

*Ты, мой лес и вода, кто обедет, а кто, как сквозняк,
проникает в тебя, кто глаголет, а кто обняк...*

И. Бродский

*“You, my forest and water! One swerves, while the other shall spout
Through your body like draught; one declares, while the first has a doubt.”*

J. Brodsky

*“Mein Vater, mein Vater, und hoerest du nicht,
Was Erlenkoenig mir leise verspricht?”*

“Sei ruhig, bleib ruhig, mein Kind!

In duerren Blaettern saeuselt der Wind.”

J. W. Goethe

CRIES AND WHISPERS IN WIND-TREE FORESTS

VINCENT DELECROIX AND ANTON ZORICH

*To the memory of Bill Thurston
with admiration for his fantastic imagination.*

ABSTRACT. We study billiard in the plane endowed with symmetric \mathbb{Z}^2 -periodic obstacles of a right-angled polygonal shape. One of our main interests is dependence of the diffusion rate of the billiard on the shape of the obstacle. We prove, in particular, that when the number of angles of a symmetric connected obstacle grows, the diffusion rate tends to zero, thus answering a question of J.-C. Yoccoz.

Our results are based on computation of Lyapunov exponents of the Hodge bundle over hyperelliptic loci in the moduli spaces of quadratic differentials, which represents independent interest. In particular, we compute the exact value of the Lyapunov exponent λ_1^+ for all elliptic loci of quadratic differentials with simple zeroes and poles.

1. INTRODUCTION

The classical wind-tree model corresponds to a billiard in the plane endowed with \mathbb{Z}^2 -periodic obstacles of rectangular shape; the sides of the rectangles are aligned along the lattice, see Figure 1.

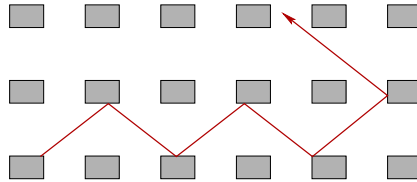


FIGURE 1. Original wind-tree model.

The wind-tree model (in a slightly different version) was introduced by P. Ehrenfest and T. Ehrenfest [Eh] about a century ago and studied, in particular, by J. Hardy and J. Weber [HaWe]. All these studies had physical motivations.

Several advances were obtained recently using the powerful technology of deviation spectrum of measured foliations on surfaces and the underlying dynamics in the moduli space. For all parameters of the obstacle and for almost all directions the trajectories are known to be recurrent [AH]; there are examples of divergent

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trajectories constructed in [D]; the non-ergodicity is proved in [FU]. It was proved in [DHL] that the diffusion rate is $\frac{2}{3}$; it does not depend either on the concrete values of parameters of the obstacle or on almost any direction and almost any starting point, see Figure 2.

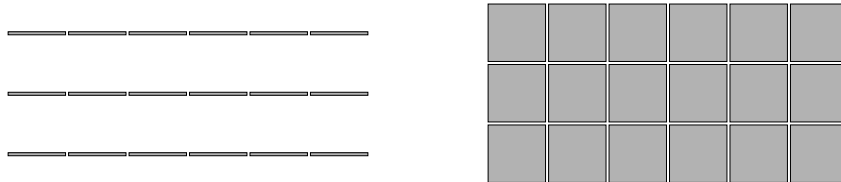


FIGURE 2. The diffusion rate $\frac{2}{3}$ does not depend on particular values of the parameters of the rectangular scatterer: it is the same for the plane with horizontal walls having tiny periodic holes, for narrow periodic corridors between “chocolate plates” and for any other periodic billiard as in Figure 1.

In other words, the maximal deviation of the trajectory from the starting point during the time t has the order of $t^{\frac{2}{3}}$ for large t in the following sense:

$$\lim_{t \rightarrow \infty} \frac{\log \text{diam}(\text{trajectory for time interval}[0, t])}{\log t} = \frac{2}{3}.$$

Thus, this behavior is quite different from the brownian motion, random walk in the plane, or billiards in the plane with periodic dispersing scatterers: for all of them the diffusion has the order \sqrt{t} (and, thus, the diffusion rate is $\frac{1}{2}$).

We address the natural question “what happens if we change the shape of the obstacle?”. We do not have ambition to solve this problem in the current paper in the most general setting. We just plant the wind-tree forest with several interesting families of obstacles and study the diffusion rate as the combinatorics of the obstacle inside the family becomes more complicated. We show, in particular, that if the obstacle is a connected symmetric right-angled polygon as on Figure 3, then the diffusion rate in the corresponding wind-tree model tends to zero as the number of corners of the obstacle grows; see Theorem 1 for more precise statement. This result gives an explicit affirmative answer to a question addressed by J.-C. Yoccoz.

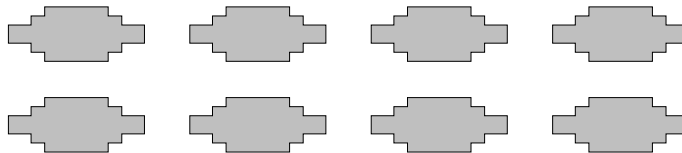


FIGURE 3. The diffusion rate in this wind-tree forest tends to zero when the number of corners of the obstacle grows.

Now, when we have showed that for certain species of wind-trees the sound in the wind-tree forest propagates “as a whisper” (in the sense that the diffusion rate tends to zero), the challenge is to prove that for certain other species it propagates “as a cry” with the diffusion rate approaching 1.

Question 1. *Are there periodic wind-tree billiards with diffusion rate arbitrary close to 1? Are there continuous families of wind-tree billiards like this? What are the shapes of the obstacles which provide diffusion rate arbitrary close to 1?*

1.1. Strategy of the proof. We develop the approach originated in the pioneering work of V. Delecroix, P. Hubert, and S. Lelièvre [DHL] who applied the results from dynamics in the moduli space to the wind-tree model. Several very deep recent advances in dynamics in moduli spaces are of crucial importance for us.

Following [DHL] we reformulate the original billiard problem in terms of the deviation spectrum for the leaves of directional measured foliations on the associated flat surface S . This part is quite elementary and straightforward.

By recent deep result of J. Chaika and A. Eskin [CkE] (based on fundamental advances of A. Eskin, M. Mirzakhani, A. Mokhammad [EMi], [EMiMo]) almost all directions on *every* flat surface are Lyapunov-generic. Combining the techniques [Fo], [Z] of deviation spectrum with the results from [DHL] we conclude that the diffusion rate of the wind-tree billiard as in Figure 3 equals to the Lyapunov exponent $\lambda^+(h^*) = \lambda^+(v^*)$ of certain very specific integer cocycles $h^*, v^* \in H^1(S, \mathbb{Z})$, where the flat surface S is considered as a point of the orbit closure $\mathcal{L}(S)$ in the ambient stratum of meromorphic quadratic differentials; the vector space $H^1(S, \mathbb{C})$ containing h^* and v^* is considered as a fiber over S of the complex Hodge bundle over $\mathcal{L}(S)$, and the Lyapunov exponents are the Lyapunov exponents of the complex Hodge bundle $H_{\mathbb{C}}^1$ over $\mathcal{L}(S)$ with respect to the Teichmüller geodesic flow.

In the current paper we intentionally focus on the family of billiards as in Figure 3 since for this family the rest of the computation is particularly transparent. Namely, the flat surface S belongs to the *hyperelliptic* locus $\mathcal{L} := \mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$. (In our particular situation, the *hyperelliptic* locus is, actually, “elliptic”: the genus of the covering surface is 1.) Moreover, applying the arguments analogous to those in [AtEZ], one immediately verifies that the family \mathcal{B} of billiards is so large (in dimension) that it is transversal to the unstable foliation in the ambient invariant submanifold \mathcal{L} , and, thus, for almost every billiard table Π in \mathcal{B} the orbit closure $\mathcal{L}(S)$ of the associated flat surface $S(\Pi)$ coincides with the entire locus \mathcal{L} .

The flat surface S has genus one, so the complex Hodge bundle $H_{\mathbb{C}}^1 = H_+^1$ has single positive Lyapunov exponent λ_1^+ . The fact that h, v are *integer* immediately implies that $\lambda^+(h) = \lambda^+(v) = \lambda_1^+$.

Formula (2.3) from [EKZ2] expresses the sum $\sum_{i=1}^g \lambda_i^+$ of all positive Lyapunov exponents of H_+^1 in terms of the degrees of zeroes (and poles) in the ambient locus and in terms of the Siegel–Veech constant $c_{area}(\mathcal{L})$. Since in our particular case the genus of the surface is equal to one, we get a formula for the individual Lyapunov exponent λ_1^+ in which we are interested.

Developing Lemma (1.1) from [EKZ2] we relate the Siegel–Veech constant $c_{area}(\mathcal{L})$ of the hyperelliptic locus $\mathcal{L} := \mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$ to the Siegel–Veech constants $c_{\mathcal{C}}(\mathcal{Q}(1^m, -1^{m+4}))$ of the underlying stratum in genus zero. Plugging in the resulting expression the explicit values of $c_{\mathcal{C}}(\mathcal{Q}(1^m, -1^{m+4}))$ obtained in the recent paper [AtEZ] and proving certain combinatorial identity for the resulting hypergeometric sum we obtain the desired explicit value of $\lambda_1^+(\mathcal{Q}^{hyp}(1^{2m}, -1^{2m}))$, which represents the diffusion rate in almost every original billiard.

