COUNTING LATTICE POINTS IN MODULI SPACES OF QUADRATIC DIFFERENTIALS

VINCENT DELECROIX, ÉLISE GOUJARD, PETER ZOGRAF, AND ANTON ZORICH

ABSTRACT

We show how to count lattice points represented by square-tiled surfaces in the moduli spaces of meromorphic quadratic differentials with simple poles on complex algebraic curves. We demonstrate the versatility of the lattice point count on three different examples, including evaluation of Masur–Veech volumes of the moduli spaces of quadratic differentials, computation of asymptotic frequencies of geodesic multicurves on hyperbolic surfaces, and asymptotic enumeration of meanders with a fixed number of minimal arcs.

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1. INTRODUCTION

Quadratic differentials on complex algebraic curves and their moduli spaces is an actively developing area of modern mathematics with close connections to algebraic geometry, dynamics, and mathematical physics (for example, this was one of the main topics at the MSRI semester program "Holomorphic Differentials in Mathematics and Physics" in the Fall 2019). In this paper we deal with the moduli spaces of meromorphic quadratic differentials with simple poles. These spaces are naturally equipped with linear coordinates (called period coordinates), volume form (called the Masur–Veech volume form), and integer lattice (whose points are represented by square-tiled surfaces). We show how to count lattice points in the moduli spaces of meromorphic quadratic differentials and apply this count to three seemingly different problems—namely, to the computation of Masur–Veech volumes of these moduli spaces, to the distribution of simple closed geodesics on hyperbolic surfaces, and to the enumeration of meanders.

A lattice point count relevant to moduli spaces $\mathcal{M}_{g,n}$ of genus g complex curves with n labeled marked points was first performed by P. Norbury in [22]. He considers a lattice in the Harer–Mumford combinatorial model of $\mathcal{M}_{g,n}$ whose points correspond to metric ribbon graphs with edges of integer length. Moreover, he shows that the top degree homogeneous part of his lattice point count polynomial is Kontsevich's volume polynomial [13] whose coefficients are given by intersection numbers of tautological ψ-classes on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ (in fact, the top degree part of Norbury's polynomial coincides, up to the factor of 2^{2g-3+n}, with the top degree homogeneous part of Mirzakhani's Weil–Petersson volume polynomial for the moduli space of bordered hyperbolic surfaces [17, 18]). He then derives a simple recursion for the lattice count polynomials that yields an elementary proof of Witten's conjecture [24].

A few words about the structure of the paper. In Section 2 we collect basic facts about moduli spaces of meromorphic quadratic differentials, and in Section 3 we establish relationships between square-tiled surfaces, geodesic multicurves on hyperbolic surfaces, and stable graphs. Section 4 contains a formula for the Masur–Veech volume of the moduli space of quadratic differentials. Our formula expresses the Masur–Veech volumes in terms of the top degree parts of Norbury's counting polynomials with explicit rational coefficients. Note that for the moduli spaces of holomorphic quadratic differentials a similar formula for the Masur–Veech volumes was obtained (without explicit evaluation) by M. Mirzakhani [19] using a different approach. In Section 5 we use the interpretation of geodesic (multi)curves on hyperbolic surfaces as square-tiled surfaces to analyze their large genus asymptotic distribution. In particular, following the ideas of M. Mirzakhani [20], we show that the nonseparating simple closed geodesics become exponentially more frequent than the separating ones when the genus of the surface grows. Finally, in Section 6 we use the correspondence between meanders and square-tiled surfaces of special type to address the problem of meander enumeration.

2. MODULI SPACES OF QUADRATIC DIFFERENTIALS AND SQUARE-TILED SURFACES

The moduli space $\mathcal{Q}_{g,n}$ of (meromorphic) quadratic differentials is defined as the set of isomorphism classes of pairs (C,q), where C is a smooth genus g complex curve with $n \geq 0$ labeled distinct marked points, and q is a meromorphic quadratic differential on C with at most simple poles at the marked points and no other poles (throughout the paper we will assume that $2g-2+n>0$). It is well known that $\mathcal{Q}_{g,n}$ is naturally isomorphic to $T^*\mathcal{M}_{g,n}$, the total space of the holomorphic cotangent bundle on the moduli space $\mathcal{M}_{g,n}$ of n-pointed genus g complex algebraic curves.

The moduli space $\mathcal{Q}_{g,n}$ is stratified according to the set $\mu = (m_1,\dots,m_k)$ of multiplicities of zeros of q (μ is a partition of $4g-4+n$). In what follows, we will mostly deal with the *principal stratum* $\mathcal{Q}(1^{4g-4+n},-1^n)$ that consists of isomorphism classes of pairs (C,q), where C is a smooth curve and q has exactly $4g-4+n$ simple zeros and n simple poles. Moreover, we will always assume that the zeros of q are labeled. The natural map $\mathcal{Q}(1^{4g-4+n},-1^n) \to \mathcal{Q}_{g,n}$ is a $(4g-4+n)!$-fold covering of its image that is open and dense in $\mathcal{Q}_{g,n}$, and its complement is closed and has positive codimension.

A nonzero differential q in $\mathcal{Q}_{g,n}$ defines a flat metric $|q|$ on the complex curve C. This metric has conical singularities at zeroes and poles of q. It defines the area function by the formula

$$A(C,q) = \int_C |q|, \tag{1}$$

so that $A(C,q)$ is finite and positive. The area function $A(C,q)$ may be viewed as a level function on the moduli space $\mathcal{Q}_{g,n}$.

