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LYAPUNOV SPECTRUM OF SQUARE-TILED CYCLIC COVERS

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ABSTRACT. A cyclic cover over $\mathbb{C}P^1$ branched at four points inherits a natural flat structure from the "pillow" flat structure on the basic sphere. We give an explicit formula for all individual Lyapunov exponents of the Hodge bundle over the corresponding arithmetic Teichmüller curve. The key technical element is evaluation of degrees of line subbundles of the Hodge bundle, corresponding to eigenspaces of the induced action of deck transformations.

CONTENTS

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1. INTRODUCTION

The current paper is an application of a long-standing project of the authors on Lyapunov exponents for the Hodge bundle of the Teichmüller geodesic flow [6]. This application was inspired by the papers [8] and [10], where G. Forni and C. Matheus observed that special cyclic covers of $\mathbb{C}P^1$ with four branch points (called square-tiled cyclic covers) induce arithmetic Teichmüller curves with peculiar properties.

The geometry and the topology of cyclic covers is well explored; some ideas used in our paper are close to ones of I. Bouw [1, 2], I. Bouw and M. Möller [3], C. McMullen [16], and of G. Forni, C. Matheus, A. Zorich [11], who studied similar cyclic covers in a similar context (see also the paper of M. Schmoll [19] where certain class of cyclic covers appear under the name "d-symmetric differentials").

In the present paper we give a simple explicit expression for *all* individual Lyapunov exponents for the Hodge bundle over the ergodic components of the Teichmüller geodesic flow associated with the aforementioned cyclic covers of $\mathbb{C}P^1$. We show that these Lyapunov exponents have a purely geometric interpretation: they are expressed in terms of degrees of line bundles contained in the eigenspace decomposition of the Hodge bundle with respect to the induced action of deck transformations. It was observed by D. Chen [4] that the sum of these Lyapunov exponents is closely related to the slope of the Teichmüller curves parameterizing square-tiled cyclic covers.

While writing this paper we did not realize how close our computation of Lyapunov exponents is to the one performed by I. Bouw and M. Möller in [3] for a seemingly different family of Teichmüller curves. The ongoing paper of A. Wright [22] clarifies the similarity of the two approaches.

1.1. **Hodge norm.** A complex structure on the Riemann surface X underlying a flat surface S of genus g determines a complex g-dimensional space of holomorphic 1-forms $\Omega(X)$ on X, and the Hodge decomposition

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \simeq \Omega(X) \oplus \bar{\Omega}(X).$$

The intersection form

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_C \omega_1 \wedge \overline{\omega_2} \tag{1.1}$$

is positive-definite on $H^{1,0}(X)$ and negative-definite on $H^{0,1}(X)$.

The projections $H^{1,0}(X) \to H^1(X,\mathbb{R})$ given by $[\omega] \mapsto [\mathrm{Re}(\omega)]$ and $[\omega] \mapsto [\mathrm{Im}(\omega)]$ are isomorphisms of vector spaces over $\mathbb{R}$. The Hodge operator $*\colon H^1(X,\mathbb{R}) \to$

$H^1(X,\mathbb{R})$ acts as the inverse of the first isomorphism composed with the second one. In other words, given $v \in H^1(X,\mathbb{R})$, there exists a unique holomorphic form $\omega(v)$ such that $v = [\mathrm{Re}(\omega(v))]$; the dual $*v$ is defined as $[\mathrm{Im}(\omega)]$.

Define the *Hodge norm* of $v \in H^1(X,\mathbb{R})$ as

$$\|v\|^2 = \langle \omega(v), \omega(v)\rangle.$$

1.2. **Gauss–Manin connection.** Passing from an individual Riemann surface to the moduli stack $\mathcal{M}_g$ of Riemann surfaces, we get vector bundles $H^1_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ and $H^1_{\mathbb{R}}$ over $\mathcal{M}_g$ with fibers $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ and $H^1(X,\mathbb{R})$, respectively, over $X \in \mathcal{M}_g$.

Using the integer lattices $H^1(X,\mathbb{Z} \oplus i\mathbb{Z})$ and $H^1(X,\mathbb{Z})$ in the fibers of these vector bundles, we can canonically identify fibers over nearby Riemann surfaces. This identification is called the *Gauss–Manin* connection. The Hodge norm *is not* preserved by the Gauss–Manin connection and the splitting $H^1_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ *is not* covariantly constant with respect to this connection.

The complex vector bundle $H^{1,0}$ carries a natural holomorphic structure, and is called the *Hodge* bundle. The underlying real smooth vector bundle is canonically isomorphic to the cohomological bundle $H^1_{\mathbb{R}}$ endowed with flat Gauss–Manin connection. Slightly abusing the language, we shall use the same name "Hodge bundle" for the flat bundle $H^1_{\mathbb{R}}$. Also, we shall use the same terminology for the pullback of $H^{1,0} \simeq H^1_{\mathbb{R}}$ under a holomorphic map from a complex algebraic curve (which will be a cover of the Teichmüller curve) to $\mathcal{M}_g$.

1.3. **Lyapunov exponents.** Informally, the Lyapunov exponents of a vector bundle endowed with a connection can be viewed as logarithms of mean eigenvalues of monodromy of the vector bundle along a flow on the base.

In the case of the Hodge bundle considered as the cohomological bundle, we take a fiber of $H^1_{\mathbb{R}}$ and pull it along a Teichmüller geodesic on the moduli space. We wait till the geodesic winds a lot and comes close to the initial point and then compute the resulting monodromy matrix $A(t)$. Finally, we compute logarithms of eigenvalues of $A^T A$, and normalize them by twice the length t of the geodesic. By the Oseledets Multiplicative Ergodic Theorem, for almost all choices of initial data (starting point, starting direction) the resulting $2g$ real numbers converge as $t \to \infty$, to limits which do not depend on the initial data. These limits $\lambda_1 \geq \cdots \geq \lambda_{2g}$ are called the *Lyapunov exponents* of the Hodge bundle along the Teichmüller flow.

The matrix $A(t)$ preserves the intersection form on cohomology, so it is symplectic. This implies that Lyapunov spectrum of the Hodge bundle is symmetric with respect to a sign interchange, $\lambda_j = -\lambda_{2g-j+1}$. Moreover, if the base of the bundle is located in the moduli space $\mathcal{H}_g$ of *holomorphic* 1*-forms*, from elementary geometric arguments it follows that one always has $\lambda_1 = 1$. Thus, the Lyapunov spectrum is defined by the remaining nonnegative Lyapunov exponents

$$\lambda_2 \geq \cdots \geq \lambda_g.$$

CONVENTION 1.1. The collection of the leading g exponents $\{\lambda_1,\dots,\lambda_g\}$ will be called the *nonnegative part* of the Lyapunov spectrum of the Hodge bundle $H^1_{\mathbb{R}}$. We warn the reader that when some of the exponents are null, the "nonnegative part" contains only half of *all* zero exponents.

1.4. **Cyclic covers.** Consider an integer $N \ge 1$ and a quadruple $(a_1,\dots,a_4)$ of integers satisfying the following conditions:

$$0 < a_i \le N, \qquad \gcd(N, a_1,\dots, a_4) = 1, \qquad \sum_{i=1}^{4} a_i \equiv 0 \pmod N. \tag{1.2}$$

Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four distinct points. By $M_N(a_1, a_2, a_3, a_4)$, we denote the closed connected nonsingular Riemann surface obtained from the one defined by the equation

$$w^N = (z-z_1)^{a_1}(z-z_2)^{a_2}(z-z_3)^{a_3}(z-z_4)^{a_4} \tag{1.3}$$

by normalization. By construction, $M_N(a_1, a_2, a_3, a_4)$ is a ramified cover over the Riemann sphere $\mathbb{C}\mathrm{P}^1$ branched over the points $z_1,\dots,z_4$. The group of deck transformations of this cover is the cyclic group $\mathbb{Z}/N\mathbb{Z}$ with a generator $T: M \to M$ given by

$$T(z, w) = (z, \zeta w), \tag{1.4}$$

where ζ is a primitive Nth root of unity (*e.g.*, $\zeta = \zeta_N = \exp(2\pi i/N)$).

