INTEGRATION ON VECTOR BUNDLES

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1. Introduction

In this paper we study the connection between different objects of integration on vector bundles and smooth supermanifolds. Certain integral transforms of Radon and Fourier type arise here naturally. The authors have been partially successful in removing the shroud of mystery from "odd" integration, connecting it with natural operations over exterior forms on bundles. Nevertheless, the authors have not made the elimination of supermathematics their goal. On the contrary, the apparatus of superanalysis affects the description of different integral transforms so well that many purely classical problems of integral geometry should be described in precisely this language. But we have intentionally not touched here on questions of integral geometry proper, so as not to make the theory of integration recounted here more complicated. These questions will be considered separately. Orientability conditions, cf. [6], are not discussed for the same reason.

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2. Bundles and Supermanifolds

We consider a supermanifold $M^{n|m}$ with local coordinates x^a, ξ^μ, $a = 1, \ldots, n$, $\mu = 1, \ldots, m$. Change of coordinates looks like this: $x^a = x^a(x', \xi') = x^a(x', 0) + \ldots$, $\xi^\mu = \xi^\mu(x', \xi') = \xi^{\mu'} T^\mu_{\mu'}(x') + \ldots$. The support of the supermanifold $M^{n|m}$, a smooth manifold M^n, is imbedded in $M^{n|m}$ and is singled out in local coordinates by the equations $\xi^\mu = 0$, $\mu = 1, \ldots, m$. As usual, by the normal bundle of the manifold M^n, imbedded in $M^{n|m}$ is meant the quotient of the tangent bundle of $M^{n|m}$ restricted to M^n by the tangent bundle of M^n. We shall denote the space of the normal bundle by $N\Pi$. The fiber of this bundle is the purely odd vector supermanifold $\mathbf{R}^{0|m}$. On $N\Pi$ change of local coordinates has the form $x^a = x^a(x', 0)$, $\xi^\mu = \xi^{\mu'} T^\mu_{\mu'}(x')$.

THEOREM 0. The supermanifolds $M^{n|m}$ and $N\Pi$ are (noncanonically) isomorphic. In other words, on any supermanifold $M^{n|m}$ one can introduce the structure of a vector bundle with even base M^n and odd fiber.

In this formulation we shall use the familiar result that one can always choose an atlas in which changes of coordinates are linear in the odd variables [1]. Thus, the classification of supermanifolds reduces to the classification of vector bundles, however the corresponding categories are not equivalent: there are more morphisms in the category of supermanifolds [8]. But where homotopy questions arise, in bordism theory [7], in cohomology theory [6], everything reduces in a certain sense to the category of bundles (cf. below).

Actually in this whole paper we shall deal only with vector bundles with even or odd fiber and even base. Many constructions carry over easily to the general case of heteroparity, but this was not exactly our goal.

3. Basic Concepts and Notation

Throughout the paper we consider the following category: the objects are smooth vector bundles over even bases with even or odd fibers, the morphisms are fiberwise linear, fiberwise injective smooth maps. The naturality of the requirement of fiberwise injectivity can be seen at many points (cf. [6] and below).

Let $N \to M$ be a bundle over an n-dimensional manifold $M = M^n$ with fiber $\mathbf{R}^m$. We shall denote by $N' \to M$ the conjugate bundle, and by $N\Pi$ the same bundle but with the opposite parity

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in the fiber. For convenience we shall call N or NΠ the normal bundle (with even or odd fiber) and N', N'Π the conormal one. We introduce local coordinates in the bundle N and corresponding coordinates in NΠ, N', and N'Π: in $N - (x^a, v^\mu)$, $x \in U \subset \mathbf{R}^n, v \in \mathbf{R}^m$, in $N\Pi - (x^a, \xi^\mu)$, $\xi \in \mathbf{R}^{0/m}$, in N', (x^a, p_μ), $p \in \mathbf{R}^{m'}$ in $N'\Pi - (x^a, \pi_\mu)$, $\pi_\mu \in \mathbf{R}^{0|m'}$. A change of coordinates has the form

$$\begin{cases} x^a = x^a(x'), \\ v^\mu = v^{\mu'} T^\mu_{\mu'}(x'), \end{cases} \quad \begin{cases} x^a = x^a(x'), \\ \xi^\mu = \xi^{\mu'} T^\mu_{\mu'}(x'), \end{cases}$$
$$\begin{cases} x^a = x^a(x'), \\ p_\mu = (T^{-1})^{\mu'}_\mu (x') p_{\mu'}, \end{cases} \quad \begin{cases} x^a = x^a(x'), \\ \pi_\mu = (T^{-1})^{\mu'}_\mu (x') \pi_{\mu'}. \end{cases}$$

Any morphism $f\colon N_1 \to N_2$ defines a morphism $f\Pi\colon N_1\Pi \to N_2\Pi$ and conversely. In coordinates the morphisms f and fΠ are given by the same matrix $U^{\mu_2}_{\mu_1}(x_1)$: $f^* x^{a_2} = (f\Pi)^* x^{a_2} = x^{a_2}(x_1)$, $f^* v^{\mu_2} = v^{\mu_1} U^{\mu_2}_{\mu_1} \times (x_1)$, $(f\Pi)^* \xi^{\mu_2} = \xi^{\mu_1} U^{\mu_2}_{\mu_1}(x_1)$. Henceforth the morphisms f and fΠ will be identified.

We consider the manifolds $G_s(N \to M)$ of all s-dimensional planes in the fibers of the bundle N, $0 \leqslant s \leqslant m$ and $G_{0|s}(N\Pi \to M)$ of 0|s-dimensional planes in the fibers of NΠ. Both manifolds are even and naturally isomorphic: $G_s(N \to M) = G_{0|s}(N\Pi \to M) = G_s$. We shall call G_s the <u>Grassmanian.</u> There are two canonical bundles over the manifold G_s: an s-dimensional one E_s and an $(m - s)$-dimensional one $E_s^\perp$. E_s is a subbundle of π^*N, the bundle induced over the Grassmanian from the bundle N over M, by means of the projection $\pi\colon G_s \to M$. A point of the Grassmanian G_s is an s-plane in the fiber of the bundle N. A fiber of the bundle E_s consists of all vectors which lie in this plane. $E_s^\perp$ is a subbundle of π^*N'. The fiber of the bundle $E_s^\perp$ is the annihilator of the fiber of E_s.