To introduce the period coordinates on $\mathcal{Q}(1^{4g-4+n},-1^n)$ consider the canonical cover of C defined by the meromorphic quadratic differential q with simple zeros and poles. More precisely, to each such pair (C,q) one associates a twofold covering $f : \hat{C} \to C$ and an abelian differential ω on $\hat{C}$, where $\hat{C} = \{(x,\omega(x)) \mid x \in C, \omega(x) \in T^*_x C, \omega(x)^2 = q(x)\}$. The curve $\hat{C}$ is smooth of genus $\hat{g} = 4g-3+n$, and the covering f is ramified precisely over zeros and poles of q. The differential ω is holomorphic on $\hat{C}$, has second order zeros at the preimages of zeros of q, and does not vanish at the preimages of poles of q

The map f is invariant under the canonical involution $\iota : \hat{C} \to \hat{C}$, $(x,\omega(x)) \mapsto (x,-\omega(x))$. The induced map $\iota_* : H_1(\hat{C},\mathbb{C}) \to H_1(\hat{C},\mathbb{C})$ allows decomposing $H_1(\hat{C},\mathbb{C})$ into the direct sum $H_1^+(\hat{C},\mathbb{C}) \oplus H_1^-(\hat{C},\mathbb{C})$ of eigenspaces corresponding to the eigenvalues ± 1 of ι_*, where $\dim_{\mathbb{C}} H_1^+(\hat{C},\mathbb{C}) = 2g$, $\dim_{\mathbb{C}} H_1^- = 6g-6+2n$. Now put $H_1^-(\hat{C},\mathbb{Z}) = H_1(\hat{C},\mathbb{Z}) \cap H_1^-(\hat{C},\mathbb{C})$ and consider the period map $P : H_1^-(\hat{C},\mathbb{Z}) \to \mathbb{C}$ defined by $\alpha \mapsto P(\alpha) = \int_\alpha \omega$, $\alpha \in H_1^-(\hat{C},\mathbb{Z})$. We can think of P as an element of $\mathrm{Hom}(H_1^-(\hat{C},\mathbb{Z}),\mathbb{C}) = H^1_-(\hat{C},\mathbb{C})$, the anti-invariant part of $H^1(\hat{C},\mathbb{C})$ with respect to the involution ι^*. This gives us *period* (or *homological*) coordinates in a neighborhood of $(C,q) \in \mathcal{Q}(1^{4g-4+n},-1^n)$.

Consider the lattice $L = \mathrm{Hom}(H_1^-(\hat{C},\mathbb{Z}),\mathbb{Z} \oplus i\mathbb{Z})$, $i^2=-1$, in the vector space $H^1_-(\hat{C},\mathbb{C})$. By definition, the points of the lattice L are those elements of $H^1_-(\hat{C},\mathbb{C})$ that take values in $\mathbb{Z} \oplus i\mathbb{Z}$ on $H_1^-(\hat{C},\mathbb{Z})$. Note that L is a sublattice in the lattice $H^1_-(\hat{C},\mathbb{Z} \oplus i\mathbb{Z}) \hookrightarrow H^1_-(\hat{C},\mathbb{C})$ of index 4^{2g} induced by the inclusion $\mathbb{Z} \oplus i\mathbb{Z} \hookrightarrow \mathbb{C}$. The

Masur–Veech volume form dV on $\mathcal{Q}(1^{4g-4+n}, -1^n)$ is defined as the linear volume form in the vector space $H^1_-(\hat{C}, \mathbb{C})$ normalized in such a way that $\mathrm{Vol}(H^1_-(\hat{C}, \mathbb{C})/L) = 1$. The form dV induces a volume form on the level sets of the area function A, and we define

$$\mathrm{Vol}\,\mathcal{Q}(1^{4g-4+n}, -1^n) = \mathrm{Vol}'\,\mathcal{Q}^{A=1/2}(1^{4g-4+n}, -1^n) = 2d\,\mathrm{Vol}\,\mathcal{Q}^{A\leq 1/2}(1^{4g-4+n}, -1^n). \tag{2}$$

Here

$$\mathcal{Q}^{A=1/2}_{g,n} = \left\{(C,q) \in \mathcal{Q}_{g,n} \mid A(C,q) = 1/2\right\}$$

is the level $A = 1/2$ hypersurface (sphere bundle on $\mathcal{M}_{g,n}$),

$$\mathcal{Q}^{A\leq 1/2}_{g,n} = \left\{(C,q) \in \mathcal{Q}_{g,n} \mid A(C,q) \leq 1/2\right\}$$

is the radius-1/2 disc bundle, and $d = 6g - 6 + 2n$ is the complex dimension of $\mathcal{Q}_{g,n}$. By a result of H. Masur [15] (cf. also W. Veech [23]), the volume of $\mathcal{Q}^{A\leq 1/2}_{g,n}$ is finite. One of the main objectives of this paper is to provide a formula for this volume in terms of the lattice point count.

Remark 1. The moduli space $\mathcal{Q}_{g,n} = T^*\mathcal{M}_{g,n}$, as a cotangent bundle, carries a canonical symplectic form that we denote by Ω. The corresponding volume form $\frac{1}{d!}\Omega^d$ is invariant under the Teichmüller flow on $\mathcal{Q}_{g,n}$, see [16]. Previously it was shown in [15, 23] that the volume form dV is also preserved by the Teichmüller flow. Since $\frac{1}{d!}\Omega^d$ and dV belong to the same Lebesgue measure class, the ergodicity of the Teichmüller flow implies that these two volume forms coincide up to a constant proportionality coefficient. We postpone the evaluation of this constant factor to another occasion.

