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PROBLEM CONTRIBUTION

Conjectural Large Genus Asymptotics of Masur–Veech Volumes and of Area Siegel–Veech Constants of Strata of Quadratic Differentials

Amol Aggarwal[1] · **Vincent Delecroix**[2] · **Élise Goujard**[3] · **Peter Zograf**[4,5] · **Anton Zorich**[6,7]

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Abstract
We state conjectures on the asymptotic behavior of the Masur–Veech volumes of strata in the moduli spaces of meromorphic quadratic differentials and on the asymptotics of their area Siegel–Veech constants as the genus tends to infinity.

Keywords Moduli space · Quadratic differential · Masur-Veech volume · Large genus asymptotics

Brief History of the Subject and State of the Art

The Masur–Veech volumes of strata in the moduli spaces of Abelian differentials and in the moduli spaces of meromorphic quadratic differentials with at most simple poles were introduced in fundamental papers (Masur 1982; Veech 1982). These papers proved that the Teichmüller geodesic flow is ergodic on each connected component of each stratum with respect to the Masur–Veech measure introduced in these papers, and that the total measure of each stratum is finite.

Masur–Veech volumes of strata of Abelian differentials

The first efficient evaluation of volumes of the strata in the moduli spaces of Abelian differentials was performed by Eskin and Okounkov (2001) twenty years later. The Masur–Veech volumes of several low-dimensional strata of Abelian differentials were computed just before that and by different methods in Zorich (2002). The algorithm of Eskin and Okounkov was implemented by Eskin in a rather efficient computer code

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✉ Anton Zorich
anton.zorich@gmail.com

Extended author information available on the last page of the article

which already at this time allowed to compute volumes of all strata of Abelian differentials up to genus 10, and volumes of some strata, like the principal one, up to genus 60 (or more). On the other hand, numerical experiments with the Lyapunov exponents produced approximate values of the Siegel–Veech constants c_{area}. This allowed Eskin and Zorich to state conjectures on asymptotic behavior of volumes of the strata in the moduli spaces of Abelian differentials and on the large genus asymptotics of the associated Siegel–Veech constants. These conjectures were stated at the end of 2003, but published in Eskin and Zorich (2015) only after publication of Eskin et al. (2014). The latter paper proved the relation between the sum of the Lyapunov exponents and the Siegel–Veech constant, and, thus, justified the prior experimental evidence.

The study of the Masur–Veech volumes of the strata of Abelian differentials was tremendously advanced in recent years. Chen, Moeller and Zagier (2018) proved the large genus asymptotic formula for the Masur–Veech volume of the principal stratum of Abelian differentials, confirming the conjectural formula from Eskin and Zorich (2015) for this particular case. Sauvaget (2018) proved the asymptotics of Masur–Veech volume conjectured in Eskin and Zorich (2015) for the minimal stratum of Abelian differentials. Aggarwal (2019a) proved the conjectural volume asymptotics for all strata by combinatorial methods. Finally, Chen, Moeller, Sauvaget and Zagier (2019a) proved the conjecture in maximal generality for all connected components of all strata of Abelian differentials combining combinatorics, geometry, and intersection theory. Sauvaget (2019) proved the asymptotic expansion in inverse powers of g for these volumes. The same authors proved in Chen et al. (2019a) very efficient recursive formula for the Masur–Veech volumes of the strata of Abelian differentials, which allows to find their exact values up to very large genera. Also, Aggarwal (2019b) and Chen et al. (2019a) proved all conjectures from Eskin and Zorich (2015) on large genus asymptotics of the Siegel–Veech constants. Masur–Veech volumes and Siegel–Veech constants of hyperelliptic connected components were computed by Athreya, Eskin and Zorich (2016). Due to these series of papers we now have comprehensive information on the Masur–Veech volumes and on Siegel–Veech constants for all strata of Abelian differentials.

Masur–Veech volumes of strata of quadratic differentials

Our knowledge of the Masur–Veech volumes and of the Siegel–Veech constants for the strata of meromorphic quadratic differentials is still relatively poor. Paper Athreya, Eskin and Zorich (2016) proved closed formulae (conjectured by Kontsevich) for the Masur–Veech volumes and for the Siegel–Veech constants in genus zero. The first reasonable table of volumes of strata of quadratic differentials was computed by Goujard (2015b, 2016), where she applied the algorithm of Eskin and Okounkov (2001) to compute volumes of all strata of dimension up to 11.