One can also consider the case when one of the ramification points z_i is located at infinity. In this case, one just skips the factor $(z-z_i)^{a_i}$ in (1.3). Notice that the integer parameter a_i is uniquely determined by three remaining numbers $(a_j)_{j\ne i}$ by relations (1.2).

Throughout this paper a *cyclic cover* will be a Riemann surface $M_N(a_1,\dots,a_4)$ with parameters $N, a_1,\dots,a_4$ satisfying relations (1.2).

1.5. **Square-tiled surface associated to a cyclic cover.** Any meromorphic quadratic differential $q(z)(dz)^2$ with at most simple poles on a Riemann surface defines a flat metric $g(z) = |q(z)|$ with conical singularities at zeroes and poles of q. Consider a meromorphic quadratic differential

$$q_0 = \frac{c_0 (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}, \qquad c_0 \in \mathbb{C} \smallsetminus \{0\}, \tag{1.5}$$

on $\mathbb{C}\mathrm{P}^1$. It has simple poles at z_1, z_2, z_3, z_4 and no other zeroes or poles. The quadratic differential q_0 defines a flat metric on a sphere obtained by identifying two copies of an appropriate parallelogram by their boundary, see Figure 1.

For a convenient choice of parameters $z_1,\dots,z_4$ and c_0 the parallelogram becomes the unit square. Metrically, we get a square pillow with four corners corresponding to the four poles of q_0.

Now consider some cyclic cover $M_N(a_1, a_2, a_3, a_4)$ and the canonical projection $p: M_N(a_1, a_2, a_3, a_4) \to \mathbb{C}\mathrm{P}^1$. Consider an induced quadratic differential $q = p^* q_0$ on $M_N(a_1, a_2, a_3, a_4)$ and the corresponding flat metric. By construction, the resulting flat surface is tiled with unit squares. In other words, we get a *square-tiled surface*.

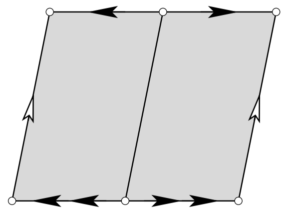


FIGURE 1. Flat sphere with four conical singularities having cone angles π

Throughout this paper we work only with such square-tiled (or, more generally, parallelogram-tiled) cyclic covers. We refer the reader to [11] for a description of the geometry, topology and combinatorics of square-tiled cyclic covers.

CONVENTION 1.2. It would be convenient to "give names" to the ramification points z_1, z_2, z_3, z_4. In other words, throughout this paper we assume that z_i, z_j are distinguishable even if $a_i = a_j$.

Varying the cross-ratio $(z_1 : z_2 : z_3 : z_4)$ (*e.g.,* keeping 3 of 4 points fixed and varying the fourth point) we obtain the moduli curve which we denote by

$$\mathcal{M}_{(a_i),N}.$$

As an abstract curve, it is isomorphic to $\mathcal{M}_{0,4} \simeq \mathbb{C}\mathrm{P}^1 - \{0,1,\infty\}$. Strictly speaking, it should be considered as a stack, because every curve $M_N(a_1, a_2, a_3, a_4)$ with named ramification points has an automorphism group canonically isomorphic to $\mathbb{Z}/N\mathbb{Z}$. The naive notion of an orbifold is not sufficient here because the moduli stack in our situation is isomorphic to the quotient of $\mathcal{M}_{0,4}$ by the *trivial* action of $\mathbb{Z}/N\mathbb{Z}$. In what follows we shall treat $\mathcal{M}_{(a_i),N}$ as a plain curve, in order to be elementary.

The curve $\mathcal{M}_{(a_i),N}$ maps onto the image of a Teichmüller disk in the moduli stack of Abelian differentials with certain multiplicities of zeroes. Forgetting the differential we obtain a Teichmüller geodesic in $\mathcal{M}_g$ where $g = g(N, a_1, \dots, a_4)$ is the genus of curve $M_N(a_1, a_2, a_3, a_4)$ (see Section 3.1 for an explicit formula for g). Hence $\mathcal{M}_{(a_i),N}$ is a Teichmüller curve (*e.g.,* in the sense of [3]). This is the main object of our study.

2. STATEMENT OF RESULTS

2.1. Splitting of the Hodge bundle.

Consider a cyclic cover

$$X = M_N(a_1, a_2, a_3, a_4)$$

over $\mathbb{C}\mathrm{P}^1$ defined by equation (1.3). Consider the canonical generator T of the group of deck transformations; let

$$T^* : H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C})$$

be the induced action in cohomology. Since $(T^*)^N = \mathrm{Id}$, the eigenvalues of T^* belong to a subset of $\{\zeta, \dots, \zeta^{N-1}\}$, where $\zeta = \exp(2\pi i/N)$. We excluded the root

$\zeta^0 = 1$ since any cohomology class invariant under deck transformations would be a pullback of a cohomology class on $\mathbb{C}\mathrm{P}^1$, and $H^1(\mathbb{C}\mathrm{P}^1) = 0$.

For $k = 1, \dots, N-1$, denote

$$V(k)(X) := \mathrm{Ker}(T^* - \zeta^k \,\mathrm{Id}) \subset H^1(X, \mathbb{C}).$$

The decomposition

$$H^1(X, \mathbb{C}) = \oplus V(k)(X)$$

is preserved by the Gauss–Manin connection, which implies that the vector bundle $H^1_{\mathbb{C}}$ over the Teichmüller curve splits into a sum of invariant subbundles $V(k)$.

Denote

$$V^{1,0}(k) := V(k) \cap H^{1,0} \qquad \text{and} \qquad V^{0,1}(k) := V(k) \cap H^{0,1}.$$

Since a generator T of the group of deck transformations respects the complex structure, it induces a linear map

$$T^* \colon H^{1,0}(X) \to H^{1,0}(X).$$

This map preserves the Hermitian form (1.1) on $H^{1,0}(X)$. This implies that T^* is a unitary operator on $H^{1,0}(X)$, and hence $H^{1,0}(X)$ admits a splitting into a direct sum of eigenspaces of T^*,

$$H^{1,0}(X) = \bigoplus_{k=1}^{N-1} V^{1,0}(k)(X). \tag{2.1}$$

The latter observation also implies that for any $k = 1, \dots, N-1$ one has $V(k) = V^{1,0}(k) \oplus V^{0,1}(k)$. The vector bundle $V^{1,0}(k)$ over the Teichmüller curve is a holomorphic subbundle of $H^1_{\mathbb{C}}$.

Denote

$$t_i(k) = \left\{ \frac{a_i}{N} k \right\}, \qquad k = 1, \dots, N-1, \tag{2.2}$$

where $\{x\}$ denotes the fractional part of x. Let

$$t(k) = t_1(k) + \cdots + t_4(k). \tag{2.3}$$

Since $(a_1 + \cdots + a_4)/N \in \{1,2,3\}$ and $\gcd(N, a_1, \dots, a_4) = 1$, we get $t(k) \in \{1,2,3\}$.

THEOREM 2.1 (I. Bouw [1, 2])**.** *For any $k = 1, \dots, N-1$, one has*

$$\dim_{\mathbb{C}} V^{1,0}(k) = \dim_{\mathbb{C}} V^{0,1}(N-k) = t(N-k) - 1, \tag{2.4}$$

$$\dim_{\mathbb{C}} V(k) = t(k) + t(N-k) - 2 \in \{0,1,2\}. \tag{2.5}$$

For the sake of completeness, we provide a proof of this theorem in Section 3.1.

It would be convenient to state the following elementary observation.