To give a geometric description of the lattice points in $\mathcal{Q}(1^{4g-4+n}, -1^n)$, we introduce a notion of square-tiled surfaces. It is possible to construct a discrete collection of meromorphic quadratic differentials with simple poles by assembling together identical flat squares in the following way. Take a finite number of oriented $1/2 \times 1/2$-squares with two opposite sides called horizontal and the other two sides called vertical. Identify pairs of sides of the squares by orientation reversing isometries in such a way that horizontal sides are glued to horizontal ones and vertical sides are glued to vertical ones. We get an oriented topological surface C without boundary, and we consider only those gluings for which C is connected. Quadratic differential dz^2 on each square is compatible with gluing and endows C with a complex structure and a nonzero quadratic differential q with at most simple poles. The total area $A(C,q)$ of C is 1/4 times the number of squares. We call such a pair (C,q) a *square-tiled surface*.

We denote by $ST(\mathcal{Q}(\mu, -1^n))$ the set of all square-tiled surfaces in the stratum $\mathcal{Q}(\mu, -1^n)$, and by $ST(\mathcal{Q}(\mu, -1^n), 2N)$ its subset consisting of surfaces made up of at most $2N$ squares.[1] Since each square-tiled surface (C,q) is glued from $1/2 \times 1/2$-squares,

1 For example, the square-tiled surface in Figure 1 is made up of 54 squares, has 3 conical points of angle 3π (corresponding to simple zeros of q), and 7 conical points of angle π (corresponding to simple poles of q). Therefore, it has genus 0 and belongs to the principal stratum $\mathcal{Q}(1^3, -1^7)$.

the periods of the abelian differential $\omega = \sqrt{q}$ on the double cover $\hat{C}$ belong to $\frac{1}{2}\mathbb{Z} \oplus \frac{i}{2}\mathbb{Z}$. In particular, this yields a natural inclusion $ST(\mathcal{Q}(1^{4g-4+n}, -1^n)) \hookrightarrow L$, where the lattice L was defined above. This inclusion is actually a bijection in each coordinate chart (in the case $g = 0$ a proof of the bijection between the set $ST(\mathcal{Q}(1^{n-4}, -1^n))$ and the lattice L can be found in **[3]**, see also **[5, APPENDIX A]**). Therefore, we have

$$\operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n) = 2(6g-6+2n) \lim_{N\to\infty} \frac{|ST(\mathcal{Q}(1^{4g-4+n}, -1^n), 2N)|}{N^{6g-6+2n}}, \tag{3}$$

where the vertical bars $|\ |$ denote the cardinality of a set. Formula (3) will serve as a starting point for our computation of $\operatorname{Vol} \mathcal{Q}_{g,n}$. Note that the volumes of $\mathcal{Q}_{g,n}$ and $\mathcal{Q}(1^{4g-4+n}, -1^n)$ are related by a simple formula

$$\operatorname{Vol} \mathcal{Q}_{g,n} = \frac{1}{(4g-4+n)!} \operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n).$$

3. CYLINDER DECOMPOSITION, MULTICURVES, AND STABLE GRAPHS

A square-tiled surface admits a decomposition into maximal horizontal cylinders filled with isometric simple closed flat geodesics. Every such maximal horizontal cylinder has at least one conical singularity on each of its two boundary components. The square-tiled surface in Figure 1 has four maximal horizontal cylinders that are highlighted by different shades:

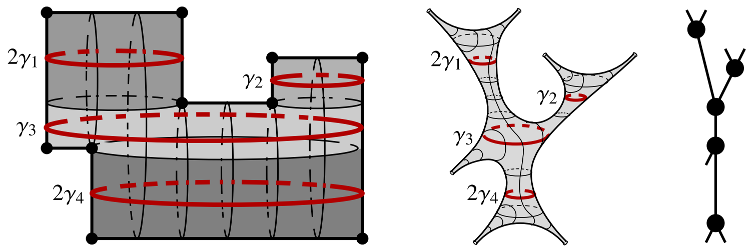


FIGURE 1 A square-tiled surface in $\mathcal{Q}(1^3, -1^7)$, and its associated multicurve and stable graph.

For a square-tiled surface (C, q), consider its decomposition into the set of k maximal horizontal cylinders. To each cylinder we associate the corresponding waist curve γ_i, $i = 1, \ldots, k$, considered up to a free homotopy. The curves γ_i are nonperipheral (i.e., none of them bounds a disc containing a single pole) and pairwise nonhomotopic. We denote the number of circular horizontal bands of squares contained in the ith maximal horizontal cylinder by h_i. The formal linear combination $\gamma = \sum_{i=1}^k h_i \gamma_i$ is called a simple closed *multicurve* on C. (For example, the multicurve associated to the square-tiled surface in Figure 1 is $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$.)

To each multicurve $\gamma = \sum_{i=1}^k h_i \gamma_i$ as above we associate its *stable graph* $\Gamma(\gamma)$. A stable graph is a graph that is dual to the nodal curve obtained from C by pinching each

γ_i to a point. Stable graphs are used to describe the natural stratification of the Deligne–Mumford boundary of the moduli space $\mathcal{M}_{g,n}$. More precisely, $\Gamma(\gamma)$ is a decorated graph whose vertices represent the components of $S \setminus \{\gamma_1 \cup \cdots \cup \gamma_k\}$ and are labeled with the genus of the corresponding component. The edges of $\Gamma(\gamma)$ correspond to the curves γ_i and connect the vertices representing the components of $S \setminus \{\gamma_1 \cup \cdots \cup \gamma_k\}$ adjacent to γ_i (which may actually be the same). Finally, $\Gamma(\gamma)$ is endowed with n "legs" (or half-edges) labeled from 1 to n. The ith leg is attached to the vertex that represents the component that contains the ith marked point of C (i.e., the position of the ith pole of q). In addition to that, it is required that at each vertex v the stability condition $2g(v) - 2 + n(v) > 0$ is satisfied, where $g(v)$ is the genus assigned to v and $n(v)$ is the degree (or valency) of v. The right picture in Figure 1 shows the stable graph associated to the multicurve γ drawn in the middle picture.