In recent paper Delecroix, Goujard, Zograf and Zorich (2019) the authors found a formula for the Masur–Veech volumes of the strata with simple zeroes and simple poles through intersection numbers of ψ-classes on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of complex curves and stated the conjecture on the large genus asymptotics of the Masur–Veech volume of the principal stratum of holomorphic quadratic differentials. Even more recent paper of Chen et al. (2019b) suggested an alternative formula for the Masur–Veech volumes of the same strata through certain very special linear Hodge integrals. The most recent paper of Kazarian (2019) provided extremely efficient recursive formula for these Hodge integrals, which allows to compute volumes of strata with simple zeroes and several simple poles in genera about 100 in split seconds. This computation corroborates the conjectural formula of the authors for these strata. According to paper (Chen et al. 2019b), work in progress Yang, Zagier and Zhang (2020) also corroborates and develops this conjecture. Moreover, as it will be explained in Sect. 2, the formula of Chen–Möller–Sauvaget and the efficient recursive algorithm of Kazarian combined together allow to extend the conjecture to a general stratum in the moduli space of meromorphic quadratic differentials with at most simple poles.

1 Conjectural Asymptotic Formula

Let $\boldsymbol{d} = (d_1, \ldots, d_n)$ be an unordered partition of a positive integer number $4g-4$ divisible by 4 into a sum $|\boldsymbol{d}| = d_1 + \cdots + d_n = 4g-4$, where $d_i \in \{-1, 0, 1, 2, \ldots\}$ for $i = 1, \ldots, n$. Denote by $\hat{\Pi}_{4g-4}$ the set of those partitions as above, which satisfy the additional requirement that the number of entries $d_i = -1$ in $\boldsymbol{d}$ is at most $\log(g)$.

Conjecture 1 *For any $\boldsymbol{d} \in \hat{\Pi}_{4g-4}$ one has*

$$\operatorname{Vol} \mathcal{Q}(d_1, \ldots, d_n) = \frac{4}{\pi} \cdot \prod_{i=1}^{n} \frac{2^{d_i+2}}{d_i+2} \cdot (1 + \varepsilon_1(\boldsymbol{d})) , \tag{1}$$

where

$$\lim_{g\to\infty} \max_{\boldsymbol{d}\in\hat{\Pi}_{4g-4}} |\varepsilon_1(\boldsymbol{d})| = 0.$$

Remark 1 We use the normalization of volumes as in Athreya, Eskin and Zorich (2016), Goujard (2016), Delecroix, Goujard, Zograf and Zorich (2019) and Chen et al. (2019b). In particular, all the zeroes and simple poles are labeled.

Conjecture 2 *For non-hyperelliptic components $\mathcal{Q}^{nonhyp}(\boldsymbol{d})$ of strata $\mathcal{Q}(\boldsymbol{d})$ of meromorphic quadratic differentials with at most simple poles, where $\boldsymbol{d} \in \hat{\Pi}_{4g-4}$, one has*

$$c_{area}\left(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\right) = \frac{1}{4} \cdot (1 + \varepsilon_2(\boldsymbol{d})) , \tag{2}$$

where

$$\lim_{g\to\infty} \max_{\boldsymbol{d}\in\hat{\Pi}_{4g-4}} |\varepsilon_2(\boldsymbol{d})| = 0.$$

It was proved in Lanneau (2008) that a stratum $\mathcal{Q}(\boldsymbol{d})$ has a hyperelliptic connected components if and only if the partition $\boldsymbol{d} = (d_1, \ldots, d_n)$ of $4g-4$, with

$d_i \in \{-1, 0, 1, \dots\}$ for $i = 1, 2, \dots, n$, has one of the following forms:

$$\begin{array}{ll}
(k_1, k_1, k_2, k_2), & \text{where both } k_1, k_2 \text{ are odd and } (k_1, k_2) \neq (-1, -1)\,, \\
(k_1, k_1, 2k_2 + 2), & \text{where } k_1 \text{ is odd and } k_2 \text{ is even}\,, \\
(2k_1 + 2, 2k_2 + 2), & \text{where both } k_1, k_2 \text{ are even}\,.
\end{array}$$

