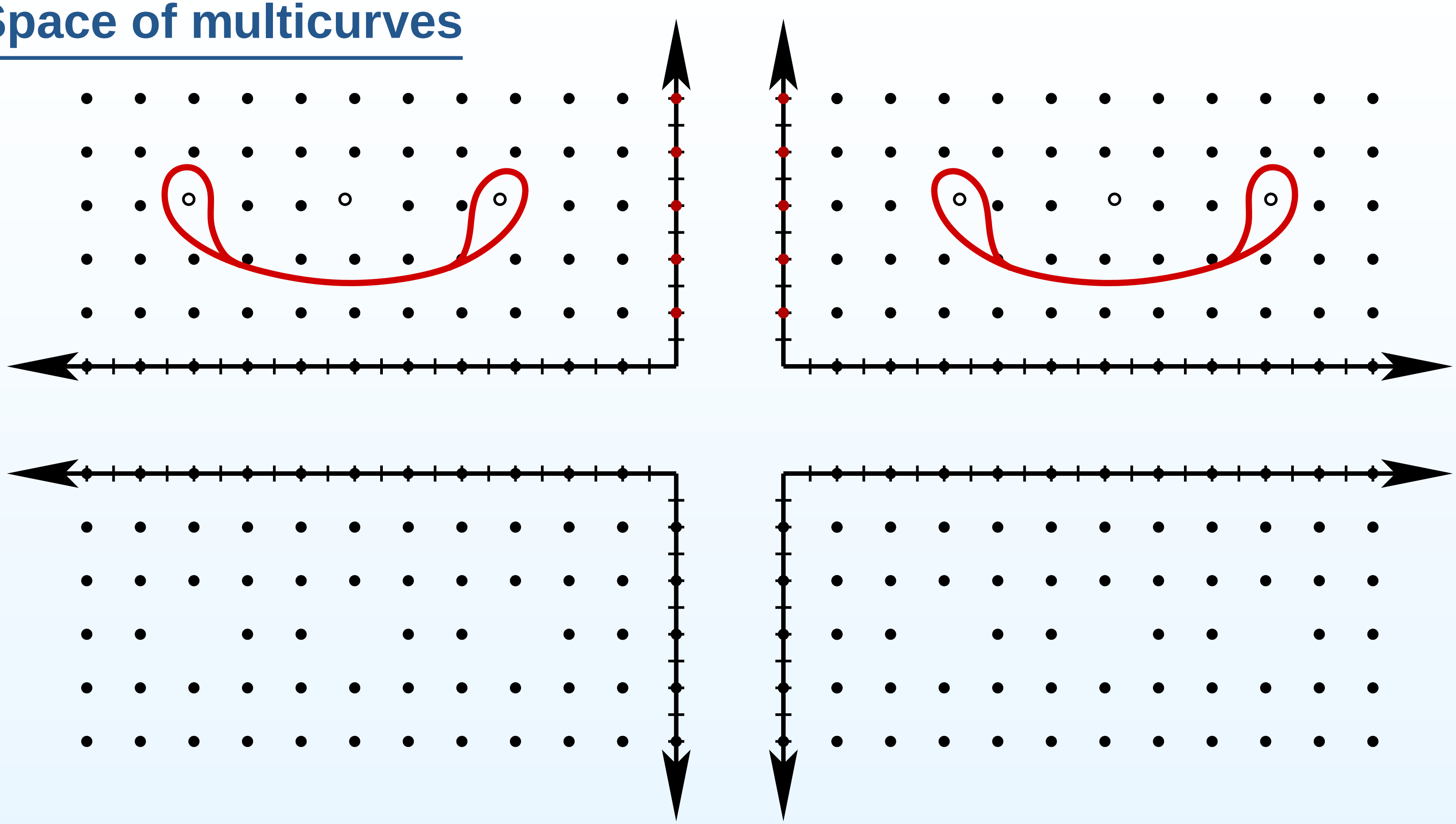

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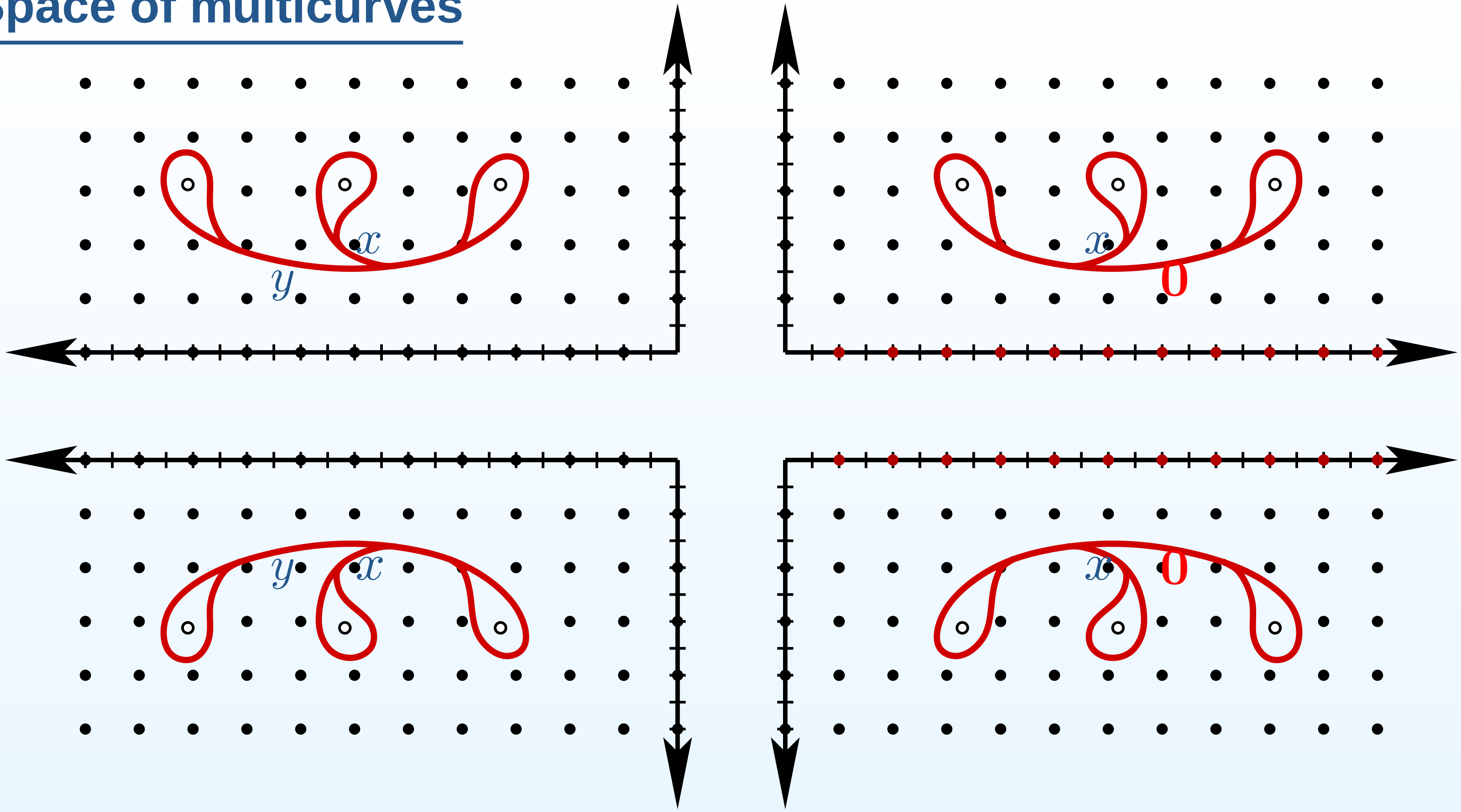

Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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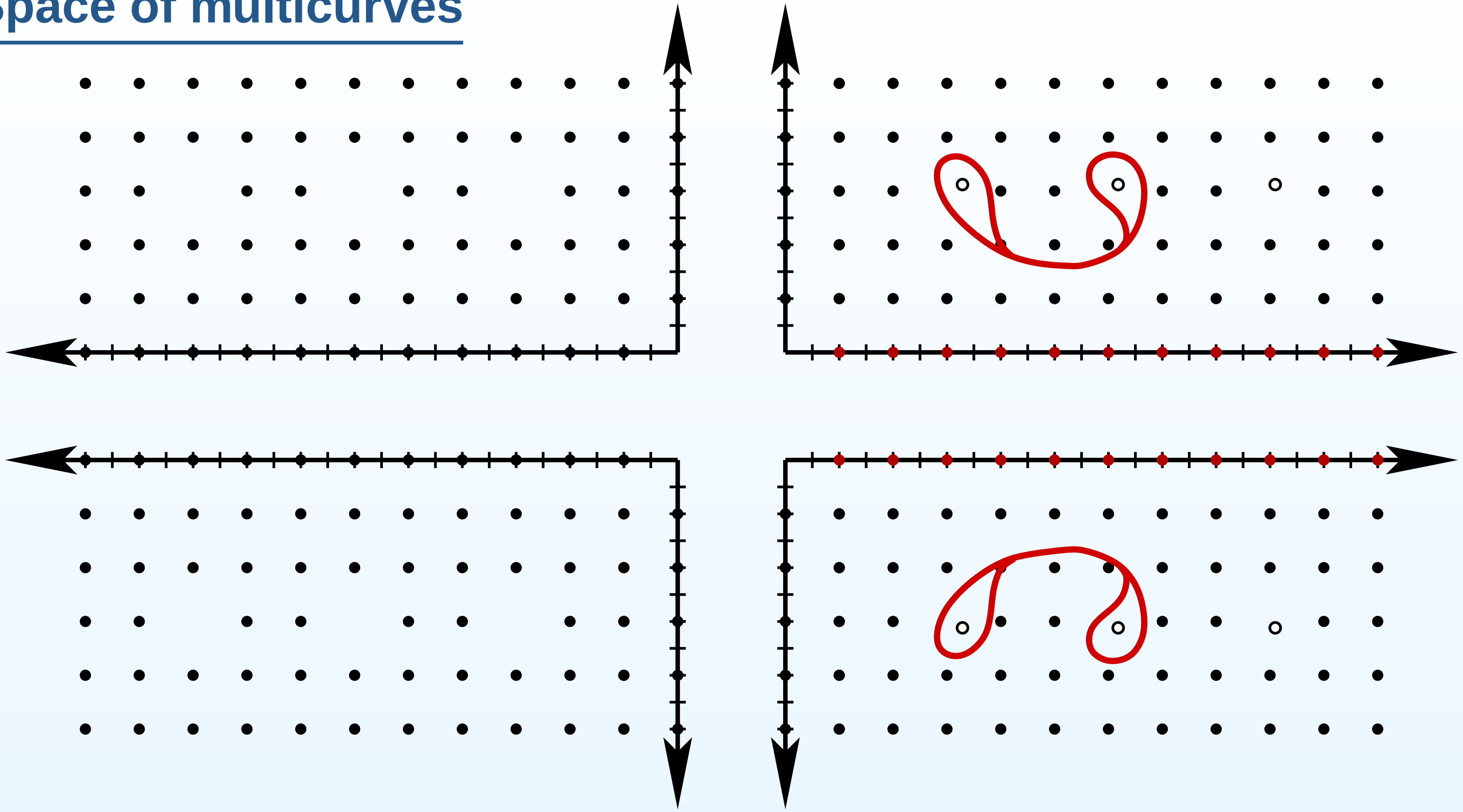

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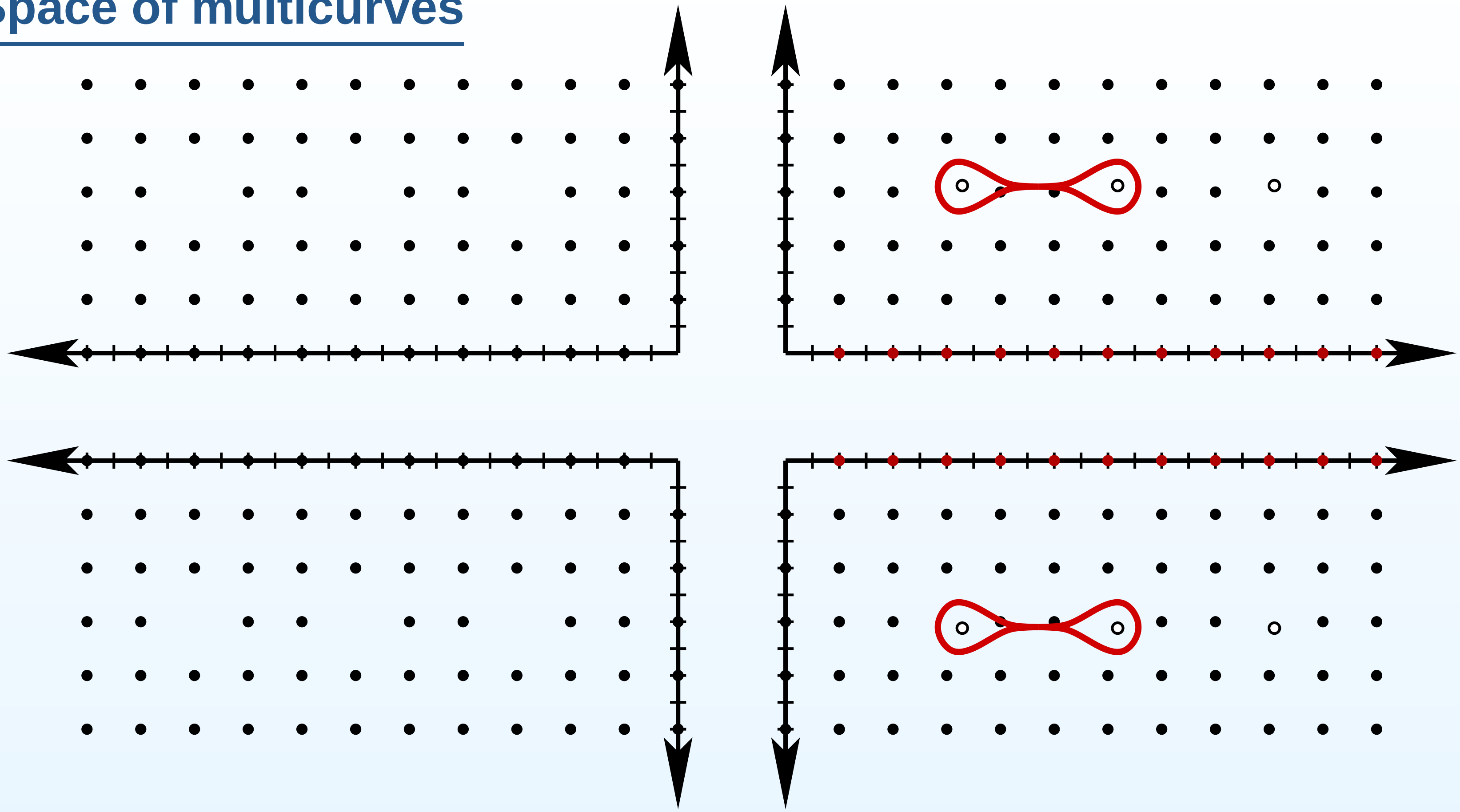

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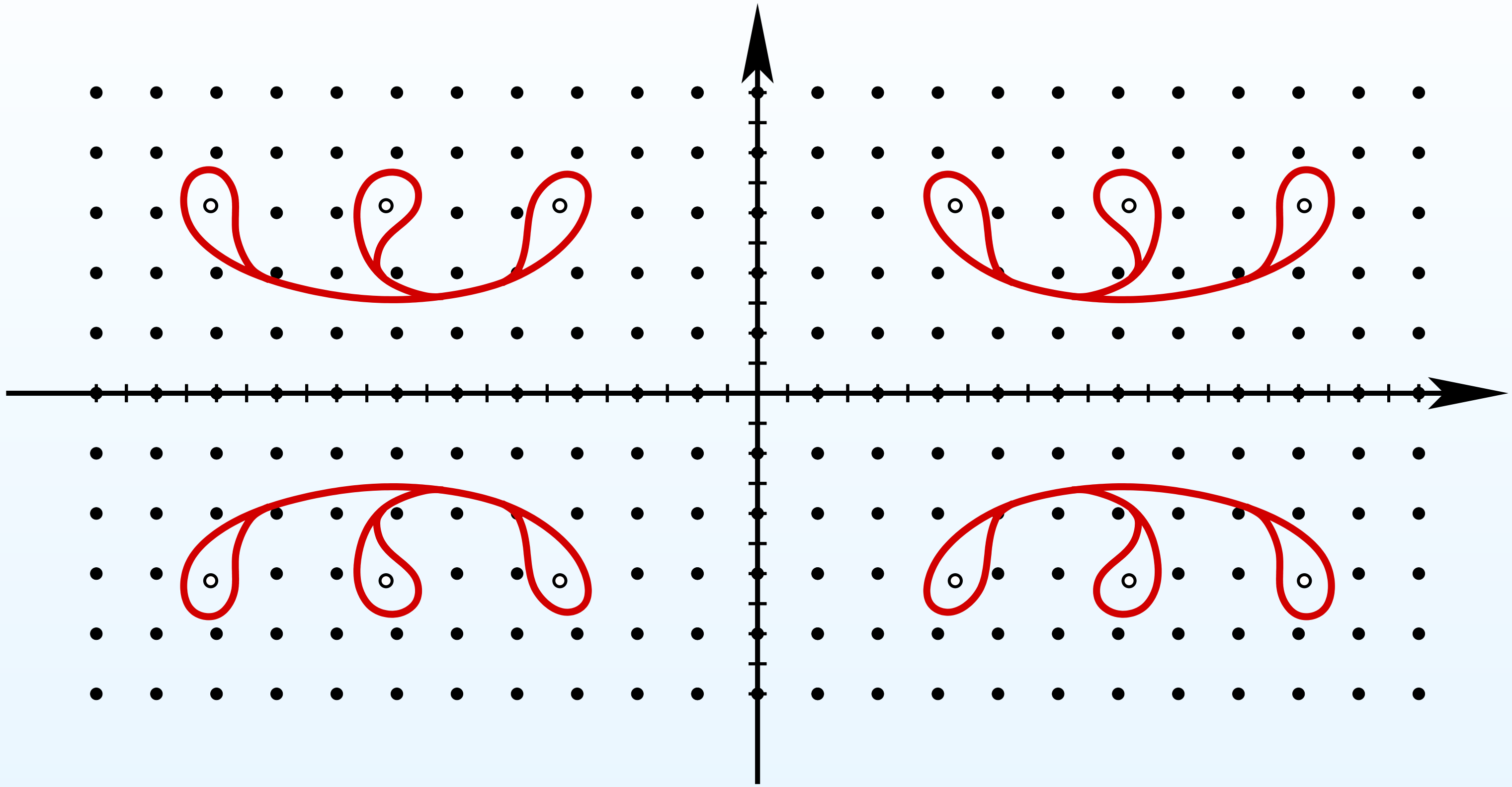

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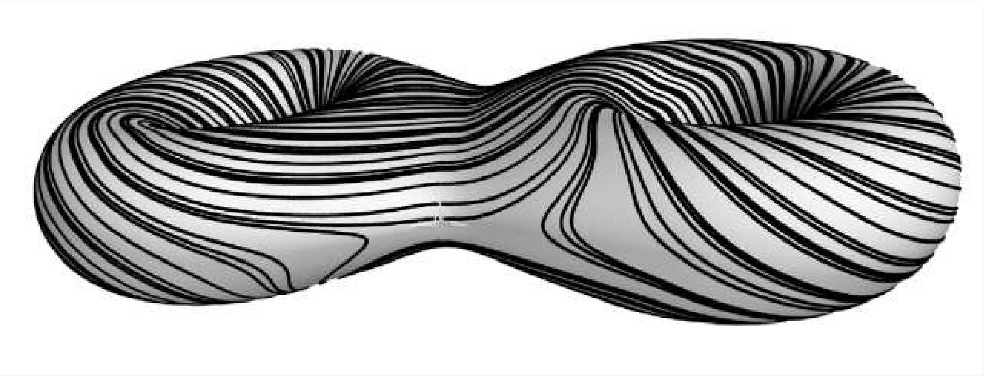
Thurston's and Mirzakhani's measures on $\mathcal{ML}_{g,n}$

Orbits of multicurves

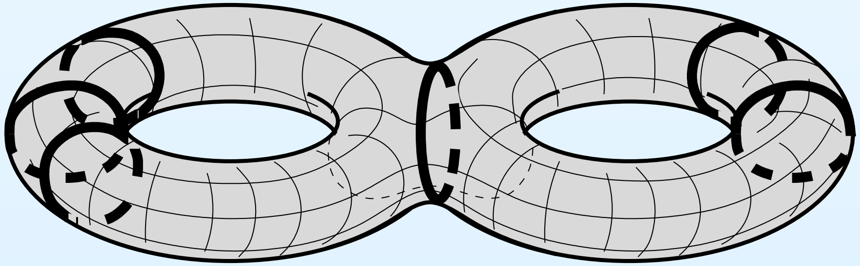
Thurston suggested to consider simple closed multicurves as integral points in the piecewise-linear space of measured laminations. All integral multicurves are partitioned in orbits under action of the mapping class group.

