Integration of pseudodifferential forms and inversion of Radon-type integral transformations

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0. Continuing work done jointly with F.F. Voronov in a series of papers, in this note we develop an integration technique on supermanifolds and describe new effects regarding the integration of pseudodifferential forms. As an application, we propose a new solution to a purely classical problem, namely, the inversion of the generalized Radon transformation for functions. We introduce a class of integral transformations that generalize the Radon transformation, and indicate a sufficient condition for their invertibility and a method of inversion by means of the proposed techniques.

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1. The Bernshtein-Leites theory of pseudodifferential forms on supermanifolds [1] parallels on the whole the usual theory of differential forms. In particular, the following assertion holds:

***Proposition* 1.** *Homotopic morphisms of supermanifolds induce identical mappings in cohomologies of pseudodifferential forms.*

But, in contrast to differential forms, *one and the same closed pseudodifferential form may yield a non-zero integral with respect to cycles of a different dimension.* Even more unexpected is the fact that such cycles exist even for "points" of $\mathbf{R}^{0|m}$.

We consider a sphere S^s embedded in the standard way in $\mathbf{R}^{s+1}$, which, in turn, is contained in $\mathbf{R}^m$. Regarding $\mathbf{R}^m$ as a vector bundle over some chosen point in $\mathbf{R}^m$, we can construct a mapping $f: TS^s \to \mathbf{R}^m$ from the tangent bundle of the sphere TS^s to the bundle $\mathbf{R}^m \to \mathrm{pt}$, which transforms the tangent plane into a sphere in a parallel plane passing through the chosen point. Interchanging the parity of the fibres in the bundle, we obtain a morphism $f\Pi$: $(TS^s)\Pi \to \mathbf{R}^{0|m}$. We denote the supermanifold $(TS^s)\Pi$ by $S^{s|s}$. We consider an arbitrary fairly rapidly decreasing function $g(v^1, \ldots, v^m)$ in $\mathbf{R}^m$. This function, regarded as a function of the even coordinates $d\xi$, that is, $g(d\xi)$, is a closed pseudodifferential form in $\mathbf{R}^{0|m}$.

***Theorem* 1.** *The integral of the closed pseudodifferential form $g(d\xi)$ over a singular supermanifold $f\Pi$: $S^{2k|2k} \to \mathbf{R}^{0|m}$ is equal to the value of g at zero multiplied by* $(-1)^k \cdot 2 \cdot (2\pi)^{2k}$.

2. In [2] an integral transformation J_s of pseudodifferential forms onto differential forms on the Grassmannian was introduced, and it was shown that this transformation is functorial and compatible with the differential and integral. In the particular case of $\mathbf{R}^{0|m}$ for pseudodifferential forms of the form $g(d\xi) \in \Omega^m(\mathbf{R}^{0|m})$ the transformation J_s: $\Omega^m(\mathbf{R}^{0|m}) \to \Omega^s(G_s(\mathbf{R}^m))$ is given by the formula J_s: $g(d\xi) \mapsto \int g(d\tau\, N - \tau\, dN) D(\tau,\ d\tau)$. Here $(\tau^1, \ldots, \tau^s) \in \mathbf{R}^{0|s}$ and N^μ_α, $\mu = 1, \ldots, m$; $\alpha = 1, \ldots, s$, are homogeneous coordinates in the Grassmann manifold $G_s(\mathbf{R}^m)$. The compatibility of J with integration means that

$$\int\limits_{f\Pi:\ S^{2k|2k} \to \mathbf{R}^{0|m}} g = \int\limits_{G_{2k}(f):\ S^{2k} \to G_{2k}(\mathbf{R}^m)} J_{2k} g. \tag{1}$$

Here $G_{2k}(f)$ is the generalized Gaussian mapping of the sphere S^{2k} in $G_{2k}(\mathbf{R}^m)$ constructed with respect to the mapping of the tangent bundle $f: TS \to \mathbf{R}^m$ of the sphere.

Remark 1. The closed differential form $(-1)^k/((2\pi)^{2k} \cdot g(0)) J_{2k} g$ realizes the Euler characteristic class of the canonical bundle over $G_{2k}(\mathbf{R}^m)$.