LEMMA 2.2. *The eigenspace $V(k)$ has complex dimension*

(i) *two if and only if $t_i(k) > 0$ for $i = 1,2,3,4$,*
(ii) *one if and only if there is exactly one $i \in \{1,2,3,4\}$ such that $t_i(k) = 0$,*
(iii) *zero if there are distinct indices $i, j \in \{1,2,3,4\}$ such that $t_i(k) = t_j(k) = 0$.*

Proof. By formula (2.5) in Theorem 2.1, one has

$$\begin{aligned}\dim_{\mathbb{C}} V(k) + 2 &= t(k) + t(N-k)\\ &= \left(\left\{\frac{a_1}{N}k\right\} + \left\{\frac{a_1}{N}(N-k)\right\}\right) + \cdots + \left(\left\{\frac{a_4}{N}k\right\} + \left\{\frac{a_4}{N}(N-k)\right\}\right),\end{aligned}$$

see (2.2) and (2.3). The statement of the lemma now follows from the following elementary remark. If $x+y$ is integer, then

$$\{x\} + \{y\} = \begin{cases} 1 & \text{when } \{x\} > 0,\\ 0 & \text{otherwise.}\end{cases} \qquad \square$$

The Teichmüller curve is not compact, but there is a canonical extension of holomorphic bundles $V^{1,0}(k)$ to orbifold vector bundles at the cusps (see the next section). Hence, we can speak about the (orbifold) degree of these bundles.

THEOREM 2.3. *For any k such that $t(N-k) = 2$, the orbifold degree of the line bundle $V^{1,0}(k)$ satisfies*

$$d(k) = \min\big(t_1(k), 1 - t_1(k), \dots, t_4(k), 1 - t_4(k)\big). \tag{2.6}$$

Moreover, if $t(N-k) = 2$ the following alternative holds: if $t(k) = 2$, then $d(k) > 0$; if $t(k) = 1$, then $d(k) = 0$.

REMARK 2.4. Note that $t(N-k) \in \{1,2,3\}$. Theorem 2.3 studies the case when $t(N-k) = 2$. Thus, there remain two complementary cases when $t(N-k) \neq 2$. Namely, when $t(N-k) = 1$ we get $\dim V^{1,0}(k) = 0$, and the bundle $V^{1,0}(k)$ is missing. When $t(N-k) = 3$, we get $\dim V^{1,0}(k) = 2$. We shall show that in this case the orbifold degree of $V^{1,0}(k)$ is equal to zero.

2.2. **On real and complex variations of polarized Hodge structures of weight 1.** Let $\mathcal{C}$ be a smooth possibly noncompact complex algebraic curve. We recall that a variation of *real* polarized Hodge structures of weight 1 on $\mathcal{C}$ is given by a real symplectic vector bundle $\mathcal{E}_{\mathbb{R}}$ with a flat connection ∇ preserving the symplectic form, such that every fiber of $\mathcal{E}$ carries a Hermitian structure compatible with the symplectic form, and such that the corresponding complex Lagrangian subbundle $\mathcal{E}^{1,0}$ of the complexification $\mathcal{E}_{\mathbb{C}} = \mathcal{E}_{\mathbb{R}} \otimes \mathbb{C}$ is *holomorphic*. The variation is said to be *tame* if all eigenvalues of the monodromy around cusps lie on the unit circle, and the subbundle $\mathcal{E}^{1,0}$ is meromorphic at cusps. For example, the Hodge bundle of any algebraic family of smooth compact curves over $\mathcal{C}$ (or an orthogonal direct summand of it) is a tame variation.

Similarly, a variation of *complex* polarized Hodge structures of weight 1 is given by a complex vector bundle $\mathcal{E}_{\mathbb{C}}$ of rank $r+s$ (where r and s are nonnegative integers) endowed with a flat connection ∇, by a covariantly constant pseudo-Hermitian form of signature (r,s), and by a holomorphic subbundle $\mathcal{E}^{1,0}$ of rank r, such that the restriction of the form to it is strictly positive. The condition of tameness is completely parallel to the real case.

Any real variation of rank $2r$ gives a complex one of signature (r,r) by the complexification. Conversely, one can associate with any complex variation

$(\mathscr{E}_{\mathbb{C}}, \nabla, \mathscr{E}^{1,0})$ of signature (r, s) a real variation of rank $2(r+s)$, whose underlying local system of real symplectic vector spaces is obtained from $\mathscr{E}_{\mathbb{C}}$ by forgetting the complex structure.

Let us assume that the variation of complex polarized Hodge structures of weight 1 has a unipotent monodromy around cusps. Then the bundle $\mathscr{E}^{1,0}$ admits a canonical extension $\overline{\mathscr{E}^{1,0}}$ to the natural compactification $\overline{\mathscr{C}}$. It can be described as follows: consider first an extension $\overline{\mathscr{E}_{\mathbb{C}}}$ of $\mathscr{E}_{\mathbb{C}}$ to $\overline{\mathscr{C}}$ as a *holomorphic* vector bundle in such a way that the connection ∇ will have only first order poles at cusps, and the residue operator at any cup is nilpotent (it is called the *Deligne extension*). Then the holomorphic subbundle $\mathscr{E}^{1,0} \subset \mathscr{E}_{\mathbb{C}}$ extends uniquely as a subbundle $\overline{\mathscr{E}^{1,0}} \subset \overline{\mathscr{E}_{\mathbb{C}}}$ to the cusps.

2.3. **Sum of Lyapunov exponents of an invariant subbundle.** Let $(\mathscr{E}_{\mathbb{R}}, \nabla, \mathscr{E}^{1,0})$ be a tame variation of polarized real Hodge structures of rank $2r$ on a curve $\mathscr{C}$ with *negative* Euler characteristic. For example, $\mathscr{C}$ could be an unramified cover of a general arithmetic Teichmüller curve, and $\mathscr{E}$ could be a subbundle of the Hodge bundle which is simultaneously invariant under the Hodge star operator and under the monodromy.

Using the canonical complete hyperbolic metric on $\mathscr{C}$ one can define the geodesic flow on $\mathscr{C}$ and the corresponding Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_{2r}$ for the flat bundle $(\mathscr{E}_{\mathbb{R}}, \nabla)$, satisfying the usual symmetry property $\lambda_{2r+1-i} = -\lambda_i$, $i = 1, \ldots, r$.

The holomorphic vector bundle $\mathscr{E}^{1,0}$ carries a Hermitian form, hence its top exterior power $\wedge^r(\mathscr{E}^{1,0})$ is a holomorphic line bundle also endowed with a Hermitian metric. Let us denote by α the curvature $(1,1)$-form on $\mathscr{C}$ corresponding to this metric, divided by $-(2\pi i)$. The form α represents the first Chern class of $\mathscr{E}^{1,0}$. Then we have the following general result.

THEOREM 2.5. *Under the above assumptions, the sum of the top r Lyapunov exponents of V with respect to the geodesic flow satisfies*

$$\lambda_1 + \cdots + \lambda_r = \frac{2\int_C \alpha}{2g_{\mathscr{C}} - 2 + c_{\mathscr{C}}}, \tag{2.7}$$

where we denote the genus of $\mathscr{C}$ by $g_{\mathscr{C}}$, and the number of hyperbolic cusps on $\mathscr{C}$ by $c_{\mathscr{C}}$.

This formula was formulated (in a slightly different form) first in [13], and then proved rigorously by G. Forni [7].

Note that a similar result holds also for *complex* tame variations of polarized Hodge structures. Namely, for a variation of signature (r, s), one has $r+s$ Lyapunov exponents with sum equal to 0:

$$\lambda_1 \geq \cdots \geq \lambda_r \geq 0 \geq \lambda_{r+1} \geq \cdots \geq \lambda_{r+s}.$$

The collection (with multiplicities) $\{\lambda_1, \ldots, \lambda_r\}$ will be called the *nonnegative part* of the Lyapunov spectrum. We claim that the sum of nonnegative exponents $\lambda_1 + \cdots + \lambda_r$ is again given by the formula (2.7).

The proof follows from the simple observation that one can pass from a complex variation to a real one by taking the underlying real local system. Both the sum of nonnegative exponents and the integral of the Chern form are multiplied by two under this procedure.