A general multicurve ρ:

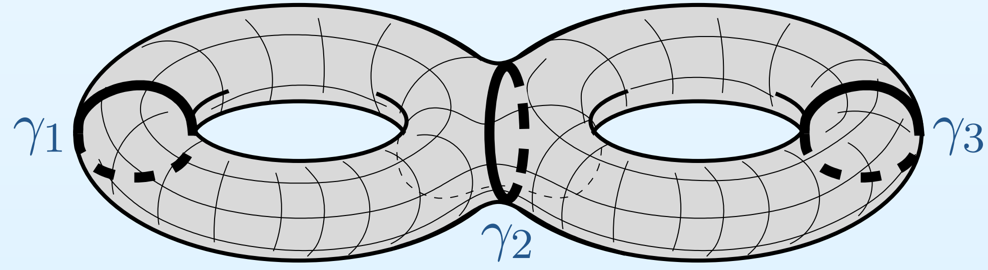


the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

$\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$

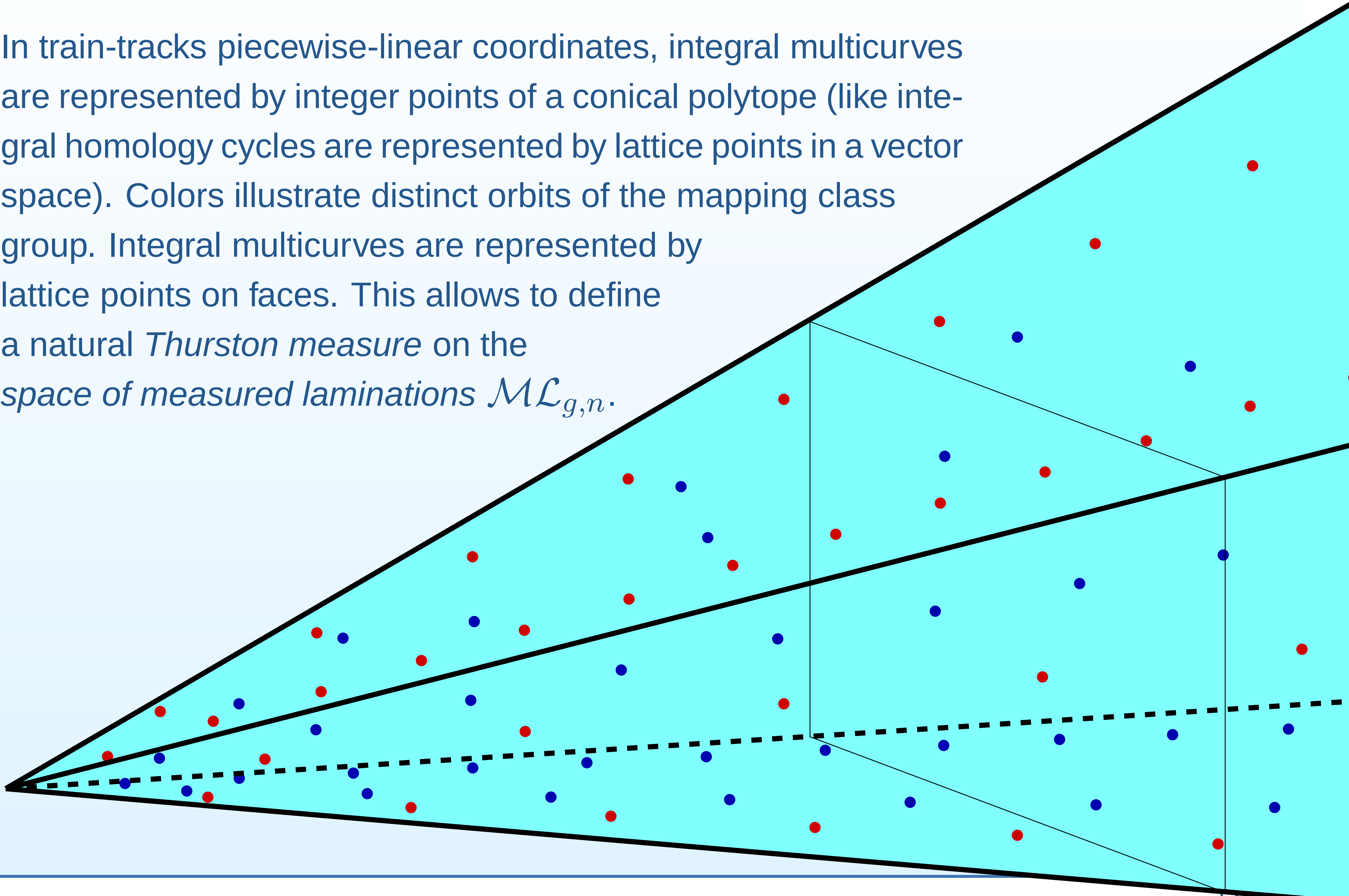


$\gamma_{reduced} = \gamma_1 + \gamma_2 + \gamma_3$



Space of multicurves

In train-tracks piecewise-linear coordinates, integral multicurves are represented by integer points of a conical polytope (like integral homology cycles are represented by lattice points in a vector space). Colors illustrate distinct orbits of the mapping class group. Integral multicurves are represented by lattice points on faces. This allows to define a natural *Thurston measure* on the *space of measured laminations* $\mathcal{ML}_{g,n}$.



Space $\mathcal{ML}_{g,n}$ and the length function

Actually, one can give sense not only to integer, and not only to rational but to *all* points of the corresponding simplicial cone and, following Bill Thurston, define a space of *measured laminations* $\mathcal{ML}_{g,n}$. We have seen that in presence of a hyperbolic metric, integral points of $\mathcal{ML}_{g,n}$ represent simple closed geodesics. Similarly, *all* other points also get geometric realization as *measured geodesic laminations* — disjoint unions of non self-intersecting infinite geodesics. Transversally they usually look like Cantor sets.

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One can extend the length function by linearity to simple closed multicurves:

$$\ell_{\sum a_i \gamma_i} := \sum a_i \ell_{\gamma_i}(X)\,.$$

By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t\cdot\lambda}(X) = t \cdot \ell_\lambda(X)$.

Thurston measure on $\mathcal{ML}_{g,n}$

Let us temporarily return to the situation when there is no hyperbolic metric on a smooth surface $S_{g,n}$.

Train track charts define **piecewise linear structure** on $\mathcal{ML}_{g,n}$.

"Integral lattice" $\mathcal{ML}_{g,n}(\mathbb{Z})$ provides canonical normalization of the linear volume form μ_{Th} in which the fundamental domain of the lattice has unit volume.

Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves, so the piecewise-linear action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the "integral lattice" $\mathcal{ML}_{g,n}(\mathbb{Z})$, and, hence, preserves the measure μ_{Th}.

Theorem (H. Masur, 1985). *The action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ is ergodic with respect to the Lebesgue measure class (i.e. any measurable subset of $\mathcal{ML}_{g,n}$ invariant under $\mathrm{Mod}_{g,n}$ has measure zero or its complement has measure zero). Any $\mathrm{Mod}_{g,n}$-invariant measure in the Lebesgue measure class is just Thurston measure rescaled by some constant factor.*

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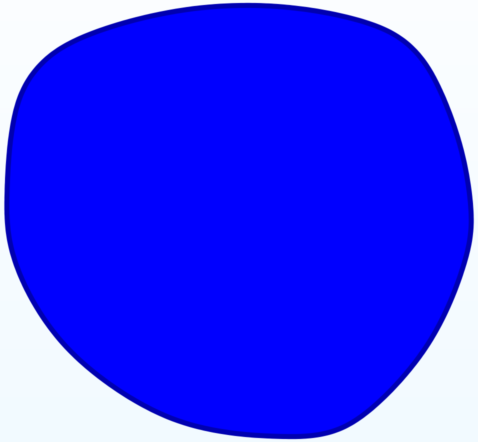
Counting the measure of a set



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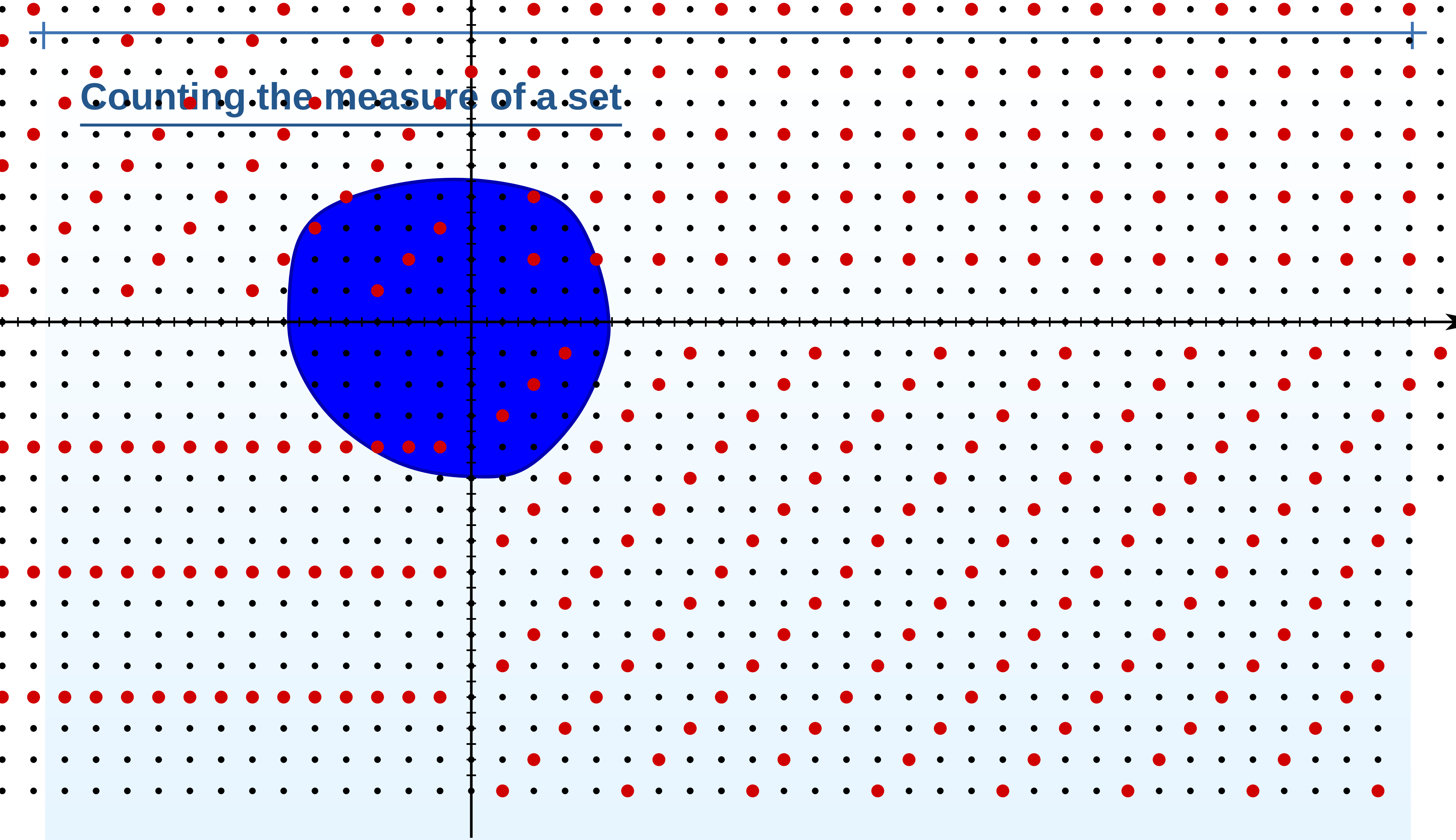

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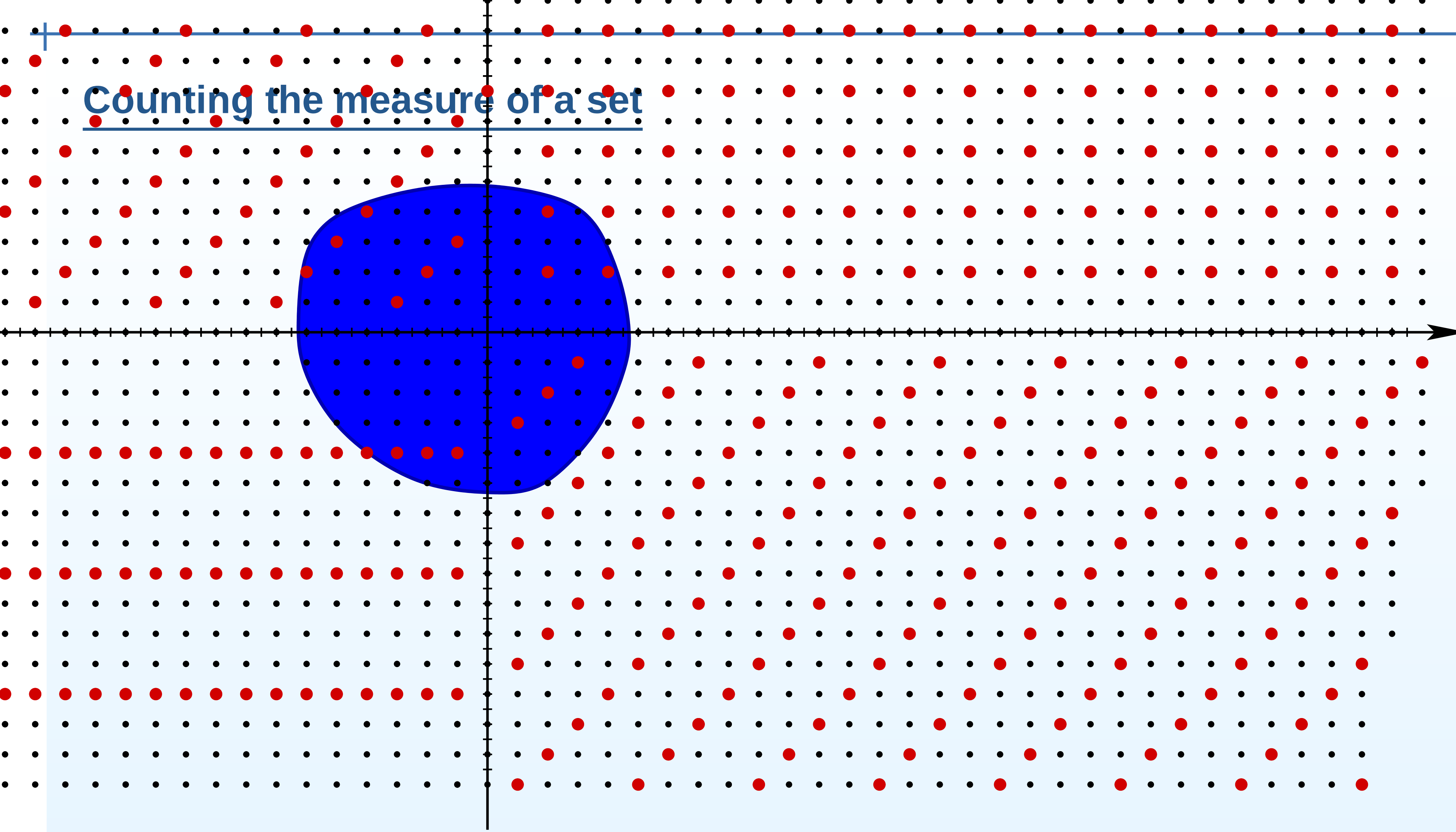

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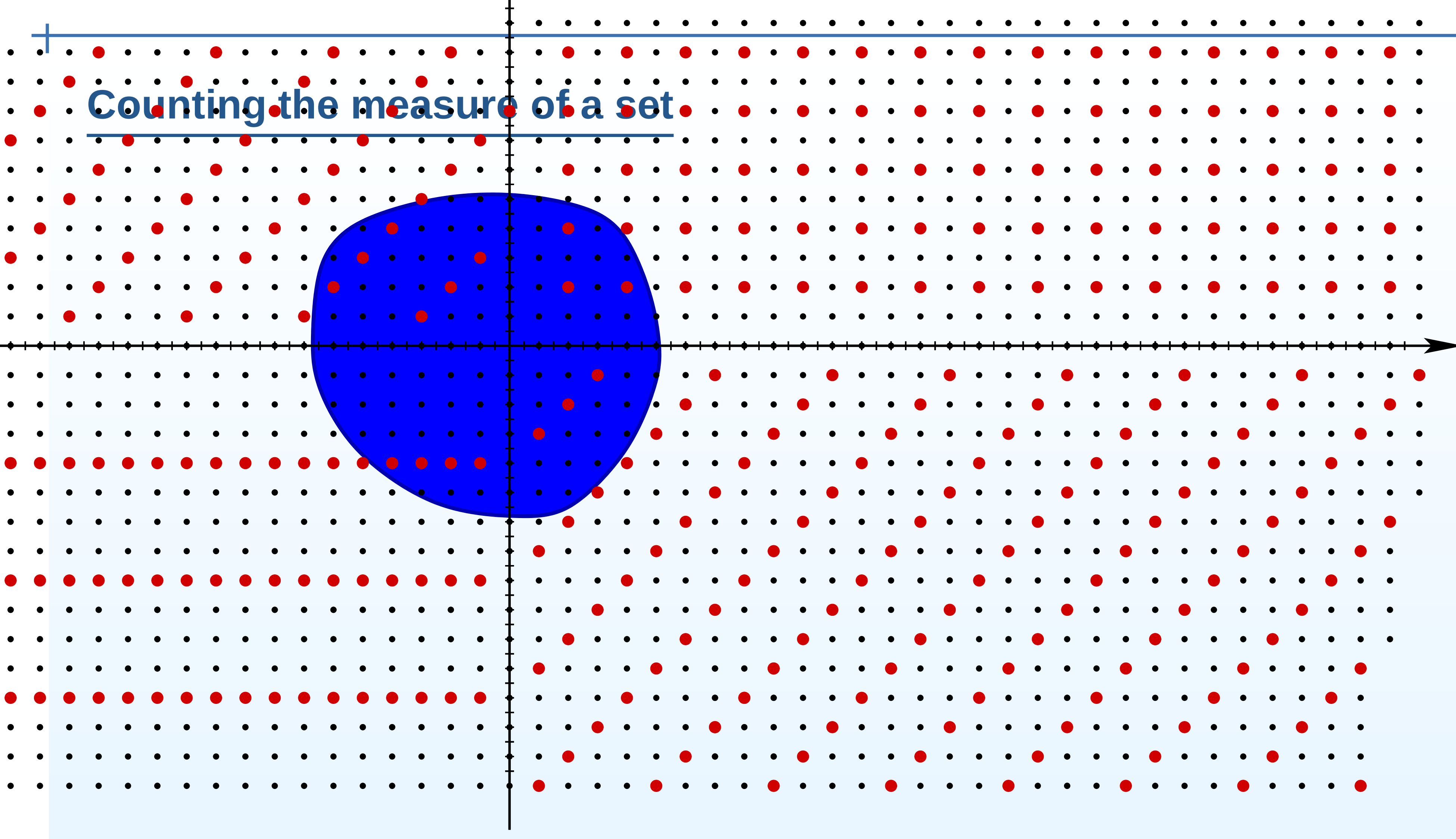

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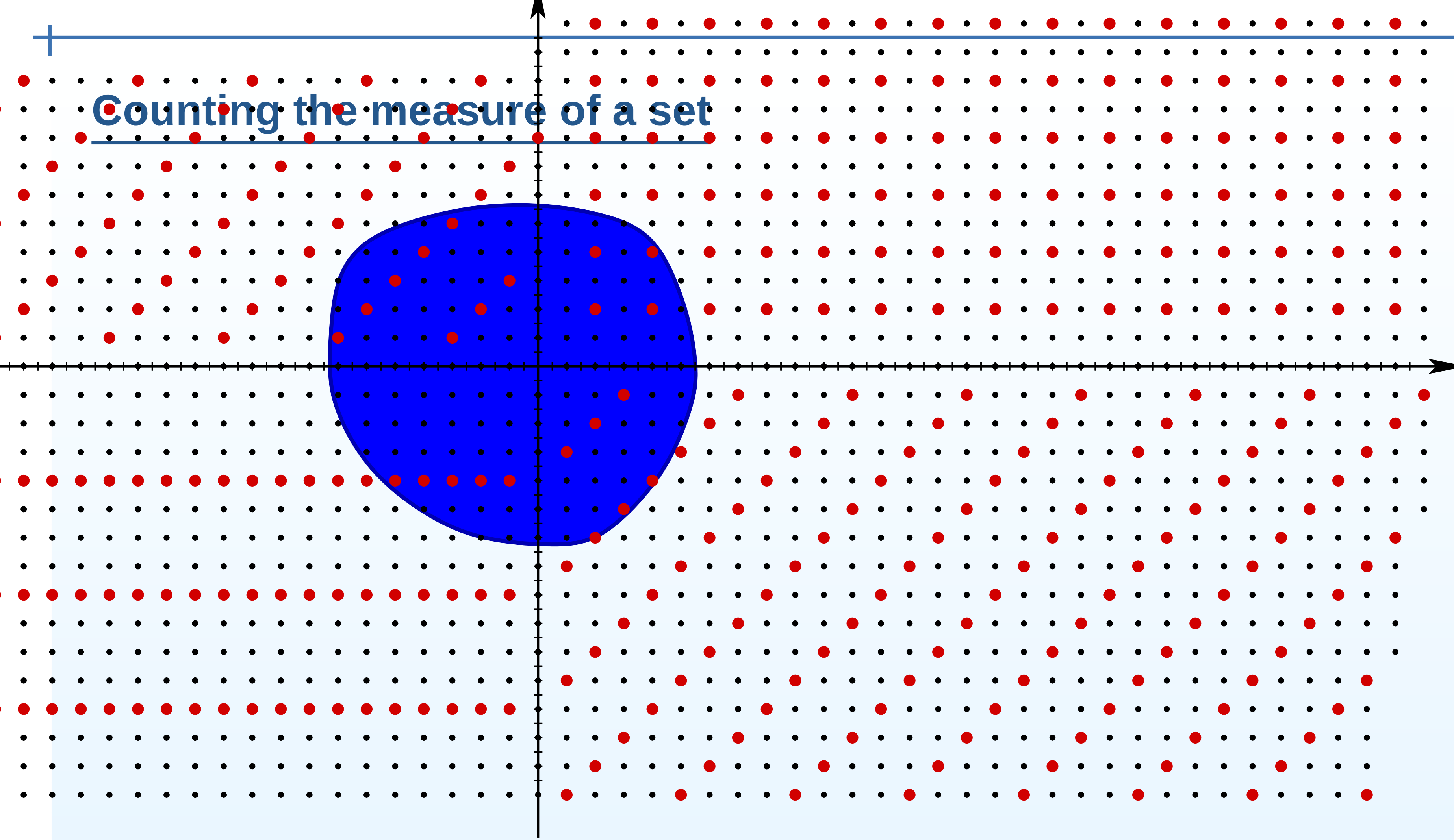

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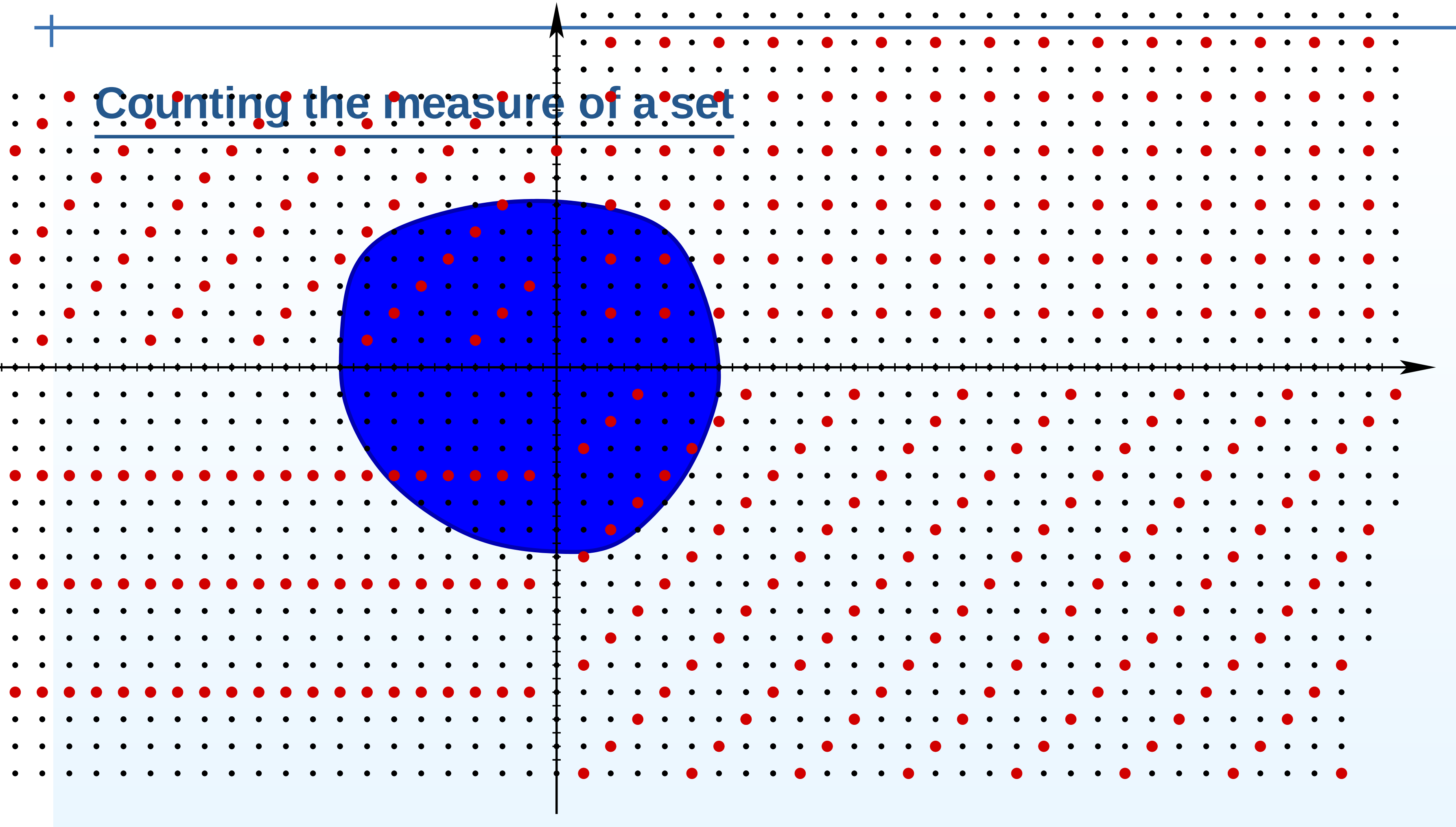