Remark 1. Our results provide certain evidence that when the genus is fixed and the number of simple poles grows, the Lyapunov exponents of the Hodge bundle tend to zero (see [GrHu] for the original conjecture).

1.2. Structure of the paper. In section 2 we state the main result in two different forms. In section 3 we show how to reduce the problem of the diffusion rate in a generalized wind-tree billiard to the problem of evaluation of the top Lyapunov exponent λ_1^+ of the complex Hodge bundle over an appropriate hyperelliptic locus of quadratic differentials. In section 3.1 we revisit the original paper [DHL] where this question is treated in all details for the original wind-tree model with periodic rectangular scatterers. We suggest, however, several simplifications. Namely, in section 3.2 we describe the hyperelliptic locus over certain stratum of meromorphic quadratic differentials in genus zero where lives the flat surface S corresponding to the wind-tree billiard and we show that the diffusion rate corresponds to the top Lyapunov exponent λ_1^+ of the complex Hodge bundle over the $\mathrm{PSL}(2, \mathbb{R})$ -orbit closure of S . Following an analogous statement in [AtEZ] we prove in section 3.3 that for almost any initial billiard table Π the $\mathrm{PSL}(2, \mathbb{R})$ -orbit closure of the associated flat surface $S(\Pi)$ coincides with the entire hyperelliptic locus. At this stage we reduce the problem of evaluation of the diffusion rate for almost all billiard table in the family to evaluation of the single positive Lyapunov exponent of the complex Hodge bundle $H_+^1 = H_{\mathbb{C}}^1$ over certain specific hyperelliptic locus.

In section 4 we evaluate this Lyapunov exponent. We start by recalling in section 4.1 the technique from [EKZ2]; we also relate the Siegel–Veech constant of the hyperelliptic locus with the Siegel–Veech constant of the corresponding stratum in genus 0. In section 4.2 we summarize the necessary material on cylinder configurations in genus zero and on the related Siegel–Veech constants from [Bo] and [AtEZ]. Finally, in section 4.3 we prove the key Theorem 2 evaluating the desired Lyapunov exponent λ_1^+ . The proof uses a combinatorial identity for certain hypergeometric sum; this identity is proved separately in section 5.

Following the title “*What’s next?*” of the conference, we discuss in appendix B directions of further research in the area relevant to the context of this paper.

2. MAIN RESULTS

Denote by $\mathcal{B}(m)$ the family of billiards such that the obstacle has $4m$ corners with the angle $\pi/2$. Say, all billiards from the original wind-tree family as in Figures 1 and 2 live in $\mathcal{B}(1)$; the billiard in Figure 3 belongs to $\mathcal{B}(3)$; the billiard in Figure 3 belongs to $\mathcal{B}(17)$.

Theorem 1. *For almost all billiard tables in the family $\mathcal{B}(m)$ and for almost all directions the diffusion rate $\delta(m)$ is the same and equals*

$$\delta(m) = \frac{(2m)!!}{(2m+1)!!}.$$

When $m \rightarrow +\infty$ $\delta(m)$ has asymptotics

$$\delta(m) = \frac{\sqrt{\pi}}{2\sqrt{m}} \left(1 + O\left(\frac{1}{m}\right) \right).$$

Here the double factorial means the product of all even (correspondingly odd) natural numbers from 2 to $2m$ (correspondingly from 1 to $2m+1$). For the original

wind-tree, when the obstacle is a rectangle, we have $m = 1$ and we get the value $\delta(1) = \frac{2}{3}$ found in [DHL].

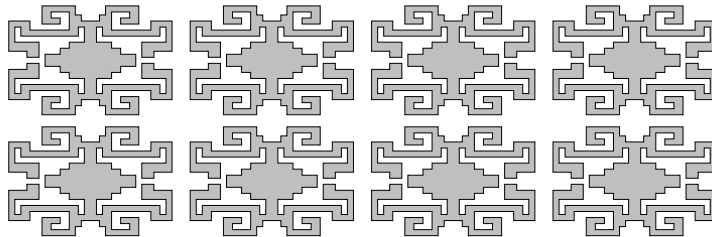


FIGURE 4. The diffusion rate depends only on the number of corners of the obstacle and not on the particular values of (almost all) length parameters nor on the particular shape of the obstacle.

Following the strategy described in section 1.1 we derive Theorem 1 from the following result.

Theorem 2. *The locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$ is connected and invariant under the action of $\mathrm{PGL}(2, \mathbb{R})$. The measure induced on $\mathcal{Q}_1^{hyp}(1^{2m}, -1^{2m})$ from the Masur–Veech measure on $\mathcal{Q}_1(1^m, -1^{m+4})$ is $\mathrm{PSL}(2, \mathbb{R})$ -invariant and ergodic under the action of the Teichmüller geodesic flow. The Lyapunov exponent $\lambda_1^+(m)$ of the Hodge bundle H_+^1 over the locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ under consideration has the following value:*

$$(2.1) \quad \lambda_1^+(m) = \frac{(2m)!!}{(2m+1)!!}.$$

Remark 2. Note that there is a very important difference between the case of $m = 1$ (corresponding to the classical wind-tree) and the cases $m \geq 2$. Namely, the locus $\mathcal{Q}^{hyp}(1^2, -1^2)$ over $\mathcal{Q}(1, -1^5)$ is *nonvarying*: the Lyapunov exponent $\lambda_1^+ = \frac{2}{3}$ for all flat surfaces in the locus $\mathcal{Q}^{hyp}(1^2, -1^2)$. For $m \geq 2$ it is not true anymore. First of all, for each integer $m \geq 2$, taking appropriate unramified covering of degree m of a flat surface in the stratum $\mathcal{Q}(1^2, -1^2)$ we get a flat surface in the hyperelliptic locus of $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$. Concretely, such surface can be built starting from the original wind-tree model with rectangles and taking a fundamental domain that is made of m copies of the unit square.

By construction, the Lyapunov exponent λ_1^+ of the resulting Teichmüller curve in $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$, and, hence, the diffusion rate δ for the corresponding wind-tree billiard does not change: $\delta = \lambda_1^+ = 2/3$. Furthermore, for $m = 2$ we were able to find examples of square tiled surfaces for which the value is neither the generic value $\delta(2) = 8/15 = 0.5333\dots$ nor $2/3 = 0.6666\dots$

permutations r and u	λ_1^+
$r = (1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12, 13, 14)$ $u = (1, 3, 13, 8, 2, 14)(4, 6, 11, 5, 10, 12)(7, 9)$	$\frac{20}{33} = 0.6060\dots$
$r = (1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12, 13, 14)$ $u = (1, 2, 3, 14, 9)(4, 13)(5, 6, 7, 11, 12)(8, 10)$	$\frac{6}{11} = 0.5454\dots$

See also the example in Appendix A with $m = 3$.

Question 2. *What are the extremal values of λ_1^+ over all closed $\mathrm{PSL}(2, \mathbb{R})$ -invariant suborbifolds in a given stratum (given locus)? Same question for the concrete hyperelliptic locus $\mathcal{Q}^{\mathrm{hyp}}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$? What are the shapes of billiards for which these values are achieved (if there are any wind-tree billiards corresponding to these invariant suborbifolds)?*

3. FROM BILLIARDS TO FLAT SURFACES

3.1. Original wind-tree revisited. Recall that in the classical case of a billiard in a rectangle we can glue a flat torus out of four copies of the billiard table and unwind billiard trajectories to flat geodesics on the resulting flat torus.

In the case of the wind-tree model we also start from gluing a flat surface out of four copies of the billiard table. The resulting surface is $\mathbb{Z} \oplus \mathbb{Z}$ -periodic with respect to translations by vectors of the original lattice. We pass to the quotient over $\mathbb{Z} \oplus \mathbb{Z}$ to get a compact flat surface without boundary. For the case of the original wind-tree billiard the resulting flat surface X is represented at Figure 5. It has genus 5; it belongs to the stratum $\mathcal{H}(2^4)$ (see section 3 of [DHL] for details).

One of the key statements of [DHL] can be stated as follows.

Let Π be the original rectangular obstacle, define the corresponding wind-tree billiard by the same symbol Π . Let $X = X(\Pi)$ be the flat surface as in Figure 5 constructed by the wind-tree billiard defined by the obstacle Π .

Consider the $\mathrm{SL}(2, \mathbb{R})$ -orbit closure $\mathcal{L}(X) \subset \mathcal{H}(2^4)$ of the flat surface X . Consider the cohomology classes $h^*, v^* \in H^1(X, \mathbb{Z})$ Poincaré-dual to cycles

$$\begin{aligned} h &= h_{00} - h_{01} + h_{10} - h_{11} \\ v &= v_{00} - v_{10} + v_{01} - v_{11} \end{aligned}$$

(see Figure 5) as elements of the fiber over the point $X \in \mathcal{L}(X)$ of the complex Hodge bundle $H_{\mathbb{C}}^1$ over $\mathcal{L}(X)$.