The Masur–Veech volumes of all hyperelliptic components $\mathcal{Q}^{hyp}(\boldsymbol{d})$ of the strata as above are computed in Proposition 3.1 in Goujard (2016). For all such partitions one has

$$\operatorname{Vol} \mathcal{Q}^{hyp}(\boldsymbol{d}) \le C_1 \cdot \frac{(2\pi)^{2g}}{(2g)!} \cdot \frac{1}{\sqrt{g}},$$

while

$$\frac{4}{\pi} \cdot \prod_{i=1}^{n} \frac{2^{d_i+2}}{d_i + 2} \ge C_2 \cdot \frac{16^g}{g^4},$$

where C_1, C_2 are universal constants. Thus, Conjecture 1 implies, in particular, that in large genera the Masur–Veech volume $\operatorname{Vol} \mathcal{Q}^{hyp}(\boldsymbol{d})$ of any hyperelliptic connected component is negligible with respect to the Masur–Veech volume $\operatorname{Vol} \mathcal{Q}(\boldsymbol{d})$ of the entire stratum.

It is proved in Lanneau (2008) and in Chen and Moeller (2014) that for $g \ge 5$ every stratum $\mathcal{Q}(\boldsymbol{d})$ is either connected or has exactly one non-hyperelliptic connected component. Thus, for high genera g, every partition $\boldsymbol{d} \in \hat{\Pi}(4g - 4)$ uniquely determines the non-hyperelliptic connected component $\mathcal{Q}^{nonhyp}(\boldsymbol{d})$ of $\mathcal{Q}(\boldsymbol{d})$.

Remark 2 The restriction on the number of simple poles contained in the definition of the set $\hat{\Pi}_{4g-4}$ admits variations. We strongly believe in validity of the above Conjectures when the number of simple poles is bounded by some constant or even grows at most logarithmically as stated above. Considerations in Sect. 3 indicate, that for any fixed $\alpha < 1$ both Conjectures might be still valid when the number of simples poles is bounded from above by g^α. However, the same considerations show that one should not expect validity of the Conjectures when the number of simple poles grows linearly in g as the genus g tends to infinity.

Corollary 1 *Applying the formula from Eskin, Kontsevich and Zorich (2014) for the sum of the Lyapunov exponents of the Hodge bundle over the Teichmüller geodesic flow in any non-hyperelliptic connected component of a stratum $\mathcal{Q}(\boldsymbol{d})$ of meromorphic quadratic differentials with $\boldsymbol{d} \in \hat{\Pi}(4g - 4)$ we get*

$$\lambda_1^+ + \dots + \lambda_g^+ = \frac{1}{24} \cdot \sum_{d_i \in \boldsymbol{d}} \frac{d_i(d_i + 4)}{d_i + 2} + \frac{\pi^2}{12} + \frac{\pi^2}{3} \cdot \epsilon_2(\boldsymbol{d}). \tag{3}$$

Remark 3 Chen, Moeller and Sauvaget (2019b) proved formulae for the Masur–Veech volume of the principal strata of meromorphic quadratic differentials and for the corresponding Siegel–Veech constant c_{area} in terms of intersection numbers. This paper also suggested analogous conjectural formulae for other strata. We hope that the algebro-geometric conjectures from Chen et al. (2019b) and the numerical Conjectures 1 and 2 might be mutually useful.

2 Numerical Evidence

Siegel–Veech constant

Numerical evaluation of Lyapunov exponents allows to compute approximate values of Siegel–Veech constants $c_{area}(\mathcal{Q}^{nonhyp}(\boldsymbol{d}))$ for connected components $\mathcal{Q}^{nonhyp}(\boldsymbol{d})$ of strata $\mathcal{Q}(\boldsymbol{d})$ in the moduli space of meromorphic quadratic differentials with at most simple poles through the following formula from Eskin et al. (2014):

$$\lambda_1^- + \cdots + \lambda_{g_{\mathit{eff}}}^- = \frac{1}{24}\cdot\sum_{d_i\in\boldsymbol{d}}\frac{d_i(d_i+4)}{d_i+2} + \frac{1}{4}\cdot\sum_{\substack{d_j\in\boldsymbol{d}\\ d_j\equiv 1 \bmod 2}}\frac{1}{d_j+2} + \frac{\pi^2}{3}\cdot c_{area}\left(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\right).$$

Here the *effective genus* g_{eff} is defined as $g_{\mathit{eff}} = \hat{g} - g$, where $\hat{g}$ is the genus of the canonical double cover $p : \hat{C} \to C$ (ramified at simple poles and at zeroes of odd orders of the quadratic differential q) such that $p^*q = \hat{\omega}^2$ is a square of a globally defined Abelian differential $\hat{\omega}$ on $\hat{C}$.