For a pair of nonnegative integers g and n with $2g - 2 + n > 0$, denote by $\mathcal{G}_{g,n}$ the set of (isomorphism classes of) stable graphs of genus g with n legs (recall that the genus of a stable graph Γ is defined as $g = \sum_{v \in V(\Gamma)} g(v) + b_1(\Gamma)$, where $V(\Gamma)$ is the set of vertices and $b_1(\Gamma)$ is the first Betty number of the graph Γ). The number of such graphs $|\mathcal{G}_{g,n}|$ is finite, though grows rapidly with g and n.[2]

For a stable graph Γ in $\mathcal{G}_{g,n}$, consider the subset $ST_{\Gamma,h}(\mathcal{Q}(1^{4g-4+n}, -1^n), 2N)$ of square-tiled surfaces with at most $2N$ squares, having Γ as the associated stable graph and $h = (h_1, \ldots, h_k)$ as the set of heights of the cylinders (in the units of 1/2). Denote by $ST_\Gamma(\mathcal{Q}(1^{4g-4+n}, -1^n), 2N)$ the analogous subset without restriction on heights. Then the contributions to $\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n)$ in (2) from these subsets are given by

$$\operatorname{Vol}(\Gamma) = 2d \cdot \lim_{N\to\infty} \frac{|ST_\Gamma(\mathcal{Q}(1^{4g-4+n}, -1^n), 2N)|}{N^d}, \tag{4}$$

$$\operatorname{Vol}(\Gamma, h) = 2d \cdot \lim_{N\to\infty} \frac{|ST_{\Gamma,h}(\mathcal{Q}(1^{4g-4+n}, -1^n), 2N)|}{N^d}, \tag{5}$$

where $d = 6g - 6 + 2n$. The results in [5] imply that for any Γ in $\mathcal{G}_{g,n}$ the above limits exist, are strictly positive, and that

$$\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n) = \sum_{\Gamma \in \mathcal{G}_{g,n}} \operatorname{Vol}(\Gamma) = \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{h \in \mathbb{N}^k} \operatorname{Vol}(\Gamma, h), \tag{6}$$

where k is the number of horizontal cylinders in C (or, equivalently, the number of edges of the stable graph Γ). Dividing both sides of (6) by $\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n)$, we see that the ratio $\operatorname{Vol}(\Gamma)/\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n)$ can be interpreted as the "asymptotic probability" that a random square-tiled surface has Γ as the stable graph associated to its horizontal cylinder decomposition.

4. FORMULA FOR THE MASUR–VEECH VOLUME

To evaluate the Masur–Veech volume of $\mathcal{Q}_{g,n}$, we introduce certain multivariate polynomials $N_{g,n}(b_1, \ldots, b_n)$ that have already appeared in the work on intersection theory

2 Table 1 below lists all stable graphs in $\mathcal{G}_{1,2}$.

of the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$, cf. [13,17,18,22]. Let $\boldsymbol{b} = (b_1, \dots, b_n)$ be a set of variables, and let $(d_1, \dots, d_n)$ be a set of nonnegative integers such that $d_1 + \dots + d_n = 3g - 3 + n$. Define the homogeneous polynomial $N_{g,n}(b_1, \dots, b_n)$ of degree $6g - 6 + 2n$ in the variables $b_1, \dots, b_n$ by the formula

$$N_{g,n}(b_1, \dots, b_n) = \frac{1}{2^{5g-6+2n}} \sum_{d_1,\dots,d_n} \frac{b_1^{2d_1} \cdots b_n^{2d_n}}{d_1! \cdots d_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \tag{7}$$

where $d_1 + \dots + d_n = 3g - 3 + n$ and $\psi_1, \dots, \psi_n$ are the tautological ψ-classes on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of curves (informally speaking, ψ_i is the class of the line bundle whose fiber over $(C, p_1, \dots, p_n)$ is the cotangent line $T^*_{p_i} C$ to C at the ith marked point p_i). For small g and n, we have

$$\begin{aligned}
N_{0,3}(b_1, b_2, b_3) &= 1, \\
N_{0,4}(b_1, b_2, b_3, b_4) &= \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2), \\
N_{1,1}(b_1) &= \frac{1}{48} b_1^2, \\
N_{1,2}(b_1, b_2) &= \frac{1}{384}(b_1^2 + b_2^2)(b_1^2 + b_2^2), \\
N_{2,1}(b_1) &= \frac{1}{1\,769\,472} b_1^8.
\end{aligned}$$

Following [3], we introduce the linear operators Y_h and Z acting on the space of polynomials in variables $b_1, \dots, b_k$. The operator Y_h is defined on monomials as

$$Y_h : \prod_{i=1}^{k} b_i^{m_i} \mapsto \prod_{i=1}^{k} \frac{m_i!}{h_i^{m_i+1}}, \tag{8}$$

and extended to arbitrary polynomials by linearity. The operator Z is defined on monomials as

$$Z : \prod_{i=1}^{k} b_i^{m_i} \mapsto \prod_{i=1}^{k} m_i! \cdot \zeta(m_i + 1), \tag{9}$$

and extended to arbitrary polynomials by linearity. In the above formula, ζ is the Riemann zeta function

$$\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s},$$

so that for any collection of positive integers $(m_1, \dots, m_k)$ we have

$$Z\left(\prod_{i=1}^{k} b_i^{m_i}\right) = \sum_{h \in \mathbb{N}^k} Y_h\left(\prod_{i=1}^{k} b_i^{m_i}\right).$$