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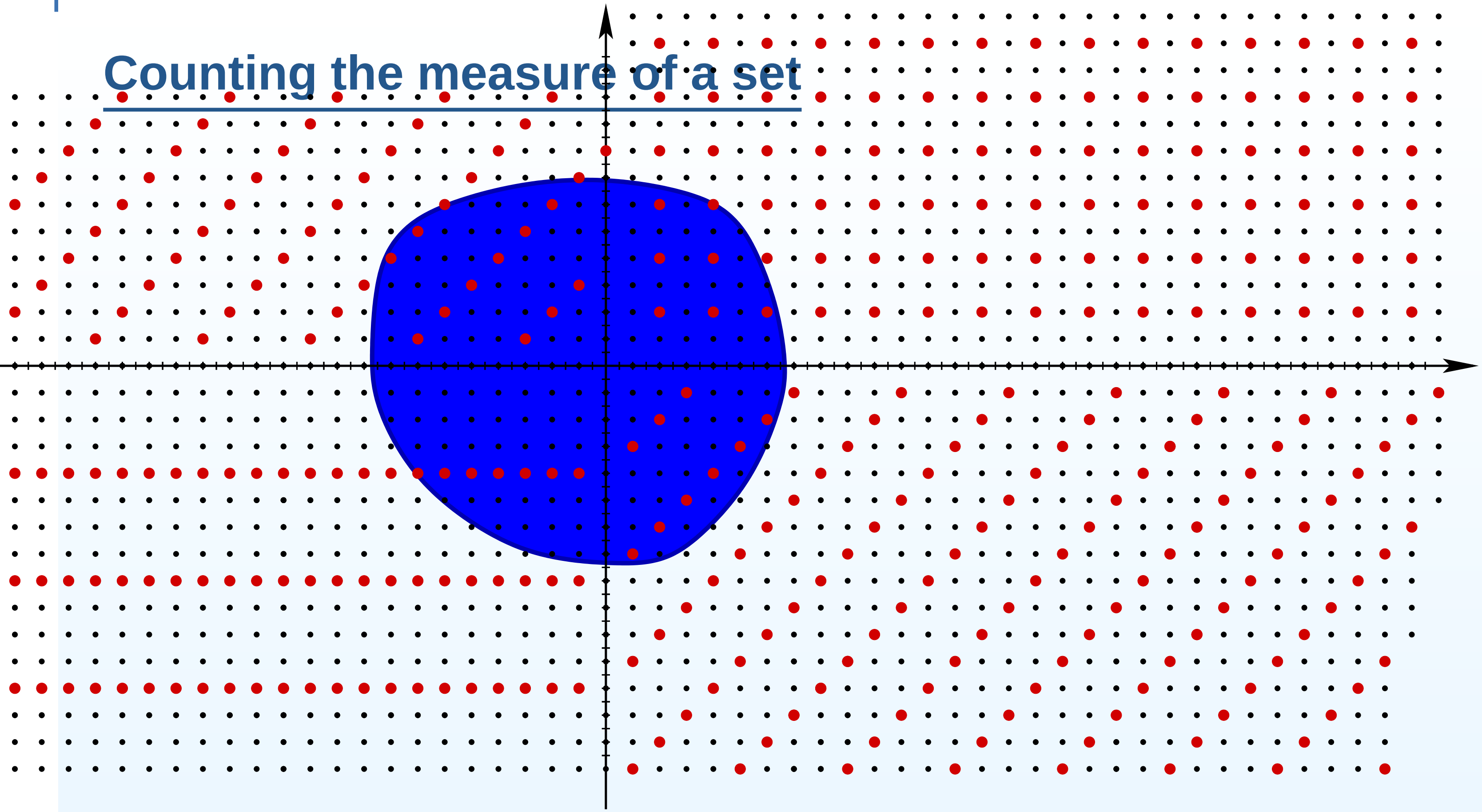

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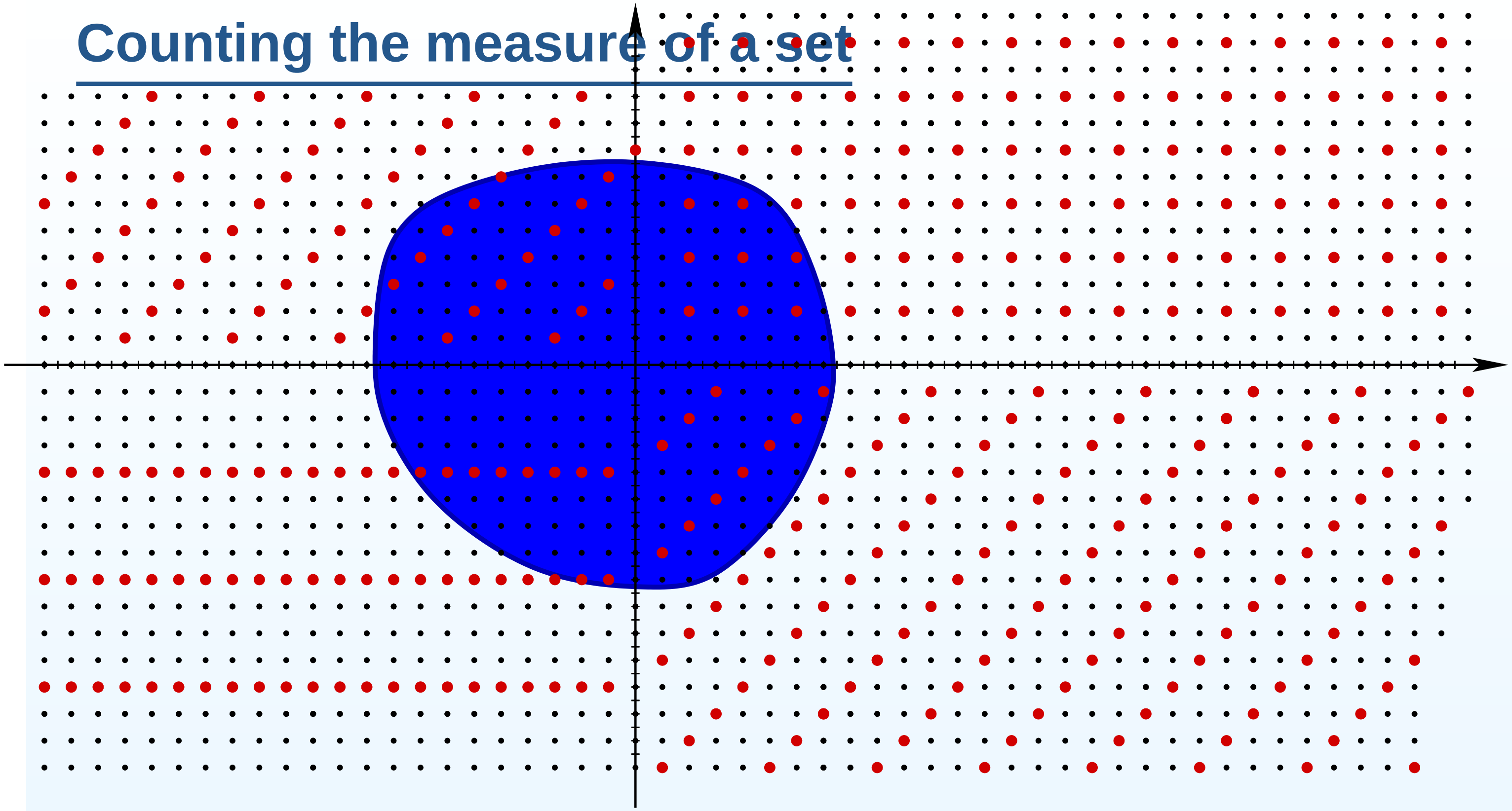

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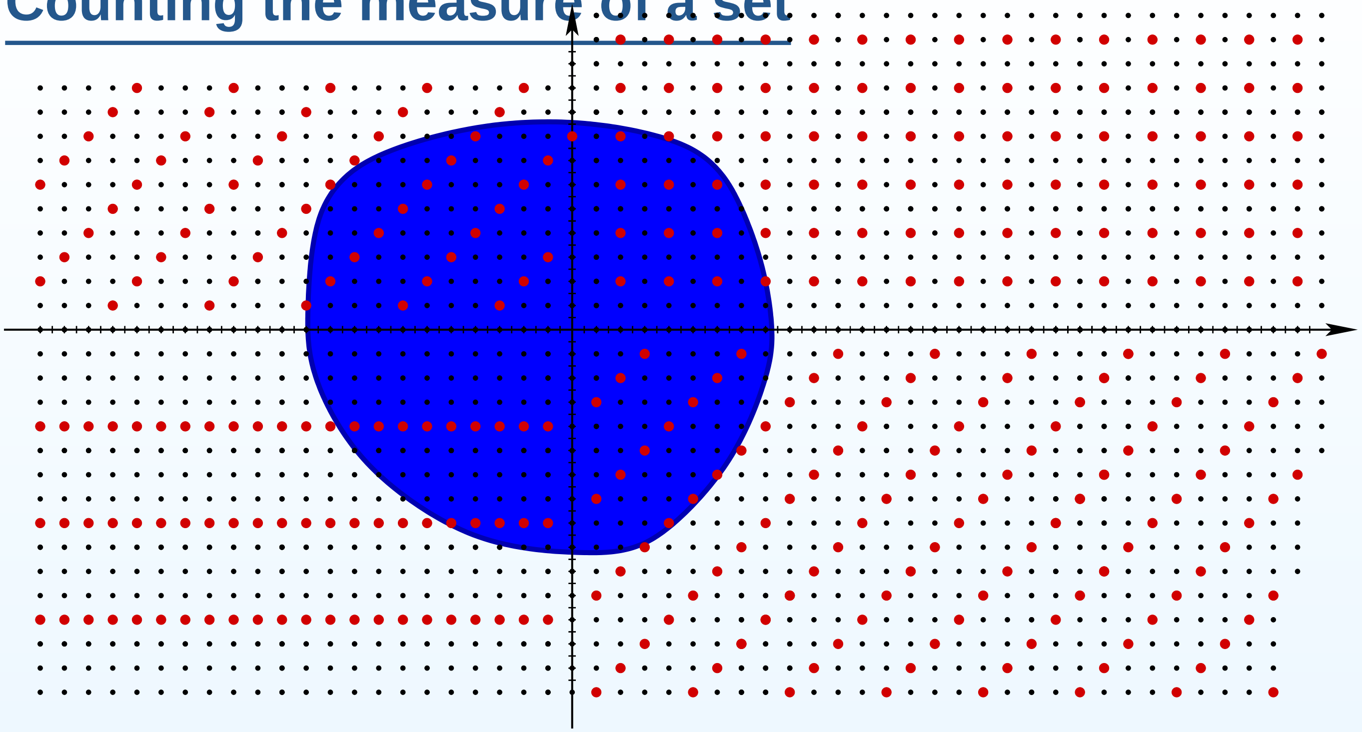

By definition, the Lebesgue measure $\mu(U)$ of a set $U \subset \mathbb{R}^n$ is defined as the limit of the normalized number of points of the ε-grid which get to U:

$$\mu(U) := \lim_{\varepsilon \to 0} \varepsilon^n \cdot \operatorname{card}(U \cap \varepsilon\mathbb{Z}^n) \, .$$

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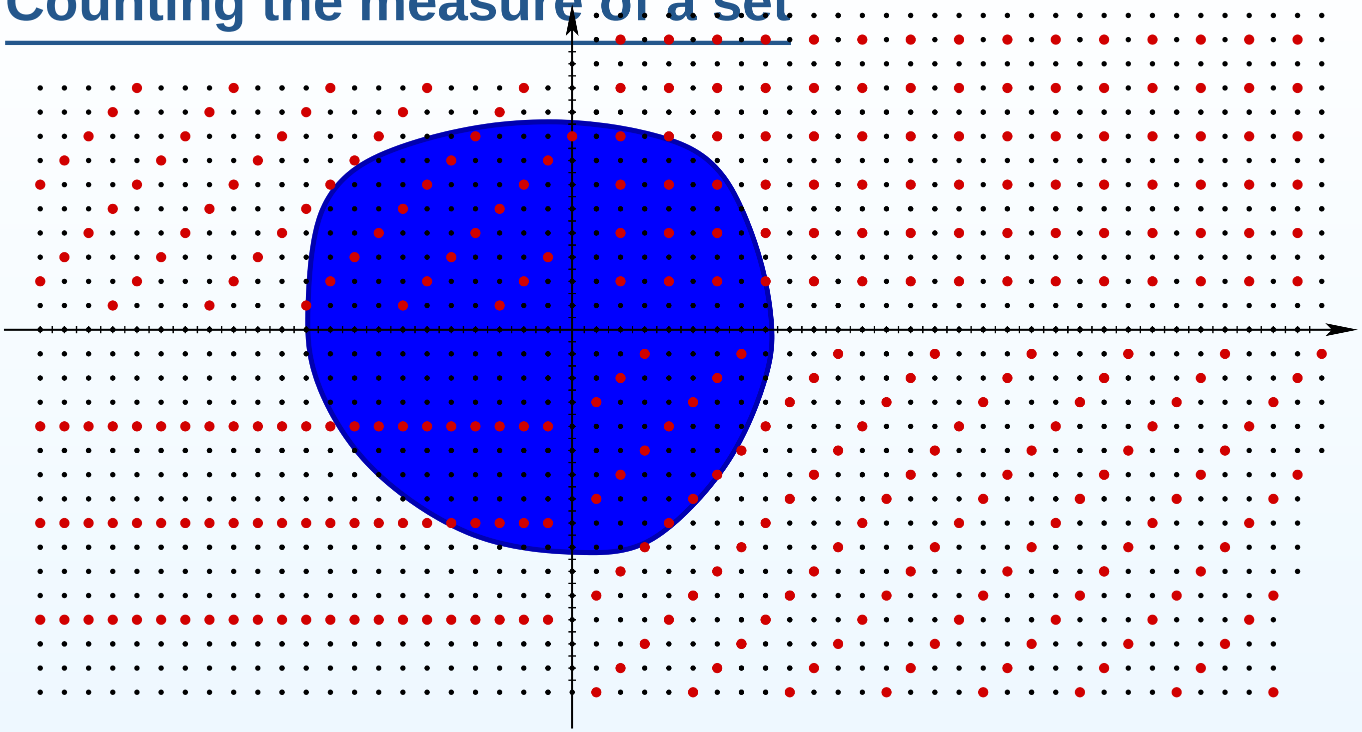

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We can fix U and scale the lattice or can fix the lattice and scale U:

$$\operatorname{card}(U \cap \varepsilon\mathbb{Z}^n) = \operatorname{card}\left(\frac{1}{\varepsilon}U \cap \mathbb{Z}^n\right)$$

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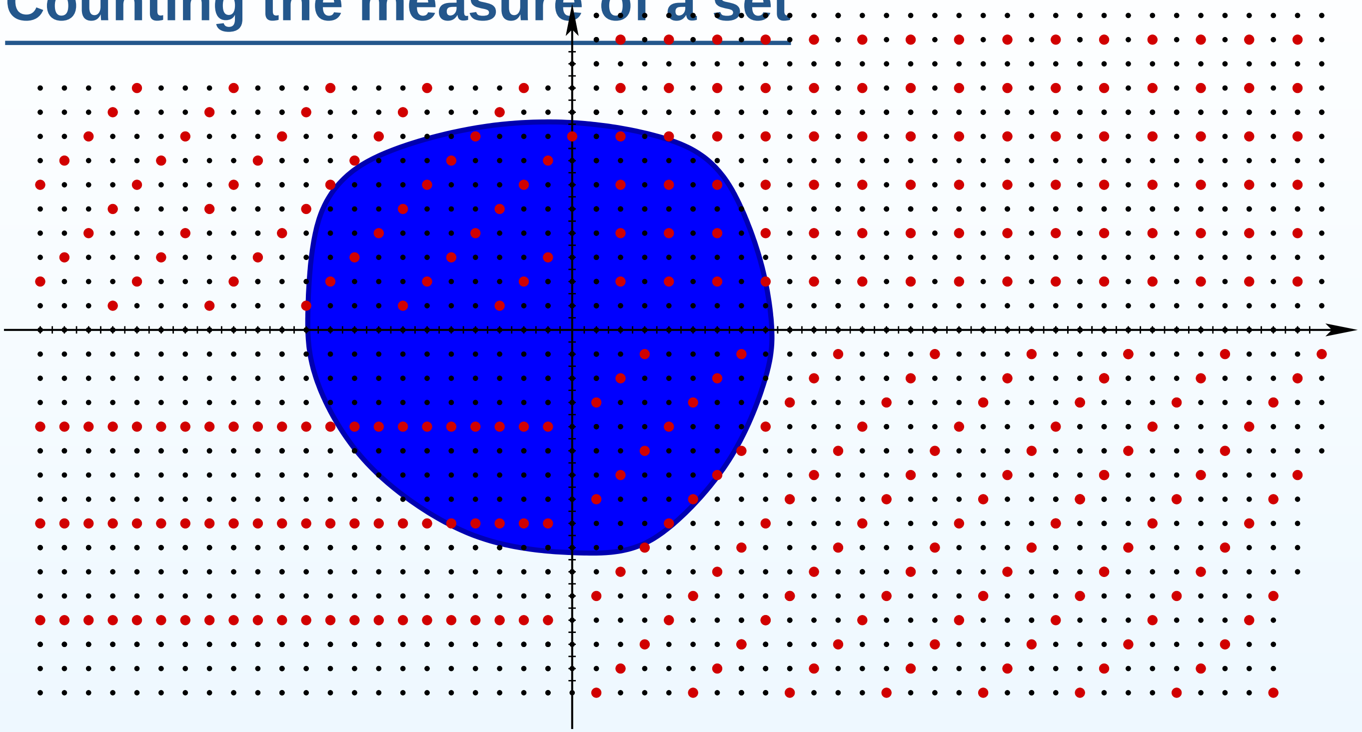
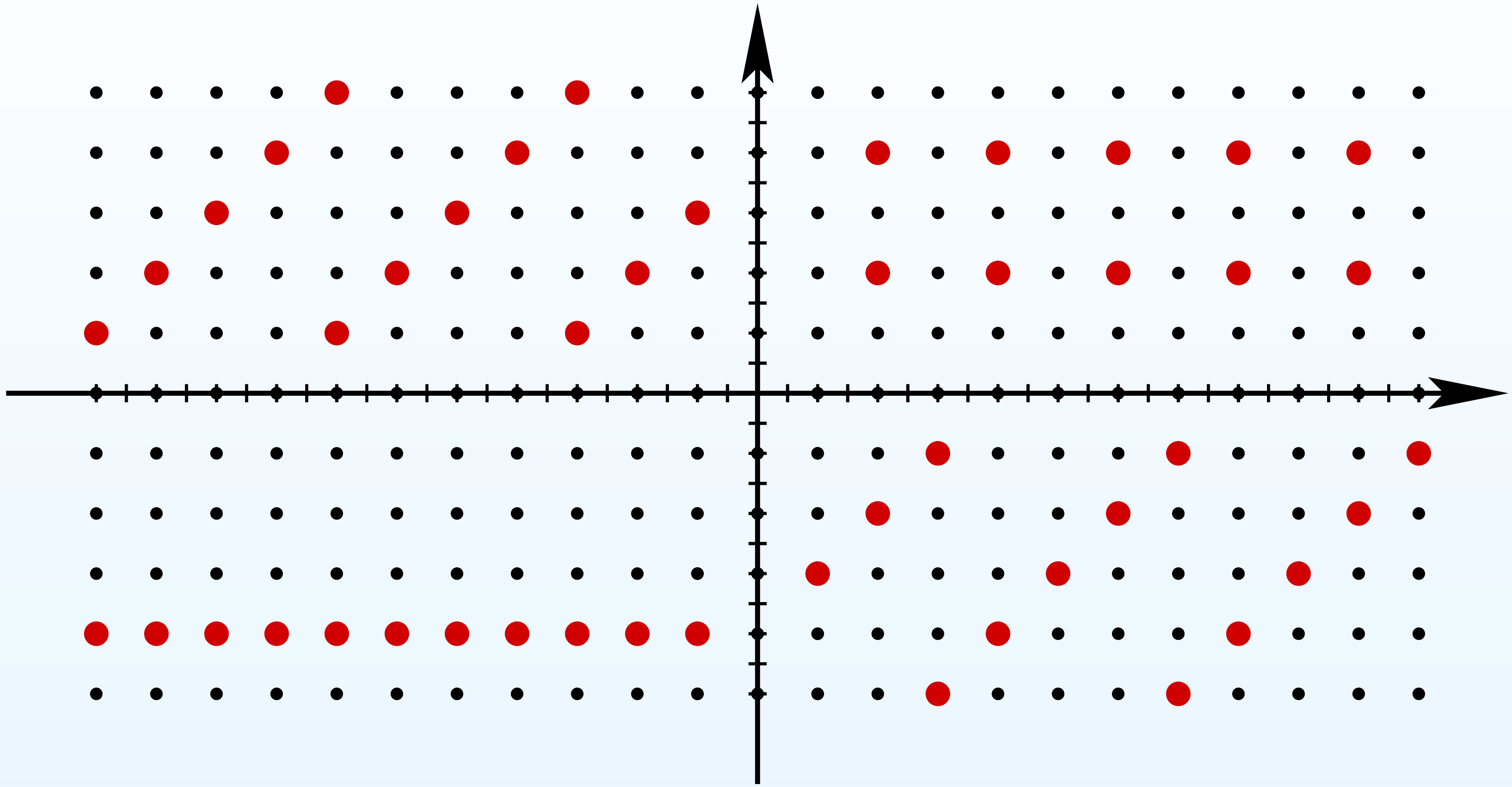


Finally, instead of using the entire lattice $\mathbb{Z}^n$ we can use any sublattice $\mathbb{L}^n \subset \mathbb{Z}^n$ having some positive *uniform density* $k > 0$ in $\mathbb{Z}^n$.

For example, the set of coprime integral points in $\mathbb{Z}^2$ has density $k = \dfrac{6}{\pi^2}$ and can be also used to define the Lebesgue measure (scaled by the factor k) in any of the two ways discussed above.

Mirzakhani's measures on $\mathcal{ML}_{g,n}$



Choose some integral multicurve γ, say, a simple closed curve on $S_{g,n}$. The subset $\mathcal{O}_\gamma := \mathrm{Mod}_{g,n} \cdot \gamma$ can be seen as an analog of a "sublattice" in $\mathcal{ML}_{g,n}(\mathbb{Z})$. The insight of Mirzakhani was to realize that replacing the discrete set $\mathcal{ML}_{g,n}(\mathbb{Z})$ with the subset $\mathcal{O}_\gamma$ we get a new measure on $\mathcal{ML}_{g,n}$ which is proportional to the Thurston measure μ_{Th} with coefficient depending only on the homotopy type of γ.