The denominator in equation (2.7) is equal to minus the Euler characteristic of $\mathcal{C}$, *i.e.*, to the area of $\mathcal{C}$ up to a universal factor 2π. The numerator also admits an algebro-geometric interpretation for variations of real Hodge structures arising as direct summands of Hodge bundles for algebraic families of curves. Namely, let us assume that the monodromy of $(\mathcal{E},\nabla)$ around any cusp is unipotent (this can be achieved by passing to a finite unramified cover of $\mathcal{C}$). Then one has the following identity (see, *e.g.*, Proposition 3.4 in [17]):

$$\int_{\mathcal{C}} \alpha = \deg \overline{\mathcal{E}^{1,0}}.$$

In general, without the assumption on unipotency, we obtain that the integral (in the numerator) is a rational number, which can be interpreted as an orbifold degree in the following way. Namely, consider an unramified Galois cover $\mathcal{C}' \to \mathcal{C}$ such that the pullback of $(\mathcal{E},\nabla)$ has a unipotent monodromy. Then the compactified curve $\overline{\mathcal{C}}$ is a quotient of $\overline{\mathcal{C}'}$ by a finite group action, and hence is endowed with a natural orbifold structure. Moreover, the holomorphic Hodge bundle on $\overline{\mathcal{C}'}$ will descend to an orbifold bundle on $\overline{\mathcal{C}}$. Then the integral of α over $\mathcal{C}$ is equal to the orbifold degree of this bundle.

The choice of the orbifold structure on $\overline{\mathcal{C}}$ is in a sense arbitrary, as we can choose the cover $\mathcal{C}' \to \mathcal{C}$ in different ways. The resulting orbifold degree does not depend on this choice. The corresponding algebro-geometric formula for the denominator given as an orbifold degree is due to I. Bouw and M. Möller [3]. In our concrete example of the Teichmüller curve $\mathcal{C} = \mathcal{M}_{(a_i),N} \simeq \mathbb{C}\mathrm{P}^1 \smallsetminus \{0,1,\infty\}$, an explicit convenient choice of the cover is the standard Fermat curve $\mathcal{C}' = F_N$ given by equation $x^N + y^N = 1$, $x, y \in \mathbb{C}^*$, with the projection map $(x,y) \mapsto x^N \in \mathbb{C} \smallsetminus \{0,1\}$.

2.4. **Lyapunov spectrum for cyclic covers.** Recall that we have a decomposition of the holomorphic Hodge bundle over $\mathcal{C} = \mathcal{M}_{(a_i),N}$ into a direct sum

$$H^{1,0} = \bigoplus_{1\le k\le N-1} V^{1,0}(k),$$

coming from the decomposition of *complex* variations of polarized Hodge structures. It induces a decomposition of the variation of *real* polarized Hodge structures

$$H^1_{\mathbb{R}} = \bigoplus_{1\le k\le N/2} W_{\mathbb{R}}(k)$$

in a way described below.

First consider the case where k is an integer such that $1 \le k \le N-1$, $k \ne N/2$. By $W(k) \subset H^1_{\mathbb{R}}$ denote the projection of the subbundle $V(k) \oplus V(N-k) \subset H^1_{\mathbb{C}}$ to the first summand in the canonical decomposition $H^1_{\mathbb{C}} = H^1_{\mathbb{R}} \oplus i H^1_{\mathbb{R}}$.

By definition $W(N-k) = W(k)$. Note that the roots of unity ζ^k and ζ^{N-k} are complex conjugate. Hence, the subspace $V(k) \oplus V(N-k)$ is invariant under complex conjugation, and, thus,

$$W_{\mathbb{C}}(k) = W_{\mathbb{C}}(N-k) = V(k) \oplus V(N-k).$$

Since the bundle $H^{1,0}$ decomposes into a direct sum (2.1) of eigenspaces, we conclude that

$$W_{\mathbb{C}}(k) = W^{1,0}(k) \oplus W^{0,1}(k),$$

where

$$W^{1,0}(k) = V^{1,0}(k) \oplus V^{1,0}(N-k) \quad \text{and} \quad W^{0,1}(k) = V^{0,1}(k) \oplus V^{0,1}(N-k).$$

The subbundle $W(k)$ of $H^1_{\mathbb{R}}$ is Hodge star-invariant.

Note that the subbundles $V(k), V(N-k) \subset H^1_{\mathbb{C}}$ of the Hodge bundle and the canonical decomposition $H^1_{\mathbb{C}} = H^1_{\mathbb{R}} \oplus i H^1_{\mathbb{R}}$ are covariantly constant with respect to the Gauss–Manin connection. Hence, the subbundle $W(k)$ is also covariantly constant with respect to the Gauss–Manin connection.

When N is even, denote by $W(N/2)$ the projection of the subbundle $V(N/2) \subset H^1_{\mathbb{C}}$ to the first summand in the canonical decomposition $H^1_{\mathbb{C}} = H^1_{\mathbb{R}} \oplus i H^1_{\mathbb{R}}$. Similar to the previous case,

$$W_{\mathbb{C}}(N/2) = V(N/2) = V^{1,0}(N/2) \oplus V^{0,1}(N/2),$$

$W(N/2) \subset H^1_{\mathbb{R}}$ is Hodge star-invariant and covariantly constant with respect to the Gauss–Manin connection.

The decomposition of $H^1_{\mathbb{C}}$ into a direct sum of the eigenspaces $V(k)$ implies a decomposition into a direct sum of variations of real polarized Hodge structures of weight 1

$$H^1_{\mathbb{R}} = \bigoplus_{1 \le k \le N/2} W(k).$$

Now everything is ready to formulate the main theorem. Recall that by definition (2.3) of $t(k)$ one has $t(k), t(N-k) \in \{1,2,3\}$. By formula (2.5) in Theorem 2.1, one has

$$\dim_{\mathbb{C}} V(k) + 2 = t(k) + t(N-k) \in \{2,3,4\}.$$

Thus, the theorem below describes all possible combinations of the values of $t(k)$ and $t(N-k)$.

THEOREM 2.6. *For any integer k such that $1 \le k < N/2$, the Lyapunov exponents of the invariant subbundle $W(k)$ of the Hodge bundle $H^1_{\mathbb{R}}$ over $\mathcal{C}$ with respect to the geodesic flow on $\mathcal{C}$ are described as follows.*

(i) *If $t(k) = 3$, then $t(N-k) = 1$ and*

$$\dim_{\mathbb{C}} V(k) = \dim_{\mathbb{C}} V^{0,1}(k) = 2.$$

If $t(N-k) = 3$, then $t(k) = 1$ and

$$\dim_{\mathbb{C}} V(k) = \dim_{\mathbb{C}} V^{1,0}(k) = 2.$$

In both cases, $\dim_{\mathbb{R}} W(k) = 4$ *and all four Lyapunov exponents of the vector bundle $W(k)$ are equal to zero.*

(ii) *If* $t(k)=2$ *and* $t(N-k)=1$*, then*

$$\dim_{\mathbb{C}} V(k)=\dim_{\mathbb{C}} V^{0,1}(k)=1.$$

If $t(N-k)=2$ *and* $t(k)=1$*, then*

$$\dim_{\mathbb{C}} V(k)=\dim_{\mathbb{C}} V^{1,0}(k)=1.$$

In both cases, $\dim_{\mathbb{R}} W(k)=2$ *and both Lyapunov exponents of the vector bundle* $W(k)$ *are equal to zero.*

(iii) *If* $t(N-k)=t(k)=2$*, then*

$$\dim_{\mathbb{C}} V^{1,0}(k)=\dim_{\mathbb{C}} V^{0,1}(k)=1,$$

and $\dim_{\mathbb{R}} W(k)=4$*. In this case, the Lyapunov spectrum of the vector bundle* $W(k)$ *is equal to* $\{2d(k), 2d(k), -2d(k), -2d(k)\}$*, where*

$$d(k)=\min\bigl(t_1(k),\,1-t_1(k),\dots,\,t_4(k),\,1-t_4(k)\bigr)>0$$

is the orbifold degree of the line bundle $V^{1,0}(k)$.