Theorem ([DHL]). *The diffusion rate in the original wind-tree billiard Π coincides with the Lyapunov exponent $\lambda(h^*) = \lambda(v^*)$ of the complex Hodge bundle $H_{\mathbb{C}}^1$ with respect to the Teichmüller geodesic flow on $\mathcal{L}(X)$.*

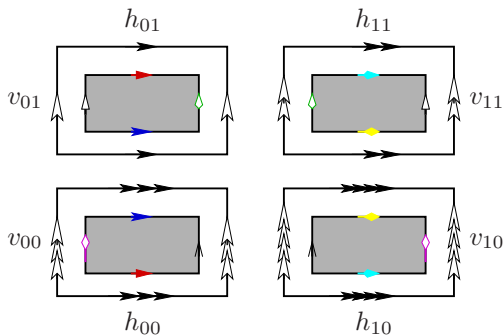


FIGURE 5. The flat surface X obtained as a quotient over $\mathbb{Z} \oplus \mathbb{Z}$ of an unfolded wind-tree billiard table.

Remark 3. Recent result in [CkE] proving that for *any* flat surface almost all directions on it are Lyapunov-generic allows to simplify part of the argument in

the proof of the above Theorem. In particular, it justifies that $\lambda(h^*)$ and $\lambda(v^*)$ are well-defined for X endowed with almost all direction.

Note that any resulting flat surface X as in Figure 5 has (at least) the group $(\mathbb{Z}/2\mathbb{Z})^3$ as a group of isometries. As three generators we can choose the isometries τ_h and τ_v interchanging the pairs of flat tori with holes in the same rows (correspondingly columns) by parallel translations and the isometry ι acting on each of the four tori with holes as the central symmetry with the center in the center of the hole.

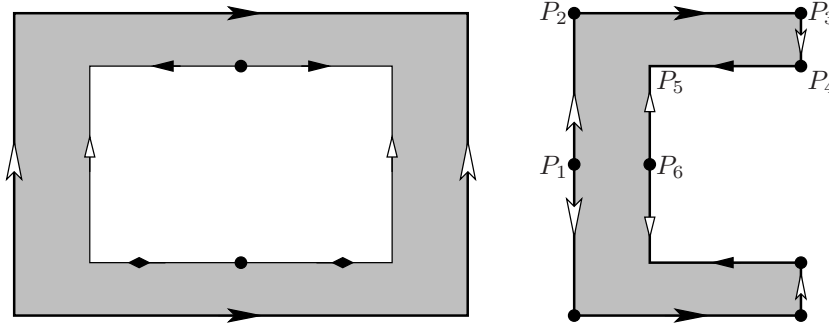


FIGURE 6. A surface \tilde{S} in the hyperelliptic locus $\mathcal{Q}^{hyp}(1^2, -1^2)$ (on the left) is a double cover over the underlying surface S in $\mathcal{Q}(1, -1^5)$ (on the right) ramified at four simple poles represented by bold dots.

Consider the quotient \tilde{S} of the flat surface \tilde{S} over the subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ of isometries spanned by τ_h and $\iota \circ \tau_v$. The resulting surface \tilde{S} as on the left side of the Figure 6 belongs to the stratum $\mathcal{Q}(1^2, -1^2)$; in particular, it has genus 1. The surface S obtained as the quotient of the original flat surface X over the entire group $(\mathbb{Z}/2\mathbb{Z})^3$ as on the right side of the Figure 6 belongs to the stratum $\mathcal{Q}(1, -1^5)$; in particular, it has genus 0. Clearly, \tilde{S} is a ramified double cover over S with ramification points at four (out of five) simple poles of the flat surface S .

Lemma 3.1. *Consider the natural projection $p : X \rightarrow \tilde{S}$. The following inclusion is valid:*

$$h^* \in p^*(H^1(\tilde{S}; \mathbb{Z})).$$

Proof. Recall that

$$h = h_{00} - h_{01} + h_{10} - h_{11},$$

where the cycles h_{ij} , $i, j = 0, 1$ are indicated in Figure 5. Now note that

$$\tau_h(h_{00}) = h_{10} \quad \iota \circ \tau_v(h_{00}) = -h_{01} \quad \iota \circ \tau_v \circ \tau_h(h_{00}) = -h_{11} \quad .$$

Thus, the cycle h is invariant under the action of the commutative group Γ spanned by τ_h and $\iota \circ \tau_v$. This implies that the Poincaré-dual cocycle h^* is also invariant under the action of Γ and, hence, is induced from some cocycle in cohomology of $\tilde{S} = X/\Gamma$. \square

Corollary 1. *Consider the orbit closure $\mathcal{M}(\tilde{S})$ of the flat surface $\tilde{S}(\Pi) \in \mathcal{Q}(1^2, -1^2)$. The diffusion rate in the original wind-tree billiard Π in vertical direction coincides*

with the positive Lyapunov exponent λ_1^+ of the complex Hodge bundle $H_{\mathbb{C}}^1$ with respect to the Teichmüller geodesic flow on $\mathcal{M}(\tilde{S})$.

Proof. From the Theorem of Delecroix–Hubert–Lelièvre cited above we know that the diffusion rate coincides with the Lyapunov exponent $\lambda(h^*)$. By the previous Lemma $h^* = p^*(\alpha)$ where $\alpha \in H^1(\tilde{S}; \mathbb{Z})$. It is immediate to see that $\lambda(h^*) = \lambda(\alpha)$, where $\lambda(\alpha)$ is already the Lyapunov exponent of the of the complex Hodge bundle $H_{\mathbb{C}}^1$ with respect to the Teichmüller geodesic flow on $\mathcal{M}(\tilde{S})$. Since $g(\tilde{S}) = 1$ the corresponding cocycle has Lyapunov exponents $\pm\lambda_1^+$. Since α is an integer covector, $\lambda(\alpha)$ cannot be strictly negative. Hence $\lambda(\alpha) = \lambda_1^+$. \square

3.2. Elliptic locus and diffusion in the generalized wind-tree billiard. The hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ in $\mathcal{Q}(1^{2m}, -1^{2m})$ is obtained by the following construction. For any flat surface S in $\mathcal{Q}(1^m, -1^{m+4})$ consider all possible quadruples of unordered simple poles. For each quadruple construct a ramified double cover with four ramification points exactly at the chosen quadruple of points. By construction the induced flat surface belongs to the stratum $\mathcal{Q}(1^{2m}, -1^{2m})$ in genus 1. Considering all possible flat surfaces S in $\mathcal{Q}(1^m, -1^{m+4})$ and all covers over all quadruples of simple poles we get the locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$.

It is immediate to see that when the obstacle is a rectangle as in the original wind-tree billiard the surface \tilde{S} constructed in section 3.1 belongs to the hyperelliptic locus $\mathcal{Q}^{hyp}(1^2, -1^2)$ over the stratum $\mathcal{Q}(1, -1^5)$. Similarly, when the obstacle has $4m$ corners with the angle $\pi/2$, the analogous surface \tilde{S} (as on the left of Figure 7) belongs to the hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$.

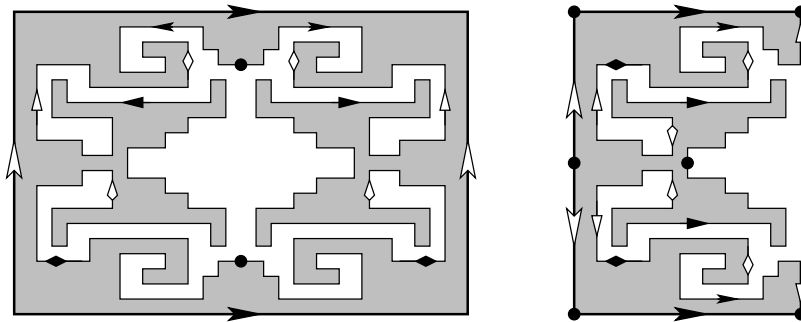


FIGURE 7. A surface \tilde{S} in the hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ is a double cover over the underlying surface S in $\mathcal{Q}(1^m, -1^{m+4})$ branched at the four simple poles represented by bold dots.

The arguments of Delecroix–Hubert–Lelièvre [DHL] extend to the more general case of obstacles Π symmetric with respect to vertical and horizontal axes with arbitrary numbers of angles $\frac{\pi}{2}$, basically, line by line with an extra simplification due to results [CkE] mentioned in Remark 3. Applying exactly the same consideration as above we prove the following statement generalizing Corollary 1 to arbitrary $m \in \mathbb{N}$.

Let Π be a connected obstacle having $4m$ corners with angle $\frac{\pi}{2}$ and $4(m-1)$ corners with angle $\frac{3\pi}{2}$ aligned in such way that all its sides are vertical or horizontal

(see the white domain in the left picture in Figure 7). Suppose that Π is symmetric under reflections over some vertical and some horizontal lines.

Proposition 1. *Consider the orbit closure $\mathcal{M}(\tilde{S}) \subset \mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ of the flat surface $\tilde{S}(\Pi)$ in the ambient hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$. The diffusion rate in the wind-tree billiard with periodic obstacles Π aligned with the lattice coincides with the positive Lyapunov exponent λ_1^+ of the complex Hodge bundle $H_{\mathbb{C}}^1$ over $\mathcal{M}(\tilde{S})$ with respect to the Teichmüller geodesic flow on $\mathcal{M}(\tilde{S})$.*

3.3. Orbit closures. Consider the original wind-tree billiard with rectangular obstacles. Such a billiard is described by five real parameters: by two lengths of the sides of the external rectangle defining the lattice; by two lengths of the sides of the inner rectangle represented by the obstacle and by the angle defining the direction of trajectory, see Figure 6. Varying continuously these parameters we obtain a continuous family of billiards.