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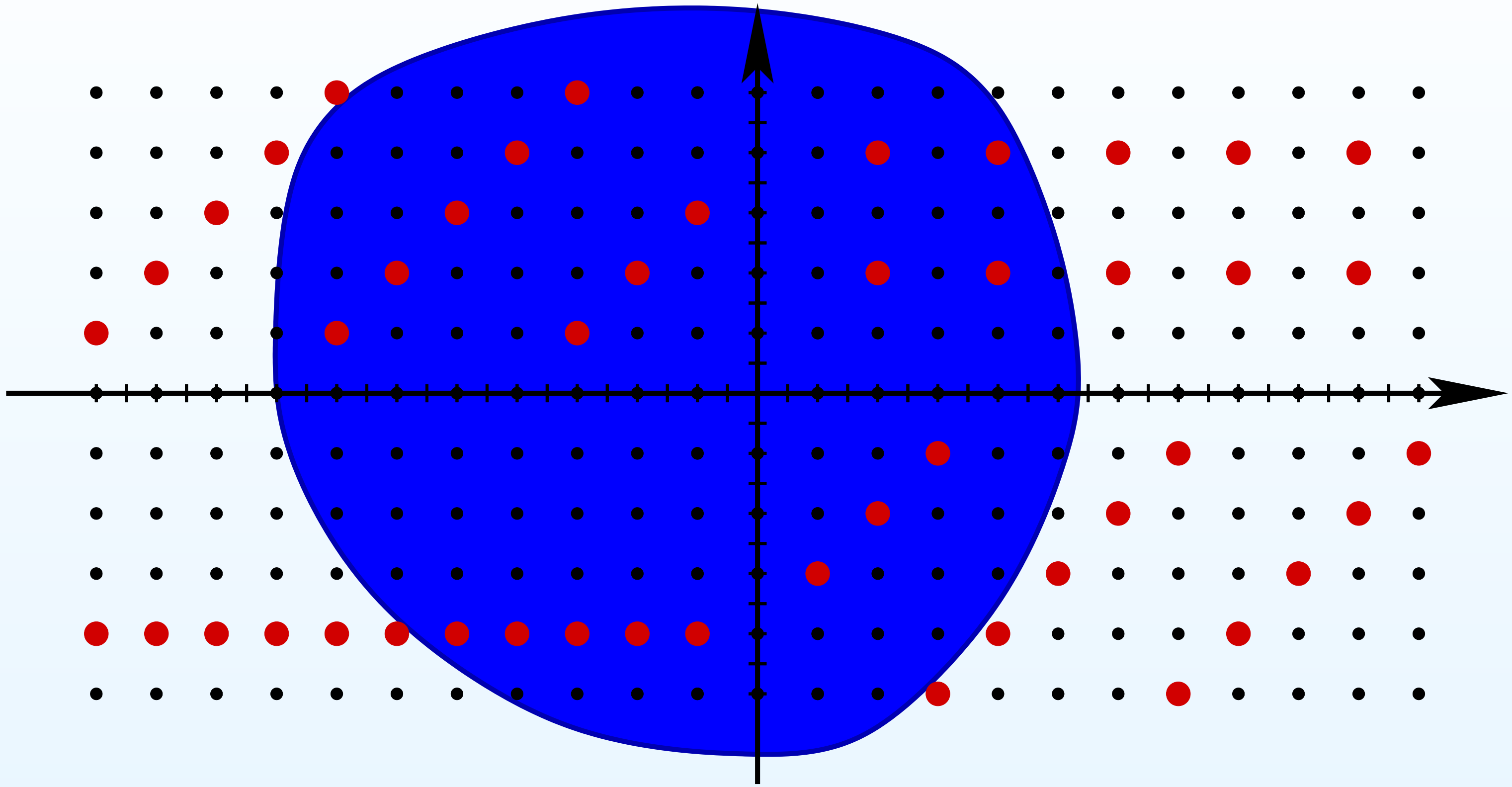

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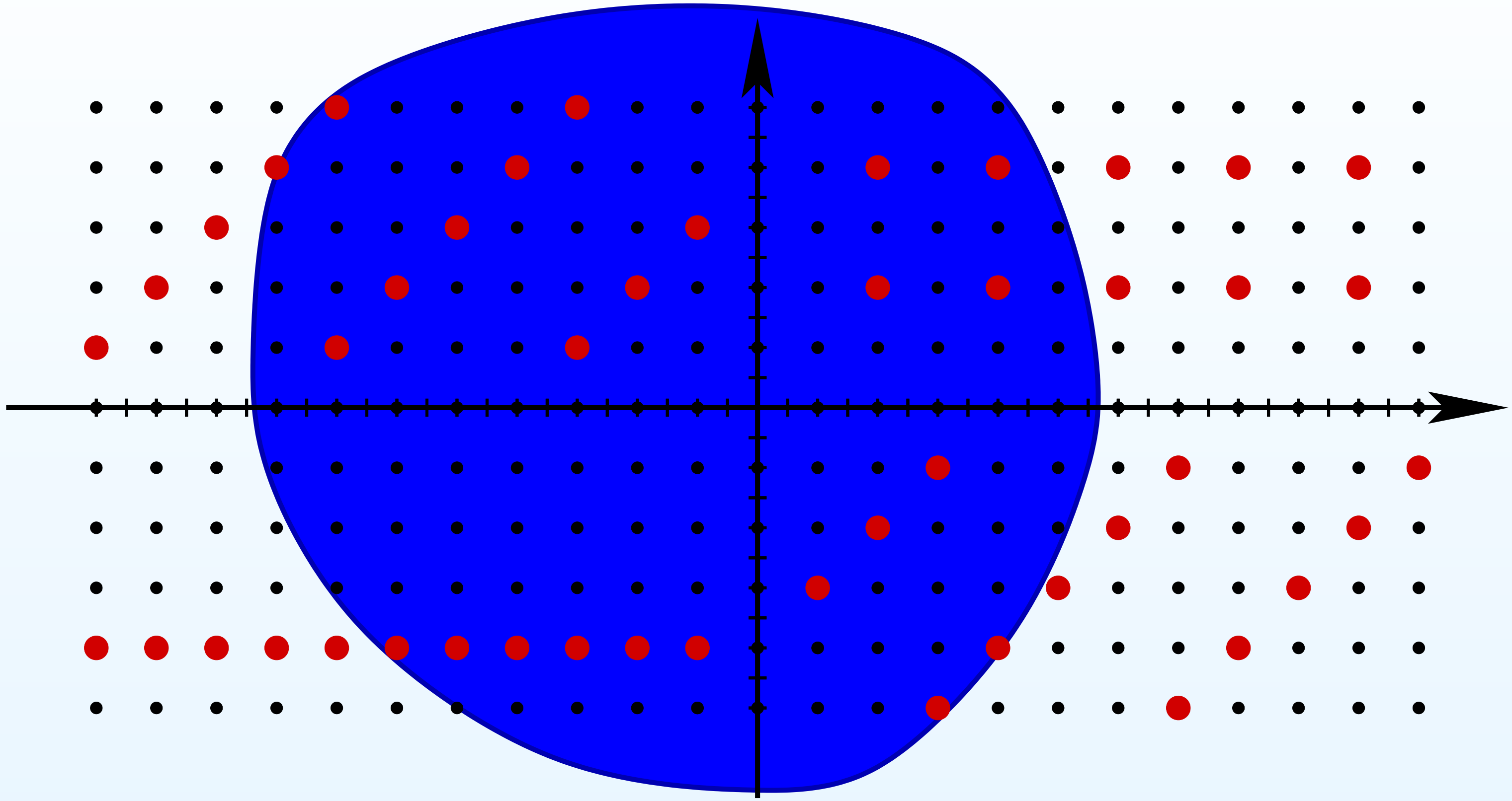

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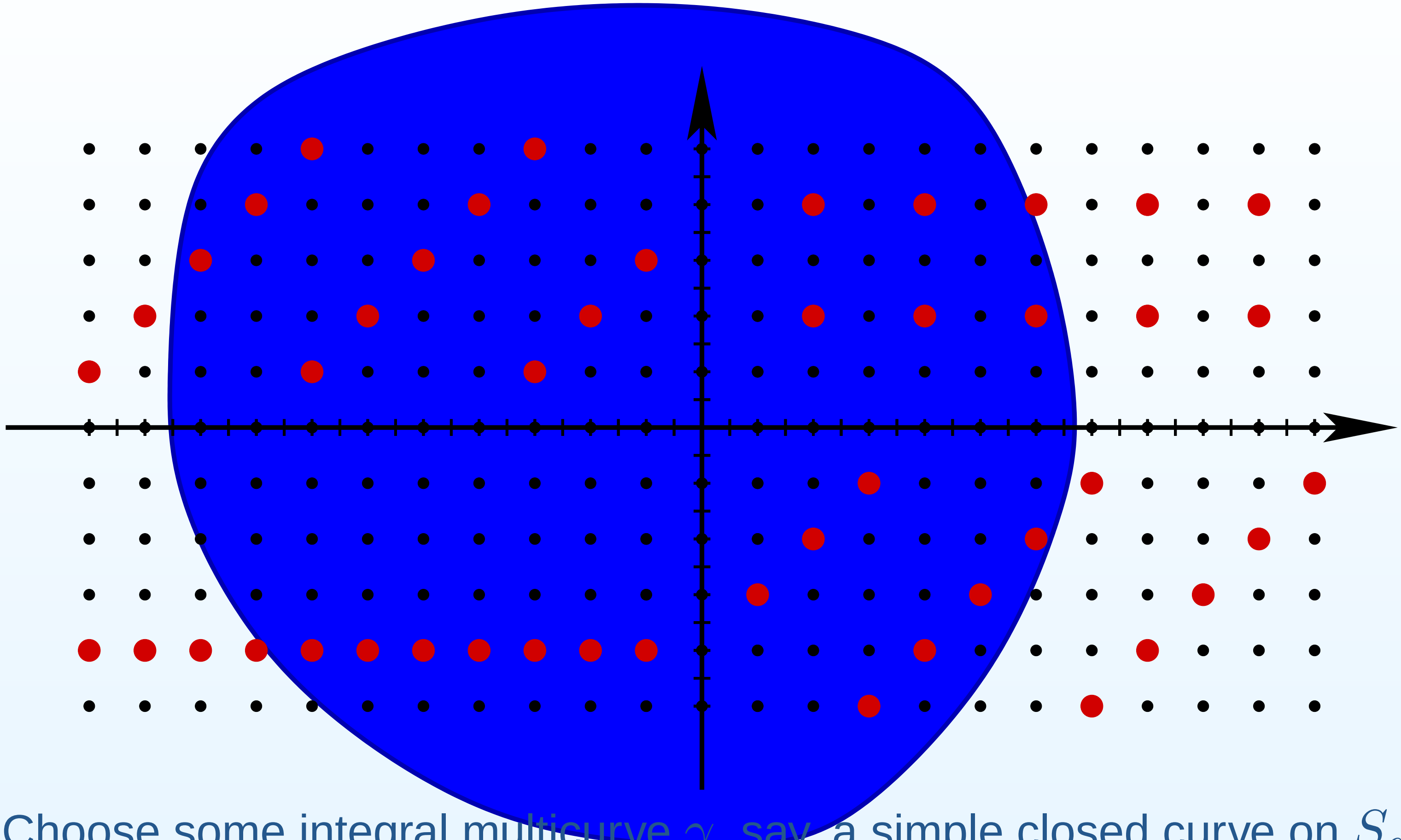

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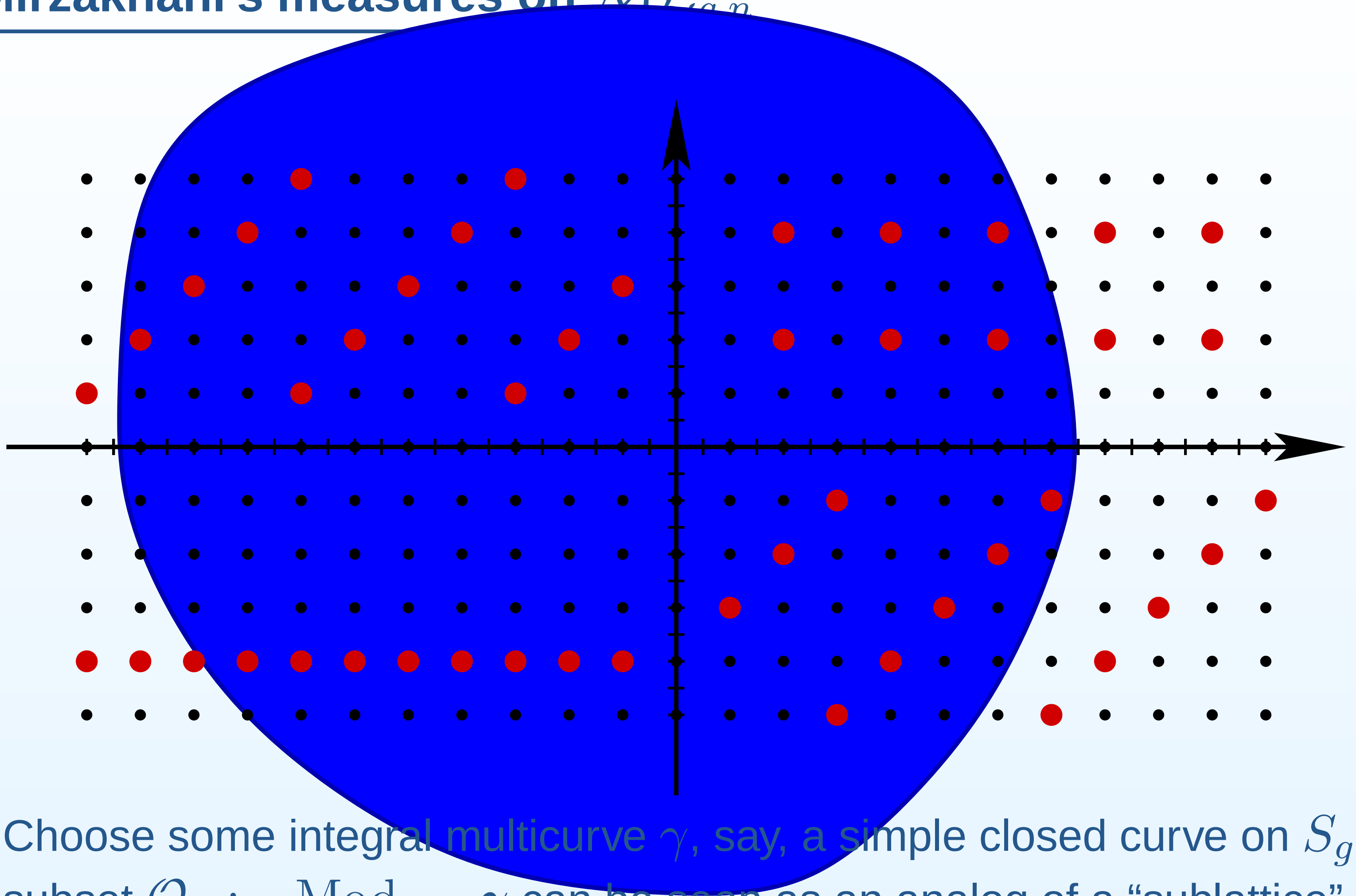

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Mirzakhani's measures on $\mathcal{ML}_{g,n}$

More formally: the Thurston measure of a subset $U \subset \mathcal{ML}_{g,n}$ is defined as

$$\mu_{\mathrm{Th}}(U) := \lim_{t\to+\infty} \frac{\operatorname{card}\{tU \cap \mathcal{ML}_{g,n}(\mathbb{Z})\}}{t^{6g-6+2n}}\,.$$

Mirzakhani defines a new measure μ_γ as

$$\mu_\gamma(U) := \lim_{t\to+\infty} \frac{\operatorname{card}\{tU \cap \mathcal{O}_\gamma\}}{t^{6g-6+2n}}\,.$$

Clearly, for any U we have $\mu_\gamma(U) \le \mu_{\mathrm{Th}}(U)$ since $\mathcal{O}_\gamma \subset \mathcal{ML}_{g,n}(\mathbb{Z})$, so μ_γ belongs to the Lebesgue measure class. By construction μ_γ is $\mathrm{Mod}_{g,n}$-invariant. Ergodicity of μ_{Th} implies that $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ where $k_\gamma = const$.

It remains to prove, however, that k_γ, which formally depends on a subsequence of times $\{t_i\}_i$, is one and the same for all subsequences; that $k_\gamma > 0$ for any topological type of a multicurve; and to compute k_γ.

Proof of the main result

Length function and unit ball

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space. One can extend the length function to simple closed multicurves $\ell_{\sum a_i\gamma_i} = \sum a_i \ell_{\gamma_i}(X)$ by linearity. By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t\cdot\lambda}(X) = t \cdot \ell_\lambda(X)$.

Each hyperbolic metric X defines its own "unit ball" B_X in $\mathcal{ML}_{g,n}$:

$$B_X := \{\lambda \in \mathcal{ML}_{g,n} \,|\, \ell_\lambda(X) \leq 1\}\,.$$

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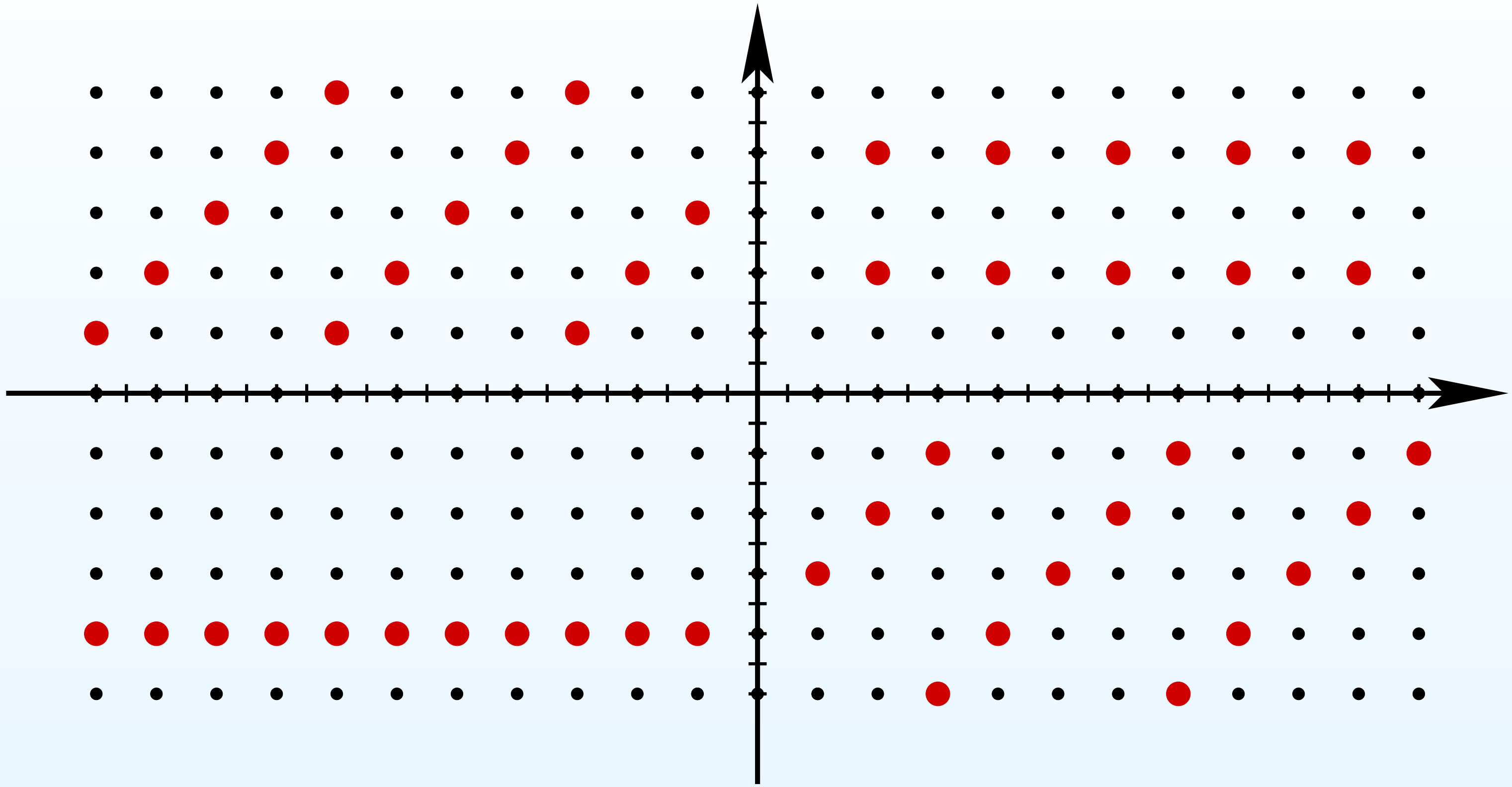

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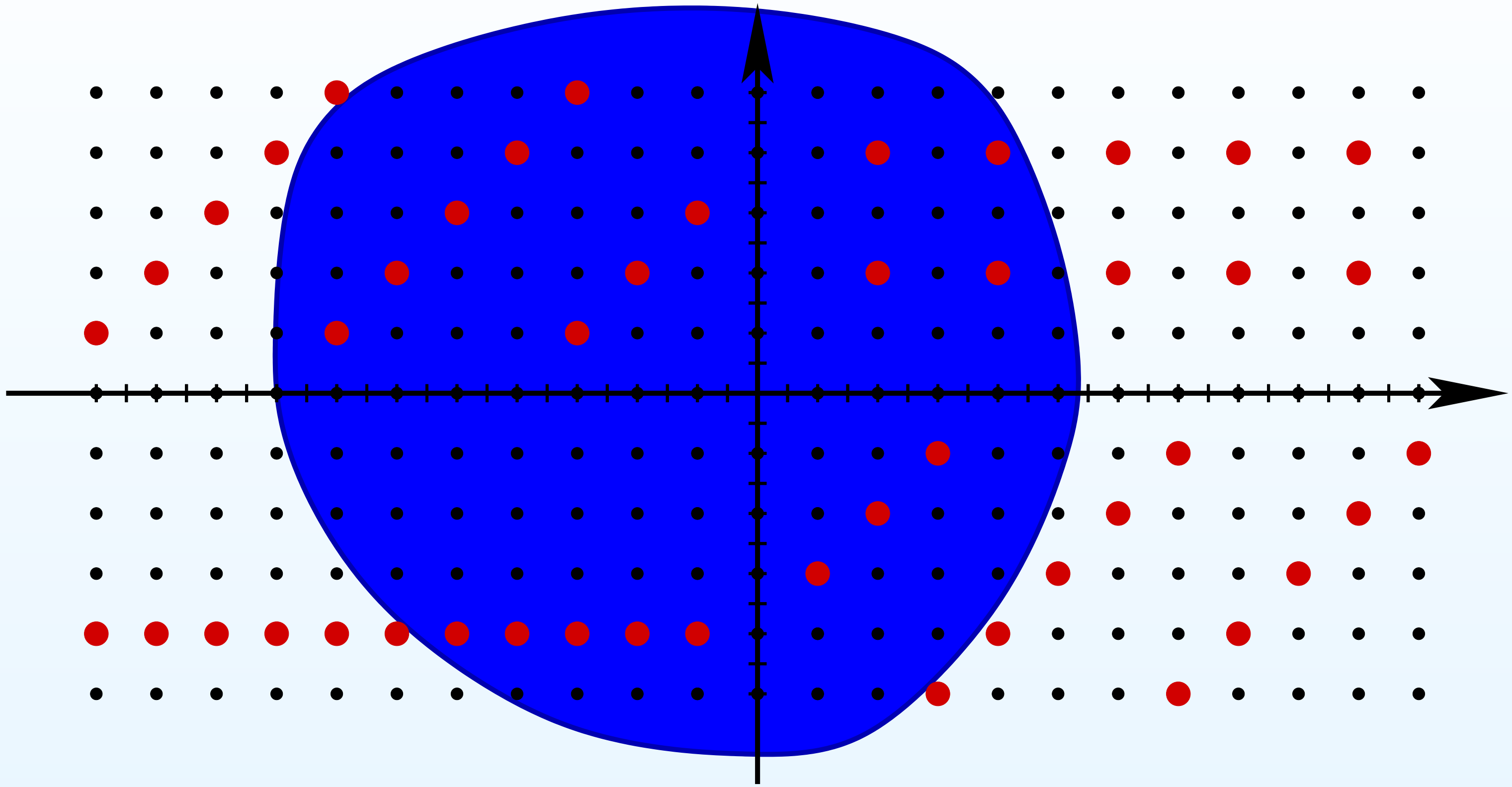