Let $N = (N^\mu_\alpha)$, $\alpha = 1, \ldots, s$, $\mu = 1, \ldots, m$ be a matrix† of rank s, $u = (u^\alpha) \in \mathbf{R}^s$, $p = (p_\mu) \in (\mathbf{R}^m)'$, $\tau = (\tau^\alpha) \in \mathbf{R}^{0|s}$, $g = (g^\beta_\alpha) \in GL(s)$. In describing G_s and the spaces connected with it it is convenient to use homogeneous coordinates: in $G_s - (x^a, N^\mu_\alpha)$, where (x, N) and (x, gN) are identified, in $E_s - (x^a, u^\alpha, N^\mu_\alpha)$ with (x, u, N) and (x, ug^{-1}, gN) identified, in $E_s\Pi - (x^a, \tau^\alpha, N^\mu_\alpha)$ with (x, τ, N) and $(x, \tau g^{-1}, gN)$ identified. The space $E_s^\perp$ is defined as a surface in π^*N', the conormal bundle induced on G_s: in π^*N', the homogeneous coordinates $(x^a, N^\pi_\alpha, p_\mu)$, and $E_s^\perp$ is defined by the equation $N^\mu_\alpha p_\mu = 0$.

The construction of the Grassmanian is functorial with respect to morphisms $f\colon N_1 \to N_2$. In particular, we shall denote by $G_s(f)$ the induced map of Grassmanians $G_s(f)\colon\ G_s(N_1 \to M_1) \to G_s(N_2 \to M_2)$, $G_s(f)^* x_2^{a_2} = x_2^{a_2}(x_1)$, $G_s(f)^* N^{\mu_2}_\alpha = N^{\mu_1}_\alpha U^{\mu_2}_{\mu_1}(x_1)$, if $f^* x_2^{a_2} = x_2^{a_2}(x_1)$, $f^*(v^{\mu_2}) = v^{\mu_1} U^{\mu_2}_{\mu_1}(x_1)$. If $P \to Q$ is an s-dimensional bundle, then we identify $G_s(P \to Q) = Q$ and for a morphism $f\colon P \to N$ of a "singular s-bundle in N," $f^* x^a = x^a(t)$, $f^* v^\mu = u^\alpha N^\mu_\alpha(t)$ by $G_s(f)$ we denote the induced Gauss map $Q \to G_s(N \to M)$, $G_s(f)^* x^a = x^a(t)$, $G_s(f)^* N^\mu_\alpha = N^\mu_\alpha(t)$ (cf. [7]).

For a supermanifold $M^{n|m}$ we shall denote by $\Omega(M^{n|m})$ the algebra of <u>pseudodifferential forms</u>‡ (cf. [3]), which looks locally like the algebra of functions of x, ξ, dx, dξ (dx being odd, dξ even). Pseudodifferential forms on even manifolds coincide with ordinary exterior forms. Compare [4, 5] for the definition of the space of r|s-forms $\Omega^{r|s}$. Locally an r|s-form $L \in \Omega^{r|s}(N\Pi)$ is a function $L = L\begin{pmatrix} x & \xi \\ K & M \\ \Lambda & N \end{pmatrix}$, $K = (K^a_i)$, $M = (M^\mu_i)$, $\Lambda = (\Lambda^a_\alpha)$, $N = (N^\mu_\alpha)$, $i = 1, \ldots, r$, $\alpha = 1, \ldots, s$ which satisfies a collection of conditions [4, 5]. Ω and $\Omega^{\cdot|s}$ are contravariant functors on the category of supermanifolds with morphisms of full rank in the odd variables.

We shall bravely use the symbols dx, dξ, etc. as competent variables, differentiate and integrate with respect to them, for example, $\int D(\xi, d\xi)$ will denote the Berezin integral with respect to the collection of variables ξ and dξ.

A delta-function of the odd variables is defined word for word like an ordinary Dirac function: for any function f(ξ) one has $\int \delta(\xi) f(\xi) D(\xi) = f(0)$. Incidentally, it is regular $\delta(\xi) = \xi^1 \xi^2 \ldots \xi^m$.

†Don't confuse the matrix N with the space of the bundle N!
‡Unfortunately it is customary to call the subalgebra $\Omega_{pol} \subset \Omega$ of functions which are polynomial in dξ, simply <u>differential forms</u> on the supermanifold $M^{n|n}$.

4. Integral Transform $I_s:\Omega(N)\to\Omega(G_s)$ of Forms on the Normal Bundle to Forms on the Grassmanian

The space $E_S = E_S(N \to M)$ has a natural structure of double bundle over N and $G_S = G_S \times (N \to M)$

$$\begin{array}{ccc} & E_s & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ N & & G_s. \end{array}$$

A point of E_S is a pair consisting of a plane in the fiber of the bundle N and a vector lying in it. The projection p_1 associates to a point of E_S this vector and the projection p_2, this plane. In coordinates: p_1: $(x, u, N) \mapsto (x, uN)$, p_2: $(x, u, N) \mapsto (x, N)$.

We consider the integral transform which corresponds to the given double bundle, $I_s = p_{2*}\circ p_1^*$: $\Omega(N) \to \Omega(G_s)$. In local coordinates the map $\omega(x, v, dx, dv) \mapsto (I_s\omega)(x, N, dx, dN)$ is defined by the formula

$$(I_s\omega)(x, N, dx, dN)=\int_{\mathbf{R}^{s|s}} D(u, du)\,\omega(x, uN, dx, du N+u\, dN). \tag{1}$$

(Here and below we consider exterior forms as pseudonorms: as functions of the coordinates and differentials.) It is easy to verify that the form $I_S\omega$ is defined in homogeneous coordinates, is GL(s)-invariant and horizontal, i.e., is really a form on G_S.