Simulations performed for numerous strata in large genera provide strong numerical evidence for conjectural asymptotics (2) from Conjecture 2. For example, experiments performed for about 20 random strata with 0, 1, 2, 3 simple poles in genera ranging from 20 to 30 give approximate values of $c_{area}(\mathcal{Q}^{nonhyp}(\boldsymbol{d}))$ varying from 0.247 to 0.259 for 10^6 iterations of fast Rauzy induction. Analogous experiments with the principal stratum $\mathcal{Q}_{g,0} = \mathcal{Q}(1^{4g-4})$ give approximate values of $c_{area}(\mathcal{Q}_{g,0})$ varying from 0.251 to 0.253 for 10^7 iterations of fast Rauzy induction for genera from 20 to 30 and approximate values of $c_{area}(\mathcal{Q}_{g,0})$ varying from 0.255 to 0.258 for 10^6 iterations of fast Rauzy induction for genera about 40. Similar simulations for the principal strata were independently performed by Fougeron.

As we explained in the introduction, combining very recent results (Chen et al. 2019b; Kazarian 2019) we can compute exact values of volumes of strata $\mathcal{Q}_{g,n} = \mathcal{Q}(1^{4g-4+n}, -1^n)$ and then compute exact values of the Siegel–Veech constants $c_{area}\big(\mathcal{Q}_{g,n}\big)$ due to the following result.

Theorem (Goujard 2015a) *Let g be a strictly positive integer, and n nonnegative integer. When $g = 1$ we assume that $n \geq 2$. Under the above conventions the following* *formula is valid:*

$$c_{area}(\mathcal{Q}_{g,n}) = \frac{1}{\mathrm{Vol}(\mathcal{Q}_{g,n})} \cdot \Bigg(\frac{1}{8} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2 \\ g_i \geq 0, n_i \geq 1, d_i \geq 1}} \frac{\ell!}{\ell_1!\ell_2!} \frac{n!}{(n_1-1)!(n_2-1)!} \cdot \frac{(d_1-1)!(d_2-1)!}{(d-1)!} \mathrm{Vol}(\mathcal{Q}_{g_1,n_1}) \times \mathrm{Vol}(\mathcal{Q}_{g_2,n_2}) + \frac{1}{16} \cdot \frac{(4g-4+n)n(n-1)}{(6g-7+2n)(6g-8+2n)} \mathrm{Vol}(\mathcal{Q}_{0,3}) \times \mathrm{Vol}(\mathcal{Q}_{g,n-1}) + \frac{\ell!}{(\ell-2)!} \frac{(d-3)!}{(d-1)!} \mathrm{Vol}(\mathcal{Q}_{g-1,n+2}) \Bigg). \tag{4}$$

Here $d = \dim_{\mathbb{C}} \mathcal{Q}_{g,n} = 6g - 6 + 2n$, $d_i = 6g_i - 6 + 2n_i$, $\ell = 4g - 4 + n$, $\ell_i = 4g_i - 4 + n_i$.

The above formula gives exact values of $c_{area}(\mathcal{Q}_{g,n})$ for pairs g, n up to $g = 250$ and larger. The analysis of the resulting data provides serious numerical evidence towards validity of Conjecture 2. In particular, for $g = 3, 4, \ldots, 250$, the Siegel–Veech constant $c_{area}(\mathcal{Q}_{g,0})$ is monotonously decreasing from 0.284275 to 0.250285. For a fixed genus g and small values $n = 0, 1, 2, \ldots$ of n, the Siegel–Veech constant $c_{area}(\mathcal{Q}_{g,n})$ does not fluctuate much. For example, the approximate values of $c_{area}(\mathcal{Q}_{250,n})$ for $n = 0, 1, \ldots, 6$ are given by the following list:

$$\{0.250285, 0.250118, 0.249992, 0.249909, 0.249867, 0.249867, 0.249909\}.$$

Volume asymptotics

The arguments towards Conjecture 1 are indirect. We start by recalling the scheme which was used to formulate analogous Conjecture in Eskin and Zorich (2015) for large genus asymptotics of Masur–Veech volumes of strata of Abelian differentials. Certain phenomena which had a status of conjectures at the early stage of the project (Eskin and Zorich 2015) are proved in the recent paper of Aggarwal (2019b). We expect that strata in the moduli spaces of meromorphic quadratic differentials have analogous geometric properties, see Conjecture 3 below.