(iv) *If* $t(N-k)=t(k)=1$*, then* $V(k)$*,* $V(N-k)$*, and* $W(k)$ *vanish.*

(v) *Finally, if* N *is even and all* a_i *are odd, then* $\dim_{\mathbb{R}} W(N/2)=2$*. In this case, the Lyapunov spectrum of the vector bundle* $W(N/2)$ *is equal to* $\{1,-1\}$.

(vi) *If* N *is even, but at least one of* a_i *is also even, then* $W(N/2)$ *vanishes.*

Proof. The Lyapunov exponents of the real flat bundle $H^1_{\mathbb{R}}$ coincide with the Lyapunov exponents of its complexification $H^1_{\mathbb{C}}$ considered as a complex flat bundle (as eigenvalues and their norms for a real matrix coincide with those of its complexification). The flat bundle $H^1_{\mathbb{C}}$ is decomposed into a direct sum of flat subbundles $V(k)$, $k=1,\dots,N-1$, underlying complex variations of polarized Hodge structures. Hence, the nonnegative part of the Lyapunov spectrum for $H^1_{\mathbb{R}}$ is the sum over k of the nonnegative parts of the spectra of the individual summands $V(k)$. For any given k we have six possibilities for the signature $(r,s)=\bigl(t(N-k)-1,\,t(k)-1\bigr)$ of the variation $V(k)$:

$$(0,0),\quad (1,0),\quad (0,1),\quad (2,0),\quad (1,1),\quad (0,2), \tag{2.8}$$

according to Theorem 2.1. In all cases except the case of signature $(1,1)$, the corresponding local system is *unitary*. Hence, all Lyapunov exponents are zero. In the nontrivial case of signature $(1,1)$, the unique nonnegative Lyapunov exponent coincides obviously with the sum of nonnegative Lyapunov exponents and hence can be calculated by formula (2.7). The denominator in this formula is equal to 1 because our Teichmüller curve $\mathscr{C}=\mathscr{M}_{(a_i),N}$ is a sphere with three punctures and has Euler characteristics -1. The numerator is twice the orbifold degree of the determinant of the holomorphic bundle $V^{(1,0)}(k)$. The latter orbifold degree is the same as the orbifold degree of the *line* bundle $V^{(1,0)}(k)$ (because $V^{(1,0)}(k)$ has rank one), and is equal to $d(k)$ by Theorem 2.3. Hence, we conclude that the contribution of the summand $V(k)$ to the nonnegative part of the Lyapunov spectrum for $H^1_{\mathbb{R}}$ for cases listed in (2.8) is given by

$$\varnothing,\quad \{0\},\quad \varnothing,\quad \{0,0\},\quad \{2d(k)\},\quad \varnothing.$$

Finally, we can find the nonnegative Lyapunov spectrum of the real variations $W(k)$, $1 \le k \le N/2$, using the fact that $W(k) \otimes \mathbb{C} = V(k) \oplus V(N-k)$ for $k < N/2$ and $W(N/2) \otimes \mathbb{C} = V(N/2)$ for even N. Also, notice that in the nontrivial case of signature $(1,1)$, one has $d(k) = d(N-k) > 0$. The whole Lyapunov spectrum for $W(k)$ is obtained from the nonnegative part by adding a copy reflected at zero. □

To illustrate Theorem 2.6, we present in Appendix D a table of quantities discussed above in a particular case of $M_{30}(3,5,9,13)$.

COROLLARY 2.7. *The nonnegative part* $\{\lambda_1,\dots,\lambda_g\}$ *of the Lyapunov spectrum of the Hodge bundle* $H^1_{\mathbb{R}}$ *over an arithmetic Teichmüller curve corresponding to a square-tiled cyclic cover* (1.3) *can be obtained by the following algorithm. Start with the empty set* $\Lambda\mathrm{Spec} = \emptyset$. *For every* $k \in \{1,\dots,N-1\}$, *compute* $t(k)$ *and proceed as follows:*

- *If* $t(k) = 3$, *then add a pair of zeroes to* $\Lambda\mathrm{Spec}$.
- *If* $t(k) = 2$, *then add a number* $2d(k)$ *to* $\Lambda\mathrm{Spec}$.
- *If* $t(k) = 1$, *then do not change* $\Lambda\mathrm{Spec}$.

The resulting unordered set $\Lambda\mathrm{Spec}$ *coincides with the unordered set* $\{\lambda_1,\dots,\lambda_g\}$.

Here the "nonnegative part" of the Lyapunov spectrum is understood in the sense of Convention 1.1.

The proof of the corollary below is absolutely elementary, so we omit it.

COROLLARY 2.8. *When* N *is even and all* a_i *are odd, the top Lyapunov exponent* $\lambda_1 = 1$ *is simple,* $1 = \lambda_1 > \lambda_2$. *All other strictly positive Lyapunov exponents of the Hodge bundle* $H^1_{\mathbb{R}}$ *over an arithmetic Teichmüller curve corresponding to a square-tiled cyclic cover* (1.3) *are strictly less than* 1 *and have even multiplicity.*

3. HODGE STRUCTURE OF A CYCLIC COVER

In Section 3.1, we construct an explicit basis of holomorphic forms on any given cyclic cover $M_N(a_1, a_2, a_3, a_4)$. All forms from this basis are eigenforms under the induced action of the group of deck transformations, which gives the description of the dimensions of $V^{1,0}(k)$ and $V^{0,1}(k)$. An analogous calculation was already performed by I. Bouw and M. Möller [2, 3], and by C. McMullen [16], so we present it here mostly for the sake of completeness.

In Section 3.4, we compute the degree of $V^{1,0}(k)$ for those k for which $V^{1,0}(k)$ is a line bundle.

3.1. **Basis of holomorphic forms.**

The contents of this section can be found in [1] and [15], as well as in the recent paper [5] citing the first two references as a source. However, to keep the current paper self-contained we present the complete proof of the lemma below.

Recall that $t(k)$ was introduced in equation (2.3), and that $\{x\}$ and $[x]$ denote fractional part and integer part of x, respectively.

LEMMA 3.1. *Consider the meromorphic form*

$$\omega = (z-z_1)^{b_1}\cdots(z-z_4)^{b_4}\frac{dz}{w^k} \tag{3.1}$$

on a cyclic cover $M_N(a_1,a_2,a_3,a_4)$. Fix k and let the integer parameters $b_1,\dots,b_4$ vary.

(i) *When $t(k)=1$, this form is not holomorphic for any parameters b_i.*

(ii) *When $t(k)=2$, the form ω_k as in (3.1) is holomorphic for $b_i(k)=\left[\frac{a_i}{N}k\right]$ and is not holomorphic for other values of b_i.*

(iii) *When $t(k)=3$, there is a two-dimensional family of holomorphic forms spanned by the forms*

$$\omega_{k,1} := (z-z_1)^{b_1}\cdots(z-z_4)^{b_4}\frac{dz}{w^k},$$
$$\omega_{k,2} := z(z-z_1)^{b_1}\cdots(z-z_4)^{b_4}\frac{dz}{w^k},$$

where $b_i(k)=\left[\frac{a_i}{N}k\right]$. Any holomorphic form ω as in (3.1) belongs to this family.

Proof. To prove the lemma, we have to study the behavior of the meromorphic form

$$\omega = (z-z_1)^{b_1}\cdots(z-z_4)^{b_4}\frac{dz}{w^k}$$

in a neighborhood of $z=z_1,\dots,z_4$ and in a neighborhood of $z=\infty$.