Starting with a more general obstacle as in Figure 7 having $4m$ corners of angle $\frac{\pi}{2}$, $4(m-1)$ corners of angle $\frac{3\pi}{2}$ and symmetric with respect to vertical and horizontal axes of symmetry we get an analogous continuous family of billiards which we denote by $\mathcal{B}(m)$. It is immediate to check that

$$(3.1) \quad \dim_{\mathbb{R}} \mathcal{B}(m) = 2m + 2$$

Here the direction of the billiard flow is not considered as a parameter of the family \mathcal{B} . For the original wind-tree billiard with rectangular obstacles this gives $\dim_{\mathbb{R}} (\mathcal{B}(1) \times \mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})^2) = 4 + 1 = 5$, as we have already seen.

Proposition 2. *For Lebesgue-almost every directional billiard in $\Pi \in \mathcal{B}(m)$ the $\mathrm{GL}(2, \mathbb{R})$ -orbit closure of $S(\Pi)$ in $\mathcal{Q}(1^m, -1^{m+4})$ coincides with the entire ambient stratum $\mathcal{Q}(1^m, -1^{m+4})$.*

Proof. The Proposition is a straightforward corollary of Lemma 3.2 below. \square

The following Lemma is completely analogous to Proposition 3.2 in [AtEZ].

Lemma 3.2. *Consider the canonical local embedding*

$$\mathcal{B}(m) \times (\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})^2) \hookrightarrow \mathcal{Q}(1^m, -1^{m+4}).$$

For almost all pairs (Π, θ) in $\mathcal{B}(m) \times (\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})^2)$ the projection of the tangent space $T_(\mathcal{B}(m) \times (\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})^2))$ to the unstable subspace of the Teichmüller geodesic flow is a surjective map.*

Proof. Let us first prove the statement for $m = 1$ and then make the necessary adjustments for the most general case. Consider a broken line following the upper part of the polygon as on the right side of figure 6 starting at the point P_0 and finishing at the corner P_4 . Turn the figure by the angle θ . Consider the associated flat surface $S \in \mathcal{Q}(1, -1^5)$. Consider the canonical orienting double cover $\hat{S} \in \mathcal{H}(2)$. The six Weierstrass points of the cover correspond to the corners of our polygonal pattern and to the two bold points on the horizontal axe of symmetry. Thus the vectors of the resulting broken line considered as complex numbers are exactly the basic half-periods of the holomorphic 1-form ω corresponding to a standard basis of cycles on the hyperelliptic surface \hat{S} (see sections 3.1 and 3.3 in [AtEZ] for more details). The first cohomology $H^1(\hat{S}; \mathbb{C})$ serve as local coordinates in $\mathcal{Q}(1, -1^5)$.

The components of the projection of the vector of periods to the space $H^1(\hat{S}; \mathbb{R})$ are of the form

$$\pm 2 \sin(\phi) |P_i P_{i+1}| \quad \text{or} \quad \pm 2 \cos(\phi) |P_i P_{i+1}| \quad \text{where } i = 1, \dots, 4.$$

Thus, for ϕ different from an integer multiple of $\pi/2$ the composition map

$$T_*(\mathcal{B}(m) \times (\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})^2)) \rightarrow H^1(\hat{S}; \mathbb{R})$$

is a surjective map.

It is clear from the proof that to extend it from $m = 1$ to arbitrary $m \in \mathbb{N}$ it is sufficient to show that the real dimension of $\mathcal{B}(m)$ coincides with the complex dimension of $\mathcal{Q}(1^m, -1^{m+4})$. Recalling (3.1) and the classical formula for the dimension of a stratum $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials

$$\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_n) = 2g + n - 2$$

we conclude that

$$\dim_{\mathbb{R}} \mathcal{B}(m) = 2m + 2 = \dim_{\mathbb{C}} \mathcal{Q}(1^m, -1^{m+4})$$

which completes the proof of the Lemma. \square

Combining Propositions 1 and 2 with the result of Chaika–Eskin [CkE] telling that for any flat surface almost all directions are Lyapunov-generic, we obtain the following Corollary, which proves the first part of Theorem 1.

Corollary 2. *For Lebesgue-almost every directional billiard in $\Pi \in \mathcal{B}(m)$ the diffusion rate in almost all directions $\theta \in (\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})^2)$ in the wind-tree billiard Π coincides with the positive Lyapunov exponent λ_1^+ of the complex Hodge bundle $H_{\mathbb{C}}^1$ over the hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ with respect to the Teichmüller geodesic flow on this locus.*

It remains to evaluate the Lyapunov exponent λ_1^+ of the complex Hodge bundle $H_{\mathbb{C}}^1$ over the hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$, which we do in the next section.

4. SIEGEL–VEECH CONSTANTS AND SUM OF THE LYAPUNOV EXPONENTS OF THE HODGE BUNDLE OVER HYPERELLIPTIC LOCI

In this section we relate the Siegel–Veech constants of an invariant hyperelliptic locus and of the underlying stratum of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 . Applying the technique from [EKZ2] and developing the results from [AtEZ] this allows us to get an explicit value for the desired Lyapunov exponent λ_1^+ of the hyperelliptic locus and thus, to prove Theorem 2.

4.1. Sum of the Lyapunov exponents of the complex Hodge bundle over hyperelliptic loci of quadratic differentials.

We need the following result

Theorem (Theorem 2 in [EKZ2]). *Consider a stratum $\mathcal{Q}_1(d_1, \dots, d_n)$ in the moduli space of quadratic differentials with at most simple poles, where $d_1 + \dots + d_n = 4g - 4$. Let \mathcal{M}_1 be any regular $\text{PSL}(2, \mathbb{R})$ -invariant suborbifold of $\mathcal{Q}_1(d_1, \dots, d_n)$.*

The Lyapunov exponents $\lambda_1^+ \geq \dots \geq \lambda_g^+$ of the complex Hodge bundle $H_{\mathbb{C}}^1 = H_{\mathbb{C}}^1$ over \mathcal{M}_1 along the Teichmüller flow satisfy the following relation:

$$(4.1) \quad \lambda_1^+ + \dots + \lambda_g^+ = \kappa + \frac{\pi^2}{3} \cdot c_{\text{area}}(\mathcal{M}_1),$$

where

$$(4.2) \quad \kappa = \frac{1}{24} \sum_{j=1}^n \frac{d_j(d_j + 4)}{d_j + 2}$$

and $c_{area}(\mathcal{M}_1)$ is the Siegel–Veech constant corresponding to the suborbifold \mathcal{M}_1 . By convention the sum in the left-hand side of equation (4.1) is defined to be equal to zero for $g = 0$.

In the context of this paper we are particularly interested in the case when \mathcal{M} is a hyperelliptic locus over some stratum of meromorphic quadratic differentials with at most simple poles in genus zero.

Let \mathcal{M}_1 be a closed $\mathrm{SL}(2, \mathbb{R})$ - (correspondingly $\mathrm{PSL}(2, \mathbb{R})$ -invariant) suborbifold in a stratum of Abelian (correspondingly quadratic) differentials; let ν be the associated $\mathrm{SL}(2, \mathbb{R})$ -ergodic (correspondingly $\mathrm{PSL}(2, \mathbb{R})$ -ergodic) measure on \mathcal{M}_1 .

Consider a locus $\tilde{\mathcal{M}}_1$ of all possible double covers of fixed profile over flat surfaces from \mathcal{M}_1 . Suppose that it is closed, connected and $\mathrm{SL}(2, \mathbb{R})$ - (correspondingly $\mathrm{PSL}(2, \mathbb{R})$ -invariant), and that $\tilde{\nu}$ is the ergodic measure on $\tilde{\mathcal{M}}$ such that ν is the direct image of $\tilde{\nu}$ with respect to the natural projection $\tilde{\mathcal{M}}_1 \rightarrow \mathcal{M}_1$.

Let c_c be an area Siegel–Veech constant associated to the counting of *multiplicity one configuration \mathcal{C} of cylinders* weighted by the area of the cylinder. (Here the notion “configuration” is understood in the sense of [MsZ] and [W2]; “multiplicity one” means that \mathcal{C} contains a single cylinder). The reader might think of \mathcal{M} as of some stratum of quadratic differentials in genus 0; then there are only two types of configurations (see section 4.2 below or the original paper [Bo]), and both configurations contain a single cylinder.

We assume that the profile of the double cover does not admit branching points inside the cylinders of the configuration \mathcal{C} . Then the configuration \mathcal{C} induces a cylinder configuration $\tilde{\mathcal{C}}$ on the double cover. Let \tilde{c} be the associated area Siegel–Veech constant. The Lemma below relates c and \tilde{c} . it is a slight generalization of Lemma 1.1 in [EKZ2].

Lemma 4.1. *If the circumference of the cylinder in the configuration has nontrivial monodromy (that is if the lift of the cylinder in the double cover is a unique cylinder twice wider), then $\tilde{c} = c/2$.*

If the circumference of the cylinder has trivial monodromy (that is, if the preimage of the cylinder consists in two cylinders isometric to the one on the base), then $\tilde{c} = 2c$.