In our case, horizontality of a form $\sigma = \sigma(x, N, dx, dN)$ is expressed by the equation $N^\mu_\beta \partial\sigma/\partial dN^\mu_\alpha = 0$. We verify directly:

$$\frac{\partial I_s\omega}{\partial\, dN^\mu_\alpha}=\frac{\partial}{\partial\, dN^\mu_\alpha}\int_{\mathbf{R}^{s|s}} D(u, du)\,\omega(x, uN, dx, duN+u\, dN)$$

$$=\int_{\mathbf{R}^{s|s}} D(u, du)\frac{\partial}{\partial\, dN^\mu_\alpha}(du^\beta N^\nu_\beta+u^\beta\, dN^\nu_\beta)\frac{\partial\omega}{\partial\, dv^\nu}(x, uN, dx, duN+udN)=\int_{\mathbf{R}^{s|s}} D(u, du)\,u^\alpha\frac{\partial\omega}{\partial\, dv^\mu}(x, uN, duN+udN).$$

From this,

$$N^\mu_\beta\frac{\partial I_s\omega}{\partial\, dN^\mu_\alpha}=\int_{\mathbf{R}^{s|s}} D(u, du)\,u^\alpha N^\mu_\beta\frac{\partial\omega}{\partial\, dv^\mu}(x, uN, dx, duN+u\, dN)=\int_{\mathbf{R}^{s|s}} D(u, du)\,u^\alpha\frac{\partial}{\partial\, du^\beta}(\omega(x, uN, dx, duN+u\, dN))=0$$

by definition of the Berezin integral.

THEOREM 1. 1. The transform I_S is functorial: for any morphism f: $N_1 \to N_2$ one has $G_s(f)^*\circ I_s=I_s\circ f^*$.

2. The degree of I_S is equal to $-s$, I_s: $\Omega^k(N)\to\Omega^{k-s}(G_s)$.

3. The transform I_S commutes with the differential: $I_s\circ d=d\circ I_s$.

4. For any "singular s-dimensional bundle" f: P → N one has $p_*\circ f^*=G_s(f)^*\circ I_s$, where p: P → Q is the projection; in particular,

$$\int_{(P, f)}\omega=\int_{(Q, G_s(f))}\omega.$$

The assertions of the theorem follow from the properties of the direct and inverse image.

5. Dual Integral Transform $I_s^\perp:\Omega(N')\to\Omega(G_s)$ of Forms on the Conormal Bundle to Forms on the Grassmanian

The double bundle

$$\begin{array}{ccc} & E_s^\perp & \\ \swarrow & & \searrow \\ N' & & G_s, \end{array}$$

where the fiber of the bundle $E_S^\perp$ over G_S consists of covectors which annihilate vectors of the fiber E_S, defines an integral transform $I_s^\perp$: $\Omega(N')\to\Omega(G_s)$, which is dual to (1), by

$$(I_s^{\perp}\omega)(x, N, dx, dN) = \int_{\mathbf{R}^{m|m}} D(p, dp)\,\delta(N^{\mu}p_{\mu})\,\delta(d(N^{\mu}p_{\mu}))\,\omega(x, p, dx, dp). \tag{2}$$

Before formulating the properties of the transformation (2), we discuss the functorial behavior of forms on N'. Any morphism $f\colon N_1 \to N_2$ decomposes into a composition $N_1 \xrightarrow{g} f^*N_2 \xrightarrow{h} N_2$, where f^*N_2 is the bundle induced on M_1 by means of f, h is the canonical fiberwise isomorphism, and g is a fiberwise injective morphism of bundles over M_1. Passage to conjugate bundles induces a diagram $N_1' \xleftarrow{g'} f^*N_2' \xrightarrow{h'} N_2'$, where g' is fiberwise surjective over M_1; hence a submersion. One can lift a form from N_2' to f^*N_2' by means of h, and then integrate along the fibers of g' (we note that this is impossible on the other side because h is generally not a submersion). Thus there is defined the inverse image $f^*\colon \Omega(N_2') \to \Omega(N_1')$. In coordinates

$$(f^*\omega)(x_1, p_1, dx_1, dp_1) = \int_{\mathbf{R}^{m_2|m_2}} D(p_2, dp_2)\,\delta(p_1 - U(x_1)p_2)\,\delta(d(p_1 - U(x_1)p_2))\,\omega\Big(x_2(x_1), p_2, dx_1\frac{\partial x_2}{\partial x_1}, dp_2\Big).$$

We note that f^* has degree $-(m_2 - m_1)$, where $m_1 \leqslant m_2$ are the dimensions of the fibers in N_1 and N_2, respectively, and commutes with the differential.

THEOREM 2. 1. The transform $I_S^{\perp}$ is functorial: for any morphism $f\colon N_1 \to N_2$ one has $G_s(f)^* \circ I_s^{\perp} = I_s^{\perp} \circ f^*$.

2. The degree of $I_S^{\perp}$ is equal to $(m - s)$, $I_s^{\perp}\colon \Omega^k(N') \to \Omega^{k-(m-s)}(G_s)$.

3. The transform $I_S^{\perp}$ commutes with the differential: $I_s^{\perp} \circ d = d \circ I_s^{\perp}$.

4. For any "singular s-dimensional bundle" $f\colon P \to N$ one has $i^* \circ f^* = G_s(f)^* \circ I_s^{\perp}$, where $i\colon Q \to P'$ is the zero section.

The assertions of the theorem follow from the properties of the direct and inverse image.