Let $\ell=\mathrm{lcm}(N,a_1)$. Consider a coordinate u in a neighborhood of a point z_1 such that

$$(z-z_1)\sim u^{\ell/a_1}.$$

Then, in a neighborhood of z_1 we have

$$w\sim(z-z_1)^{a_1/N}\sim u^{\ell/N}$$

and

$$\omega\sim u^{b_1\ell/a_1-k\ell/N+(\ell/a_1-1)}\,du.$$

Thus, the form ω is holomorphic in a neighborhood of z_1 if and only if the integer

$$\frac{\ell}{a_1}b_1-\frac{\ell}{N}k+\left(\frac{\ell}{a_1}-1\right)$$

is nonnegative, which is equivalent to the inequality

$$b_1+1\geq\frac{a_1}{N}k+\frac{a_1}{\ell},$$

which is in turn equivalent to

$$b_1+1>\frac{a_1}{N}k.$$

Similarly, the conditions

$$b_i+1>\frac{a_i}{N}k \qquad \text{for } i=1,2,3,4$$

are necessary and sufficient for ω to be holomorphic in neighborhoods of z_1, $\dots, z_4$. The latter conditions are equivalent to

$$b_i \ge \left[\frac{a_i}{N}k\right] \qquad \text{for } i = 1,2,3,4, \tag{3.2}$$

where $[x]$ denotes the integer part of x.

Consider now a coordinate v in a neighborhood of ∞ such that

$$z \sim \frac{1}{v}.$$

Then, in a neighborhood of ∞, we have

$$w \sim z^{\sum a_i/N} \sim v^{-\sum a_i/N}$$

and

$$\omega \sim v^{-\sum b_i + \sum a_i k/N - 2}\, dv.$$

Thus, the form ω is holomorphic at ∞ if and only if the integer

$$-\sum b_i + \sum a_i \frac{k}{N} - 2$$

is nonnegative, which is equivalent to the following inequality

$$2 + \sum_{i=1}^{4} b_i \le \sum_{i=1}^{4} \frac{a_i}{N} k. \tag{3.3}$$

This inequality, together with (3.2), implies

$$2 + \sum_{i=1}^{4} \left[\frac{a_i}{N}k\right] \le \sum_{i=1}^{4} \frac{a_i}{N} k,$$

or equivalently,

$$\sum_{i=1}^{4} \left\{\frac{a_i}{N}k\right\} \ge 2.$$

Passing to the notations in (2.2) and (2.3), we can rewrite the latter inequality as

$$t(k) \ge 2. \tag{3.4}$$

Note that for any $k = 1, \dots, N-1$, we have $t(k) \in \{1,2,3\}$. The inequality (3.4) implies that for those k for which $t(k) = 1$, there are no integer solutions $b_1, \dots, b_4$ of the system of inequalities (3.2)–(3.3).

For those k, for which $t(k) = 2$, there is a single integer solution $b_1, \dots, b_4$ of the system of inequalities (3.2)–(3.3), namely $b_i = \left[\frac{a_i}{N}k\right]$, $i = 1,2,3,4$.

Finally, when $t(k) = 3$, there is a "basic" integer solution $b_i = \left[\frac{a_i}{N}k\right]$, where $i = 1,2,3,4$, of the system of inequalities (3.2)–(3.3), and four other solutions, where precisely one of the integers b_i of the basic solution is augmented by one. The resulting holomorphic forms span a two-dimensional family described in the statement of Lemma 3.1. □

LEMMA 3.2. *For any cyclic cover $M_N(a_1, a_2, a_3, a_4)$, the holomorphic forms constructed in Lemma 3.1 for $k = 1, \dots, N-1$, form a basis of the space of holomorphic forms.*

Proof. The statement of the lemma is equivalent to the following identity:

$$\sum_{k=1}^{N-1}(t(k)-1)=g. \tag{3.5}$$

It is easy to check (see [11]) that the genus g of $M_N(a_1,a_2,a_3,a_4)$ is expressed in terms of parameters N and a_i as

$$g=N+1-\frac{1}{2}\sum_{i=1}^{4}\gcd(a_i,N),$$

and we can rewrite relation (3.5) as

$$\sum_{i=1}^{4}\sum_{k=1}^{N-1}\left\{\frac{a_i}{N}k\right\}=(N-1)+\Big(N+1-\frac{1}{2}\sum_{i=1}^{4}\gcd(a_i,N)\Big).$$

Thus, to prove (3.5), it is sufficient to prove that

$$\sum_{k=1}^{N-1}\left\{\frac{a_i}{N}k\right\}=\frac{1}{2}\big(N-\gcd(a_i,N)\big)\qquad\text{for } i=1,\dots,4. \tag{3.6}$$

For any integers $a,N>0$, a sequence

$$\left\{\frac{a}{N}\right\},\ \left\{2\frac{a}{N}\right\},\ \dots$$

is periodic with period $T=N/\gcd(N,a)$. A collection of numbers

$$\left\{\left\{\frac{a}{N}\right\},\left\{2\frac{a}{N}\right\},\dots,\left\{T\frac{a}{N}\right\}\right\}$$

within each period considered as *unordered* set coincides with the set

$$\left\{\frac{0}{T},\frac{1}{T},\frac{2}{T},\dots,\frac{T-1}{T}\right\}$$

which forms an arithmetic progression. The sum of numbers in this latter set equals $(T-1)/2$. Hence,

$$\begin{aligned}\sum_{k=1}^{N-1}\left\{\frac{a}{N}k\right\}&=\gcd(N,a)\sum_{k=1}^{T}\left\{\frac{a}{N}k\right\}\\&=\frac{\gcd(N,a)}{2}\left(\frac{N}{\gcd(N,a)}-1\right)=\frac{1}{2}\big(N-\gcd(a,N)\big),\end{aligned}$$

which proves (3.6). □

Proof of Theorem 2.1. Now, let us show how relations (2.4) and (2.5) in Theorem 2.1 follow from Lemmas 3.1 and 3.2.

Equation (1.4) describing the action of the group of deck transformations implies that all forms constructed in Lemma 3.1 are eigenforms; namely, the form (3.1) is an eigenform with the eigenvalue $\zeta^{-k}=\zeta^{N-k}$. By Lemma 3.2, they form a basis of the space of holomorphic forms. Hence, the forms (3.1) give a basis of $V^{1,0}(N-k)$ for each individual k. Combined with Lemma 3.1, this observation implies

$$\dim_{\mathbb{C}}V^{1,0}(k)=t(N-k)-1.$$

Clearly, the complex conjugate of an eigenform (3.1) is an eigenform with a conjugate eigenvalue. Hence, the complex conjugates of (3.1) form a basis in the space of antiholomorphic forms and

$$\dim_{\mathbb{C}} V^{1,0}(k) = \dim_{\mathbb{C}} V^{0,1}(N-k).$$

This implies that

$$\dim_{\mathbb{C}} V(k) = t(k) + t(N-k) - 2 \in \{0,1,2\}. \qquad \square$$

3.2. **Extension of the Hodge bundle to cusps: general approach.** Consider a holomorphic family X_ε of smooth complex curves of genus greater than one over a punctured disk, $\varepsilon \in \mathscr{D} \smallsetminus \{0\}$. It follows from the semistable reduction theorem of Deligne and Mumford that this family extends to a holomorphic family of stable curves over the entire disk $\mathscr{D}$ if and only if the induced monodromy on the bundle $H^1_{\mathbb{C}}$ is unipotent.

Geometrically, such an extension can be described as follows. The complex curves X_ε considered as Riemann surfaces endowed with the hyperbolic metric get pinched at some fixed collection $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ of short hyperbolic geodesics such that for small ε the thick components $X_{\varepsilon,j}$ of $X_\varepsilon - \Gamma$ are "geometrically close" to the irreducible components $X_{0,j}$ of the stable curve X_0 endowed with the canonical hyperbolic metric with cusps at the nodal points. The limiting curve X_0 is a regular smooth complex curve if and only if and only if no hyperbolic geodesic gets pinched, so Γ is empty in this particular case.

The Deligne extension of the Hodge bundle has the following fiber at $\varepsilon = 0$. It consists of holomorphic 1-forms on the smooth locus of the stable curve X_0 which have simple poles at double points of X_0 and such that the sum of two residues at any double point vanishes.

