Integral transformations of pseudodifferential forms

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0. Introduction

A rigorous theory of integration on a supermanifold $M^{n|m}$ ought to contain "forms" that can be integrated on $r|s$-dimensional surfaces, the differential d on "forms", and the Stokes formula connecting the differential and the integral. Moreover, the cohomology $H^{r|s}(M^{n|m})$ ought to be richer than the cohomology of the support $H^*(M^n, \mathbf{R})$. This set of requirements leads naturally to the theory of $r|s$-forms constructed by the authors in [3].

In supergeometry there is also a type of object that can be integrated on $r|s$-dimensional surfaces, namely pseudodifferential forms [2]. It becomes clear that the application of some natural integral transformations reduces their integration to that of $r|s$-forms, which in a well-defined sense solves the problem of uniqueness for regular integration theory.

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1. $r|s$-forms.

We generalize the definition of an $r|s$-form [3] so as to adapt various types of $r|s$-form to integration over surfaces $\Gamma^{r|s}$ with various conditions of orientability of the support Γ^r and a normal fibration to it ν^s.

Definition 0. $\mathrm{Ber}_{(\alpha,\beta)}\begin{pmatrix} A & \mathrm{B} \\ \Gamma & D\end{pmatrix} := (\mathrm{sign}\det A)^\alpha (\mathrm{sign}\det D)^\beta \,\mathrm{Ber}\begin{pmatrix} A & \mathrm{B} \\ \Gamma & D\end{pmatrix}$, where $\alpha, \beta = 0, 1$.

Remark 1. Four different Berezinians $\mathrm{Ber}_{\alpha,\beta}$ correspond to the four connected components of the supergroup $\mathrm{GL}(n \mid m)$: $\pi_0(\mathrm{GL}(n \mid m)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ (compare with $\pi_0(\mathrm{GL}(n)) = \mathbf{Z}_2$).

Definition 1. By an $r|s$-form of the type (α, β) with $\alpha, \beta = 0, 1$ we understand a function

$L = L\begin{pmatrix} \overset{x}{K} & \overset{\xi}{\mathrm{M}} \\ \Lambda & N\end{pmatrix} = L\begin{pmatrix} z \\ V\end{pmatrix}$ (see [3]) satisfying the condition

$$L\left(\begin{pmatrix} A & \mathrm{B} \\ \Gamma & D\end{pmatrix}\cdot\begin{pmatrix} \overset{x}{K} & \overset{\xi}{\mathrm{M}} \\ \Lambda & N\end{pmatrix}\right) = \mathrm{Ber}_{\alpha,\beta}\begin{pmatrix} A & \mathrm{B} \\ \Gamma & D\end{pmatrix}\cdot L\begin{pmatrix} \overset{x}{K} & \overset{\xi}{\mathrm{M}} \\ \Lambda & N\end{pmatrix}$$

and the equations

$$\frac{\partial^2 L}{\partial V_F^A\,\partial V_G^B} - (-1)^{\bar{B}(\bar{F}+\bar{G})+\bar{F}\bar{G}}\frac{\partial^2 L}{\partial V_G^A\,\partial V_F^B} = 0. \tag{1}$$

Here, the parity of the upper index corresponds to the parity of the coordinate, and the parity of the lower index corresponds to the parity of the line to which it gives its number.

We list the basic facts concerning the theory of $r|s$-forms. Cohomologies of $r|s$-forms are homotopically invariant; there exists a canonical pairing with the theory of bordisms of supermanifolds; there is a well-defined natural homomorphism into the real cohomology of the Grassmannian of s-dimensional planes in fibres of the normal fibration (M^n, ν^m) to the support; there is a spectral sequence with a term $E_2^{p,q|s} = H^p(M; H^{q|s}(\mathbf{R}^{0|m}))$; as Novikov stated, the constructed theory is analogous in a well-defined sense to the extraordinary cohomology theories (see [4]); the "point" cohomology $H^{q|s}(\mathbf{R}^{0|m})$ is estimated from below by some characteristic classes in $H^q(G_s(m))$. A more detailed exposition of the theory of cohomology of $r|s$-forms and of the distinctive theory of bordisms of supermanifolds that we have constructed will not be discussed here.