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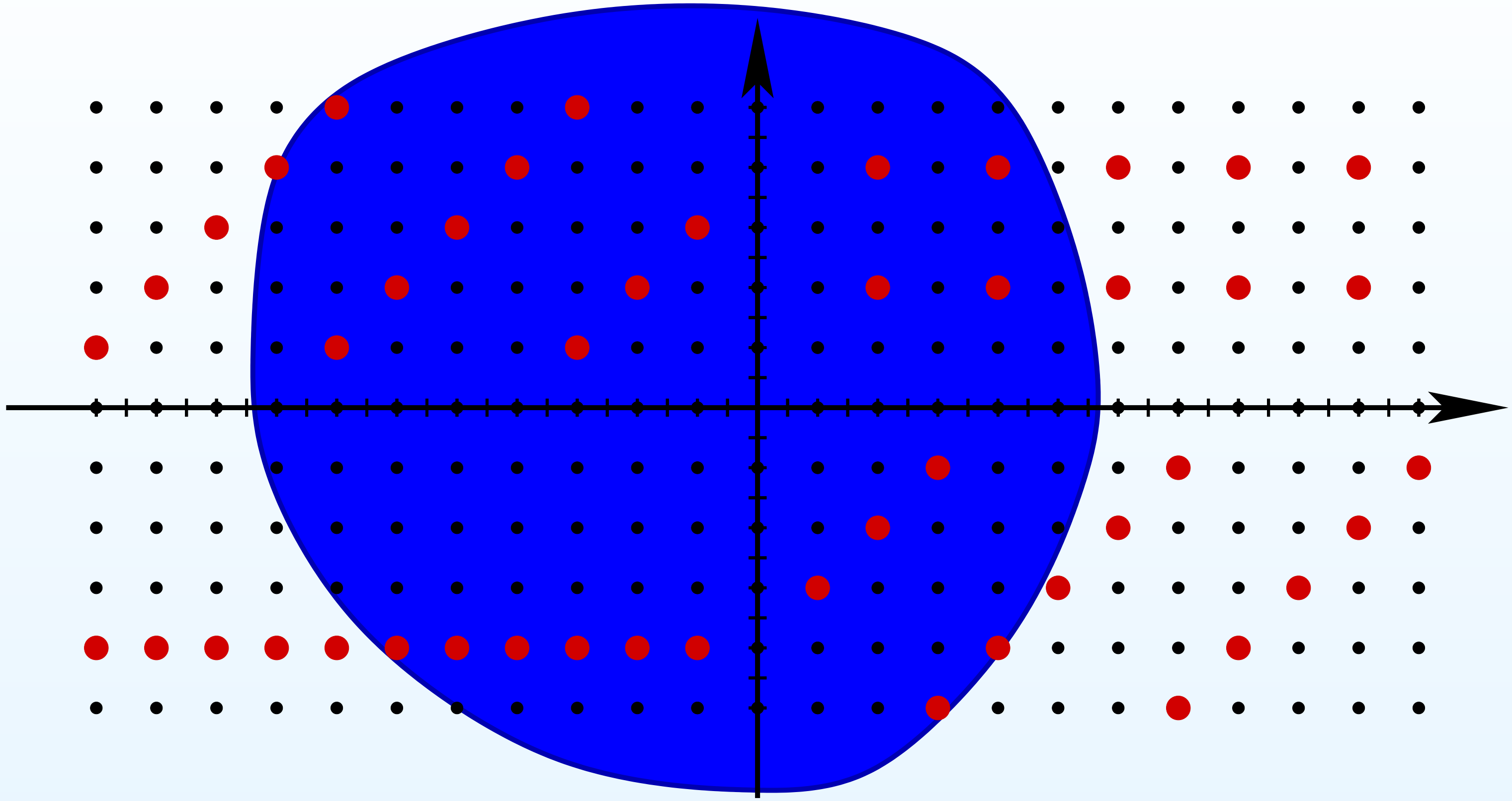

By definition of μ_{Th}, the Thurston volume of the unit ball is equal to the normalized number of integral points in a “ball of radius L” associated to X:

$$\mu_{\mathrm{Th}}(B_X) = \lim_{L\to+\infty} \frac{\mathrm{card}\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell_\lambda(X) \leq L\}}{L^{6g-6+2n}}\,.$$

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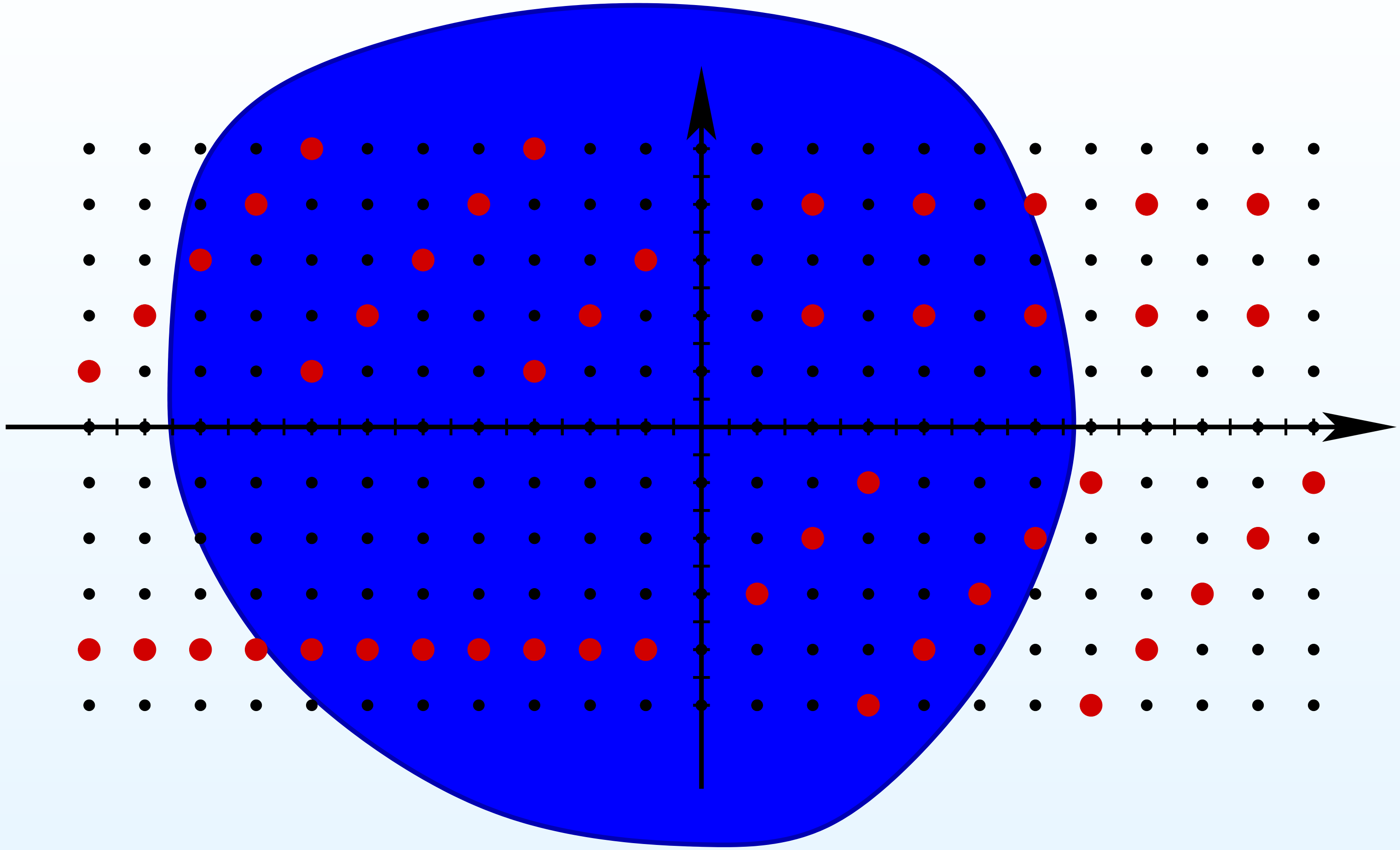

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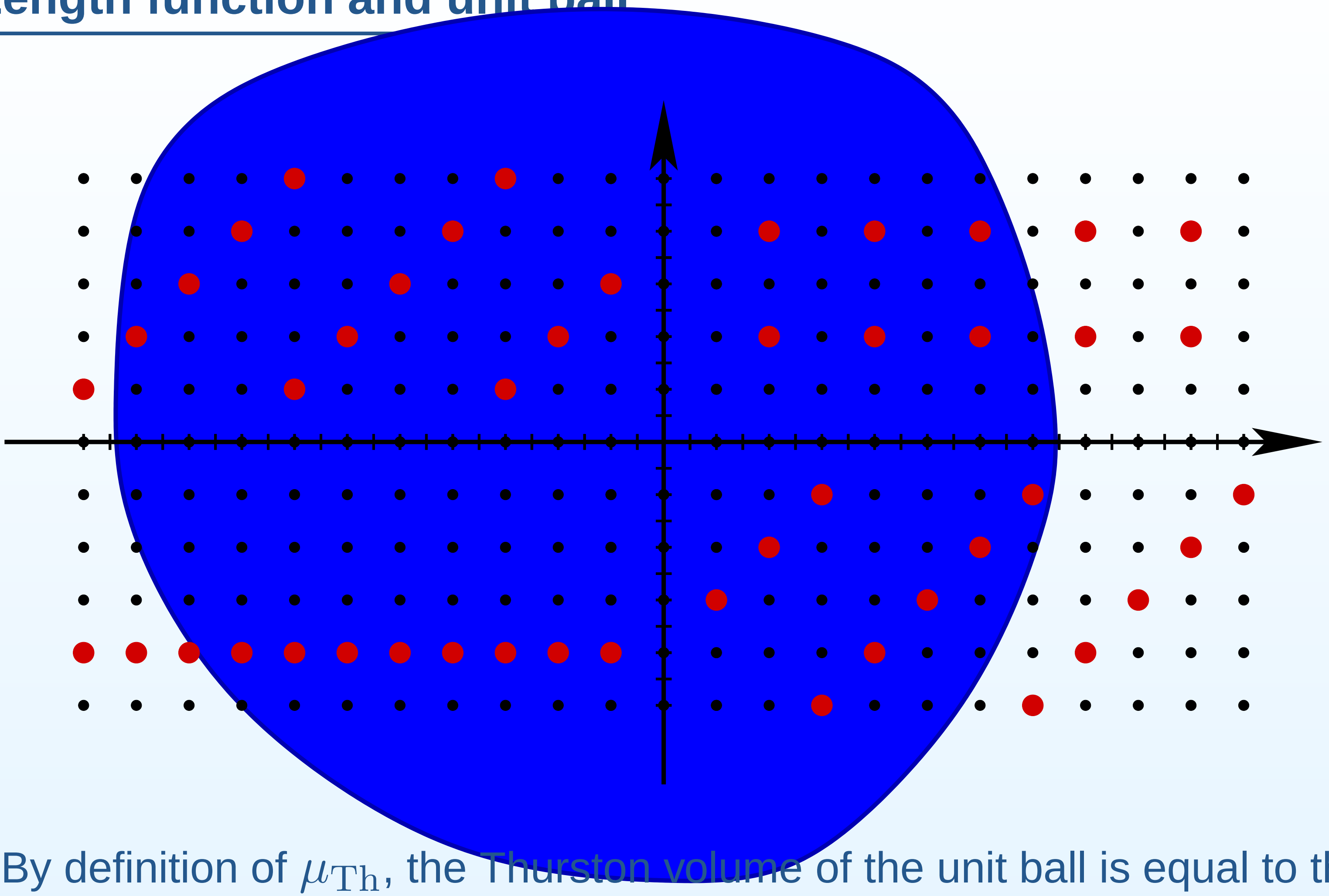

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$$\mu_{\mathrm{Th}}(B_X) = \lim_{L \to +\infty} \frac{\mathrm{card}\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \,|\, \ell_\lambda(X) \leq L\}}{L^{6g-6+2n}}.$$

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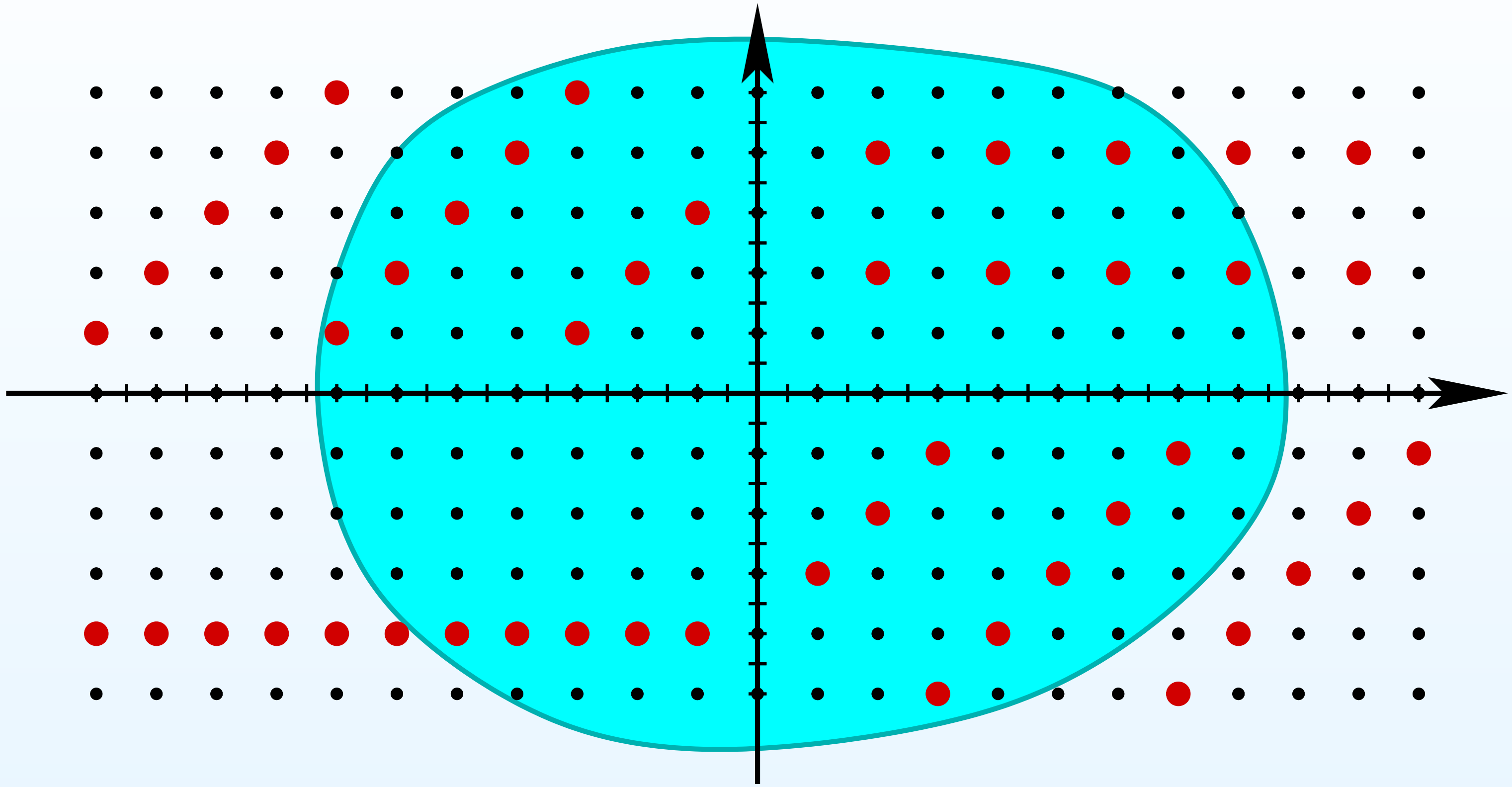

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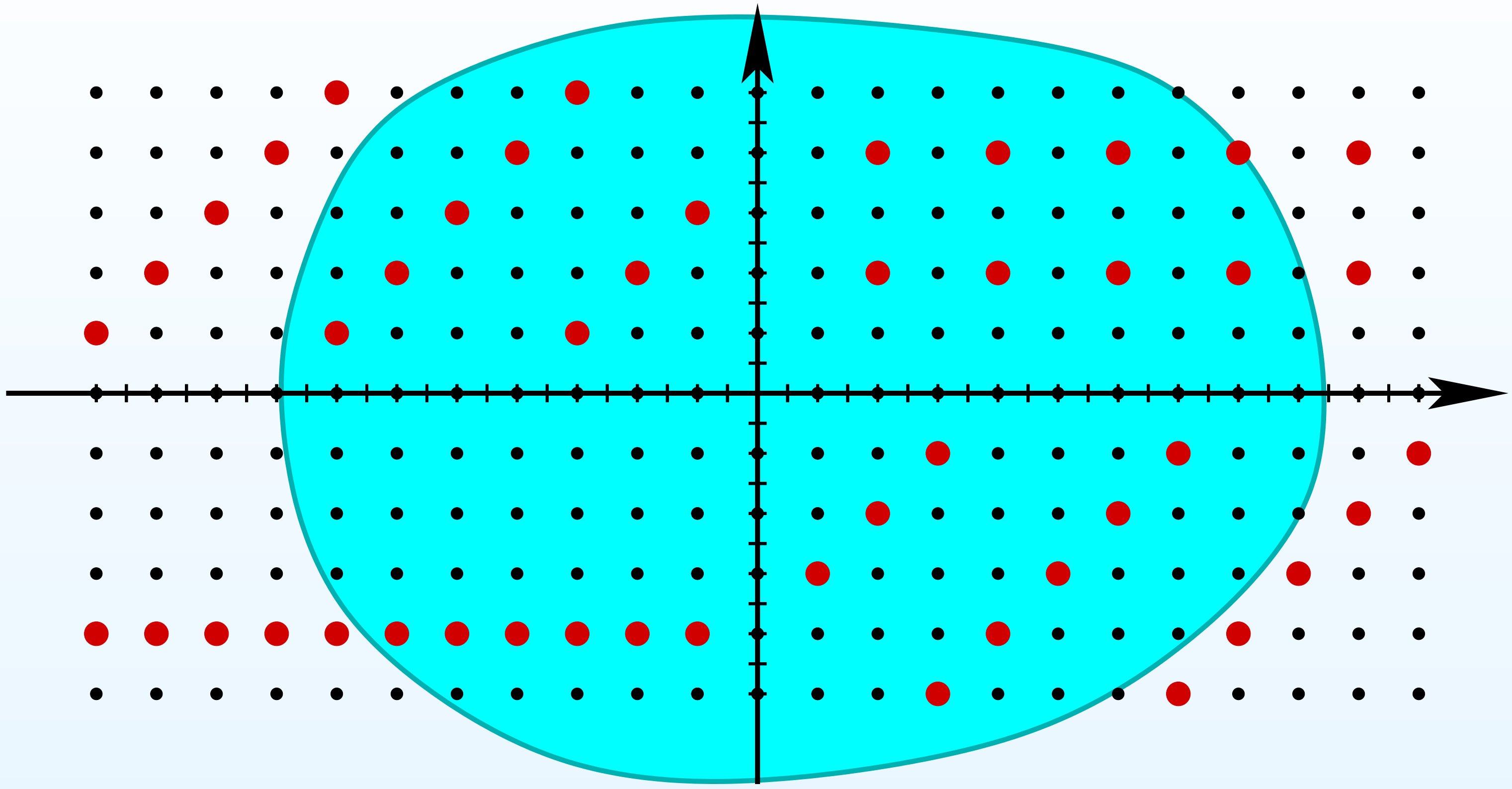

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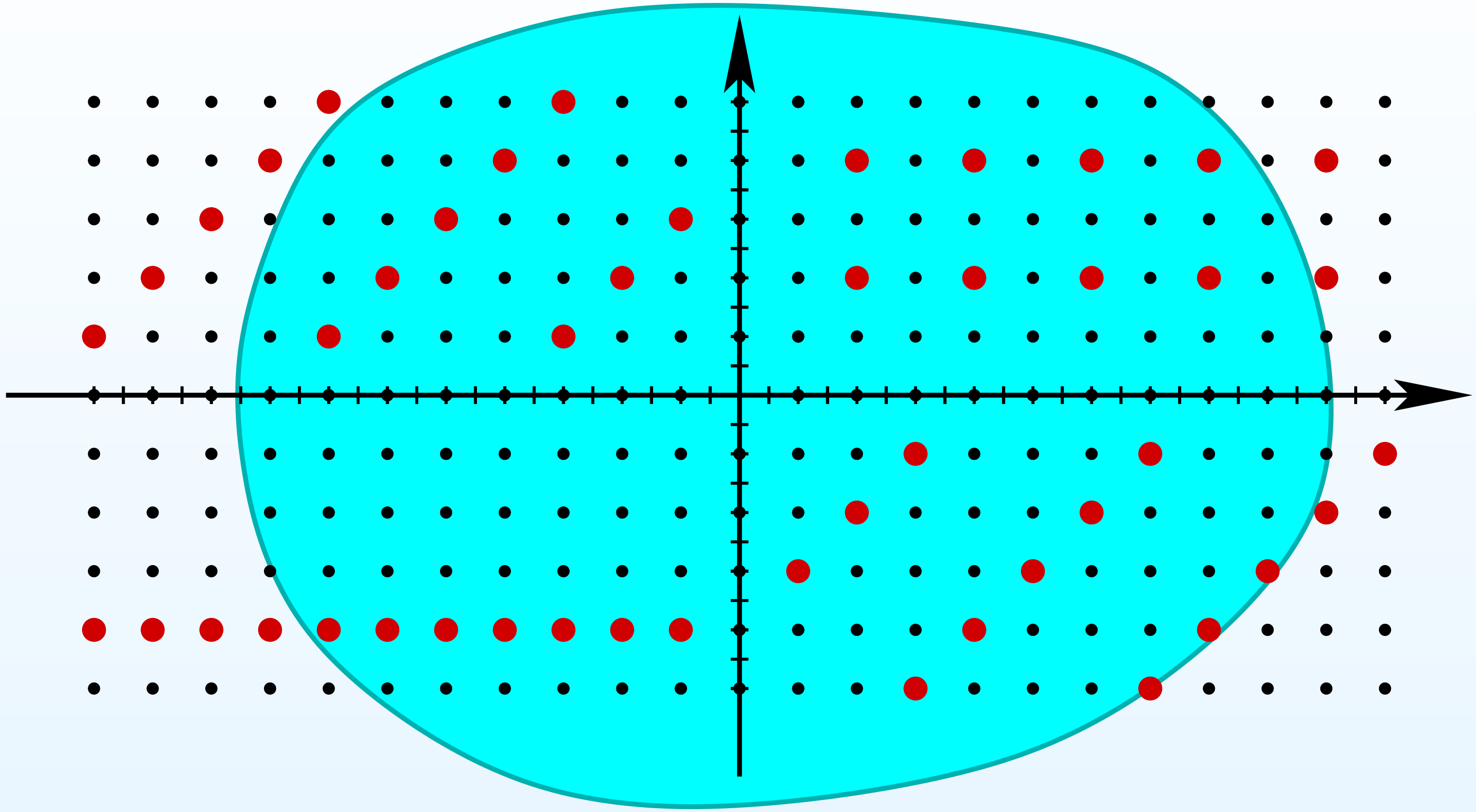