Suppose that in addition we are given a nontrivial holomorphic section ω of the Hodge bundle $H^{1,0}$ over the punctured disk $\mathscr{D} \smallsetminus \{0\}$, where $\omega_\varepsilon \in H^{1,0}(X_\varepsilon)$. According to the generalization of the semistable reduction theorem of Deligne and Mumford, for any such section and for each irreducible component $X_{0,j}$, we have an integer number $d = d(j,\omega)$ such that the holomorphic 1-form $\varepsilon^{-d}\omega_\varepsilon$ restricted to the thick component $X_{\varepsilon,j}$ tends to some well-defined nontrivial meromorphic form $\tilde{\omega}_{0,j}$ on the desingularized irreducible component $X_{0,j}$. The limiting 1-form $\tilde{\omega}_{0,j}$ is allowed to have poles (possibly of order greater than one) only at the nodal points of $X_{0,j}$, see [6] for details. Note that in general the integers $d(j,\omega)$ vary from one irreducible component $X_{0,j}$ to the other. Define $d(\omega) \in \mathbb{Z}$ to be the minimum of integers $d(j,\omega)$ over all components $X_{0,j}$. The limit at $\varepsilon \to 0$ of $\varepsilon^{-d(\omega)}\omega_\varepsilon$ is a form with at most simple poles at double points, possibly vanishing at some components $X_{0,j}$ of the special fiber X_0, and nonvanishing on at least one component $X_{0,k}$.

Let us assume that the line subbundle of $H^{1,0}$ generated by the section ω is the subbundle $\mathscr{E}^{1,0}$ corresponding to a direct summand $\mathscr{E}_{\mathbb{C}}$ of the variation of complex polarized Hodge structures $H_{\mathbb{C}}$, with signature $(1,1)$. Then, outside of the cusp, ω gives a section of the canonical extension of $\mathscr{E}^{1,0}$ to the cusp. The latter extension is obviously a direct summand of the canonical extension of

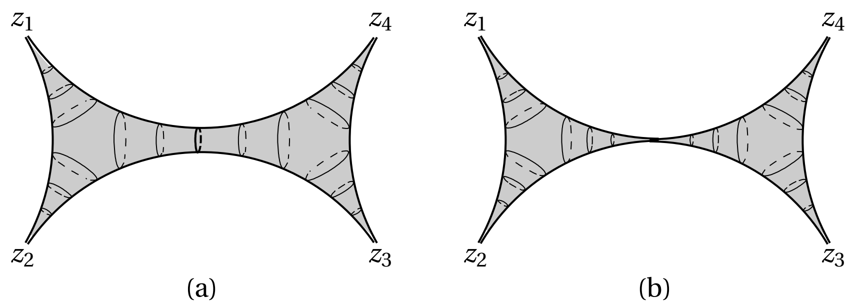


FIGURE 2. Degeneration of $\mathbb{C}\mathrm{P}^1$ with 4 marked points

$H^{1,0}$. Therefore, the order of zero of the section ω of $\overline{\mathscr{E}^{1,0}}$ at $\varepsilon = 0$ is equal to $d(\omega)$.

Finally, if our family does not allow a semistable reduction (*i.e.,* the monodromy is not unipotent), then one can pass to a finite cyclic cover of certain order $n \geq 1$ of the punctured disk, and reduce the question to the unipotent case. Let us denote by $\tilde{\omega}$ the pullback of the section ω to the covering. Then the orbifold order of vanishing at $\varepsilon \to 0$ of ω is the order of zero $d(\tilde{\omega})$ corresponding to the covering, divided by n. One can replace this formula by a "local" one. Namely, for each component $X_{0,j}$ of a special fiber of a semistable reduction obtained by passing to a finite covering, we associate a *rational* number $r(j,\omega) = d(j,\tilde{\omega})/n$ equal to the fractional order of the zero in ε, and then consider the minimum over all components.

After completing the paper [6], we learned that analogous results were simultaneously and independently obtained by S. Grushevsky and I. Krichever in [12], and by J. Smillie in [21].

3.3. **Extending the line bundles $V^{1,0}(k)$ to the cusps.** Before passing to the proof of Theorem 2.3 let us discuss how the general setting described in the previous section applies to our concrete situation.

Recall that by Convention 1.2, all the points z_i are "named" and, thus, are distinguishable. Hence, the arithmetic Teichmüller curve $\mathscr{C} = \mathscr{M}_{(a_i),N}$ can be identified with the space $\mathscr{M}_{0,4}$ of configurations of ordered quadruples of distinct points $\{z_1, z_2, z_3, z_4\}$ on the Riemann sphere considered up to a holomorphic automorphism preserving the "names" (see [11] for details). We fix three points z_1, z_2, and z_4 equal to 0, 1, and ∞, respectively. Now, our cyclic cover is defined by a complex parameter $z_3 \notin \{0,1,\infty\}$ which serves as a coordinate on the compactified Teichmüller curve $\overline{\mathscr{C}} \simeq \overline{\mathscr{M}_{0,4}} \simeq \mathbb{C}\mathrm{P}^1$. This curve is a base of the universal curve. Under the chosen normalization, a fiber X_{z_3} of the universal curve over a point $z_3 \in \mathscr{C}$ is defined by equation

$$w^N = z^{a_1}(z-1)^{a_2}(z-z_3)^{a_3}. \tag{3.7}$$

We warn the reader of a possible confusion: we have $\mathbb{C}P^1$ constantly appearing in two different roles. On the one hand the Teichmüller curve $\mathcal{C}$ is isomorphic to $\mathbb{C}P^1$ with three cusps. On the other hand, the fiber X_{z_3} of the universal curve over $\mathcal{C}$ has the structure of a cyclic cover, $X_{z_3} \simeq M_N(a_1, a_2, a_3, a_4) \to \mathbb{C}P^1 \simeq Y_{z_3}$, and we have $\mathbb{C}P^1 \simeq Y_{z_3}$ in the base of this cover. We mostly work with $\mathbb{C}P^1$ in its second appearance: it can be viewed as a fiber of the associated universal curve.

Our goal is to calculate the orbifold degree of holomorphic vector bundle $V^{1,0}(k)$ on the curve $\mathcal{M}_{(a_i),N} \simeq \mathcal{M}_{0,4}$, in the case when it is a *line* bundle. Recall that we have a canonical section of $V^{1,0}(k)$ given by one of basis elements of $H^{1,0}$. Hence, the degree coincides with the sum of multiplicities of zeroes of this section. All zeroes and poles appear only at cusp points.

A point $z_3 \in \mathcal{C}$ of the Teichmüller curve approaches one of the three cusps when z_3 tends to 0, 1, or ∞. The induced monodromy on the bundle $H^1_{\mathbb{C}}$ is not unipotent. Therefore, we should go to an appropriate cover $\mathcal{C}' \to \mathcal{C}$ of the Teichmüller curve (*e.g.,* the punctured Fermat curve, see the last sentence in 2.3). The exact nature of this cover is irrelevant.

The stable curve which arises as the limit as z_3 goes to the cusp on $\overline{\mathcal{C}}$ is a cover (possibly ramified at the double point and marked points) of a semistable genus zero curve with four marked points, which is the limit of Y_{z_3}. The stable curves (with four marked points) Y_0, Y_1 and Y_∞ over the three cusps have the same structure, see Figure 2(b). Each of them has two components, where each component is a $\mathbb{C}P^1$ endowed with three marked points; the two components are glued together by identifying a pair of the marked points.

Geometrically, the degeneration of the fiber Y_{z_3} near the cusp can be described as follows. The fiber $Y_{z_3} \simeq \mathbb{C}P^1$ has 4 marked points, namely, $0, 1, \infty, z_3$. The corresponding hyperbolic surface is a topological sphere with four cusps (a cusp at each of the marked points). Such hyperbolic surface can be glued from two identical pairs of pants, where each pair of pants has two cusps and a nontrivial geodesic boundary curve; the boundaries are glued together by a hyperbolic isometry, see Figure 2(a). When z_3 approaches one of the points $0, 1, \infty$ the close hyperbolic geodesic, serving as the common waist curve of the pairs of pants, gets pinched, and at the limit we get two identical pairs of pants, each having three cusps, see Figure 2(b).