2. The integration of pasudodifferential forms.

It is well known that pseudodifferential forms can be integrated over $r|s$-dimensional surfaces [2]; the integration is possible only over surfaces $\Gamma^{r|s}$ in which, apart from the support Γ^r, the normal bundle ν^s to the support is oriented, which had not been pointed out before.

The pseudodifferential forms integrable over $r|s$-dimensional surfaces are functorial only with respect to morphisms of supermanifolds of full rank with respect to odd variables.

Restricting ourselves to the category of vector bundles (with even and odd fibres), we have been able to construct various integral transformations of Radon type preserving the differential and the integral: from $r|s$-forms, differential and pseudodifferential forms into forms on the Grassmannian of the bundle; the last two can apparently be connected with Fourier-type transformations. But there is an integral transformation [1] that acts directly in the category of supermanifolds.

3. The Baranov-Shvarts transformation.

In [1] a map was constructed that associates with a pseudodifferential form $\omega = \omega(x, \xi, dx, d\xi)$ an $r|s$-density L by the following formula:

$$L\begin{pmatrix} \overset{x}{K} & \overset{\xi}{M} \\ \Lambda & N \end{pmatrix} = \int_{R^{s|r}} \omega(x, \xi, \pi K + p\Lambda, \pi M + pN)\, D(p, \pi). \tag{2}$$

Here, the row $(dx, d\xi)$ in the argument of ω is replaced by the row $(\pi, p)\cdot\begin{pmatrix} K & M \\ \Lambda & N\end{pmatrix}$, and the integration is carried out with respect to the parameters p^α ($\alpha = 1, ..., s$) and π^i ($i = 1, ..., r$). (We note that in [1] a minor inaccuracy was allowed: strictly speaking, the object obtained as a result of the transformation (2) is not a density in the sense of [1]; this object is an *$r|s$-density of the type* (0, 1).) It is easy to verify that the transformation (2) defined in terms of the coordinates (x, ξ) does not depend on the choice of the coordinate system.

We shall call the transformation (2) a *Baranov-Shvarts transformation.* We shall denote by $\lambda^{r|s}\omega$ the result of its application to the form ω. The Baranov-Shvarts transformation is the superanalogue of the Radon transformation of functions.

Theorem 1. *The $r|s$-density that is the image of the Baranov-Shvarts transformation satisfies the equations* (1), *that is, it is an $r|s$-form* (*of the type* (0, 1)).

Theorem 2. *The pseudodifferential form ω is integrable on any $r|s$-dimensional surface if and only if the Baranov-Shvarts transformation $\lambda^{r|s}\omega$ is defined. Moreover, for any surface $\Gamma^{r|s}$,*

$$\int_{\Gamma^{r|s}} \omega = \int_{\Gamma^{r|s}} \lambda^{r|s}\omega.$$

If the transformation $\lambda^{r|s}$ is defined for a pseudodifferential form ω, then for its differential $d\omega$ the transformation $\lambda^{r+1|s}$ is also defined, and $d\lambda^{r|s}\omega = \lambda^{r+1|s}d\omega$.

The proof of the theorem is obtained by direct computation. We note that the fact that the transformation $\lambda^{r|s}$ commutes with the integral was stated in [1]. Indeed, by definition, the integral of a pseudodifferential form over an $r|s$-surface, as an integral of the function ω with respect to the variables t, τ, dt, $d\tau$, is the composition of the transformation of ω into an $r|s$-form (integration with respect to dt and $d\tau$) and of the integral of the $r|s$-form $\lambda^{r|s}\omega$ (integration with respect to t and τ).

4. Conclusion.

Theorems 1 and 2 show that the theory of integration of pseudodifferential forms finds a natural interpretation in the theory of $r|s$-forms.

We should mention that Theorem 2 is mostly due to the first author, while Theorem 1 is due to the second author.

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