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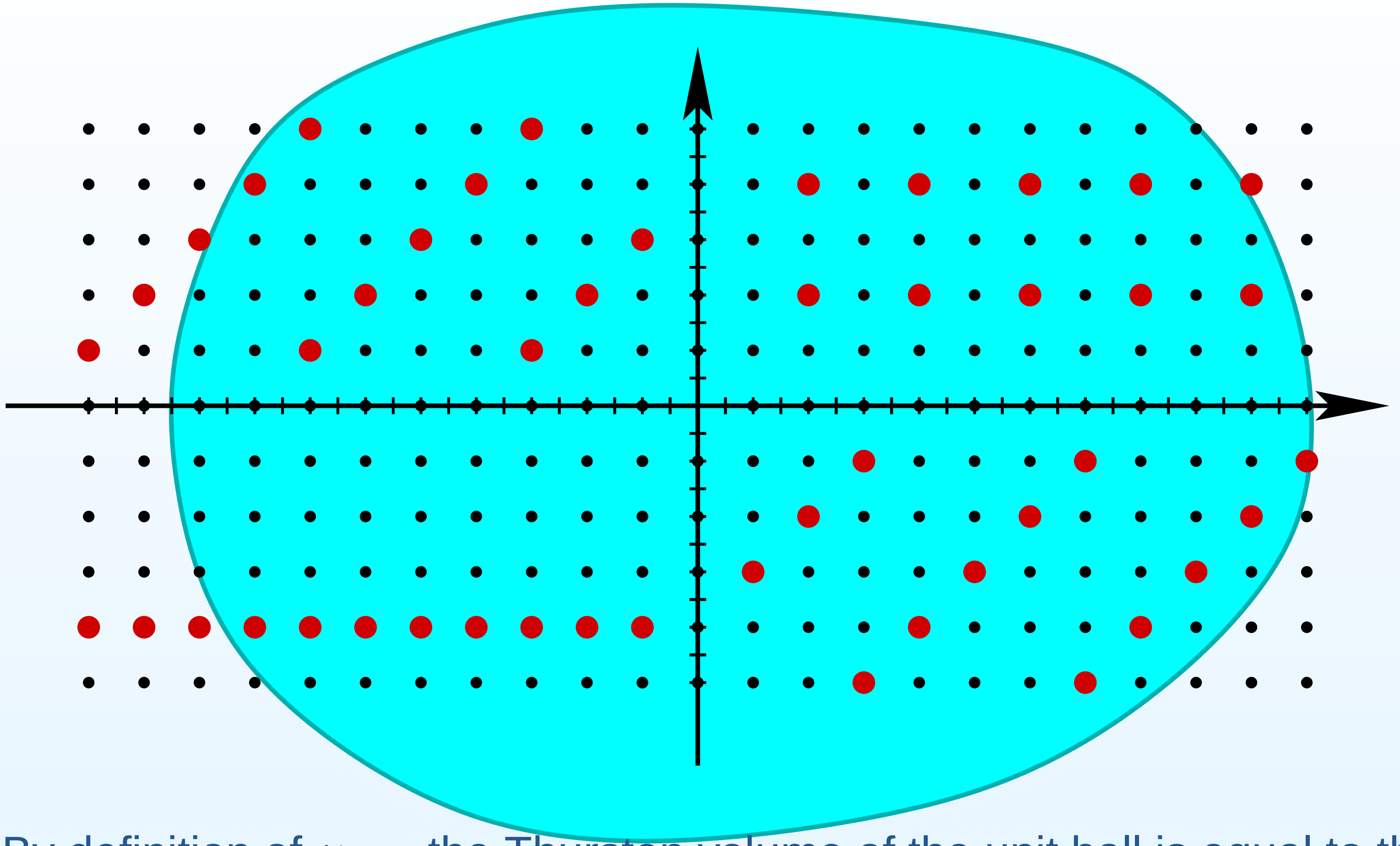

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Summary of notations

- X — a hyperbolic surface in $\mathcal{M}_{g,n}$.
- $s_X(L,\gamma)$ — the number of geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.
- $P(L\,,\gamma) := \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX$ — the polynomial in L providing the *average* number of geodesic multicurves of topological type $[\gamma]$ and of hyperbolic length at most L over all hyperbolic surfaces $X \in \mathcal{M}_{g,n}$.
- $c(\gamma)$ — the coefficient of the leading term $L^{6g-6+2n}$ of the polynomial $P(L\,,\gamma)$.

- $B(X)$ — "Unit ball" in $\mathcal{ML}_{g,n}$ defined by means of the length function $\ell_X(\alpha)$, where $\alpha \in \mathcal{ML}_{g,n}$.
- $\mu_{\mathrm{Th}}(B(X)) := \lim_{L\to+\infty} \dfrac{\operatorname{card}\{L\cdot B_X \cap \mathcal{ML}(\mathbb{Z})\}}{L^{6g-6+2n}}$ is the Thurston measure of the unit ball $B(X)$
- $\mu_\gamma(B(X)) := \lim_{L\to+\infty} \dfrac{\operatorname{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\!\cdot\gamma\}}{L^{6g-6+2n}}$ is the Mirzakhani measure of the unit ball $B(X)$ defined by the sublattice $\mathrm{Mod}_{g,n}\!\cdot\gamma \subset \mathcal{ML}(\mathbb{Z})$.

Main counting results

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here the quantity $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X (it is the Thurstom measure of the unit ball B_X in the metric X); $b_{g,n}$ is a global constant depending only on g and n (which is the average value of $B(X)$ over $\mathcal{M}_{g,n}$); $c(\gamma)$ depends only on the topological type of γ (expressed in terms of the Witten–Kontsevich correlators).

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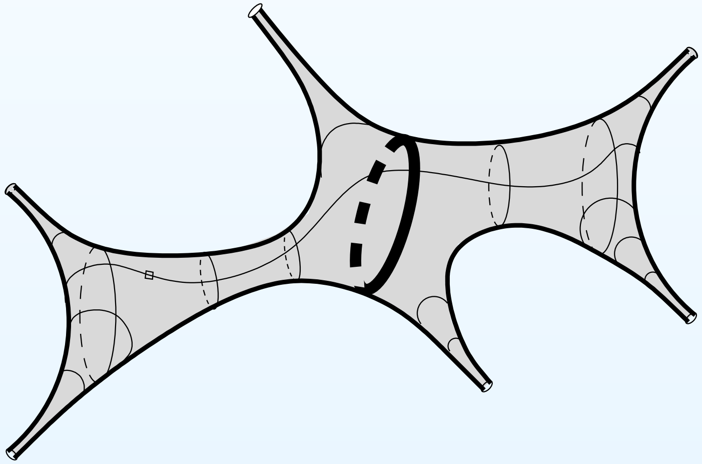
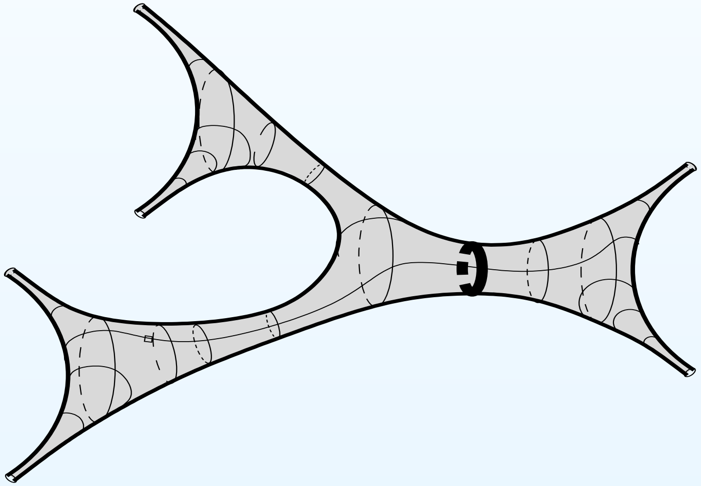
Corollary (M. Mirzakhani, 2008). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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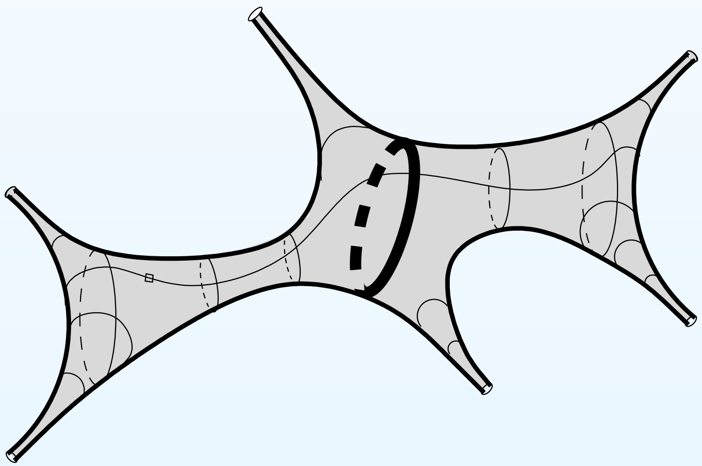
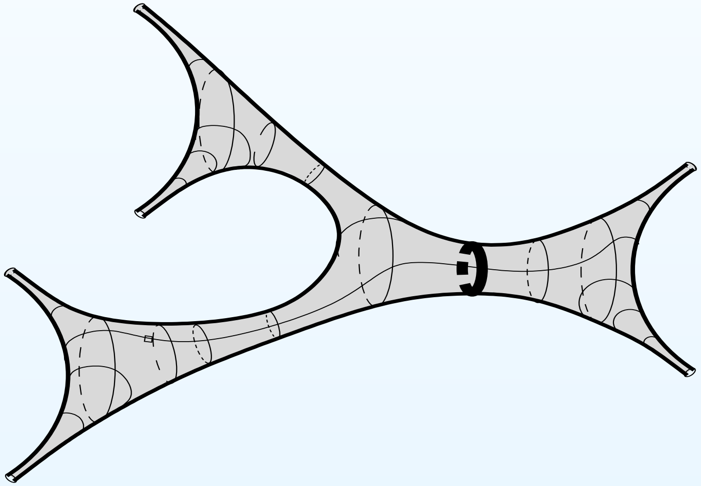


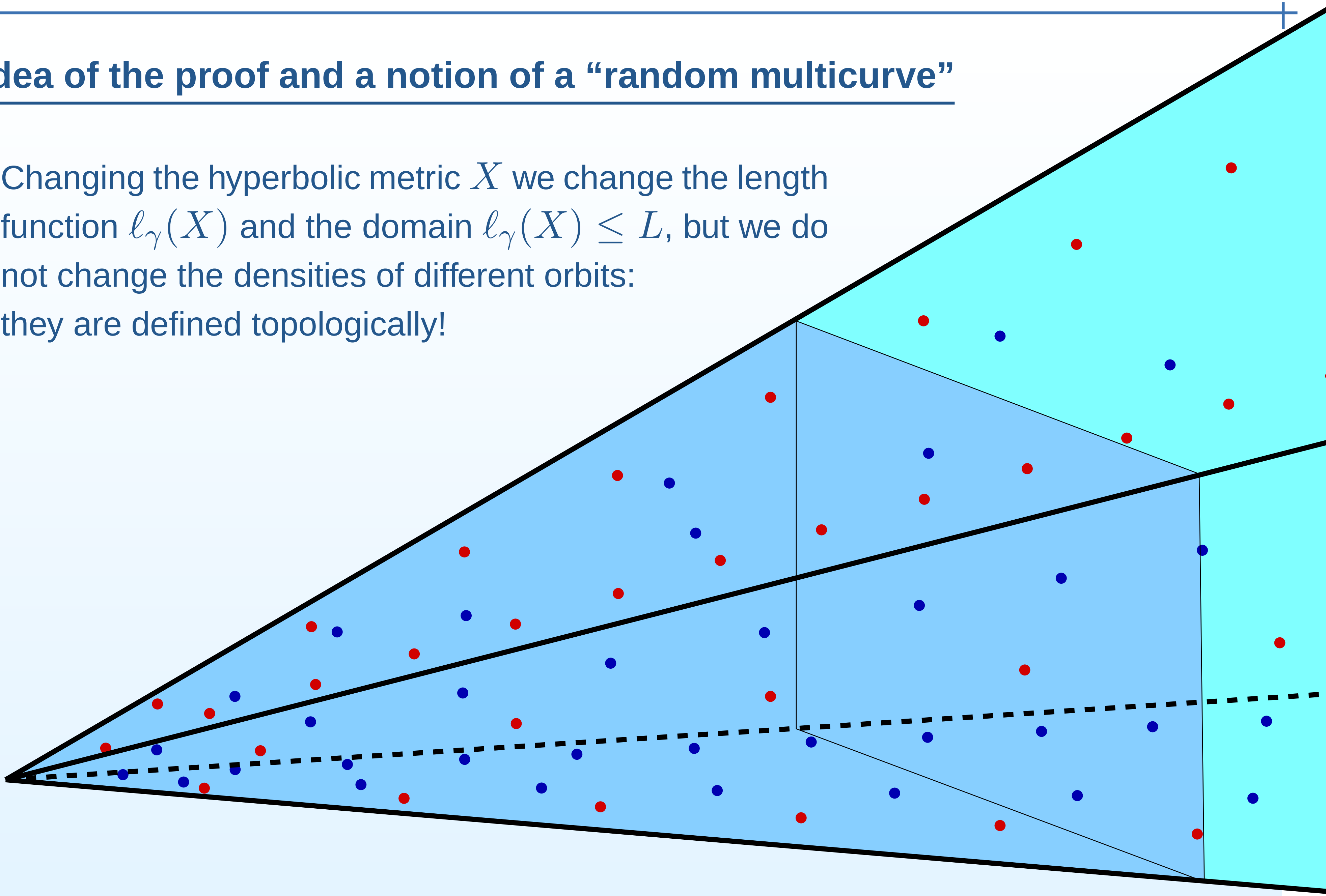


Example. (M. Mirzakhani, 2008); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by other means.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}\,.$$

Idea of the proof and a notion of a "random multicurve"

Changing the hyperbolic metric X we change the length function $\ell_\gamma(X)$ and the domain $\ell_\gamma(X) \leq L$, but we do not change the densities of different orbits: they are defined topologically!



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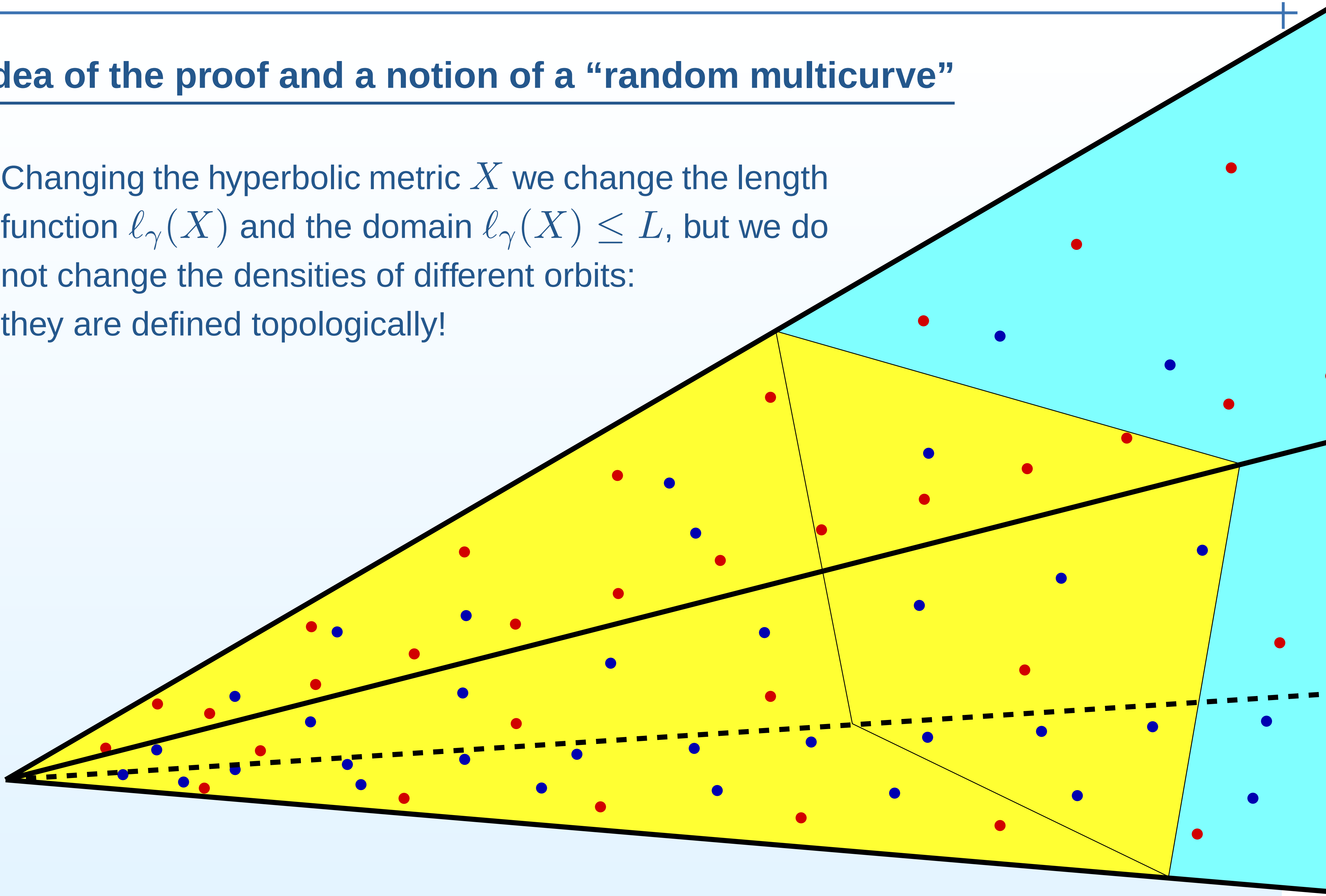

More honest idea of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the "unit ball" B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\mathrm{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma\cdot\mu_{\mathrm{Th}}(B_X)\,dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B_X)\,dX =$$

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so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzakhani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

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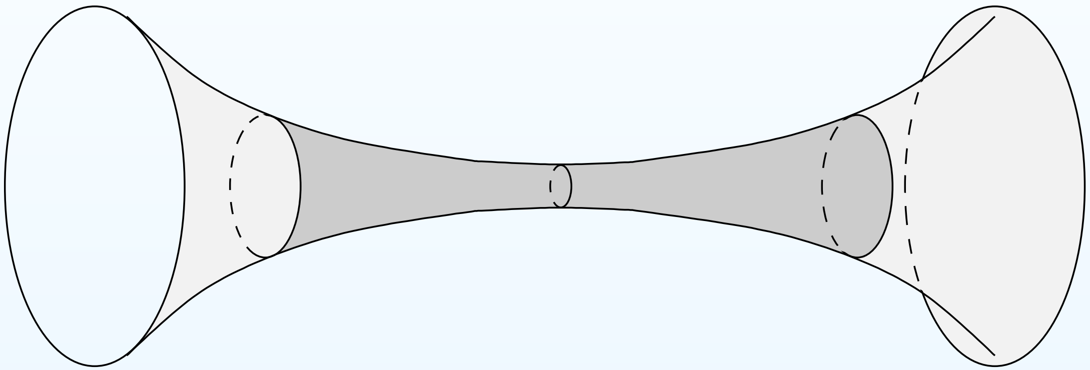
Collar lemma

Collar Lemma

Collar Lemma Informally. *Short closed geodesics on hyperbolic surfaces have large tubular neighborhoods which are topological cylinders.*



Corollary of Collar Lemma (P. Buser). *Let γ, δ be transversally intersecting closed geodesics on a closed hyperbolic surface S. Assume that γ is simple. Then*

$$\sinh\left(\tfrac{1}{2}\ell(\gamma)\right) \cdot \sinh\left(\tfrac{1}{2}\ell(\delta)\right) > 1\,.$$

A version of Collar Lemma is valid in noncompact case.

Collar Lemma: full version

Collar Lemma (as in the book of P. Buser). *Let S be a compact Riemann surface of genus $g \geq 2$, and let $\gamma_1, \ldots \gamma_m$ be pairwise disjoint simple closed geodesics on S. Then the following hold.*

i) $m \leq 3g - 3$.

ii) *There exist simple closed geodesics $\gamma_{m+1}, \ldots, \gamma_{3g-3}$ which, together with $\gamma_1, \ldots \gamma_m$ decompose S into pair of pants.*

iii) *The collars*

$$\mathcal{C}(\gamma_i) = \{p \in S \mid \operatorname{dist}(p, \gamma_i) \leq w(\gamma_i)\}$$

of widths

$$w(\gamma_i) = \operatorname{acsinh}\left(1/\sinh\left(\frac{1}{2}\ell(\gamma_i)\right)\right)$$

are pairwise disjoint for $i = 1, \ldots, 3g - 3$.

iv) *Each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times S^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell^2(\gamma_i)\cosh^2\rho\, dt^2$.*

Integrability of $F(X)$ for $\mathcal{M}_{1,1}$

Given $x, y, L > 0$, consider the set $A_{x,y}(L)$ defined by

$$A_{x,y}(L) = \{(m,n) \,|\, mx + ny \le L\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+ \,.$$

Following M. Mirzakhani notice that for $L > 1$ one has

$$\operatorname{card} A_{x,y}(L) \le 3\left(\frac{L^2}{x \cdot y} + \frac{L}{\min(x,y)} + 1\right) \le 3L^2\left(1 + \frac{1}{x \cdot y} + \frac{1}{\min(x,y)} + 1\right).$$

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Excercise. Using this observation and Collar Lemma prove that for any $X \in \mathcal{M}_{1,1}$, any sufficiently small ε and any $1 \leq L$ one has

$$\frac{b_X(L)}{L^2} \leq \frac{const(\varepsilon)}{\ell_\gamma(X)}\,,$$

where $\ell_\gamma(X)$ is the unique closed geodesic shorter than ε.

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Let $\mathrm{sys}(X)$ be the *systole* of X — the length of the shortest closed geodesic on X.
Using Fenchel–Nielsen coordinates prove that $F(X) = \frac{1}{\mathrm{sys}(X)}$ is integrable over $\mathcal{M}_{1,1}$.

Integration over $\mathcal{M}_{1,1}$

Mirzakhani observed that having a continuous function $f_\gamma(X)$ on $\mathcal{M}_{1,1}$ of the form

$$f_\gamma(X) = \sum_{[\alpha]\in \mathrm{Mod}_{1,1}\cdot[\gamma]} f(\ell_X(\alpha))$$

we can integrate it over $\mathcal{M}_{1,1}$ as follows

$$\int_{\mathcal{M}_{1,1}} \sum_{[\alpha]\in \mathrm{Mod}_{1,1}\cdot[\gamma]} f(\ell_\alpha(X))\, dX = \int_{\mathcal{M}_{1,1}^{\gamma}} f(\ell_\alpha(X))\, dl\, d\tau =$$

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Note that our counting function $s_X(L,\gamma)$ is exactly of this form with $f = \chi([0,L])$. In this particular case we get

$$P(L,\gamma) := \int_{\mathcal{M}_{1,1}} s_X(L,\gamma)\, dX = \int_0^\infty \chi([0,L])\, l\, dl = \int_0^L \ell d\ell = \frac{L^2}{2}\,.$$

and thus $c(\gamma) = \lim_{L\to+\infty} \frac{P(L,\gamma)}{L^2} = \frac{1}{2} > 0$.