For a stable graph Γ with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$, we associate a homogeneous polynomial P_Γ of degree $6g - 6 + 2n$ as follows. To every edge $e \in E(\Gamma)$ we assign a formal variable b_e. For a vertex $v \in V(\Gamma)$ of weight $g(v)$ and valency $n(v)$, we assign the variable b_e to each half of the edge e incident to v, and we assign 0 to each leg of Γ. We denote by $\boldsymbol{b}_v$ the resulting collection of $n(v)$ variables. More specifically, if an edge e is a loop joining v to itself, b_e enters $\boldsymbol{b}_v$ twice; if an edge e joins v to a distinct vertex, b_e enters $\boldsymbol{b}_v$ once; the remaining entries of $\boldsymbol{b}_v$ correspond to legs (they are represented by zeros). To each vertex $v \in V(\Gamma)$ we associate the polynomial $N_{g(v),n(v)}(\boldsymbol{b}_v)$, where $N_{g,n}$ is defined in (7). We define P_Γ by the formula

$$P_\Gamma = \frac{2^{6g-5+2n} \cdot (4g-4+n)!}{(6g-7+2n)!} \cdot \frac{1}{2^{|V(\Gamma)|-1}} \cdot \frac{1}{|\operatorname{Aut}(\Gamma)|} \cdot \prod_{e \in E(\Gamma)} b_e \cdot \prod_{v \in V(\Gamma)} N_{g(v),n(v)}(\boldsymbol{b}_v). \tag{10}$$

Lemma 1. *The contributions* $\operatorname{Vol}(\Gamma)$ *and* $\operatorname{Vol}(\Gamma, h)$ *to* $\operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n)$*, given by formulas* (4) *and* (5)*, respectively, satisfy the relations*

$$\operatorname{Vol}(\Gamma) = Z(P_\Gamma) \quad \textit{and} \quad \operatorname{Vol}(\Gamma, h) = Y_h(P_\Gamma). \tag{11}$$

Substituting these expressions into formula (6), we immediately obtain

Theorem 1. *The Masur–Veech volume of the principal stratum* $\mathcal{Q}(1^{4g-4+n}, -1^n)$ *is given by the formula*

$$\operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n) = \sum_{\Gamma \in \mathcal{G}_{g,n}} Z(P_\Gamma) = \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{h \in \mathbb{N}^{|E(\Gamma)|}} Y_h(P_\Gamma). \tag{12}$$

Table 1 illustrates the computation of the polynomials P_Γ, as well as the contributions $\operatorname{Vol}(\Gamma)$ to the Masur–Veech volume $\operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n)$, in the simplest nontrivial case of $(g, n) = (1, 2)$. To make the computations tractable, we follow the structure of formula (10). The first numerical factor $\frac{32}{3}$ in the first line of each calculation in the right column of Table 1 is $\frac{2^{6g-5+2n}(4g-4+n)!}{(6g-7+2n)!}$ evaluated at $(g, n) = (1, 2)$; it is common for all stable graphs in $\mathcal{G}_{1,2}$. The second numerical factor is $\frac{1}{2^{|V(\Gamma)|-1}}$. The third numerical factor is $\frac{1}{|\operatorname{Aut}(\Gamma)|}$ (while the vertices and edges of Γ are not labeled, the automorphism group $\operatorname{Aut}(\Gamma)$ respects the decoration of the graph). The resulting value

$$\operatorname{Vol} \mathcal{Q}(1^2, -1^2) = \left(\frac{8}{45} + \frac{1}{135} + \frac{2}{27} + \frac{2}{27} \right) \cdot \pi^4 = \frac{\pi^4}{3}$$

matches that previously found in **[11]** by implementing a completely different algorithm of **[10]**.

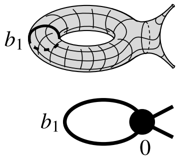
For $g = 0$, the formula for the Masur–Veech volumes simplifies considerably. As it was shown in **[3]**, for all $n \geq 4$,

$$\operatorname{Vol} \mathcal{Q}(1^{n-4}, -1^n) = 4 \left(\frac{\pi^2}{2} \right)^{n-3} \tag{13}$$

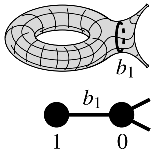
($\operatorname{Vol} \mathcal{Q}_{0,3} = 4$ and $\operatorname{Vol} \mathcal{Q}_{1,1} = \frac{2}{3}\pi^2$ by convention).

Table 2 below provides the volumes of principal strata $V_{g,n} = \operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n)$ for small g and n (some additional data can be found in **[7]**).

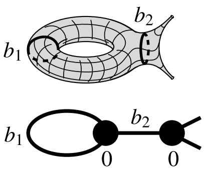
Remark 2. An alternative formula for the Masur–Veech volume of the principal stratum $\mathcal{Q}(1^{4g-4+n}, -1^n)$ was obtained in **[4]**. The formula is based on the intersection theory and expresses $\operatorname{Vol} \mathcal{Q}(1^{4g-4+n}, -1^n)$ as a weighted sum of certain linear Hodge integrals over the



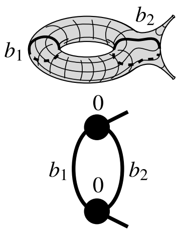
$$\begin{aligned} &\tfrac{32}{3} \cdot 1 \cdot \tfrac{1}{2} \cdot b_1 \cdot N_{0,4}(b_1, b_1, 0, 0) \\ &\quad = \tfrac{16}{3} \cdot b_1 \cdot (\tfrac{1}{4}(2b_1^2)) = \tfrac{8}{3} \cdot b_1^3 \\ &\quad \overset{Z}{\longmapsto} \quad \tfrac{8}{3} \cdot 3! \cdot \zeta(4) = \tfrac{8}{45}\pi^4 \end{aligned}$$