We want to turn the reader's attention to the difference between the formulas in point 4 of this theorem and the preceding one. The formula $p_* \circ f^* = G_s(f)^* \circ I_s$ has the following geometric meaning: in order to integrate the form ω on the bundle N over the singular s-bundle (P, f), it is necessary to transform it into the form $I_S\omega$ on the s-th Grassmanian $G_S(N \to M)$ and to integrate the form obtained over the singular manifold $[Q, G_S(f)]$ in the Grassmanian. In order to give analogous meaning to the formula $i^* \circ f^* = G_s(f)^* \circ I_s^{\perp}$, it is necessary to agree to properly understand the integral of a form on the conjugate bundle. In particular, the integral over the space N itself looks like this: we take a form on N', restrict it to the base, and integrate.

6. Integral Transform $J_s\colon \Omega(N\Pi) \to \Omega(G_s)$ of Pseudoforms to Forms on the Grassmanian and the Dual Transform $J_s^{\perp}\colon \Omega(N'\Pi) \to \Omega(G_s)$

The normal bundle NΠ with even fiber is indistinguishable from the bundle N from the point of view of transition functions. The difference becomes perceptible in considering functions and forms on the total space. We recall that a pseudodifferential form on a supermanifold with coordinates x, ξ is a function of the variables x, ξ, dx, dξ [3]. Since $d\xi^{\mu}$ in contrast with dv^{μ} is an even variable and the dependence on it is not generally polynomial, the algebra Ω(NΠ) does not carry any natural grading by the degrees of dx and dξ, analogous to the grading of exterior forms on N. However, the bundle structure on NΠ lets us introduce a rather unexpected grading on Ω(NΠ):

Definition 1. A pseudodifferential form ω = ω(x, ξ, dx, dξ) on the space NΠ has degree k, if $\# dx + m - \# \xi = k$ in its decomposition with respect to ξ^{μ} and dx^a, where m is the dimension of the fiber, and $\#\xi$ etc. denotes the total degree in the given variable.

Proposition 1. The degree of a pseudodifferential form is well-defined (independent of the choice of coordinates in the bundle NΠ). Thus, the space Ω(NΠ) decomposes into a direct sum: $\Omega(N\Pi) = \bigoplus_{k=0}^{n+m} \Omega^k(N\Pi)$, where dim NΠ = n|m. The differential $d = dx^a \frac{\partial}{\partial x^a} + d\xi^{\mu} \frac{\partial}{\partial \xi^{\mu}}$ has degree +1, $d\colon \Omega^k(N\Pi) \to \Omega^{k+1}(N\Pi)$, and multiplication has degree $-m$, $\Omega^k(N\Pi) \otimes \Omega^l(N\Pi) \to \Omega^{k+l-m}(N\Pi)$; the parity of the form ω differs from its grading by m mod 2.

In passing from the bundle NΠ to an arbitrary supermanifold $M^{n|m}$ the grading we have introduced is not preserved, but becomes the associated increasing filtration.

The operator of fiberwise integration (direct image) of pseudodifferential forms is defined analogously to the purely even case and is given by the formula $\int D(w, dw)$, where w^A are the coordinates in the fiber of the submersion. Its degree (when it makes sense) is equal to the even part of the dimension of the fiber, taken with a minus sign. We note, incidentally, that the inverse image f^* for pseudodifferential forms has nonzero degree generally, and equal to the difference of the odd dimensions of the domain of values and the domain of definition of f.

We consider the double bundle

$$\begin{array}{ccc} & E_s\Pi & \\ \swarrow & & \searrow \\ N\Pi & & G_s, \end{array}$$

and in complete analogy with the purely even situation we define the integral transform $J_s\colon\ \Omega(N\Pi) \to \Omega(G_s)$. In coordinates $\omega(x, \xi, dx, d\xi) \mapsto (J_s\omega)(x, N, dx, dN)$:

$$(J_s\omega)(x, N, dx, dN) = \int_{\mathbf{R}^{s|s}} D(\tau, d\tau)\,\omega(x, \tau N, dx, d\tau N - \tau\, dN).$$

THEOREM 3. 1. The transform J_s is functorial: for any morphism $f\colon N_1 \to N_2$ one has $G_s(f)^* \circ J_s = J_s \circ f^*$.

2. The degree of J_s is equal to $-(m-s)$, $J_s\colon \Omega^k(N\Pi) \to \Omega^{k-(m-s)}(G_s)$.

3. The transform J_s commutes with the differential: $J_s \circ d = d \circ J_s$.

4. For any "singular 0|s-dimensional bundle" $f\colon P\Pi \to N\Pi$ the formula $p_* \circ f^* = G_s(f)^* \circ J_s$, where $p\colon P\Pi \to Q$ is the projection, is true; whence, for any $\omega \in \Omega(N\Pi)$ one has

$$\int_{(P\Pi,\, f)} \omega = \int_{(Q,\, G_s(f))} J_s\omega.$$

The assertions of the theorem follows from the properties of the direct and inverse image of pseudodifferential forms.

In complete analogy with the way the transform $I_s^{\perp}$, dual to I_s was constructed, one can construct the transform

$$J_s^{\perp}\colon \Omega(N'\Pi) \to \Omega(G_s), \tag{3}$$

dual to the transform J_s. It corresponds to the double bundle

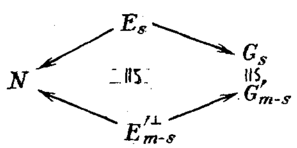
$$\begin{array}{ccc} & E_s^{\perp}\Pi & \\ \swarrow & & \searrow \\ N'\Pi & & G_s \end{array}$$

and in coordinates (x^a, π_μ) on $N'\Pi$, is given by the formula $\omega(x, \pi, dx, d\pi) \mapsto (J_s^{\perp}\omega)(x, N, dx, dN)$:

$$(J_s^{\perp}\omega)(x, N, dx, dN) = \int_{\mathbf{R}^{m|m}} D(\pi, d\pi)\,\delta(N\pi)\,\delta(d(N\pi))\,\omega(x, \pi, dx, d\pi).$$

The transform $J_s^{\perp}$ has degree $-s$, while the rest of its properties are parallel to the properties of $I_s^{\perp}$ described in Theorem 2.