Proof. The second case corresponds to Lemma 1.1 in [EKZ2]. The first case is proved analogously. \square

In the same setting, let κ be the expression (4.2) in degrees of singularities of the flat surfaces in the invariant manifold \mathcal{M} and $\tilde{\kappa}$ be analogous expression in degrees of singularities of the double covers of fixed profile in the invariant manifold $\tilde{\mathcal{M}}$.

It would be convenient to introduce the following notations

$$\begin{aligned} \Delta\tilde{\kappa} &:= \tilde{\kappa} - 2\kappa \\ \Delta\tilde{c}_{area} &:= c_{area}(\tilde{\mathcal{M}}_1) - 2c_{area}(\mathcal{M}_1). \end{aligned}$$

Lemma 4.2. *For any ramified double covering the degrees of zeroes of the quadratic differential on the underlying surface and of the induced quadratic differential on*

the double cover satisfy the following relation:

$$(4.3) \quad \Delta\tilde{\kappa} = \tilde{\kappa} - 2\kappa = \frac{1}{4} \sum_{\substack{\text{ramification} \\ \text{points}}} \frac{1}{d_j + 2}.$$

Proof. The non ramified zeros cancel out. For the other zeroes (and poles) a singularity of degree d_j at the ramification point gives rise to a zero of degree $2d_j + 2$. Hence,

$$\begin{aligned} \tilde{\kappa} - 2\kappa &= \frac{1}{24} \sum_{\substack{\text{ramification} \\ \text{points}}} \left(\frac{(2d_j + 2)(2d_j + 6)}{2d_j + 4} - 2 \frac{d_j(d_j + 4)}{d_j + 2} \right) \\ &= \frac{1}{12} \sum_{\substack{\text{ramification} \\ \text{points}}} \frac{(d_j + 1)(d_j + 3) - d_j(d_j + 4)}{d_j + 2} \\ &= \frac{1}{12} \sum_{\substack{\text{ramification} \\ \text{points}}} \frac{3}{d_j + 2}. \end{aligned}$$

□

The following notational Lemma would help to simplify certain bulky computations.

Lemma 4.3. *For any locus $\tilde{\mathcal{M}}_1$ of double coverings of fixed profile as above over a $\text{PSL}(2, \mathbb{R})$ -invariant orbifold \mathcal{M}_1 in some stratum of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 the sum of Lyapunov exponents*

$$\Lambda^+ = \lambda_1^+ + \dots + \lambda_g^+$$

satisfies the following relation

$$(4.4) \quad \Lambda^+ = \Delta\tilde{\kappa} + \frac{\pi^2}{3} \cdot \Delta\tilde{c}_{\text{area}}.$$

Proof. For any invariant submanifold \mathcal{M}_1 in a stratum of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 the sum of Lyapunov exponents is null, so formula (4.1) gives

$$0 = \kappa + \frac{\pi^2}{3} \cdot c_{\text{area}}(\mathcal{M}_1).$$

By the same formula (4.1) gives we have

$$\Lambda^+ = \tilde{\kappa} + \frac{\pi^2}{3} \cdot c_{\text{area}}(\tilde{\mathcal{M}}_1).$$

Extracting from the latter relation twice the previous one we obtain the desired relation (4.4). □

4.2. Configurations for the strata in genus zero and corresponding Siegel–Veech constants (after Boissy and Athreya–Eskin–Zorich). In this section we recall briefly the results from [Bo] describing configurations of periodic geodesics for flat surfaces in genus zero, and the results from [AtEZ] providing the values of the corresponding Siegel–Veech constants. By $Q(d_1, \dots, d_k)$ we denote a stratum of meromorphic quadratic differentials with at most simple poles, where $d_i \in \{-1, 1, 2, \dots\}$ denote all zeroes and poles, and $\sum_{i=1}^k d_i = -4$.

A “pocket”. In this configuration we have a single cylinder filled with closed regular geodesics, such that the cylinder is bounded by a saddle connection joining a fixed pair of simple poles P_{j_1}, P_{j_2} on one side and by a separatrix loop emitted from a fixed zero P_i of order $d_i \geq 1$ on the other side.

By convention, the affine holonomy associated to this configuration corresponds to the closed geodesic and *not* to the saddle connection joining the two simple poles. (Such a saddle connection is twice as short as the closed geodesic.)

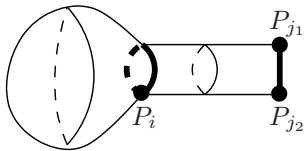


FIGURE 8. A “pocket” configuration with a cylinder bounded on one side by a saddle connection joining two simple poles, and by a saddle connection joining a zero to itself on the other side.

By Theorem 4.5 and formula (4.28) in [AtEZ], the Siegel–Veech constant $c_{j_1, j_2; i}^{pocket}$ corresponding to this configuration has the form

$$(4.5) \quad c_{j_1, j_2; i}^{pocket} = \frac{d_i + 1}{(k - 4)} \cdot \frac{1}{2\pi^2}.$$

One can consider the union of several configurations as above fixing the pair of simple poles P_{j_1}, P_{j_2} but considering *any* zero P_i on the boundary of the cylinder. By Corollary 4.7 and formula (4.36) in [AtEZ], the resulting Siegel–Veech constant c_{j_1, j_2}^{pocket} corresponding to this configuration has the form

$$(4.6) \quad c_{j_1, j_2}^{pocket} = \frac{1}{2\pi^2}.$$

A “dumbbell”. For the second configuration we still have a single cylinder filled with closed regular geodesics. But this time the cylinder is bounded by a separatrix loop on each side. We assume that the separatrix loop bounding the cylinder on one side is emitted from a fixed zero P_i of order $d_i \geq 1$ and that the separatrix loop bounding the cylinder on the other side is emitted from a fixed zero P_j of order $d_j \geq 1$.

Such a cylinder separates the original surface S in two parts; let $P_{i_1}, \dots, P_{i_{k_1}}$ be the list of singularities (zeroes and simple poles) which get to the first part and $P_{j_1}, \dots, P_{j_{k_2}}$ be the list of singularities (zeroes and simple poles) which get to the second part. In particular, we have $i \in \{i_1, \dots, i_{k_1}\}$ and $j \in \{j_1, \dots, j_{k_2}\}$. We assume that S does not have any marked points. Denoting as usual by d_k the order of the singularity P_k we can represent the sets with multiplicities $\alpha := \{d_1, \dots, d_k\}$ as a disjoint union of the two subsets

$$\{d_1, \dots, d_k\} = \{d_{i_1}, \dots, d_{i_{k_1}}\} \sqcup \{d_{j_1}, \dots, d_{j_{k_2}}\}.$$

(Recall that $\{d_1, \dots, d_k\}$ denotes all zeroes and poles.) This information is considered to be part of the configuration.

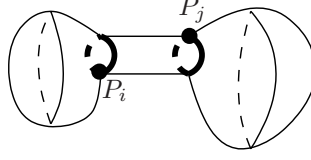


FIGURE 9. A “dumbbell” composed of two flat spheres joined by a cylinder. Each boundary component of the cylinder is a saddle connection joining a zero to itself.

By Theorem 4.8 and equation (4.38) in [AtEZ] the corresponding Siegel–Veech constant $c^{dumbell}$ is expressed as follows:

$$(4.7) \quad c_{i,j}^{dumbell} = \frac{(d_i + 1)(d_j + 1)}{2} \cdot \frac{(k_1 - 3)!(k_2 - 3)!}{(k - 4)!} \cdot \frac{1}{\pi^2}.$$

According to [Bo] and [MsZ], almost any flat surface S in any stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles different from the pillowcase stratum $\mathcal{Q}_1(-1^4)$ does not have a single regular closed geodesic not contained in one of the two families described above.

Finally, by Corollary 4.10 from [AtEZ] (generalizing the theorem of Vorobets from [Vo]) the Siegel–Veech constant c_{area} is expressed in terms of the above Siegel–Veech constants as follows. For any stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of meromorphic quadratic differentials with simple poles on \mathbb{CP}^1 the Siegel–Veech constant c_{area} is expressed in terms of the Siegel–Veech constants of configurations as follows:

$$(4.8) \quad c_{area} = \frac{1}{k - 3} \cdot \sum_{\substack{\text{Configurations } \mathcal{C} \\ \text{containing a cylinder}}} c_{\mathcal{C}}.$$

4.3. Proof of Theorem 2. Now everything is ready for the proof of Theorem 2.

Proof. The connectedness of our hyperelliptic locus follows from the fact that it can be seen as a \mathbb{C}^* -bundle over the space $\mathcal{C}(m, m, 4)$ of configurations of points on \mathbb{CP}^1 considered up to a modular transformation. More precisely, $\mathcal{C}(m, m, 4)$ can be seen as a set of configurations of $2m + 4$ distinct points arranged into three groups (“colored into three colors”) of cardinalities $\{m, m, 4\}$, where the points inside each group are named. The first group represents the simple zeroes, the second one — unramified simple poles, and the last one — the ramified simple poles. Clearly, such space is connected.

By the results of H. Masur [Ms] and of W. Veech [Ve], the Teichmüller geodesic flow on the underlying stratum is ergodic, and, moreover, “sufficiently hyperbolic”. Thus, the induced flow on any connected finite cover is also ergodic. The hyperelliptic locus $\mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$ is a finite cover, and as we have just proved it is connected. This proves the ergodicity of the Teichmüller flow on the hyperelliptic locus.