$$\begin{aligned} &\tfrac{32}{3} \cdot \tfrac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{0,3}(0, 0, b_1) \\ &\quad = \tfrac{16}{3} \cdot b_1 \cdot (\tfrac{1}{48} b_1^2) \cdot (1) = \tfrac{1}{9} \cdot b_1^3 \\ &\quad \overset{Z}{\longmapsto} \quad \tfrac{1}{9} \cdot 3! \cdot \zeta(4) = \tfrac{1}{135}\pi^4 \end{aligned}$$



$$\begin{aligned} &\tfrac{32}{3} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot N_{0,3}(b_1, 0, 0) \\ &\quad = \tfrac{8}{3} \cdot b_1 b_2 \cdot (1) \cdot (1) = \tfrac{8}{3} \cdot b_1 b_2 \\ &\quad \overset{Z}{\longmapsto} \quad \tfrac{8}{3} \cdot (\zeta(2))^2 = \tfrac{2}{27}\pi^4 \end{aligned}$$



$$\begin{aligned} &\tfrac{32}{3} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(0, b_1, b_2) \cdot N_{0,3}(b_1, b_2, 0) \\ &\quad = \tfrac{8}{3} \cdot b_1 b_2 \cdot (1) \cdot (1) = \tfrac{8}{3} \cdot b_1 b_2 \\ &\quad \overset{Z}{\longmapsto} \quad \tfrac{8}{3} \cdot (\zeta(2))^2 = \tfrac{2}{27}\pi^4 \end{aligned}$$

TABLE 1 Computation of $\operatorname{Vol} \mathcal{Q}(1^2, -1^2) = 2! \cdot \operatorname{Vol} \mathcal{Q}_{1,2}$. The left column lists the multicurves and their associated stable graphs Γ, while the right column gives the polynomials P_Γ and the corresponding volume contributions $\operatorname{Vol}(\Gamma) = Z(P_\Gamma)$.

g	n	$V_{g,n}$
1	3	$\frac{11}{60}\pi^6$
1	4	$\frac{1}{10}\pi^8$
1	5	$\frac{163}{3\,024}\pi^{10}$
2	0	$\frac{1}{15}\pi^6$
2	1	$\frac{29}{840}\pi^8$
2	2	$\frac{337}{18\,144}\pi^{10}$
3	0	$\frac{115}{33\,264}\pi^{12}$
4	0	$\frac{2\,106\,241}{11\,548\,293\,120}\pi^{18}$

TABLE 2 Numerical values of Masur–Veech volumes of low-dimensional principal strata.

moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. The subsequent papers [12, 25] provided efficient algorithms for computing these Hodge integrals that allowed computing $\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n)$ for large enough values of g and n reasonably fast. In particular, this approach yields the same data as in Table 2.

Remark 3. A conjectural large genus asymptotic formula for the Masur–Veech volume of any stratum in the moduli space $\mathcal{Q}_{g,n}$ was proposed in [2]. For the principal stratum $\mathcal{Q}(1^{4g-4+n}, -1^n)$, it reads

$$\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n) \approx \frac{4}{\pi} \cdot 2^n \cdot \left(\frac{8}{3}\right)^{4g-4+n} \quad \text{as } g \to \infty. \tag{14}$$

Formula (14) was recently proven in [1] on the basis of (12) and a uniform large genus estimate for intersection numbers of ψ-classes on the moduli space $\overline{\mathcal{M}}_{g,n}$.

More details concerning the results of this section can be found in [8].

5. RANDOM SQUARE-TILED SURFACES AND RANDOM MULTICURVES

Here we discuss statistical geometry of simple closed hyperbolic multicurves. The research in this direction was inspired by the pioneering work of M. Mirzakhani [19–21]. The new tool that we bring in is a natural bijection between square-tiled surfaces and multicurves described in Section 3.

To give a flavor of these results, let us take a complex curve C of genus g with n punctures endowed with the compatible complete hyperbolic metric. Let $\gamma = \sum_{i=1}^{k} h_i \gamma_i$ be a multicurve on C consisting of pairwise disjoint primitive simple closed geodesics γ_i. Furthermore, denote by ℓ the hyperbolic length function, and put $L = \sum_{i=1}^{k} h_i \ell(\gamma_i)$ to be the total length of γ.

Denote by $\mathcal{ML}_{g,n}(\mathbb{Z})$ the lattice of integer points in the space of measured laminations on C (i.e., the set of all simple closed geodesic multicurves). We say that two multicurves have the same topological type if they belong to the same orbit of the mapping class group $\operatorname{Mod}_{g,n}$ in $\mathcal{ML}_{g,n}(\mathbb{Z})$. By definition, the asymptotic probability that a random multicurve belongs to the orbit $\operatorname{Mod}_{g,n} \cdot \gamma$ is

$$P_{g,n}(\gamma) = \lim_{L\to\infty} \frac{|\{\gamma' \in \operatorname{Mod}_{g,n} \cdot \gamma \mid \ell(\gamma') \le L\}|}{|\{\gamma' \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell(\gamma') \le L\}|}. \tag{15}$$

M. Mirzakhani [20] expressed the probability $P_{g,n}(\gamma)$ in terms of the intersection numbers of tautological ψ-classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$. Her formula implies, in particular, that $P_{g,n}(\gamma)$ depends only on topology of the pair C, γ and is independent of the complex structure or hyperbolic metric on C. We have the following refinement of her result, cf. [8]:

Theorem 2. *The asymptotic probability $P_{g,n}(\gamma)$ is given in terms of the volume contribution $\operatorname{Vol}(\Gamma, h)$ by the formula*

$$P_{g,n}(\gamma) = \frac{\operatorname{Vol}(\Gamma, h)}{\operatorname{Vol}\mathcal{Q}(1^{4g-4+n}, -1^n)}, \tag{16}$$

where Γ is the stable graph corresponding to the multicurve γ.