We note that the Grassmanian G_s the canonical bundle E_s, and all its neighbors— $E_s^{\perp}$, $E_s\Pi$, $E_s^{\perp}\Pi$, can be considered in two representations: the Grassmanian G_s is canonically isomorphic to the Grassmanian of (m − s)-dimensional coplanes G'_{m-s}; the space E_s is canonically isomorphic to the space $E'^{\perp}_{m-s}$ of the bundle with base G'_{m-s} and fiber the annihilator of a coplane from G'_{m-s}, etc. Each of the transforms described can be written in two versions, depending on whether we want to get the image in coordinates on G_s or G'_{m-s} for example I_s, after identification of G_s and G'_{m-s}, coincides with $I'^{\perp}_{m-s}$



This explains the difference in the notation for I, J and $I^{\perp}$, $J^{\perp}$. If we write the images of $I^{\perp}$ and $J^{\perp}$ in coordinates on G'_{m-s} then all the formulas become identical. It is remarkable that there exist integral transforms of Fourier type, cross-woven with transforms with the opposite parity in the fibers: I_s with $J_s^{\perp}$ and $I_s^{\perp}$ with J_s. More about them below.

7. Baranov–Shvarts Transform $\lambda^{\cdot|s}:\Omega(N\Pi)\to\Omega^{\cdot|s}(N\Pi)$ of Pseudonorms to $\cdot|s$-Forms

In [1] an integral transform was introduced, which associates with a pseudodifferential form on a supermanifold, an r|s-density; it turned out [5] that as a result of the transform one actually gets an r|s-form.

The Baranov–Shvarts transform $\omega(x,\xi,dx,d\xi)\mapsto\lambda^{r|s}\omega\begin{pmatrix}x&\xi\\K&M\\\Lambda&N\end{pmatrix}$ is given by the formula

$$(\lambda^{r|s}\omega)\begin{pmatrix}x&\xi\\K&M\\\Lambda&N\end{pmatrix}=\int\limits_{\mathbf{R}^{s|r}}D(dt,d\tau)\,\omega(x,\xi,dt\cdot K+d\tau\cdot\Lambda,dt\cdot M+d\tau\cdot N). \tag{4}$$

The transform has natural geometric meaning: when the pseudoform is integrated over an r|s-surface, it is first turned into an r|s-form by (4), and then integrated; this is a literal deciphering of the definition of the integral of a pseudoform. By Theorem 3 the transform $\lambda^{r|s}$: $\Omega^k(N\Pi)\to\Omega^{r|s}(N\Pi)$ for $k\neq r+m-s$ leads to r|s-forms, all of whose integrals over linear r|s-surfaces are equal to zero. Hence it is reasonable to consider the Baranov–Shvarts transform $\lambda^{r|s}$ only on $\Omega^{r+m-s}(N\Pi)$. From this point of view the transform $\lambda^{\cdot|s}=\Sigma\,\lambda^{r|s}$ has degree $-(m-s)$, where the degree of an r|r-form is considered to be r.

It is worth noting that one and the same pseudoform $\omega\in\Omega^k(N\Pi)$ can make a contribution to integrals over r|s-surfaces with different r and s provided $r-s=k-m$. For example, the only nonzero cohomology class of pseudoforms in $\mathbf{R}^{0|m}$, $c=[\exp(-(d\xi^1)^2-\ldots-(d\xi^m)^2)]$, has degree m, and for any $s<m/2$ there exists a 2s|2s-dimensional cycle Γ in $\mathbf{R}^{0|m}$ such that $\int_\Gamma c\neq 0$ (cf. [5]).

We collect the information on $\lambda^{\cdot|s}$ into a theorem (cf. Theorem 2 of [5]).

THEOREM 4. 1. The transform $\lambda^{\cdot|s}$ is functorial: for any morphism $f: N_1\to N_2$ one has $f^*\circ\lambda^{\cdot|s}=\lambda^{\cdot|s}\circ f^*$.

2. The Baranov–Shvarts transform has degree $-(m-s)$:

$$\lambda^{k-(m-s)|s}:\Omega^k(N\Pi)\to\Omega^{k-(m-s)|s}(N\Pi).$$

3. The transform $\lambda^{\cdot|s}$ commutes with the differential: $\lambda^{\cdot|s}\circ d=d\circ\lambda^{\cdot|s}$.

4. For any "singular 0|s-dimensional bundle $f: P\Pi\to N\Pi$ one has the equation $\int\limits_{(p\Pi,f)}\omega=\int\limits_{(p\Pi,f)}\lambda^{\cdot|s}\omega$.

8. Integral Transform $k_s:\Omega^{\cdot|s}(N\Pi)\to\Omega(G_s)$ of $\cdot|s$-Forms to Forms on the Grassmanian

The direct image of r|s-forms, defined on a 0|s-bundle with coordinates (y, ζ) is given by the formula

$$p_*:L\begin{pmatrix}y&\zeta\\K&M\\\Lambda&N\end{pmatrix}\mapsto\int L\begin{pmatrix}y&\zeta\\K&0\\0&E\end{pmatrix}D(\zeta)=(p_*L)(y,K),$$

where $(p_*L)(y,K)$ is a differential form on the base of the bundle.