Having explained the scheme in the case of Abelian differentials (where everything is proved by now) we describe necessary adjustments which we use for the case of quadratic differentials. In this latter case, part of results is still conjectural.

Stating volume asymptotics conjecture for Abelian differentials

Eskin, Masur and Zorich (2003) provided a formula for the Siegel–Veech constant $c_{area}(\mathcal{H}(m))$ of any (connected component of any) stratum $\mathcal{H}(m)$ in the moduli space of Abelian differentials. This formula expressed $c_{area}(\mathcal{H}(m))$ through Masur–Veech volumes of the associated *principal boundary* strata normalized by $\mathrm{Vol}\,\mathcal{H}(m)$. It was conjectured that in large genera a dominant part of the contribution to $c_{area}(\mathcal{H}(m))$ comes from the boundary strata corresponding to simplest degenerations (represented by so-called *configurations of multiplicity one*). Neglecting contributions of more sophisticated boundary strata one obtains particularly simple approximate formula for $c_{area}(\mathcal{H}(m))$ as an explicit weighted sum of ratios of volumes of simplest boundary strata over the volume of the stratum $\mathcal{H}(m)$. The constant asymptotic value $c_{area}(\mathcal{H}(m)) \approx \frac{1}{2}$ in this expression leads to the formula

$$\operatorname{Vol} \mathcal{H}(m_1, \dots, m_n) = const \cdot \prod_{i=1}^{n} \frac{1}{m_i + 1} \cdot (1 + \varepsilon_3(m)) \,, \tag{5}$$

where $\varepsilon_3(m) \to 0$ as $g \to +\infty$ uniformly for all partitions m of $2g - 2$ into a sum of positive integers.

Evaluation of the universal constant *const* in the above formula is a separate nontrivial problem. Since the Masur–Veech volumes of strata appear in the approximate formula for $c_{area}(\mathcal{H}(m))$ only in ratios of the volume of some smaller stratum over the volume of the original stratum, the global normalization constant cancels out in every ratio. In the case of Abelian differentials, the constant *const* was guessed from numerics. Namely, in the particular case of the principal stratum, the original formula of Eskin and Okounkov (2001) allows to compute $\operatorname{Vol} \mathcal{H}(1^{2g-2})$ for genus $g = 60$ and higher, which allowed to guess the correct value $const = 4$. Asymptotics (5) was rigorously proved for particular cases in Chen, Moeller and Zagier (2018) and in Sauvaget (2018) and then in Aggarwal (2019a) and Chen et al. (2019a) in general case.

We address the reader interested in more details to Zorich (2019) where the notion of *configuration of multiplicity one* is explained in detail, and where the contributions of such configurations to all possible Siegel–Veech constants, including $c_{area}(\mathcal{H}(m))$, are computed in full details. The conjecture that the total contribution of more complicated configurations to $c_{area}(\mathcal{H}(m))$ becomes negligible in large genera is recently proved by Aggarwal (2019b).

Stating volume asymptotics conjecture for quadratic differentials

Developing technique from Eskin et al. (2003) and using the topological description of the principal boundary strata in the moduli spaces of quadratic differentials given in Masur and Zorich (2008), Goujard (2015a) obtained an expression for the Siegel–Veech constant $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ through Masur–Veech volumes of $\mathcal{Q}^{nonhyp}(\boldsymbol{d})$ and of its principal boundary. Formula (4) is a particular case of this more general formula.

Analogously to the situation with Abelian differentials, we expect that the total contribution to $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ coming from the principal boundary strata different from the simplest ones becomes negligible in high genera. We state this conjecture as Conjecture 3 below. Extracting from the formula of Goujard for $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ the contribution coming from these simplest degenerations we get an approximate formula for $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ through a weighted sum of ratios of volumes of the boundary strata divided by the volume of the stratum under consideration. In the particular case of the strata $\mathcal{Q}_{g,n}$ this corresponds to removing from (4) the summands containing products of the volumes, see Sect. 3 for more details.

Similarly to the case of Abelian differentials, expression (1) is the unique expression, up to a global normalization factor, such that for any $\boldsymbol{d} \in \hat{\Pi}_{4g-4}$ the resulting weighted sum tends to the asymptotic value $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big) \approx \frac{1}{4}$. We provide this weighted sum for $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ in Conjecture 3 below.