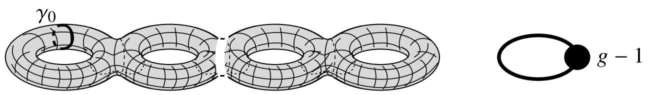


FIGURE 2 The non-separating curve γ_0 and the associated stable graph Γ_0.

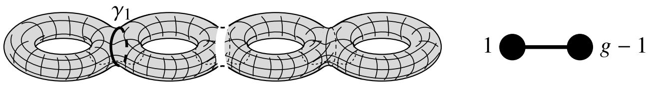


FIGURE 3 The separating curve γ_1 and the associated stable graph Γ_1.

Let us now present a consequence of Theorem 2 for $n = 0$, $g \geq 2$. In this case there is just one topological type of nonseparating simple closed geodesics γ_0 as in Figure 2, and $[g/2]$ topological types of separating closed geodesics $\gamma_1, \ldots, \gamma_{[g/2]}$ as in Figure 3, where γ_i splits the complex curve C into two parts of genera i and $g - i$, respectively. Then the following asymptotic formula holds:

$$\frac{\sum_{i=1}^{[g/2]} P_{g,0}(\gamma_i)}{P_{g,0}(\gamma_0)} \approx \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{2^{2g}} \quad \text{as } g \to \infty. \tag{17}$$

In plain words, on a compact hyperbolic surface of large genus, nonseparating simple closed curves are exponentially more frequent than separating ones. The proof of formula (17) comprises evaluation of individual contributions of particularly simple stable graphs displayed in Figures 2 and 3 and analysis of their large genus asymptotic behavior using an explicit closed-form expression for the intersection numbers $\int_{\overline{\mathcal{M}}_{g,2}} \psi_1^{d_1} \psi_2^{d_2}$ obtained in [26], cf. [8] for details.

In order to go beyond the case of simple closed curves, a much more involved asymptotic analysis of intersection numbers of ψ-classes is needed that is performed in full generality in [1, 6].

6. SQUARE-TILED SURFACES AND ENUMERATION OF MEANDERS

Here we apply the lattice point count to the enumeration problem of meanders. A meander is a configuration in the plane that consists of a straight line and a simple closed curve transversely intersecting it, considered up to isotopy (see Figure 4 for an example). Meanders naturally appear in various areas of mathematics and theoretical physics (for instance, they provide a model of polymer folding). A brief introduction to meanders can be found in [14]

Enumeration of meanders is a long-standing difficult combinatorial problem. Let $\mathcal{M}(N)$ be the number of meanders with $2N$ crossings. Then, conjecturally,

$$\mathcal{M}(N) \approx \text{const} \cdot R^N N^\alpha \quad \text{as } N \to \infty.$$

There is a plausible prediction that $\alpha = -\frac{29+\sqrt{145}}{12}$ coming from its interpretation as the critical exponent in a two dimensional conformal field theory with central charge $c = -4$, see [9]. However, even a hypothetical value of R is not known.

The situation becomes more tractable if we impose on meanders an additional topological restriction. We call an arc *minimal* if it connects two adjacent intersections. The maximal arc connecting the first and the last intersections, if present, is also treated as a minimal arc at infinity (for instance, the meander displayed on Figure 4 has 6 minimal arcs including the one at infinity).

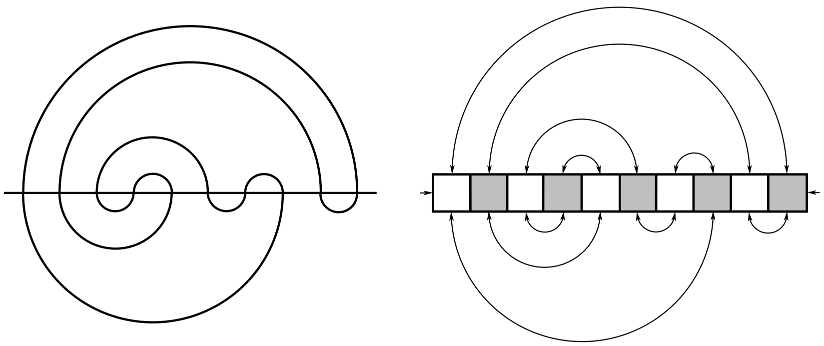


FIGURE 4 A meander with 10 crossings and 6 minimal arcs, and the corresponding square-tiled surface in $\mathcal{Q}(1^2, 0, -1^6)$ (pairs of sides connected with arrowed arcs are identified).

Denote by $\mathcal{M}_n(N)$ the number of meanders with not more than $2N$ crossings and exactly n minimal arcs (it is not hard to see that $n \geq 4$). Then we have

Theorem 3. *For any fixed $n \geq 4$, the number of meanders $\mathcal{M}_n(N)$ satisfies the following asymptotic formula:*

$$\mathcal{M}_n(N) = \frac{4}{n!(n-4)!}\left(\frac{2}{\pi^2}\right)^{n-3}\binom{2n-4}{n-2}^2\frac{N^{2n-5}}{4n-10} + o(N^{2n-5}) \quad \text{as } N \to \infty. \tag{18}$$

Below we sketch a derivation of formula (18). To begin with, we establish a relationship between meanders and certain square-tiled surfaces. A meander with $2N$ crossings may be viewed as a 4-regular plane graph, and its dual graph is a quadrangulation of the sphere made up of $2N$ squares. We can think of these squares to be identical of size $1/2 \times 1/2$ as above. Since this quadrangulation respects the horizontal and vertical sides of the squares, it gives rise to a genus 0 square-tiled surface (C, q), where the quadratic differential q has exactly n simple poles. Besides the horizontal cylinder decomposition of C considered earlier, we define its vertical cylinder decomposition in the same way. Clearly, a square-tiled surface associated with a meander has exactly one horizontal and one vertical cylinders, both of maximal circumference $2N$ (in the units of $1/2$), see Figure 4. At last, we need to mark a point on C that corresponds to the infinity (by convention, we put it at the endpoint of a vertical side of one of the squares). Thus, we get a correspondence between the set of meanders with $2N$ crossings and n minimal arcs on one side and the set of genus 0 square-tiled surfaces glued from $2N$ squares with exactly n simple poles, one marked point, one horizontal and one vertical cylinders of maximal circumference $2N$ on the other (note that the latter carries no labeling of either poles or zeros).[3] This correspondence is generically two-to-one for large N since, for a regular vertex of a square-tiled surface, there are two vertical edges incident to it, and either of them can be chosen as a distinguished edge at infinity.