Let us compute Δc_{area}^{pocket} , see section 4.1. When both simple poles P_{j_1}, P_{j_2} involved in the “pocket” configuration are unramified points or when they are both ramified, the holonomy of the hyperelliptic cover $\tilde{S} \rightarrow S$ along the perimeter of

the cylinder is trivial, so by Lemma 4.1 such configurations do not contribute to $\Delta\tilde{c}_{area}^{pocket}$. By Lemma 4.1, a configuration when one of P_{j_1}, P_{j_2} is ramified and the other one is nonramified, contributes to $\Delta\tilde{c}_{area}^{pocket}$ with a weight $-\frac{3}{2}$. Since we have 4 ramified poles and m nonramified, there are $4m$ such configurations. Applying (4.6) and (4.8) with $k = m + (m + 4)$ we get

$$(4.9) \quad \frac{\pi^2}{3} \cdot \Delta\tilde{c}_{area}^{pocket} = \frac{\pi^2}{3} \cdot 4m \cdot \left(-\frac{3}{2}\right) \cdot \left(\frac{1}{2m+1} \cdot \frac{1}{2\pi^2}\right) = -\frac{m}{2m+1}.$$

Let us proceed to computation of $\Delta\tilde{c}_{area}^{dumbbell}$. When the number of ramified simple poles on two parts of the “dumbbell” is even, the holonomy of the cover along the perimeter of the cylinder is trivial, so by Lemma 4.1 such configurations do not contribute to $\Delta\tilde{c}_{area}^{dumbbell}$. By Lemma 4.1, a configuration when the number of ramified simple poles on each side of the dumbbell is odd contributes to $\Delta\tilde{c}_{area}^{dumbbell}$ with a weight $-\frac{3}{2}$. To compute the number of such configurations we remark that we have to split m named simple zeroes into two groups of m_1 and $m - m_1$ ones; we also have to split 4 ramified simple poles into 1 and 3; finally we have to split m unramified simple poles into $m_1 + 1$ and $m - m_1 - 1$ ones to have in total $m_1 + 2$ simple poles on one side of the “dumbbell” and $m - m_1 + 2$ on the other side. Note that the fact that there is a single ramified simple pole on one part and 3 ramified simple poles on the other makes our count of configurations asymmetric. Finally note that we have to chose one of m_1 zeroes to be located at the boundary of the cylinder on one side and one of $m - m_1$ zeroes to be located at the boundary of the cylinder on the other side.

For any given m_1 , where $1 \leq m_1 \leq m - 1$ our count gives

$$\binom{m}{m_1} \cdot \binom{4}{1} \cdot \binom{m}{m_1 - 1} \cdot (m_1 \cdot (m - m_1))$$

Applying the general formulae (4.7) and (4.8) to $c_{i,j;area}^{dumbbell}$; taking into consideration that in our particular case we have $d_i = d_j = 1$; $k_1 = m_1 + (m_1 + 2)$; $k_2 = (m - m_1) + (m - m_1 + 2)$; and $k = m + (m + 4)$; and applying (4.8) we get

$$(4.10) \quad \begin{aligned} \frac{\pi^2}{3} \Delta\tilde{c}_{area}^{dumbbell} &= \\ &= \frac{\pi^2}{3} \sum_{m_1=1}^{m-1} \left(\binom{m}{m_1} \cdot \binom{4}{1} \cdot \binom{m}{m_1 - 1} \cdot (m_1 \cdot (m - m_1)) \right) \cdot \left(-\frac{3}{2}\right) \cdot \\ &\cdot \frac{1}{(2m+4) - 3} \cdot \frac{(1+1)(1+1)}{2} \cdot \frac{((2m_1+2) - 3)! \cdot ((2m - 2m_1 + 2) - 3)!}{(2m+4) - 4!} \cdot \frac{1}{\pi^2} = \\ &= -\frac{1}{2m+1} \cdot \sum_{m_1=1}^{m-1} \binom{m}{m_1} \binom{m}{m_1 - 1} \cdot \frac{(2m_1)! \cdot (2m - 2m_1)!}{(2m)!} = \\ &= -\frac{1}{2m+1} \cdot \sum_{m_1=1}^{m-1} \frac{\binom{m}{m_1} \binom{m}{m_1 - 1}}{\binom{2m}{2m_1}} \end{aligned}$$

Summing up (4.9) and (4.10) and applying the standard convention

$$\binom{m}{m+1} := 0 \quad \text{and} \quad \binom{m}{0} := 1$$

we obtain

$$(4.11) \quad \frac{\pi^2}{3} \Delta \tilde{c}_{area} = -\frac{1}{2m+1} \sum_{m_1=0}^m \frac{\binom{m}{m_1} \binom{m}{m_1+1}}{\binom{2m}{2m_1}} = -1 + \frac{(2m)!!}{(2m+1)!!}.$$

where the second equality uses the combinatorial identity (5.2) proved in Proposition 3.

By formula (4.3) we have

$$(4.12) \quad \Delta \tilde{\kappa} = \frac{1}{4} \cdot \sum_{i=1}^4 \frac{1}{(-1+2)^i} = 1$$

Plugging the results (4.11) and (4.12) of our calculation in the general formula (4.4) for the sum Λ^+ we get

$$\Lambda^+ = 1 + \left(-1 + \frac{(2m)!!}{(2m+1)!!} \right).$$

It remains to note that the stratum $\mathcal{Q}(1^{2m}, -1^{2m})$ corresponds to genus one, so the spectrum of Lyapunov exponents of H_+^1 contains a single entry and $\Lambda^+ = \lambda_1^+$, which proves Theorem 2. \square

Corollary 3. *The Lyapunov exponent $\lambda_1^+(m)$ tends to zero as m tends to infinity. More precisely,*

$$(4.13) \quad \delta(m) = \frac{\sqrt{\pi}}{2\sqrt{m}} \left(1 + O\left(\frac{1}{m}\right) \right).$$

Proof. Rewriting the double factorials in terms of usual factorials as

$$\frac{(2m)!!}{(2m+1)!!} = \frac{((2m)!!)^2}{(2m+1)!} = \frac{1}{2m+1} \cdot 2^{2m} \frac{(m!)^2}{(2m)!},$$

and applying the Stirling's formula

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right),$$

to both factorials and simplifying the resulting expression we get

$$\frac{(2m)!!}{(2m+1)!!} = \frac{\sqrt{\pi m}}{2m+1} \left(1 + O\left(\frac{1}{m}\right)\right).$$

Clearly, the latter expression tends to zero as m tends to infinity. \square

5. COMBINATORIAL IDENTITIES

Proposition 3. *For any $m \in \mathbb{N}$ the following identities hold*

$$(5.1) \quad \sum_{k=0}^m \frac{\binom{m}{k} \binom{m}{k}}{\binom{2m}{2k}} = \frac{(2m)!!}{(2m-1)!!} = 4^m \frac{(m!)^2}{(2m)!}$$

$$(5.2) \quad \sum_{k=0}^m \frac{\binom{m}{k} \binom{m}{k+1}}{\binom{2m}{2k}} = 2m + 1 - \frac{(2m)!!}{(2m-1)!!}$$

$$(5.3) \quad \sum_{k=0}^m \frac{\binom{m}{k} \binom{m+1}{k+1}}{\binom{2m}{2k}} = 2m + 1.$$

Proof of Proposition 3. First note that

$$\binom{m+1}{k+1} = \binom{m+1}{k} + \binom{m}{k}.$$

Thus, the expression in the left-hand side of (5.3) is the sum of the corresponding expressions in the left-hand sides of (5.1) and (5.2). Hence, any two out of three identities (5.1)–(5.3) imply the remaining one.

Proof of identity (5.3). Developing binomial coefficients into factorials in (5.3) and moving the common factorials to the right-hand side, we can rewrite this identity as

$$(5.4) \quad \sum_{k=0}^m \frac{(2k)!}{k!(k+1)!} \cdot \frac{(2m-2k)!}{(m-k)!(m-k)!} \stackrel{?}{=} \binom{2m+1}{m}.$$

Denote the sum in the left-hand side of (5.4) by $s_3(m)$. We prove the latter identity by induction. For $m = 0$ it clearly holds. Assuming that (5.4) holds for some integer m we are going to prove that

$$(5.5) \quad (m+2) \cdot s_3(m+1) - 2(2m+3) \cdot s_3(m) = 0.$$

Since the right-hand side of (5.4) satisfies the same relation (which is an exercise), this completes the step of induction. It remains to prove (5.5) under assumption (5.4).

$$\begin{aligned}
& (m+2) \cdot s_3(m+1) - 2(2m+3) \cdot s_3(m) = \\
&= (m+2) \sum_{k=0}^{m+1} \frac{(2k)!}{k!(k+1)!} \cdot \frac{(2m+2-2k)!}{(m+1-k)!(m+1-k)!} - \\
&\quad - 2(2m+3) \sum_{k=0}^m \frac{(2k)!}{k!(k+1)!} \cdot \frac{(2m-2k)!}{(m-k)!(m-k)!} = \\
&= (m+2) \frac{(2m+2)!}{(m+1)!(m+2)!} + \sum_{k=0}^m \frac{(2k)!}{k!(k+1)!} \cdot \frac{(2m-2k)!}{(m-k)!(m-k)!} \cdot \\
&\quad \cdot \left((m+2) \frac{(2m+2-2k)(2m+1-2k)}{(m+1-k)(m+1-k)} - 2(2m+3) \right) = \\
&= 2 \frac{(2m+1)!}{(m+1)!m!} + \sum_{k=0}^m \frac{(2k)!}{k!(k+1)!} \cdot \frac{(2m-2k)!}{(m-k)!(m-k)!} \cdot \left(-2 \frac{k+1}{m+1-k} \right) = \\
&= 2 \binom{2m+1}{m} - 2 \sum_{k=0}^m \frac{(2k)!}{k!k!} \cdot \frac{(2m-2k)!}{(m-k)!(m-k+1)!} = \\
&= 2 \binom{2m+1}{m} - 2 \sum_{j=0}^m \frac{(2(m-j))!}{(m-j)!(m-j)!} \cdot \frac{(2j)!}{j!(j+1)!} = \\
&= 2 \binom{2m+1}{m} - 2s_3(m) = 0.
\end{aligned}$$

where the last equality is the induction assumption. Identity (5.4) and hence identity (5.3) is proved.