We consider the double bundle of Sec. 6 and we define the integral transform $k_s:\Omega^{\cdot|s}\times(N\Pi)\to\Omega(G_s)$, which corresponds to this double bundle.† In coordinates it looks like this:

$$(k_sL)\begin{pmatrix}x&N\\K&R\end{pmatrix}=\int\limits_{\mathbf{R}^{0|s}}D(\tau)\cdot L\begin{pmatrix}x&\tau N\\K&\tau R\\0&N\end{pmatrix}, \tag{5}$$

†The special case of k_s for $N\Pi=\mathbf{R}^{0|m}$ was used in the study of the analog of the cohomology of a point in [6].

where $K = (K_i^a)$, $R = (R_{i\alpha}^{\mu})$, $i = 1, \ldots, r$, $\tau N = (\tau^{\alpha} N_{\alpha}^{\mu})$, $\tau R = (\tau^{\alpha} R_{i\alpha}^{\mu})$; it is convenient here for us to represent an r-form on the Grassmanian as a function on the vectors (K_i, R_i), where K_i corresponds to $\partial x/\partial t^i$ and R_i to $\partial N/\partial t^i$. One can verify that it is well-defined: independent of the choice of coordinates in the bundle N, $k_s L$ will be GL(s)-invariant and horizontal, i.e., a real r-form on G_s.

THEOREM 5. 1. The transform k_s is functorial: for any morphism $f\colon \bar{N}_1 \to N_2$ one has $k_s \circ f^* = G_s(f)^* \circ k_s$.

2. The degree of k_s is equal to zero: $k_s\colon \Omega^{r|s}(N\Pi) \to \Omega^r(G_s)$.

3. The transform k_s commutes with the differential: $k_s \circ d = d \circ k_s$.

4. For any "singular 0|s-dimensional bundle" $f\colon P\Pi \to N\Pi$ with base Q, one has $\int_{(P\Pi, f)} L = \int_{(Q, G_s(f))} k_s L$.

9. Fourier Transform F between Pseudodifferential Forms of $\Omega(N\Pi)$ and Forms on the Conormal Bundle $\Omega(N')$

The general definition of the Fourier transform extends naturally to Lie supergroups. For an Abelian supergroup $\mathbf{R}^{n|m}$ the dual group will be the conjugate space $(\mathbf{R}^{n|m})'$, and the Fourier transform is given by the formula

$$(Ff)(k, \lambda) = \int_{\mathbf{R}^{n|m}} D(x, \xi)\, e^{-ix^a k_a - i\xi^{\mu}\lambda_{\mu}} f(x, \xi),$$

where (x, ξ) are coordinates in $\mathbf{R}^{n|m}$, (k, λ) in $(\mathbf{R}^{n|m})'$, and measures on $\mathbf{R}^{n|m}$ are identified with functions with the help of the coordinate volume element $D(x, \xi)$. The properties of the Fourier transform on $\mathbf{R}^{n|m}$ are determined by the properties of the usual Fourier integral and the properties of the transform F in the purely odd case $\mathbf{R}^{0|m}$.

Proposition 2. On the purely odd supergroup $\mathbf{R}^{0|m}$ the Fourier transform F has the following properties:

1. The transform F is invertible, and the inverse transform has the form

$$(F^{-1}g)(\xi) = i^m (-1)^{\frac{m(m+1)}{2}} \int_{(\mathbf{R}^{0|m})'} D(\lambda)\, e^{i\xi^{\mu}\lambda_{\mu}} g(\lambda).$$

2. One has the formulas

$$F\left(P\left(\frac{\partial}{\partial \xi}\right) f(\xi)\right) = P(i\lambda)(Ff)(\lambda), \qquad F(P(\xi) f(\xi)) = P\left(-i\frac{\partial}{\partial \lambda}\right)(Ff)(\lambda),$$
$$F^{-1}\left(P\left(\frac{\partial}{\partial \lambda}\right) f(\lambda)\right) = P(i\xi)(F^{-1}f)(\xi), \quad F^{-1}(P(\lambda) f(\lambda)) = P\left(-i\frac{\partial}{\partial \xi}\right)(F^{-1}f)(\xi).$$

3. Up to factors, the Fourier transform F coincides with the Hodge operators $* : \bigwedge \times ((\mathbf{R}^m)') \to \bigwedge(\mathbf{R}^m)$, if one identifies functions on $\mathbf{R}^{0|m}$ with elements of the exterior algebra $\bigwedge((\mathbf{R}^m)')$, and the volume element $D(\xi)$ with the basis $\lambda_1, \ldots, \lambda_m$ in $\bigwedge^m(\mathbf{R}^m)$.

4. The transform F can be calculated by the formula

$$(Ff)(\lambda) = i^m (-1)^{\frac{m(m+1)}{2}} f\left(-i\frac{\partial}{\partial \lambda}\right) \lambda_1 \ldots \lambda_m.$$

The hodge operator can be normalized so that it coincides exactly with the Fourier transform. Then the familiar properties of the operator * will follow from the formulas in point 2.

It is known that the Fourier transform carries multiplication into convolution and conversely. For the group $\mathbf{R}^{0|m}$ the operation of convolution has degree $-m$ in the natural grading by powers of ξ:

$$\bigwedge^k((\mathbf{R}^m)') \otimes \bigwedge^l((\mathbf{R}^m)') \to \bigwedge^{k+l-m}((\mathbf{R}^m)'),$$

it is associative and commutative in the graded sense.

We pass to the bundles N, N', NΠ, and N'Π. For the conormal bundle N' each fiber has the structure of a vector space with coordinates p_{μ}. We consider the space $\mathbf{R}^{m|m}$ with coordinates p_{μ}, dp_{μ}. It is clear from the change of coordinate formulas in N' that $\mathbf{R}^{m|m}$ is endowed with a canonical structure of supermanifold with distinguished volume element D(p, dp). We endow the conjugate space $(\mathbf{R}^{m|m})'$ with coordinates which we shall denote by $d\xi^\mu$ (even) and ξ^μ (odd). It is clear from the change of coordinate formulas that ξ^μ can be identified with the coordinates in the fiber of NΠ and $d\xi^\mu$ with their differentials!