Uniform density of coprime integer points

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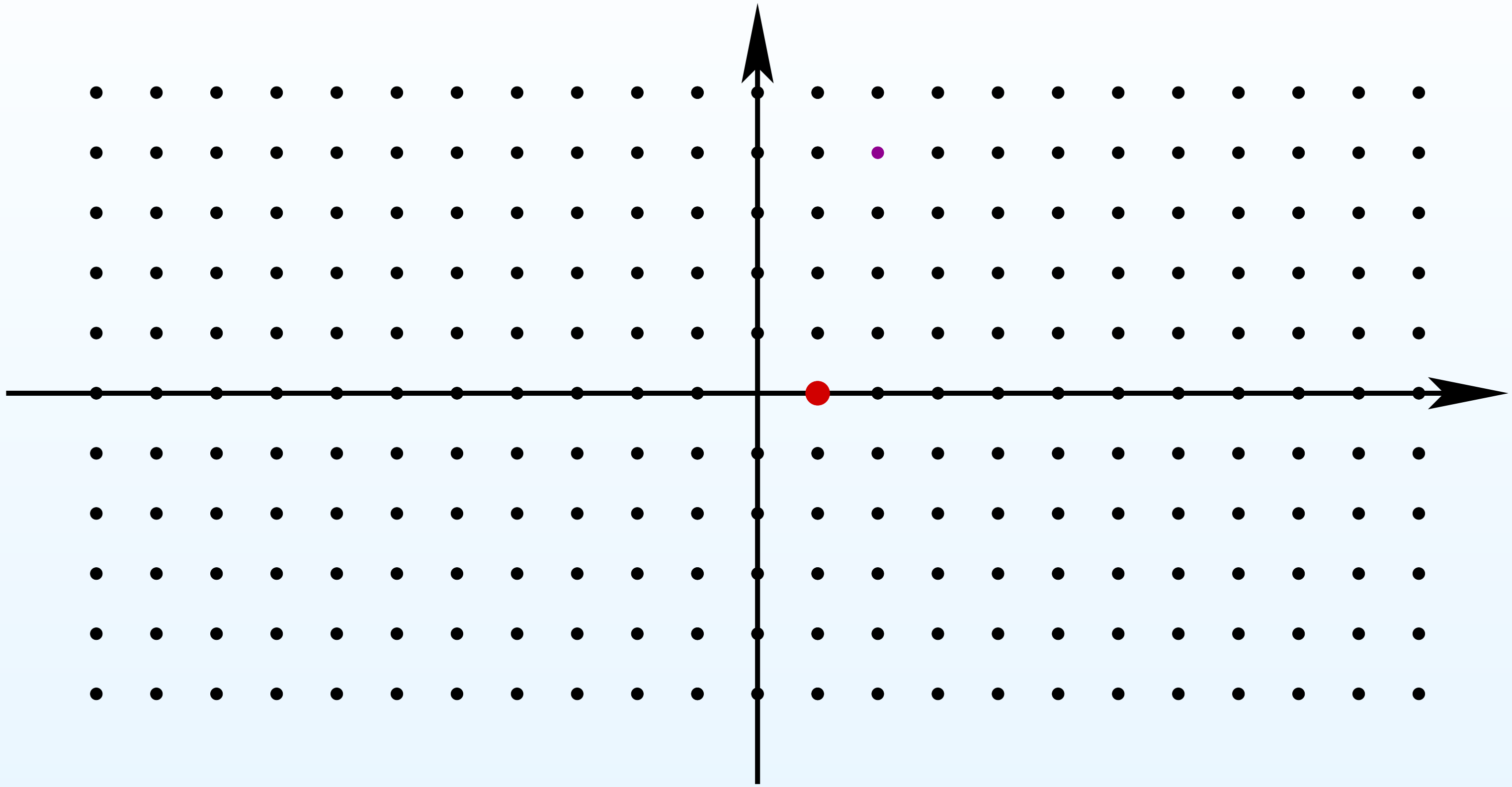
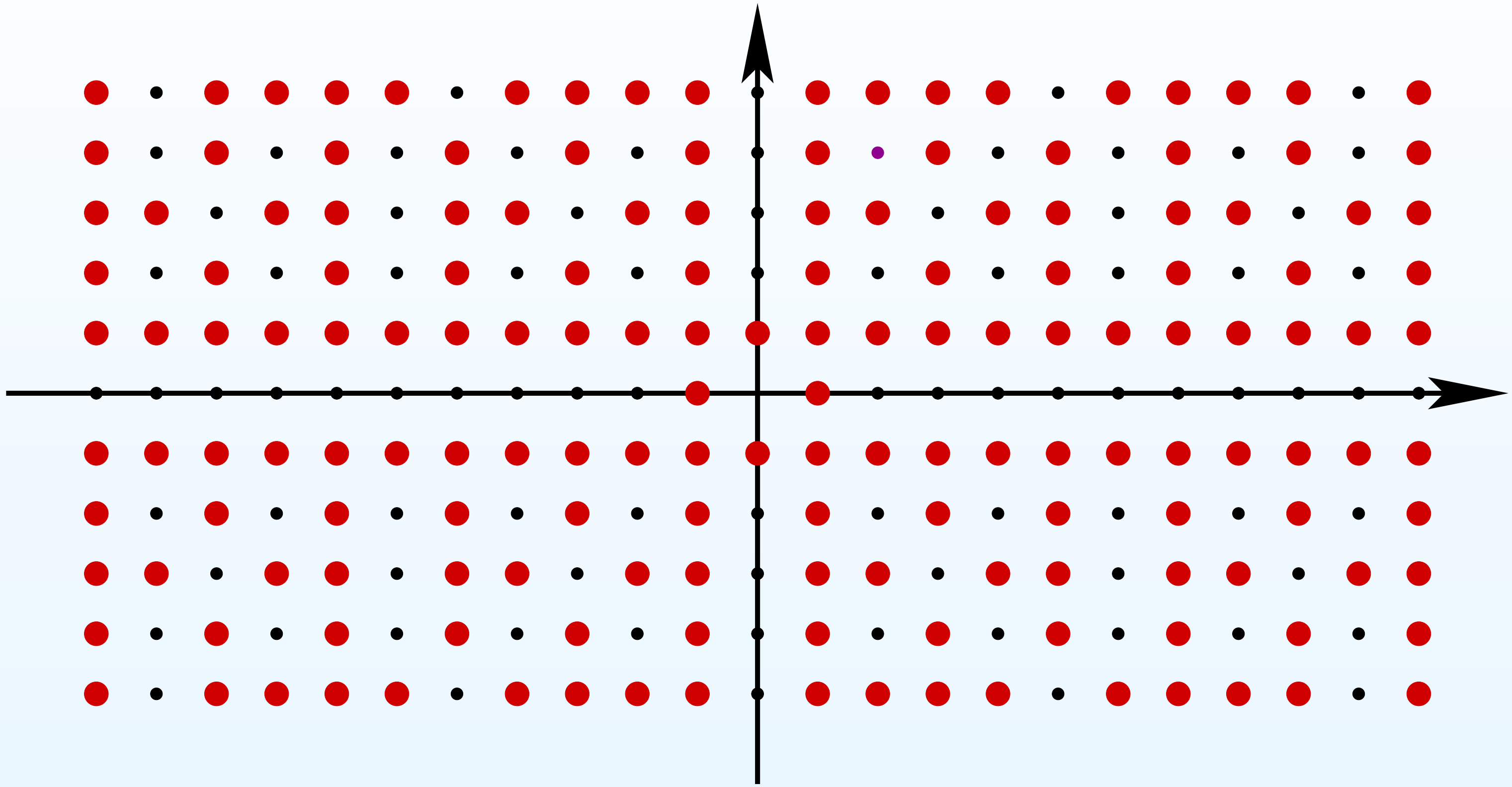


The set of *coprime* points $(p,q) \in \mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ such that $\mathrm{pgcd}(p,q) = 1$ is an $\mathrm{SL}(2,\mathbb{Z})$ orbit of $(1,0)$. In train-track coordiantes it can be identified with the mapping class group orbit $\mathrm{Mod}_{1,1} \cdot [\gamma]$ of a simple closed curve γ in $\mathcal{ML}_{1,1}$. We have proved that this subset has nonzero *uniform density* $k_1 = \frac{c(\gamma)}{b_{1,1}}$ in the ambient lattice of all integral measured laminations.

Uniform density of coprime integer points



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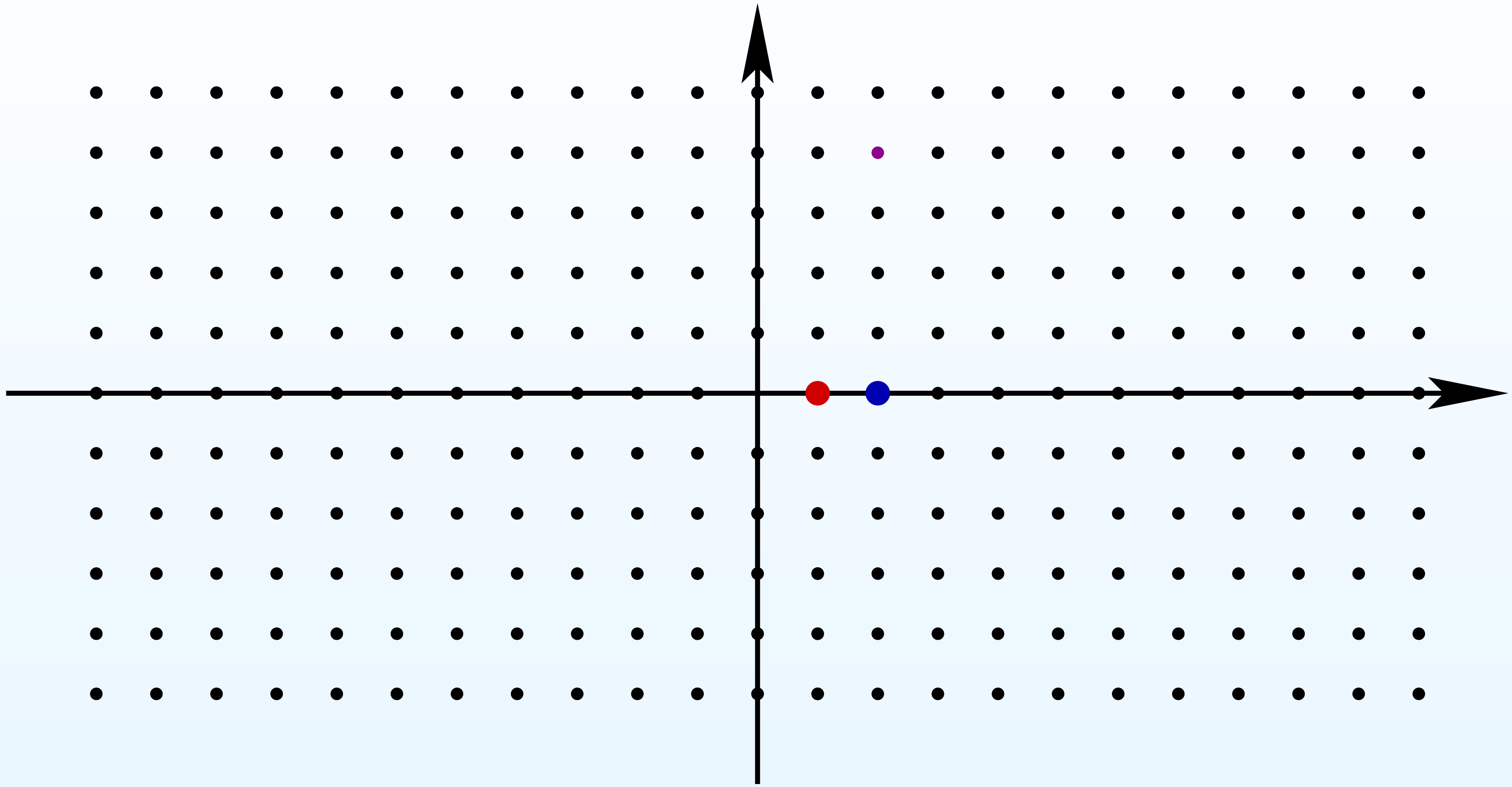
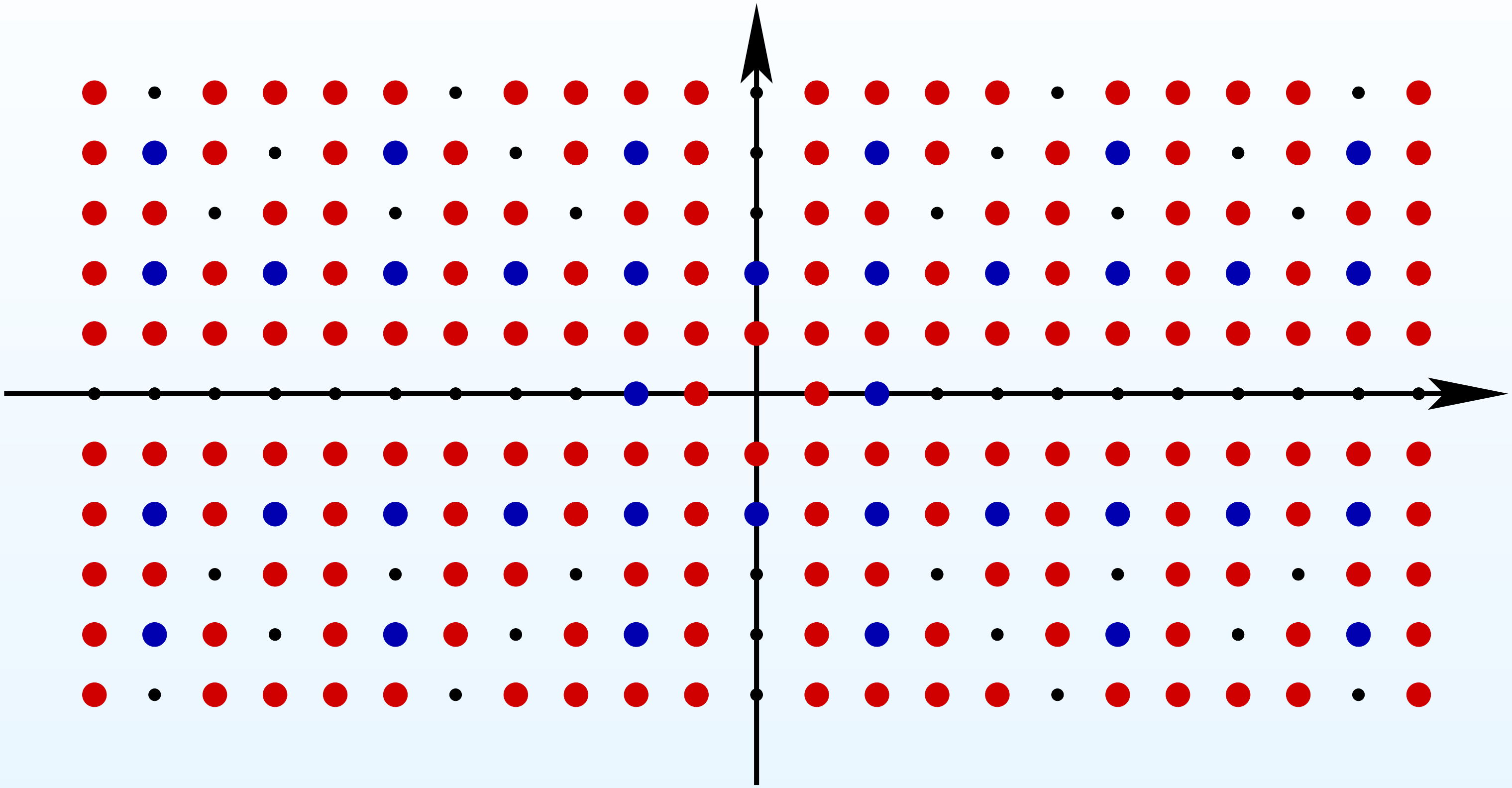


The set of points $(p,q) \in \mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ such that $\mathrm{pgcd}(p,q) = 2$ is an $\mathrm{SL}(2,\mathbb{Z})$ orbit of $(2,0)$. It can be obtained from the orbit of $(1,0)$ by proportional dilatation with coefficient 2. Thus, this new subset also has nonzero *uniform density* $k_2 = \frac{1}{2^2} \cdot k_1$.

Uniform density of coprime integer points



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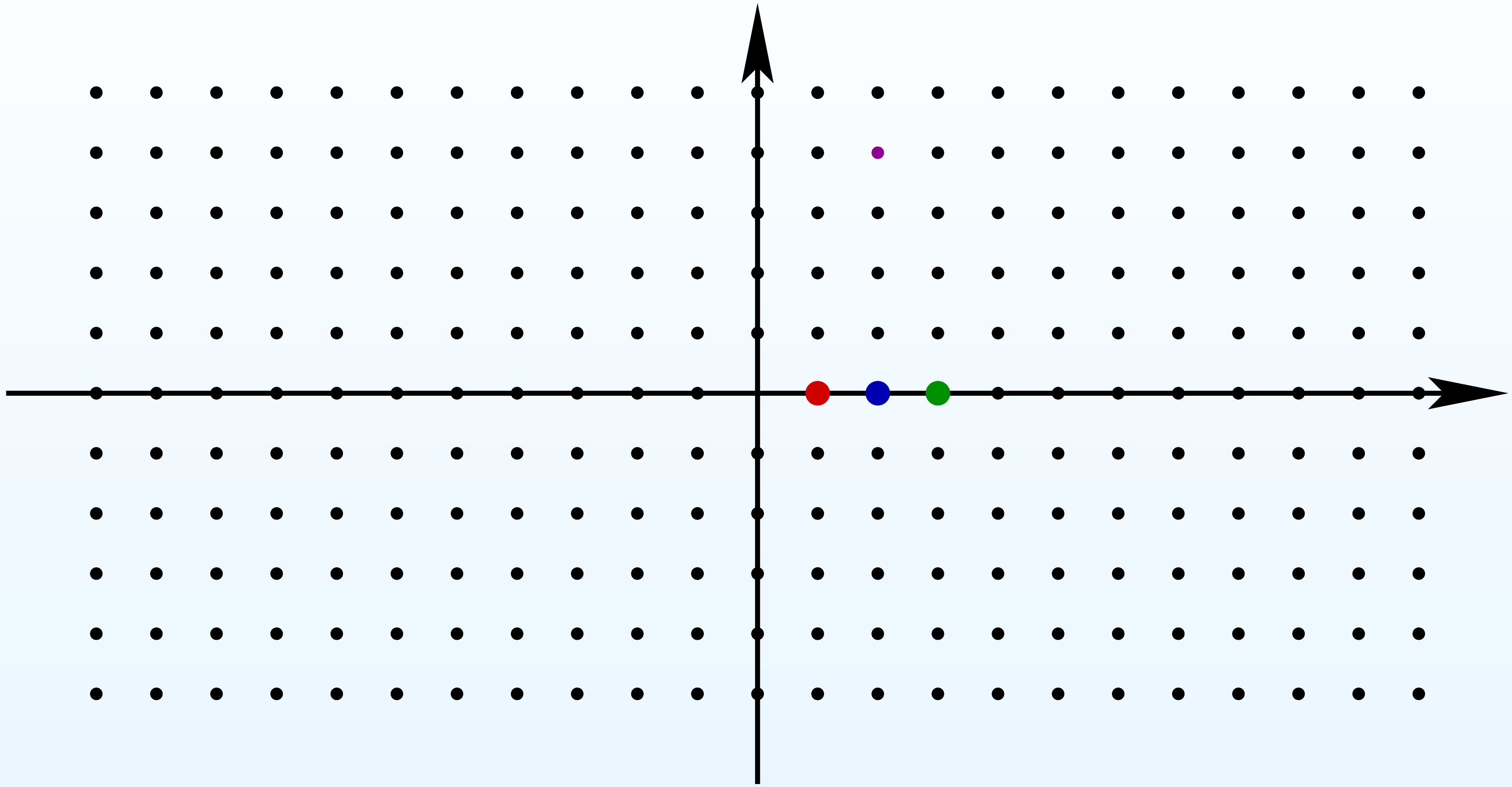

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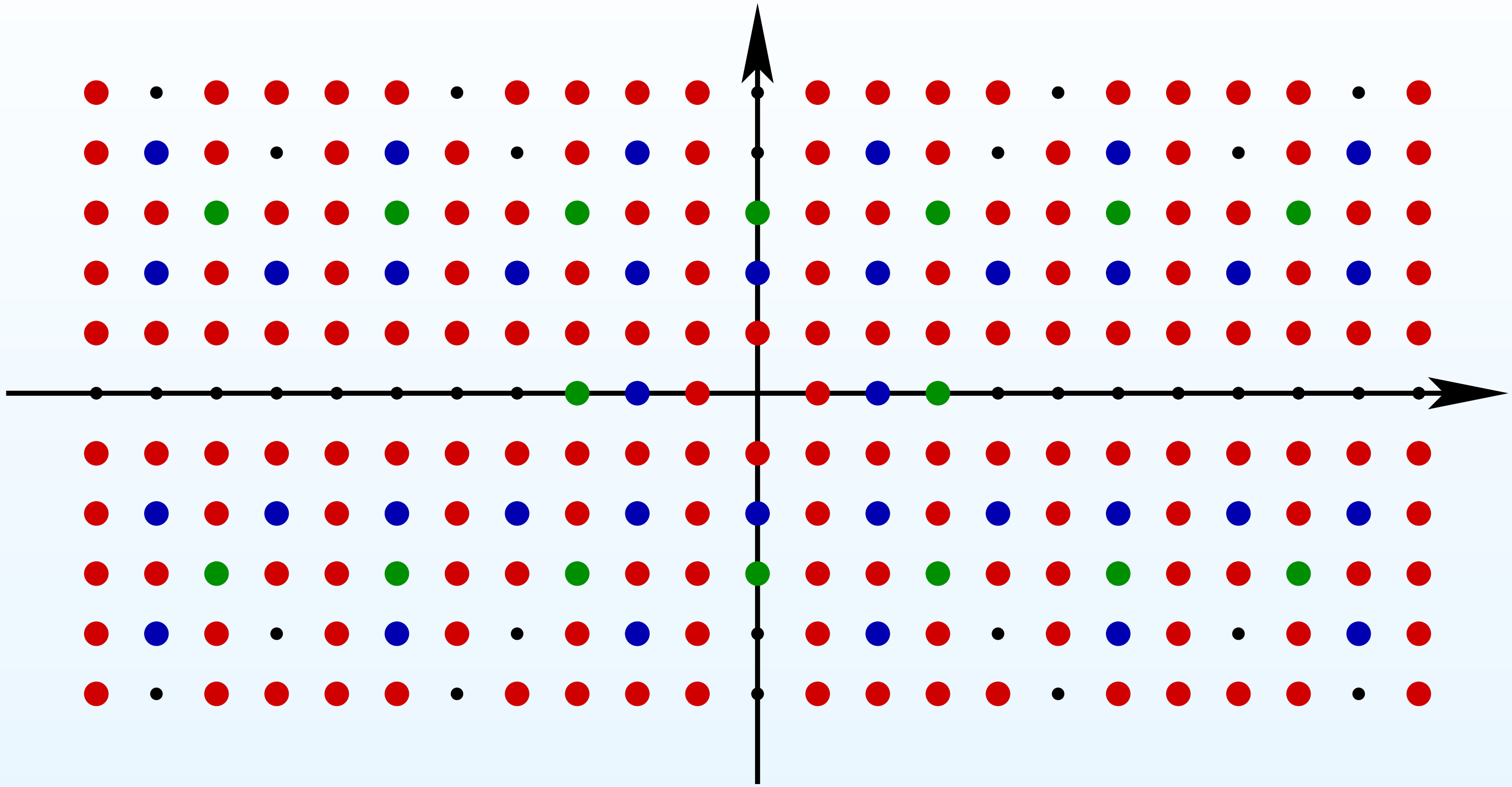
Uniform density of coprime integer points