Proof of identity (5.1). Developing binomial coefficients into factorials in (5.1) and simplifying common factorials in the right and left hand side of (5.1), we can rewrite this identity as

$$\sum_{j=0}^m \binom{2j}{j} \binom{2m-2j}{m-j} = 4^m,$$

which is identity (3.90) in [Go].

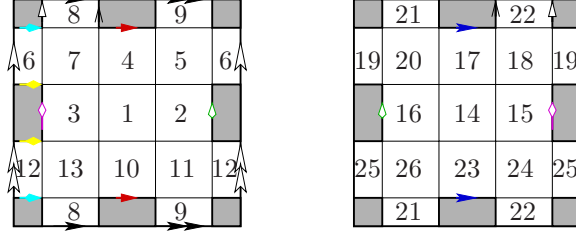
Proposition 3 is proved. \square

APPENDIX A. REMOVING SOME SQUARES IN THE WIND-TREE MODEL

Let us consider the periodic wind-tree models with square obstacles. If we remove periodically one obstacle out of four, we can still perform the same construction as before and end up with a surface in $\mathcal{Q}^{hyp}(1^6, -1^6)$ over $\mathcal{Q}(1^3, -1^7)$.

Namely, unfolding the wind-tree in Figure 10 and taking the quotient over $\mathbb{Z} \oplus \mathbb{Z}$ we get a compact translation surface represented in Figure 11. This translation surface corresponds to the wind-tree in Figure 10 exactly in the same way as the compact flat surface in Figure 5 corresponds to the original wind-tree in Figure 1.

As in the previous examples, the flat surface X in Figure 11 has the group $(\mathbb{Z}/2\mathbb{Z})^3$ as a group of isometries (we have already seen exactly the same group of symmetries for the surface in Figure 5). As before, we can choose as generators

FIGURE 12. The flat surface $\hat{S} = X/\tau_v$.

Now, following the strategy of section 3.1 we pass to the second quotient $\tilde{S} = X/\langle\tau_h, \tau_v\rangle \in \mathcal{Q}^{hyp}(1^6, -1^6)$ (on the left of Figure 13) followed by the quotient $S = X/\langle\tau_h, \tau_v, \iota\rangle \in \mathcal{Q}(1^3, -1^{3+4})$ (on the right of Figure 13).

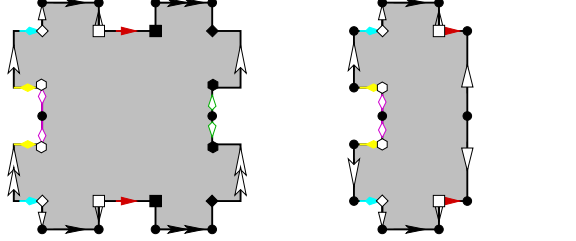


FIGURE 13. The flat surfaces $S = X/\langle\tau_h, \tau_v\rangle \in \mathcal{Q}^{hyp}(1^6, -1^6)$ (on the left) and $S = X/\langle\tau_h, \tau_v, \iota\rangle \in \mathcal{Q}(1^3, -1^{3+4})$ (on the right). We inverse shadowing at this picture with respect to Figures 12 and 11: now we shadow the surface, and not the obstacles. Tiny black discs at the vertices represent the simple poles; the other vertices correspond to simple zeroes.

The surface $\hat{S} \in \mathcal{H}(2^6)$ is the orienting double cover of the surface \tilde{S} . Hence, the Lyapunov spectrum (A.1) of the arithmetic Teichmüller curve $\mathcal{L}(\hat{S})$ is the union of the spectra of Lyapunov exponents of the arithmetic Teichmüller curve $\mathcal{L}(\tilde{S})$:

$$\{\lambda_1(\hat{S}), \dots, \lambda_7(\hat{S})\} = \{\lambda_1^-(\tilde{S}), \dots, \lambda_6^-(\tilde{S})\} \cup \{\lambda_1^+(\tilde{S})\}.$$

By formula (2.4) from [EKZ2] we have

$$(\lambda_1^-(\tilde{S}) + \dots + \lambda_6^-(\tilde{S})) - \lambda_1^+(\tilde{S}) = \frac{1}{4} \cdot \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j + 2} = \frac{1}{4} \cdot 6 \cdot \left(\frac{1}{3} + 1\right) = 2.$$

Together with (A.1) this implies that

$$\lambda_1^+(\tilde{S}) = \frac{1}{2} \left(\frac{3088}{1053} - 2 \right) = \frac{491}{1053}.$$

and, hence, that the diffusion rate for the square-tiled wind-tree as in in Figure 10 is given by

$$\delta = \lambda_1^+(\tilde{S}) = \frac{491}{1053}.$$

We shall see that in all the other cases, removing some of the obstacles out of every repetitive block of 2×2 obstacles in the wind-tree model with any parameters a and b we keep the diffusion rate $\delta = \lambda_1^+ = 2/3$. There are three cases to consider:

- (1) removing two obstacles which are in the same row or in the same column;
- (2) removing two obstacles which are on the same diagonal;
- (3) removing three obstacles.

In the first and in the last case we can just choose a new fundamental domain of the rectangular lattice (duplicating it in the first case and choosing it 2×2 bigger in the third case) to reduce the situation to the original wind-tree with different parameters. We have seen that the diffusion rate in the original wind-tree as in Figure 1 does not depend neither on the parameters of the lattice, nor on the parameters of the obstacle. Hence in both cases $\delta = \lambda_1^+ = 2/3$.

Let us consider the case where we remove two obstacles on the same diagonal as in Figure 14. In this case, we can modify the construction a little bit. We unfold the billiard as before, but then we quotient the resulting periodic surface by the integer sublattice spanned by the integer vectors $(1, 1)$ and $(1, -1)$ in $\mathbb{Z} \oplus \mathbb{Z}$ and not by the entire lattice $\mathbb{Z} \oplus \mathbb{Z}$ as before. The quotient belongs to the same locus as the original wind-tree. We deduce that we again have $\delta = \lambda_1^+ = 2/3$.

Let us emphasize that the above construction is very specific to the sublattice $\mathbb{Z}(1, 1) \oplus \mathbb{Z}(1, -1)$ which is invariant under reflexions by the horizontal and vertical axes. In such situation the quotient keeps a $(\mathbb{Z}/2\mathbb{Z})^3$ group of symmetry.

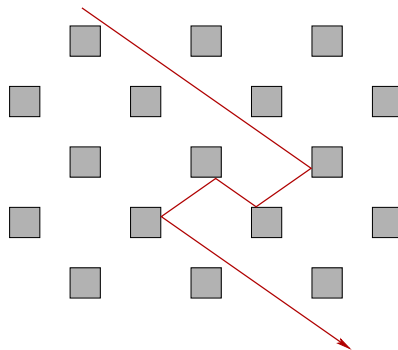


FIGURE 14. A square-tiled wind-tree where we regularly remove every second square obstacle has the original diffusion rate $\delta = 2/3$.

APPENDIX B. WHAT IS NEXT?

The name of the conference and personality of Bill Thurston suggest to discuss what are the current directions of research in the area. We focus on those aspects which are somehow related to the current paper. The selection represents our particular taste and might be subjective. These problems are, certainly, known in our community; some of them are already subjects of intensive investigation, so we do not claim novelty in this discussion.

The Magic Wand Theorem of Eskin–Mirzakhani–Mohammadi enormously amplified the importance of the classification of the orbit closures. Ideally, any problem about an individual flat surface (say, diffusion rate for a wind-tree model with obstacles of some specific “rational shape”) should be solved as follows: touch the

corresponding surface with the Magic Wand and find the corresponding orbit closure \mathcal{L} . Now touch the complex Hodge bundle over \mathcal{L} with the Magic Wand and find its irreducible component containing the two integer cocycles responsible for the diffusion. Find or estimate the corresponding Lyapunov exponents and the problem is solved. This strategy evokes three problems: how to find an orbit closure? How to find irreducible components of the Hodge bundle? How to estimate the top Lyapunov exponent of a given invariant flat subbundle?

B.1. Classification of invariant suborbifolds.