Conclusion: the fiberwise Fourier transform is well-defined and turns forms on the conormal bundle into pseudoforms on the normal bundle with odd fiber. Using the Leibniz formula we get

$$(F\omega)(x, \xi, dx, d\xi) = \int_{\mathbf{R}^{m|m}} D(p, dp)\, e^{-id(\xi^\mu p_\mu)}\omega(x, p, dx, dp). \tag{6}$$

THEOREM 6. 1. The Fourier transform F is functorial: for any morphism $f: N_1 \to N_2$ one has $f^* \circ F = F \circ f^*$.

2. The degree of the Fourier transform is equal to zero: $F: \Omega^k(N') \to \Omega^k(N\Pi)$.

3. The transform F commutes with the differential: $F \circ d = d \circ F$.

4. The Fourier transform F is an interlacing operator for the integral transforms $I_s^\perp$ and J_s: $J_s \circ F = I_s^\perp$. Thus one has connected integration of pseudodifferential forms and forms on the conormal bundle.

Proof. Functoriality. We verify the commutativity of the diagram: let $\omega \in \Omega(N')$; then

$$(Ff^*\omega)(x_1, \xi_1, dx_1, d\xi_1) = \int_{\mathbf{R}^{m_1|m_1}} D(p_1, dp_1) e^{-id(\xi^{\mu_1}p_{\mu_1})} f^*\omega(x_1, p_1, dx_1, dp_1)$$

$$= \int_{\mathbf{R}^{m_1|m_1}} D(p_1, dp_1)\, e^{-id(\xi^{\mu_1}p_{\mu_1})} \int_{\mathbf{R}^{m_2|m_2}} D(p_2, dp_2)\, \delta(p_1 - U(x_1)\, p_2)$$

$$\times \delta(d(p_1 - U(x_1)\, p_2))\, \omega(x_2(x_1),\ p_2, dx_1 \frac{\partial x_2}{\partial x_1}, dp_2) =$$

$$= \int_{\mathbf{R}^{m_2|m_2}} D(p_2, dp_2) e^{-id(\xi^{\mu_1}U_{\mu_1}^{\mu_2}(x_1)p_{\mu_2})}\, \omega\left(x_2(x_1), p_2, dx_1 \frac{\partial x_2}{\partial x_1}, dp_2\right)$$

$$= (F\omega)(x_2(x_1), \xi_1 U(x_1), dx_1 \frac{\partial x_2}{\partial x_1}, d\xi_1 U(x_1) - \xi_1\, dx_1 \frac{\partial U}{\partial x_1}) = (f^*F\omega)(x_1, \xi_1, dx_1, d\xi_1).$$

One can consider the transformation F as the composition of the ordinary Fourier integral with respect to p and the "odd Fourier integral" with respect to dp; since the latter turns monomials with $\# dp = k$ into monomials with $\# \xi = m - k$, it is obvious that the degree of F is equal to 0 – recall the definition of the grading of pseudoforms.

We verify the commutativity with the differential.

$$F\, d\omega = F\left(dx^a \frac{\partial\omega}{\partial x^a} + dp_\mu \frac{\partial\omega}{\partial p_\mu}\right) = dx^a \frac{\partial}{\partial x^a}(F\omega) + \left(-i\frac{\partial}{\partial\xi^\mu}\right)(id\xi^\mu)F\omega = dx^a \frac{\partial}{\partial x^a}(F\omega) + d\xi^\mu \frac{\partial}{\partial\xi^\mu}(F\omega) = dF\omega.$$

Finally, we prove assertion 4. Let $\omega \in \Omega(N')$; then

$$(J_sF\omega)(x, N, dx, dN) = \int_{\mathbf{R}^{s|s}} D(\tau, d\tau) \int_{\mathbf{R}^{m|m}} D(p, dp)\, e^{-id(\tau N^\mu p_\mu)}\omega(x, p, dx, dp)$$

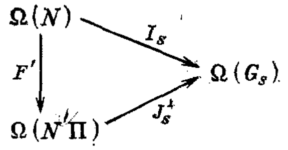
$$= \int_{\mathbf{R}^{s|s}} D(\tau, d\tau) \int_{\mathbf{R}^{m|m}} D(p, dp) e^{-i(d\tau N^\mu p_\mu - \tau d(N^\mu p_\mu))}\omega(x, p, dx, dp)$$

$$= (-1)^{ms} \int_{\mathbf{R}^{m|m}} D(p, dp) \int_{\mathbf{R}^{s|s}} D(\tau, d\tau)\, e^{-id\tau(N^\mu p_\mu)} e^{i\tau d(N^\mu p_\mu)}\omega(x, p, dx, dp)$$

$$= (-1)^{ms}(2\pi)^s i^s (-1)^{\frac{s(s+1)}{2}} \int_{\mathbf{R}^{m|m}} D(p, dp)\, \delta(Np)\, \delta(d(Np))\, \omega(x, p, dx, dp)$$

$$= (-1)^{ms}(2\pi)^s i^s (-1)^{\frac{s(s+1)}{2}} (I_s^\perp\omega)(x, N, dx, dN).$$

The factor which arises depends on the normalization of the Fourier transform. An analogous Fourier transform F' exists between forms on N and N'Π. For the transform F' all the assertions analogous to the assertions of Theorem 6 are valid. It interlaces the integral transforms I_s and $J_s^\perp$: the diagram



is commutative. Thus we have connected integration of differential forms and pseudodifferential forms on the conormal bundle with odd fiber.

The fact that the Fourier transforms connect forms on conjugate bundles with opposite parity in the fiber is remarkable! (It is easy to see from the change of coordinate formulas that there exist in all two bilinear combinations of coordinates and differentials which lead to an invariant even kernel.)

In the algebras $\Omega(N')$ and $\Omega(N\Pi)$ there are two associative commutative operations: ordinary multiplication [which has degree 0 for $\Omega(N')$ and $-m$ for $\Omega(N\Pi)$] and multiplication on the base in combination with fiberwise convolution [of degree $-m$ for $\Omega(N')$ and 0 for $\Omega(N\Pi)$]. The Fourier transform is an isomorphism of algebras with respect to the multiplications of different names. If one considers Ω as algebra of functions on certain supermanifolds, it is clear that the Fourier transform is not generated by any map of these supermanifolds: it is necessary that multiplication go into multiplication.