As in the case of Abelian differentials, this approach does not allow to find the global constant factor in the asymptotic formula (1). The corresponding factor $\frac{4}{\pi}$ was originally conjectured in Delecroix, Goujard, Zograf and Zorich (2019) for the large genus asymptotics of the Masur–Veech volume of the principal stratum of holomorphic quadratic differentials. This conjecture is based on fine geometric considerations combined with elaborate computations.

Combining, on one hand, the formula from Chen et al. (2019b) for the same volume and, on the other hand, a very efficient recursion from Kazarian (2019) for the linear Hodge integrals involved in this formula, one obtains a very serious numerical evidence for validity of our conjecture for the strata with only simple zeroes and simple poles, when the number of simple poles is small enough.

For any pair g, n of nonnegative integers define

$$\operatorname{Vol}^{appr} \mathcal{Q}_{g,n} = \frac{4}{\pi} \cdot 2^n \cdot \left(\frac{8}{3}\right)^{4g-4+n} .$$

This value corresponds to the right-hand side of expression (1) applied to the partition $\boldsymbol{d} = (1^{4g-4+n}, -1^n)$, where the factor $\big(1 + \varepsilon_1(\boldsymbol{d})\big)$ is omitted. The sequence of ratios $\dfrac{\operatorname{Vol} \mathcal{Q}^{appr}_{g,0}}{\operatorname{Vol} \mathcal{Q}_{g,0}}$ is monotonously decreasing in the range $g = 3, 4, \ldots, 250$, from $\dfrac{\operatorname{Vol} \mathcal{Q}^{appr}_{3,0}}{\operatorname{Vol} \mathcal{Q}_{3,0}} \approx 1.01892$ to $\dfrac{\operatorname{Vol} \mathcal{Q}^{appr}_{250,0}}{\operatorname{Vol} \mathcal{Q}_{250,0}} \approx 1.00027$. For any fixed genus g and small values $n = 0, 1, 2, \ldots$ of n, the volumes do not fluctuate much. For example, the approximate values of $\dfrac{\operatorname{Vol} \mathcal{Q}^{appr}_{250,n}}{\operatorname{Vol} \mathcal{Q}_{250,n}}$ for $n = 0, 1, \ldots, 5$ are given by the following list:

$$\{1.00027406, 1.00027477, 1.00027493, 1.00027481, 1.00027469, 1.00027484\} .$$

In particular, these numerical data corroborates our conjectural value $\frac{4}{\pi}$ of the universal normalizing constant.

Remark 4 Analogous analysis of numerical data was independently performed by Chen, Moeller and Sauvaget who used the algorithm from Yang, Zagier and Zhang (2020).

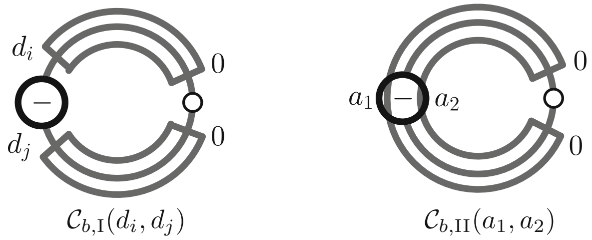
Range of validity of the conjecture

Goujard (2015b) computed exact values of the Masur–Veech volumes of strata in the moduli space of quadratic differentials for strata of dimension at most 11. The ratio $\operatorname{Vol}^{appr} \mathcal{Q}(\boldsymbol{d}) / \operatorname{Vol} \mathcal{Q}(\boldsymbol{d})$ evaluated for strata $\mathcal{Q}(\boldsymbol{d})$ of genera 5 and 6 of dimension 11 varies from 1.037 to 1.135, which suggests that formula (1) gives reasonable approximation of the Masur–Veech volume already for all strata in relatively small genera.

3 Contribution of Configurations of Multiplicity One to $c_{area}(\mathcal{Q}^{nonhyp}(\boldsymbol{d}))$

We refer the reader to Masur and Zorich (2008) and to Goujard (2016) for the notions "configurations of homologous saddle connections" and "principal boundary strata".

Conjecture 3 *The Siegel–Veech constant $c_{area}(\mathcal{Q}^{nonhyp}(\boldsymbol{d}))$ of non-hyperelliptic components of all strata $\mathcal{Q}(\boldsymbol{d})$ with $\boldsymbol{d} \in \hat{\Pi}_{4g-4}$, is asymptotically supported on the following set of configurations:*



where d_i, d_j, a_1, a_2 are such that $d_i, d_j \geq 1$, $\{d_i, d_j\} \subset \{d_1, \ldots, d_n\}$, where $i \neq j$, $a_1, a_2 \geq 0$, $a_1 + a_2 \geq 3$ and $a_1 + a_2 + 2 \in \{d_1, \ldots, d_n\}$.