3. The canonical s-dimensional vector bundle over the Grassmann manifold $G_s(\mathbf{R}^m)$ is a subbundle of the trivial m-dimensional bundle. The total space of the corresponding factor-bundle over $G_s(\mathbf{R}^m)$, denoted by $H_s(\mathbf{R}^m)$, is isomorphic to the space of affine s-planes in $\mathbf{R}^m$. We consider the double fibering

$$\mathbf{R}^m \leftarrow \mathbf{R}^m \times G_s(\mathbf{R}^m) \to H_s(\mathbf{R}^m), \tag{2}$$

where on the left we have a projection and on the right a mapping of the factorization with respect to a subbundle, that is, a vector bundle with fibre $\mathbf{R}^s$. We consider an integral transformation R_s of the rapidly decreasing functions $g(v)$ in $\mathbf{R}^m$, corresponding to the double fibering (2) and defined by $R_s g = \hat{g}(v, N) := \int g(v + uN)du$. Here v^μ are the coordinates in $\mathbf{R}^m$, u^α the coordinates in the fibre $\mathbf{R}^s$ of the canonical bundle over $G_s(\mathbf{R}^m)$, and N^μ_α homogeneous coordinates in $G_s(\mathbf{R}^m)$. We see that the integral transformation R_s corresponding to (2) is in fact the Radon transformation.

In the formula for $\hat{g}$ we replace v by $d\xi$ and u by $d\tau$, that is, $\hat{g}(d\xi, N) = \int g(d\xi + d\tau\, N)d(d\tau)$. We now consider the transformation $\varkappa$: $\hat{g}(d\xi, N) \mapsto \int \hat{g}(-\tau\, dN, N) \times D(\tau)$. Then $\varkappa\hat{g} = \int g(d\tau\, N - \tau\, dN)D(\tau,\ d\tau) = J_s g$. Using Theorem 1 and (1), we find that

$$\int_{G_s(f):S^{2k}\to G_{2k}(\mathbf{R}^m)} \varkappa\hat{g} = (-1)^k \cdot 2 \cdot (2\pi)^{2k} g(0).$$

Changing the chosen point in the affine space $\mathbf{R}^m$, we can reconstruct the value of g at an aribtrary point of $\mathbf{R}^m$ in terms of the image of the Radon transformation $\hat{g}$. In the end we obtain

$$(3)\qquad g(v_0) = (-1)^k/(2\cdot(2\pi)^{2k})\cdot \int_{G_{2k}(f):\ S^{2k}\to G_{2k}(\mathbf{R}^m)} \left(\int_{\mathbf{R}^{0|2k}} \hat{g}(v_0 - du\, dN,\ N)\, D\,(du)\right).$$

If we rewrite (3) by writing the differential form under the inner integral in terms of components, we then obtain the Gel'fand-Gindikin-Graev-Shapiro inversion formula (see [3]).

4. We consider a pair of manifolds with vector bundles (M_1, ν_1^s) and (M_2, ν_2^m) of dimensions s and m, respectively, connected by a layerwise injective, layerwise linear morphism of bundles $f: E(M_1, \nu_1^s) \to E(M_2, \nu_2^m)$. Then ν_1^s is a subbundle of the bundle $(f^*\nu_2)$ induced on M_1 from ν_2^m. At the same time, the factor-bundle with total space $E(M_1, (f^*\nu_2)/\nu_1)$ is defined. The diagram

$$(4)\qquad E\,(M_2,\ \nu_2) \leftarrow E\,(M_1,\ f^*\nu_2) \rightarrow E\,(M_1,\ f^*\nu_2/\nu_1)$$

(where on the right-hand side we have a mapping of a layer factorization and on the left-hand side a layer isomorphism) defines an integral transformation of forms or functions on $E(M_2, \nu_2)$ into forms or functions on $E(M_1, f^*\nu_2/\nu_1)$. The double fibering (2) is a special case of such a diagram. In addition, we now require that the basis mapping $M_1 \to M_2$ should be a fibering with fibre F. Then the integral transformation of functions defined by (4) reduces in essence to the family of transformations defined by the diagrams

$$(5)\qquad \mathbf{R}^m_{(x)} \leftarrow \mathbf{R}^m_{(x)} \times F_{(x)} \rightarrow E\,(F_{(x)},\ \mathbf{R}^m/\nu^s\,|_F).$$

Here $\mathbf{R}^m_{(x)}$ is the fibre of the bundle (M_2, ν_2^m) suspended over the point $x \in M_2$.

***Theorem* 2.** *If s is even and the Euler class of the bundle $(F, \nu^s|_F)$ is non-zero, then the integral transformation corresponding to the diagram* (4) *is invertible.*

The inversion scheme is exactly the same as before. We single out the interesting construction of the Euler class, which is used in the inversion process.

***Theorem* 3.** *The analogue of the transformation J_{2k} corresponding to the diagram* (5) *maps a pseudodifferential form $g(d\xi)$ into the differential form $J_{2k}g \in \Omega^{2k}(F)$, which realizes the characteristic Euler class of the bundle $(F,\ \nu_1^{2k}|_F)$ up to the coefficient $(-1)^k/(2\cdot(2\pi)^{2k}\cdot g(0))$.*

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