Problem 1. *Classify all $\mathrm{GL}(2, \mathbb{R})$ -invariant suborbifolds in $\mathcal{H}(d_1, \dots, d_n)$.*

The invariant suborbifolds in the strata of quadratic differentials are not mentioned since replacing any flat surface corresponding to a quadratic differential by its canonical ramified double cover on which the induced quadratic differential becomes a global square of a holomorphic 1-form we get an associated invariant suborbifold in the corresponding stratum of Abelian differentials.

In genus 2 the problem is solved by C. McMullen [McM1]. There are very serious advances in this problem for the stratum $\mathcal{H}(4)$, due to Aulicino, Nguyen and Wright [ANW] for $\mathcal{H}^{odd}(4)$ and Nguyen and Wright [ANW] for $\mathcal{H}^{hyp}(4)$. Certain finiteness results for the number of primitive Teichmüller curves are obtained by Bainbridge and Möller [BM] and by Matheus and Wright [MaWr]. Situation in the Prym loci is very well understood due to the work of Lanneau and Nguyen [LN1], [LN2] and [LN3].

The Theorem of Eskin–Mirzakhani–Mohammadi states that such suborbifolds correspond to complex affine subspaces in cohomological coordinates. Due to result of Avila–Eskin–Möller [AEMö] the projection of such affine subspace to absolute cohomology should be a symplectic subspace: the restriction of the intersection form to this subspace is nondegenerate. Results of A. Wright [W1] impose conditions on the field of definition. At some point all these constraints started to seem so restrictive that there was a doubt whether any really new invariant suborbifolds are left? Namely, there is a separate question of classification of Teichmüller curves (both arithmetic and non arithmetic), and there are plenty invariant suborbifolds which can be obtained from Teichmüller curves or from connected components of the strata by various ramified covering constructions. The question was whether there is anything else (with exception for some sporadic series of invariant suborbifolds specific for very small genera).

A recent example of an invariant submanifold of enigmatic origin in $\mathcal{H}(6)$ (which does not fit any of the known schemes mentioned above) was recently found by M. Mirzakhani and A. Wright. For a brief overview of the state of the art in this direction see [W3].

B.2. Classification of invariant subbundles of the complex Hodge bundle.

The complex Hodge bundle $H_{\mathbb{C}}^1$ over the moduli space \mathcal{M}_g of curves has the homology space $H^1(C; \mathbb{C})$ as a fiber over the point represented by the curve C . This bundle can be pulled back to any stratum $\mathcal{H}(m_1, \dots, m_n)$.

Problem 2. *Find the decomposition of the complex Hodge bundle $H_{\mathbb{C}}^1$ over any given $\mathrm{GL}(2, \mathbb{R})$ -invariant suborbifold into irreducible $\mathrm{GL}(2, \mathbb{R})$ -equivariant subbundles.*

The fact that such decomposition exists is proved by S. Filip [Fi1]. Here “irreducible” should be understood in the sense that there is no further splitting even when we pass to any finite ramified cover of the $GL(2, \mathbb{R})$ -invariant suborbifold.

The problem is meaningful already for the strata! It is absolutely frustrating, but we do not have a proof that the only equivariant subbundles of $H_{\mathbb{C}}^1$ over a connected component of any stratum is the tautological subbundle and its symplectic complement. We do not know either whether $H_{\mathbb{R}}^1$ over any connected component of any stratum of quadratic differentials is irreducible in this sense.

The current tools allow, in principal, to prove the latter two facts more or less by hands for some low-dimensional strata. Namely, one can start with an arithmetic Teichmüller curve, compute certain number of monodromy matrices and then use technique of [MMöY] or the method of Eskin as in [FMZ] to prove irreducibility of the complex Hodge bundle over the Teichmüller curve. As a consequence of [Fi1] one gets the irreducibility over over the ambient stratum. However, what is really needed is some general proof for all strata at once.

A related question is

Problem 3. *What groups are realizable as Zariski closures of leafwise monodromy groups of equivariant irreducible blocks of the complex Hodge bundle $H_{\mathbb{C}}^1$ over $GL(2, \mathbb{R})$ -invariant suborbifolds?*

The original guess of Forni–Matheus–Zorich [FMZ] states that this group is always $SU(p, q)$ for appropriate p and q . The paper of S. Filip [Fi2] shows that general Hodge-theoretical arguments admit a priori larger list (including some more sophisticated representations of $SU(p, q)$). However, it is not clear which of these groups (representations) are realizable as Zariski closures of leafwise monodromy groups of equivariant subbundles of the complex Hodge bundle over $GL(2, \mathbb{R})$ -invariant suborbifolds (and not just over some flat subbundles of the complex Hodge bundle over abstract submanifolds of the moduli space).

B.3. Estimates for individual Lyapunov exponents. Paper [EKZ2] provides a formula for the sum of the positive Lyapunov exponents of the complex Hodge bundle over along the Teichmüller geodesic flow. Though there is no reason to hope for exact values of individual Lyapunov exponents, paper [Yu] of Fei Yu conjectures that partial sums of Lyapunov exponents might be estimated through Chern classes of holomorphic vector bundles over Teichmüller curves normalized by the Euler characteristics of Teichmüller curves. More generally (though less precisely):

Problem 4. *Study extremal properties of the “curvature” of the Lyapunov subbundles compared to holomorphic subbundles of the Hodge bundle. Estimate the individual Lyapunov exponents.*

For example, estimates for λ_1^+ over hyperelliptic locus in the principal stratum and estimates for λ_1^+ over the entire principal stratum $Q(1, \dots, 1)$ of holomorphic quadratic differentials would provide estimates for the diffusion rate in certain families of wind-tree billiards.

B.4. Siegel–Veech constants in terms of an adequate intersection theory. The sum of positive Lyapunov exponents of the complex Hodge bundle over an $SL(2, \mathbb{R})$ -invariant suborbifold \mathcal{L} in a stratum $\mathcal{H}_1(m_1, \dots, m_n)$ is expressed

in [EKZ2] as

$$\lambda_1 + \cdots + \lambda_g = \frac{1}{12} \sum_{i=1}^n \frac{m_i(m_i + 2)}{m_i + 1} + \frac{\pi^2}{3} \cdot c_{area}(\mathcal{L}),$$

where $c_{area}(\mathcal{L})$ is the Siegel–Veech constant of \mathcal{L} . Currently there are two formulae for $c_{area}(\mathcal{L})$ for two extremal cases of \mathcal{L} . When \mathcal{L} is a connected component of a stratum, the Siegel–Veech constant $c_{area}(\mathcal{L})$ is expressed as a polynomial in volumes of simpler *principal boundary strata* (normalized by the volume of the initial stratum); see [EMsZ] for the strata of Abelian differentials and [Gj] for the strata of quadratic differentials. When \mathcal{L} is a Teichmüller curve, $c_{area}(\mathcal{L})$ is expressed as the integral of the Chern class of the determinant bundle over \mathcal{L} normalized by the Euler characteristic of \mathcal{L} , see [K1], [BwMo], [EKZ1]. The challenge is to construct a bridge between these two cases:

Problem 5. *Express $c_{area}(\mathcal{L})$ in terms of an appropriate intersection theory.*

Here we do not mean some kind of asymptotic limit formulae, but something in the spirit of ELSV-formula for Hurwitz numbers, see [ELSV].

There is certain resemblance between the “hyperbolic regime” studied in [Mi1]–[Mi3] by M. Mirzakhani and the “flat regime” studied in [EMsZ]. M. Mirzakhani used dynamics on moduli space to relate the length functions of simple geodesics on hyperbolic surfaces to the Weil–Peterson volumes of the moduli spaces $\mathcal{M}_{g,n}$ of punctured Riemann surfaces, and also to relate the Weil–Peterson volumes to the intersection numbers of tautological line bundles over $\mathcal{M}_{g,n}$.

Morally, we have somehow similar situation in a parallel flat world. The step of relating the counting functions for simple *flat* geodesics (for the flat metrics in the same conformal class as the original hyperbolic metric) to polynomial in volumes of the strata with respect to Masur–Veech volume form is already performed in [EMsZ] and [Gj]. The challenge is to accomplish the second step and to relate these volumes to an adequate intersection theory.

Actually, certain parallel between hyperbolic and flat world manifests in further aspects. For example, there are conjectural simple asymptotic formulae for large genera for the Weil–Peterson volumes [Mi3] and for Masur–Veech volumes. About ten years ago A. Eskin and one of the authors conjectured a very simple and explicit asymptotic formula for the Masur–Veech volume. For the principal stratum $\mathcal{H}(1^{2g-2})$ this conjecture was recently proved in [CMöZ] by D. Chen, M. Möller, and D. Zagier.

B.5. Dynamics on other families of complex varieties. One more challenging direction of study is dynamics of the complex Hodge bundle over geodesic flows over moduli spaces of higher-dimensional complex manifolds.

Problem 6. *Study dynamics of the Hodge bundle over geodesic flows on other families of compact varieties. Are there other dynamical systems (compared to billiards in rational polygons) which admit renormalization leading to dynamics on families of complex varieties?*

Some experimental results for families of Calabi–Yau varieties are recently obtained by M. Kontsevich [K2]. S. Filip studied in [Fi3] families of K3-surfaces.

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The numerical computations with square-tiled surfaces have been done with the computer software [S⁺09].

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