It turns out that the connection introduced in the bundle N now lets us construct an isomorphism of supermanifolds with algebras of functions $\Omega(N\Pi)$ and $\Omega(N)$.

Definition 2 (cf. [3]). Let $M^{n|m}$ be a supermanifold with coordinates x, ξ. Then $\hat{M}^{n|m}$ is a supermanifold of dimension $n + m|n + m$ with coordinates x, ξ, dx, $d\xi$.

The "roof" $\hat{}$ is the covariant functor which is isomorphic to the functor $T\Pi$ (tangent bundle with reversed parity, cf. [8]). We have already used it implicitly in the construction of the fiberwise Fourier transform.

We define a connection in the bundle N with the help of the local $\mathfrak{gl}(s)$-valued 1-forms $A = A(x, dx) = (dx^a A^{\mu}_{\nu a})$. Under change of coordinates

$$A = T^{-1}A'T - T^{-1}dT.$$

We set

$$\nabla v^{\mu} := dv^{\mu} + v^{\nu} dx^a A^{\mu}_{\nu a},$$

$$\nabla \xi^{\mu} := d\xi^{\mu} - \xi^{\nu} dx^a A^{\mu}_{\nu a}.$$

We note that ∇v is odd and $\nabla\xi$ even. One can consider the variables $(x, v, dx, \nabla v)$ and $(x, \xi, dx, \nabla\xi)$ as new local coordinates in $\hat{N}$ and $\hat{N\Pi}$. The supermanifolds $\hat{N}$ and $\hat{N\Pi}$ have identical dimension.

THEOREM 7. The equation of coordinates $v^{\mu} = \nabla\xi^{\mu}$, $\xi^{\mu} = \nabla v^{\mu}$ defines an isomorphism of supermanifolds $\hat{N}$ and $\hat{N\Pi}$. This isomorphism identifies the volume elements

$$D(x, \xi, dx, d\xi) \text{ and } D(x, v, dx, dv).$$

The degree of the induced isomorphism of algebras $\Omega(N)$ and $\Omega(N\Pi)$ is equal to $\pm m$

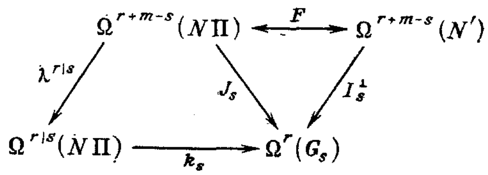
Unfortunately, the isomorphism described between pseudodifferential forms and forms on the normal bundle does not preserve the differential. Thus, it turns out to be much less useful than the Fourier transform.

10. Basic Diagram

We gather the results found together. We consider a singular bundle $f: P\Pi \to N\Pi$ with odd $0|s$-dimensional fiber and r-dimensional base Q. Which objects can be integrated over it? They are $r|s$-forms and pseudoforms (of degree $r + m - s$). Moreover, over the corresponding singular manifold in the Grassmanian one can integrate differential r-forms on G_s. We showed above that differential forms on the conormal bundle N' are a distinctive object of integration

The objects of integration described are connected with one another.

THEOREM 8. The diagram



which consists of functorial transforms which commute with the differential and integral, is commutative.

Thus, integration of pseudodifferential forms is the same as integration of forms on the conormal bundle according to the following rule: the lift to P' and the integral over the zero section $Q \to P'$. The isomorphism here is given by the Fourier transform F. The Baranov–Shvarts transform λ reduces integration of pseudoforms to integration of the corresponding r|s-forms. Finally, the transformations k_s, J_s, and $I_s^{\perp}$ let us consider, instead of integrals of r|s-forms, pseudodifferential forms, and forms on N', integrals of differential forms on the Grassmanian G_s over the Gaussian singular manifold $[Q, G_s(f)]$.

One formulates the dual problem naturally. We note that the subcomplexes of forms on the Grassmanian G_s, which arise as images of the integral transforms are essentially different in the direct and dual diagram.

* * *

In comparison with ordinary spaces, vector bundles have a rich supplementary structure, thanks to which several different integration theories arise for them. It turns out that all these integration theories split into two classes, which one can conditionally call Cartan–de Rham integration and Berezin integration; both "even" and "odd" theories belong to each class. A close connection is established here between the objects of integration belonging to one class, which is preserved under maps and which preserves the differential and integral.

LITERATURE CITED

1. M. A. Baranov and A. S. Shvarts, "Cohomology of supermanifolds," Funkts. Anal. Prilozhen., 18, No. 3, 69-70 (1984).
2. F. A. Berezin, Introduction to Algebra and Analysis with Anticommuting Variables [in Russian], Moscow State Univ. (1983).
3. I. N. Bernshtein and D. A. Leites, "How to integrate differential forms on supermanifolds," Funkts. Anal. Prilozhen., 11, No. 3, 70-71 (1977).
4. F. F. Voronov and A. V. Zorich, "Complexes of forms on a supermanifold," Funkts. Anal. Prilozhen., 20, No. 2, 58-59 (1986).
5. F. F. Voronov and A. V. Zorich, "Integral transforms of pseudodifferential forms," Usp. Mat. Nauk, 41, No. 6, 167-168 (1986).
6. F. F. Voronov and A. V. Zorich, "Cohomology of supermanifolds and integral geometry," Dokl. Akad. Nauk SSSR, 298, No. 3, 528-533 (1988).
7. F. F. Voronov and A. V. Zorich, "Bordism theory and homotopy properties of supermanifolds," Funkts. Anal. Prilozhen., 21, No. 3, 77-78 (1987).
8. Yu. I. Manin, Gauge Fields and Complex Geometry [in Russian], Nauka, Moscow (1984).