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Uniform density of coprime integer points

The disjoint union of all these orbits gives all the lattice

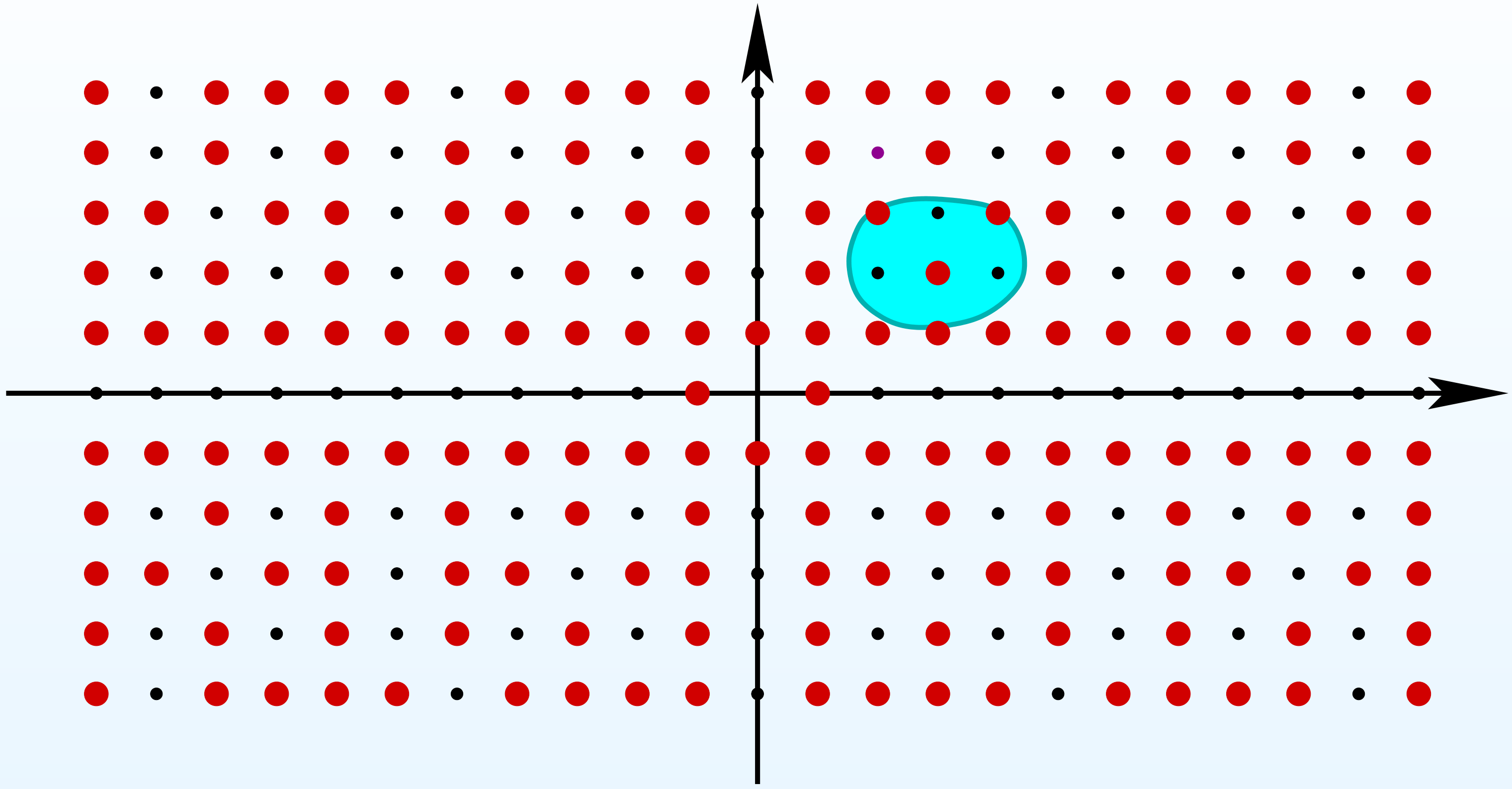
$$\sqcup_{n=1}^{\infty} \mathrm{SL}(2, \mathbb{Z}) \cdot (n, 0) = \mathbb{Z} \oplus \mathbb{Z}\,.$$

Thus the sum of the densities gives the full density, $k_1 + k_2 + \cdots = 1$. Since $k_n = \frac{1}{n^2} \cdot k_1$, we get

$$k_1(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots) = 1$$

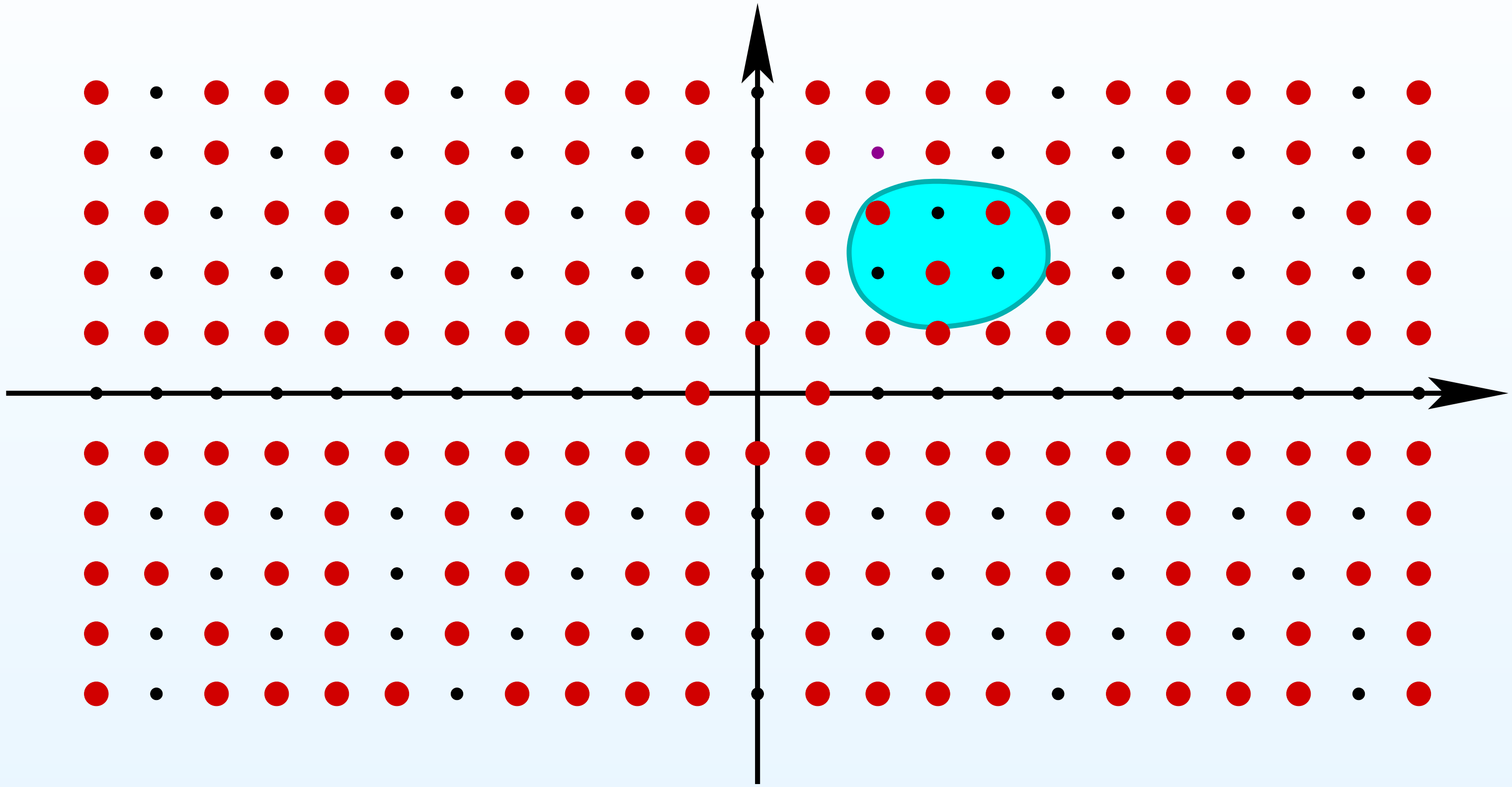
and, hence, $k_1 = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.

Uniform density of coprime integer points



Note that though we can measure the Lebesgue measure of a set by counting the number of coprime points inside its dilatations (which is the definition of a *uniform density*), it is known that going exponentially far from the origin, i.e. at the distance of the order e^R or more one can find islands of radius R without a single coprime point, where R is any positive number.

Uniform density of coprime integer points



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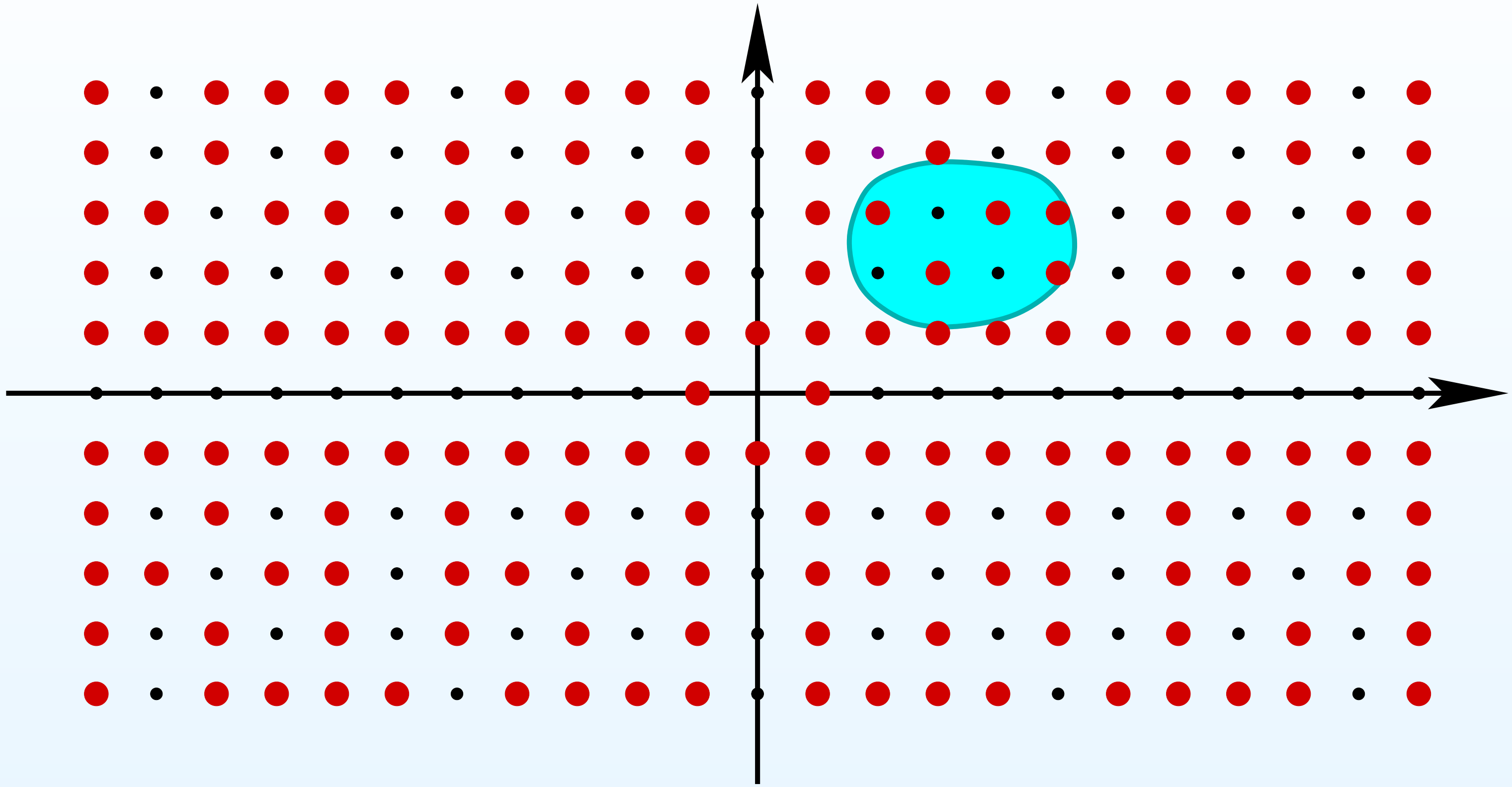

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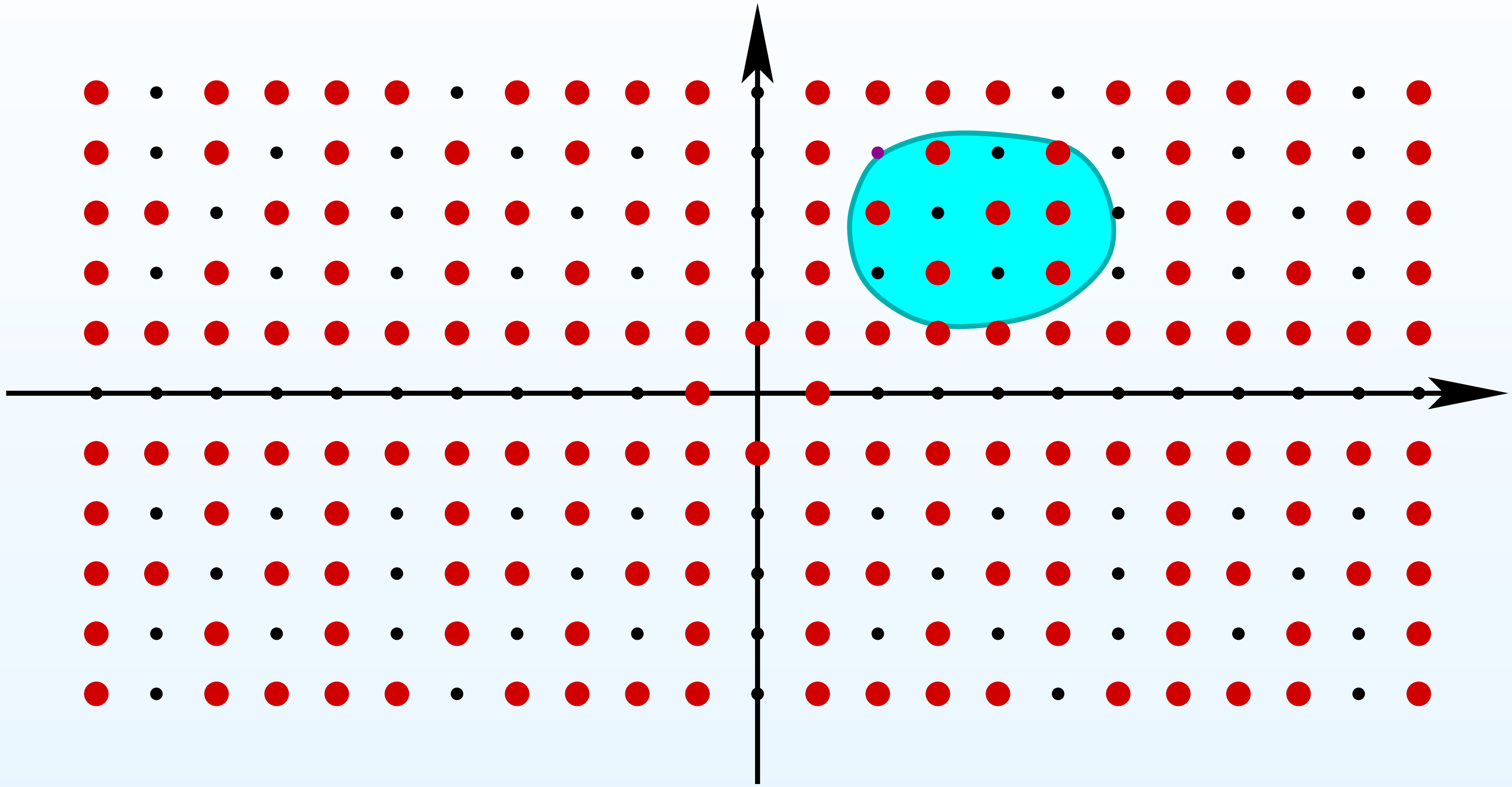

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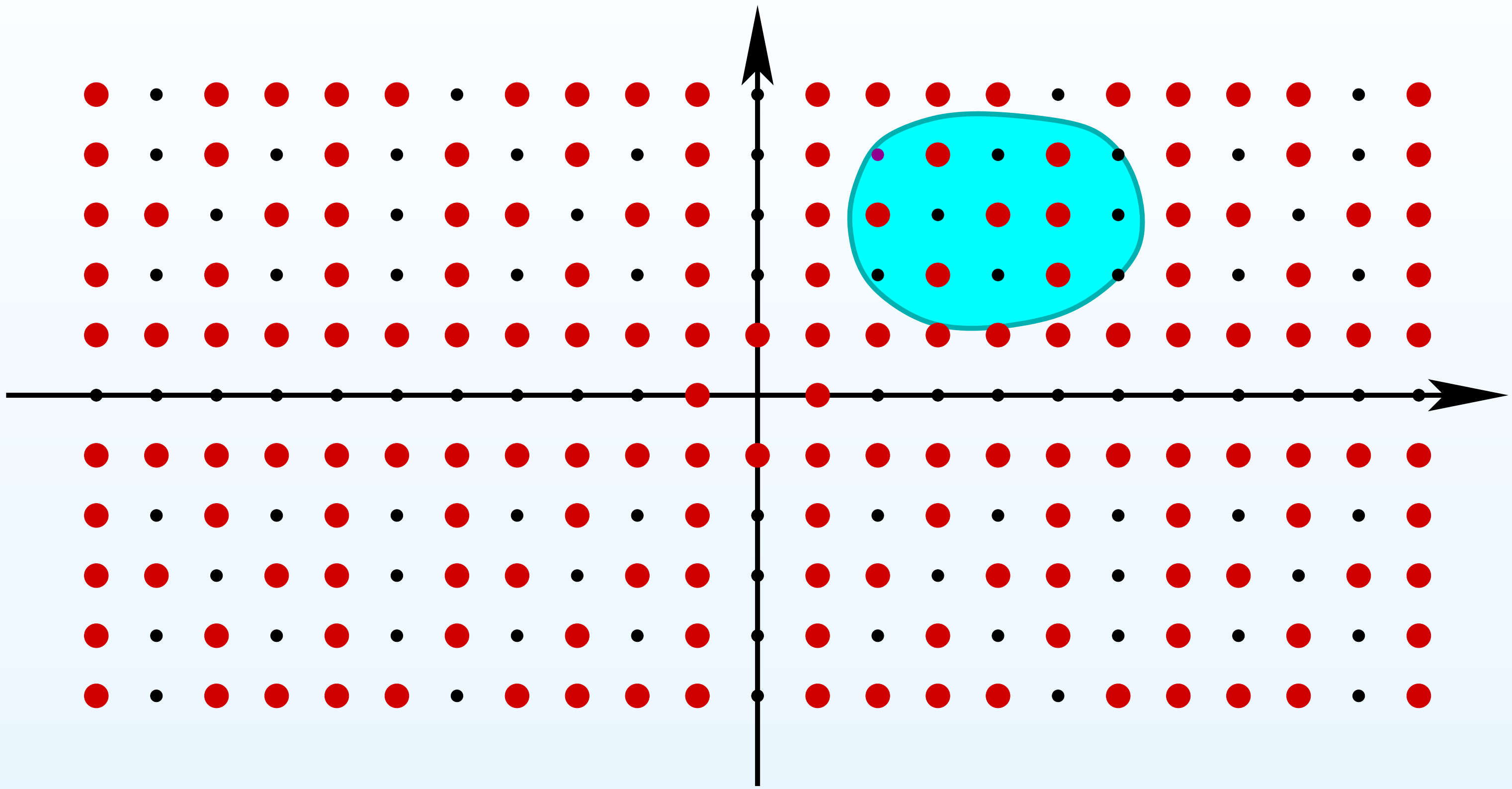

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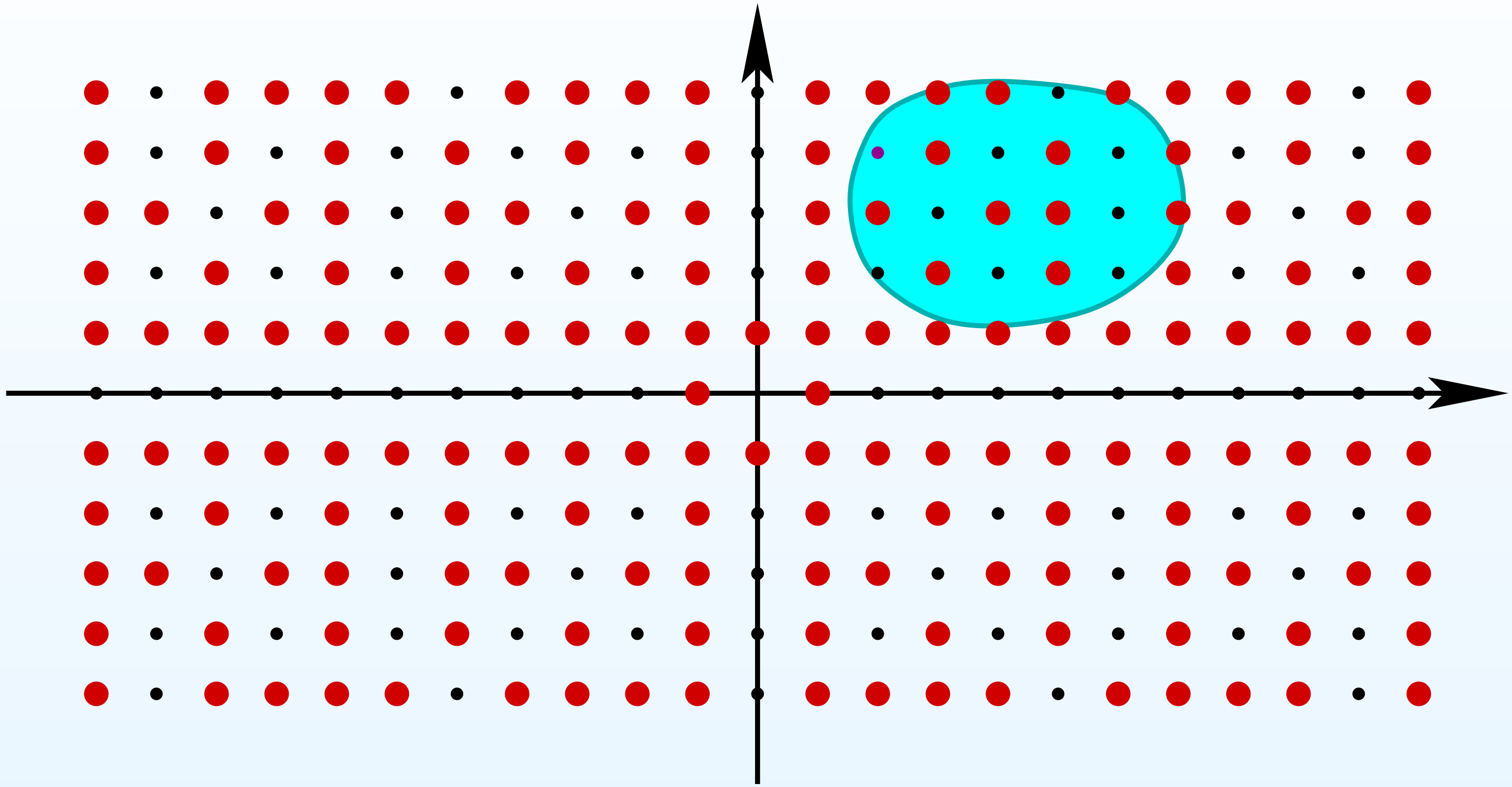

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