Corollary 2 *Assume that both Conjectures 1 and 3 hold. Then, Conjecture 2 holds for all partitions $d \in \hat{\Pi}_{4g-4}$ satisfying the additional restriction that at most $\log(g-1)-2$ of the d_i are equal to -1.*

Proof First note that

$$c_{area}(\mathcal{Q}(0, d_1, \ldots, d_n)) = c_{area}\left(\mathcal{Q}^{nonhyp}(d_1, \ldots, d_n)\right).$$

Hence, it is sufficient to prove Corollary 2 for partitions which do not contain zero entries (representing the marked points).

Choose any pair of indices $1 \leq i < j \leq n$ such that $d_i, d_j \geq 1$. Applying Theorem 1 from Goujard (2015a), we obtain the following expression for the contribution $c_{area}(\mathcal{C}_{b,\mathrm{I}}(d_i, d_j))$ of the configuration $\mathcal{C}_{b,\mathrm{I}}(d_i, d_j)$ to $c_{area}(\mathcal{Q}^{nonhyp}(\boldsymbol{d}))$:

$$c_{area}(\mathcal{C}_{b,\mathrm{I}}(d_i, d_j))=\frac{(d-3)!}{(d-1)!} \cdot 2d_i d_j \cdot \frac{\operatorname{Vol} \mathcal{Q}_{g-1}(d_1, \ldots, d_i - 2, \ldots, d_j - 2, \ldots, d_n)}{\operatorname{Vol} \mathcal{Q}_g(d_1, \ldots, d_n)},$$

where $d = \dim_{\mathbb{C}} \mathcal{Q}_g(d_1, \ldots, d_n) = 2g+n-2$. Note that the bold symbol $\boldsymbol{d}$ denotes a partition in $\hat{\Pi}_{4g-4}$, while the symbol d denotes the complex dimension of the stratum $\mathcal{Q}(\boldsymbol{d})$.

Assuming that $g \gg 1$ and using Conjecture 1 we obtain

$$c_{area}(\mathcal{C}_{b,\mathrm{I}}(d_i, d_j)) \sim \frac{(d_i+2)(d_j+2)}{2^3 d^2}. \tag{6}$$

Choose any index i in $\{1, \dots, n\}$ such that $d_i \geq 3$ (if such index exists). By Theorem 1 from Goujard (2015a), for any pair of nonnegative integers a_1, a_2 satisfying $a_1 + a_2 = d_i - 2$, the contribution $c_{area}(\mathcal{C}_{b,\mathrm{II}}(a_1, a_2))$ of the configuration $\mathcal{C}_{b,\mathrm{II}}(a_1, a_2)$ to $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ has the following form:

$$c_{area}(\mathcal{C}_{b,\mathrm{II}}(a_1, a_2)) = \frac{(d-3)!}{(d-1)!} \cdot 2(a_1 + a_2) \cdot \frac{\mathrm{Vol}\, \mathcal{Q}_{g-1}\,(d_1, \dots, d_i - 4, \dots d_n)}{\mathrm{Vol}\, \mathcal{Q}_g\,(d_1, \dots, d_n)},$$

so, assuming that $g \gg 1$ and using Conjecture 1, we obtain the following contribution to $c_{area}\big(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\big)$ of all pairs a_1, a_2 corresponding to the fixed d_i as above:

$$\begin{aligned} \frac{1}{2} \sum_{a_1=0}^{d_i-2} c_{area}(\mathcal{C}_{b,\mathrm{II}}(a_1, a_2)) &\sim \frac{1}{2^4 d^2}(d_i - 1)(d_i + 2) \\ &= \frac{1}{2^4 d^2}\Big((d_i + 2)^2 - 3(d_i + 2)\Big). \end{aligned} \tag{7}$$

We use notation $\mathcal{Q}(d_1, \dots d_n) = \mathcal{Q}(-1^{\mu_{-1}}, 1^{\mu_1}, 2^{\mu_2}, \dots)$ to record the multiplicities of entries $-1, 1, 2, \dots$ in the partition $\boldsymbol{d}$.