Now we want to realize these square-tiled surfaces as lattice points in a moduli space of quadratic differentials. Denote by $\mathcal{Q}_{g,n,1}$ the moduli space of pairs (C,q), where C is a genus g complex curve with $n+1$ labeled marked points and q is a quadratic differentials with at most simple poles at the first n marked points and regular at the last one (in fact, $\mathcal{Q}_{g,n,1} = p^*T^*\mathcal{M}_{g,n}$, where $p : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ is the forgetful map). Let $\mathcal{Q}(1^{4g-4+n}, 0, -1^n)$ be the (principal) stratum of quadratic differentials with $4g-4+n$ labeled simple zeros, so that the natural map $\mathcal{Q}(1^{4g-4+n}, 0, -1^n) \to \mathcal{Q}_{g,n,1}$ is a $(4g-4+n)!$-fold covering of its image. Period coordinates on $\mathcal{Q}(1^{4g-4+n}, 0, -1^n)$ are defined as in Section 2, the only difference is that there is an additional coordinate given by the integral of ω along a path connecting two preimages of the marked point. The Masur–Veech volume form is also well defined in this case.

Consider three nested sets of square-tiled surfaces defined as follows:

- the set $ST(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N)$ of all square-tiled surfaces in $\mathcal{Q}(1^{n-4}, 0, -1^n)$ made up of at most $2N$ squares;
- the subset $ST_1(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N) \subset ST(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N)$ of surfaces with one horizontal cylinder of maximal circumference;
- the subset $ST_{1,1}(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N) \subset ST_1(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N)$ of surfaces with one horizontal and one vertical cylinders of maximal circumference.

Since square-tiled surfaces are uniformly distributed in $\mathcal{Q}(1^{n-4}, 0, -1^n)$ relative to the Masur–Veech volume form, we get the asymptotics

$$\left|ST\big(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N\big)\right| = c(n)\frac{N^d}{2d} + o(N^d) \quad \text{as } N \to \infty, \tag{19}$$

where $d = 2n-5 = \dim_{\mathbb{C}} \mathcal{Q}(1^{n-4}, 0, -1^n)$ and $c(n) = \operatorname{Vol} \mathcal{Q}(1^{n-4}, 0, -1^n)$, cf. formula (3). By a more subtle argument, we obtain that

$$\left|ST_1\big(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N\big)\right| = c_1(n)\frac{N^d}{2d} + o(N^d) \quad \text{as } N \to \infty, \tag{20}$$

$$\left|ST_{1,1}\big(\mathcal{Q}(1^{n-4}, 0, -1^n), 2N\big)\right| = c_{1,1}(n)\frac{N^d}{2d} + o(N^d) \quad \text{as } N \to \infty. \tag{21}$$

As it was shown in [5], the coefficients $c_1(n)$ and $c_{1,1}(n)$ also exist and satisfy the condition $c(n) > c_1(n) > c_{1,1}(n) > 0$. Though we cannot access $c_{1,1}(n)$ directly, we can use the fun-

3 Actually, such a square-tiled surface additionally admits a proper chessboard coloring, but we will not need it here.

damental fact proven in [5] that the numbers of horizontal and vertical cylinders of a random square-tiled surface are asymptotically uncorrelated when $N \to \infty$. In particular, this yields

$$\frac{c_{1,1}(n)}{c_1(n)} = \frac{c_1(n)}{c(n)}, \tag{22}$$

so that $c_{1,1}(n) = \frac{c_1(n)^2}{c(n)}$. The coefficients $c(n)$ and $c_1(n)$ can be computed explicitly. By (13), we have

$$c(n) = \operatorname{Vol} \mathcal{Q}(1^{n-4}, 0, -1^n) = 2 \operatorname{Vol} \mathcal{Q}(1^{n-4}, -1^n) = 8\left(\frac{\pi^2}{2}\right)^{n-3}, \tag{23}$$

and, by a direct combinatorial argument in [5],

$$c_1(n) = 4\binom{2n-4}{n-2}. \tag{24}$$

Note that the contribution to $\mathcal{M}_n(N)$ from meanders whose associated square-tiled surfaces lie in the principle stratum $\mathcal{Q}(1^{n-4}, 0, -1^n)$ becomes predominant as $N \to \infty$. This implies that for $N \to \infty$ we have

$$\begin{aligned} \mathcal{M}_n(N) &= \frac{2c_{1,1}(n)}{n!(n-4)!} \frac{N^d}{2d} + o(N^d) \\ &= \frac{4}{n!(n-4)!}\left(\frac{2}{\pi^2}\right)^{n-3}\binom{2n-4}{n-2}^2 \frac{N^{2n-5}}{4n-10} + o(N^{2n-5}) \end{aligned}$$

as claimed.

Remark 4. The techniques outlined in this section can be applied to asymptotic enumeration of pairs of multicurves on surfaces of arbitrary genus satisfying certain topological restrictions. The details will appear elsewhere.

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