A problem of Novikov on the semiclassical motion of an electron in a uniform almost rational magnetic field

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In the absence of a magnetic field an electron in a lattice Γ moves in accordance with quantum mechanics with constant quasi-momenta p_1, p_2, p_3 defined modulo a vector of the inverse lattice and has the dispersion law $\varepsilon(p_1, p_2, p_3)$. Thus, the set of quasi-momenta ([2], [3]) is a point of the torus $T^3 = \mathbf{R}^3/\Gamma$. In a weak magnetic field physicists consider the semiclassical motion of an electron in the space of quasi-momenta given by the Hamiltonian $\varepsilon = \varepsilon(p + \frac{e}{c}A)$, where A is the vector potential. Let $p' = p + \frac{e}{c}A$. The motion of an electron in a uniform magnetic field H is defined in the space p' as a section of the surface $\varepsilon(p') = \text{const}$ by a plane orthogonal to the magnetic field (see [1], §6). The study of the topological properties of these trajectories is a particular case of the general problem concerning the topological structure of level surfaces of a closed 1-form on a compact manifold (the Novikov problem).

A non-singular level surface $\varepsilon(p') = \text{const}$ is denoted by M^2 and is called a Fermi surface. The restriction of the form $\Omega = H_1\,dp_1' + H_2\,dp_2' + H_3\,dp_3'$ with constant coefficients to a Fermi surface is denoted by $\omega = \Omega|_{M^2}$. We assume that ω has only non-degenerate critical points. We define the degree of irrationality of a closed 1-form ω on a compact manifold as the number of rationally independent integrals over cycles and denote it by $r(\omega)$. In our case $r(\omega) \leqslant r(\Omega) \leqslant 3$. When $r(\omega) = 1$, then there exists a smooth map $f_\omega : M^2 \to S^1$ whose critical points are non-degenerate. The inverse image of a point under this map is the fibre $\omega = 0$. A bordism $W \subset M^2$ with Morse function f_ω on it is the inverse image of an arc of S^1. This means that the non-singular fibres of ω, $r(\omega) = 1$, are smooth compact one-dimensional submanifolds of M^2, that is, the fibre of ω is diffeomorphic to a finite collection of circles. On the universal cover $\mathbf{R}^3$ the level surfaces of ω, $r(\omega) = 1$, are either closed or periodic. Note that any fibre of ω is contained in a fibre of Ω, which is a torus T^2.

Suppose that $r(\Omega) = 1$ and that to each critical value of f_ω there corresponds just one critical point on M^2.

Theorem. *For any constant form Φ in $\mathbf{R}^3$ sufficiently close to Ω, a level surface of the form $\varphi = \Phi|_{\hat{M}^2}$ lies in a strip of finite width on the plane $\Phi = 0$. The cover over the Fermi surface is denoted by $\hat{M}^2$.*

As already remarked, the map $f_\omega : M^2 \to S^1$ determines a decomposition of M^2 into a composition of elementary bordisms. We now consider the connected component W of the elementary bordism with a critical point of saddle-type. The boundary of W consists of three components, each of which is a realization of a cycle in T^3.

Lemma. *At least one of the three cycles is homologous to zero in T^3.*

Suppose, for instance, that the cycles α and β lie on one level surface and that neither is homologous to zero in T^3. Since α and β do not intersect and both are embedded in T^2 (the fibre $\Omega = 0$), we see that $\alpha = \pm\beta$. Then γ is either 0 or 2α. But since γ has no self-intersections, it can be embedded in T^2 (fibre $\Omega = 0$) in which we can realize the cycle α, so that the possibility $\gamma = 2\alpha$ is excluded. This proves the lemma.

We now choose on each elementary bordism exactly one connected component of a level surface of ω, which realizes a cycle homologous to zero in T^3. Then for any form φ near to ω there exist closed level surfaces close to the one chosen for ω, which also realize cycles homologous to zero in the torus. We cut the Fermi surface along such fibres of the form φ. We claim that for any connected component N the image of the map $i_* : H_1(N, \mathbf{Z}) \to H_1(T^3, \mathbf{Z})$ induced by the inclusion $i : N \to T^3$ has at most two generators. This means that for any connected components N of the "cut" Fermi surface M^2 there are two parallel planes in $\mathbf{R}^3$ such that the cover $\hat{N}$ over N lies between them. Since any fibre $\varphi = 0$ belongs to some connected component N, it is contained in $\mathbf{R}^3$ in the strip formed by the intersection of the plane $\Phi = 0$ with the chosen pair of planes.

This completes the proof of the theorem.

References

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