Equation (6) implies that

$$\begin{aligned} 2^3 d^2 \sum_{\substack{1 \le i < j \le n \\ d_i, d_j \ge 1}} c_{area}(\mathcal{C}_{b,\mathrm{I}}(d_i, d_j)) \sim \sum_{1 \le i < j \le n} (d_i + 2)(d_j + 2) \\ - \mu_{-1} \sum_{i=1}^{n} (d_i + 2) + \frac{\mu_{-1}(\mu_{-1} + 1)}{2}. \end{aligned} \tag{8}$$

Equation (7) implies that

$$\begin{aligned} 2^4 d^2 \sum_{\substack{1 \le i \le n \\ d_i \ge 3}} \sum_{\substack{a_1 + a_2 = d_i - 2 \\ a_1, a_2 \ge 0}} c_{area}(\mathcal{C}_{b,\mathrm{II}}(a_1, a_2)) \sim \sum_{i=1}^{n} (d_i + 2)^2 - 3 \sum_{i=1}^{n} (d_i + 2) \\ -(\mu_{-1} + 9\mu_1 + 16\mu_2) + 3(\mu_{-1} + 3\mu_1 + 4\mu_2). \end{aligned} \tag{9}$$

Note that

$$\sum_{i=1}^{n} (d_i + 2) = \left(\sum_{i=1}^{n} d_i\right) + 2n = 4g - 4 + 2n = 2d, \tag{10}$$

and, in particular, $\mu_{-1} + \mu_1 + \mu_2 + \dots = n < d$, which implies that

$$\frac{2\mu_{-1} - 6\mu_1 - 12\mu_2}{d^2} \to 0 \quad \text{as} \quad d \to +\infty. \tag{11}$$

Note also, that by assumption, the number of simple poles in a stratum is much smaller than the genus g. Since $g < d$ we get

$$\frac{\mu_{-1}}{d} \to 0 \quad \text{as} \quad d \to +\infty.$$

We conclude that for partitions $\boldsymbol{d}$ as in the statement of Corollary 2 with $\mu_0 = 0$ we get the following asymptotics as $g \to +\infty$:

$$\begin{aligned} &c_{area}\left(\mathcal{Q}^{nonhyp}(\boldsymbol{d})\right) \\ &\sim \sum_{\substack{1 \le i < j \le n \\ d_i, d_j \ge 1}} c_{area}(\mathcal{C}_{b,\mathrm{I}}(d_i, d_j)) + \sum_{\substack{1 \le i \le n \\ d_i \ge 3}} \sum_{\substack{a_1 + a_2 = d_i - 2 \\ a_1, a_2 \ge 0}} c_{area}(\mathcal{C}_{b,\mathrm{II}}(a_1, a_2)) \\ &\sim \frac{1}{4} \frac{1}{(2d)^2} \left(\left(\sum_{i=1}^{n} (d_i + 2) \right)^2 - 3 \sum_{i=1}^{n} (d_i + 2) \right) = \frac{1}{4} \frac{1}{(2d)^2} \left((2d)^2 - 3 \cdot 2d \right) \sim \frac{1}{4}, \end{aligned}$$

where the first equivalence is the statement of Conjecture 3 and the second equivalence is the combination of (8)–(11). □

Remark 5 Note that in this proof we used much weaker restriction on the growth rate of the number of simple poles, namely it was sufficient to have $\mu_{-1} = o(g)$ as $g \to +\infty$.

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Affiliations

Amol Aggarwal[1] · Vincent Delecroix[2] · Élise Goujard[3] · Peter Zograf[4,5] · Anton Zorich[6,7]

Amol Aggarwal
agg_a@math.harvard.edu

Vincent Delecroix
20100.delecroix@gmail.com

Élise Goujard
elise.goujard@gmail.com

Peter Zograf
zograf@pdmi.ras.ru

1 Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA

2 LaBRI, Domaine universitaire, 351 cours de la Libération, 33405 Talence, France

[3] Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351 cours de la Libération, 33405 Talence, France

[4] St. Petersburg Department, Steklov Math. Institute, Fontanka 27, St. Petersburg 191023, Russia

[5] Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O. 29B, St. Petersburg 199178, Russia

[6] Center for Advanced Studies, Skoltech, Moscow, Russia

[7] Institut de Mathématiques de Jussieu-Paris Rive Gauche, Case 7012, 8 Place Aurélie Nemours, 75205 Paris Cedex 13, France