By assumption of Theorem 2.3, we consider only those values of k, for which one has $t(N-k) = 2$. By formula (2.4) from Theorem 2.1, this implies that $\dim_{\mathbb{C}} V^{1,0}(k) = 1$. By Lemma 3.1 the fiber l_{z_3} of the corresponding line bundle $l = V^{1,0}(k)$ is spanned by the holomorphic form

$$\omega = z^{b_1}(z-1)^{b_2}(z-z_3)^{b_3}\frac{dz}{w^k}, \qquad b_i = [a_i \cdot k/N],\ i = 1,2,3. \tag{3.8}$$

To compute the orbifold degree of $l = V^{1,0}(k)$ it is sufficient to compute the divisor of the section ω. Since outside of the cusps the section ω is nonzero, we have to compute the degrees of zeroes or poles of the extension of the section ω at the three cusps 0, 1, ∞ of the Teichmüller curve $\mathcal{C}$.

Following the general approach presented in the previous section, we introduce a small local parameter ε in a neighborhood of a cusp on $\mathscr{C}$. Let, for example, the cusp correspond to the point 0. For each component $X_{0,j}$ of the stable curve X_0, we need to find an appropriate fractional power $r(j,\omega)\in\mathbb{Q}$ of ε such that the form $\varepsilon^{-r(j,\omega)}\cdot\omega$ on the corresponding thick part $X_{\varepsilon,j}$ of X_ε tends to a nontrivial meromorphic form on the chosen group of components of the stable curve $X_{0,j}$ as $\varepsilon\to 0$.

As will clear from the calculation, we do not need to individually consider all irreducible components of the stable curve X_0. We use the projection $X_0\to Y_0$ map between the corresponding stable curves to organize the components of X_0 into two groups corresponding to preimages of the two components, $Y_{0,1}$ and $Y_{0,2}$, of the stable curve Y_0, see Figure 2(b). All components $X_{0,j_1},\dots,X_{0,j_m}$ in the same group corresponding to $Y_{0,\alpha}$, $\alpha\in\{1,2\}$, share the same rational number $r^{(\alpha)} := r(j_1,\omega)=\cdots=r(j_m,\omega)$. The minimum $\min(r^{(1)},r^{(2)})$ of the resulting two rational numbers corresponding to the two components $Y_{0,1}$, $Y_{0,2}$ of Y_0 determines the order of a zero or pole of the meromorphic section ω of $V^{1,0}$ at the cusp 0 in $\mathscr{C}$. The situation for the two remaining cusps, 1 and ∞, in $\mathscr{C}$ is completely analogous.

3.4. **Computation of degrees of line bundles.** In this section, we prove Theorem 2.3. The notations $t_i(k)$ and $t(k)$, which we use in the proof, were introduced in equations (2.2) and (2.3), respectively.

Proof of Theorem 2.3. Let us choose $z_3=\varepsilon\to 0$. The base of the cover $X_{z_3}\simeq M_N(a_1,a_2,a_3,a_4)\to\mathbb{C}\mathrm{P}^1$ splits into two Riemann spheres, where 0 and z_3 stay in one component, and 1 and ∞ belong to the other one, see Figure 2(b).

Introducing coordinates $\tilde z=\varepsilon^{-1}z$ and $\tilde w=\varepsilon^{-(a_1+a_3)/N}w$ in the first component we see that

$$\omega=\varepsilon^{b_1+b_3+1-k(a_1+a_3)/N}\,\tilde\omega_\varepsilon,$$

where a holomorphic form $\tilde\omega_\varepsilon$ tends to a nontrivial meromorphic form $\tilde\omega_0$ when $z_3=\varepsilon\to 0$. Here and below "trivial" means everywhere null.

REMARK 3.3. Formally, we should introduce $n := N/\gcd(a_1+a_3,N)$ and pass to a ramified n-fold cover $\hat{\mathscr{D}}\to\mathscr{D}$ of a neighborhood $\mathscr{D}$ of the cusp 0 at $\mathscr{C}$, so $\varepsilon=\delta^n$. Then $\tilde z=\delta^{-n}z$ and $\tilde w=\delta^{-(a_1+a_3)n/N}w$, where the powers of δ are already integer; see the comments about the orbifold degree in Sections 2.3 and 3.2.

On the other component, ω tends to a nontrivial form when z_3 tends to zero without any renormalization. Hence, our section has singularity of order $\min(0,\,b_1+b_3+1-k(a_1+a_3)/N)$ at this cusp. We proved in Lemma 3.1 that when $t(N-k)=2$, we have $b_i=\left[\frac{a_i}{N}k\right]$. Thus, we can rewrite the above expression as

$$\min\Big(0,\,b_1+b_3+1-\frac{a_1+a_3}{N}\,k\Big)=\min\big(0,\,1-(t_1(k)+t_3(k))\big). \tag{3.9}$$

Similarly, the order of the singularity of the section ω at the cusp $z_3=1$ is equal to

$$\min\big(0,\,1-\big(t_2(k)+t_3(k)\big)\big). \tag{3.10}$$

Finally, when $z_3 \to \infty$ the base of the cover $M_N(a_1, a_2, a_3, a_4) \to \mathbb{C}\mathrm{P}^1$ once again splits into two Riemann spheres, where 0 and 1 stay in one component, and z_3 and ∞ belong to the other component. Let $z_3 = 1/\varepsilon$, and consider the behavior of ω on the first component, containing 0 and 1, as ε tends to 0.

The equation (3.7) of the curve implies that $w \sim \varepsilon^{-a_3/N}$. The equation (3.8) of the form implies that

$$\omega = \varepsilon^{(-b_3+a_3k/N)}\phi_\varepsilon,$$

where the meromorphic form ϕ_ε tends to a nontrivial meromorphic form ϕ_0 on the first component when $z_3^{-1} = \varepsilon \to 0$.

Introduce the coordinate $\tilde{z} = 1/(\varepsilon z)$ on the second component. The equation (3.7) of the curve implies that $w \sim \varepsilon^{-(a_1+a_2+a_3)/N}$. The equation (3.8) of the form implies that

$$\omega = \varepsilon^{(-b_1-b_2-b_3-1+k(a_1+a_2+a_3)/N)}\psi_\varepsilon,$$

where the meromorphic form ψ_ε tends to a nontrivial meromorphic form ψ_0 on the second component when $z_3^{-1} = \varepsilon \to 0$.

By Lemma 3.1 we have $b_i = \left[\frac{a_i}{N}k\right]$. Applying the notations (2.2) for $t_i(k) = \left\{\frac{a_i}{N}k\right\}$ we conclude that the section ω of the line bundle $V^{1,0}(k)$ has singularity of order

$$\min\big(t_3(k),\, t_1(k)+t_2(k)+t_3(k)-1\big)$$

at ∞. Taking into consideration that $t(k) = t_1(k)+t_2(k)+t_3(k)+t_4(k) = 2$, we can rewrite the latter expression as

$$\begin{aligned}\min\big(t_3(k),\, t_1(k)+t_2(k)+t_3(k)-1\big) &= t_3(k)+\min\big(0,\, t_1(k)+t_2(k)-1\big)\\ &= t_3(k)+\min\big(0,\, 1-(t_3(k)+t_4(k))\big).\end{aligned} \tag{3.11}$$

Summing up (3.9), (3.10), and (3.11) we see that the orbifold degree of the section ω is equal to

$$\begin{aligned} &t_3(k)+\min\big(0,\, 1-(t_1(k)+t_3(k))\big)\\ &\quad+\min\big(0,\, 1-(t_2(k)+t_3(k))\big)+\min\big(0,\, 1-(t_4(k)+t_3(k))\big).\end{aligned} \tag{3.12}$$

Formula (2.6) for the orbifold degree of the line bundle $V^{1,0}(k)$ stated in Theorem 2.3 now follows from the relation (3.13) proved in the elementary lemma below.

LEMMA 3.4. *For any quadruple of numbers $t_i \in [0,1]$, satisfying the relation $t_1 + t_2 + t_3 + t_4 = 2$, the following identity holds:*

$$\begin{aligned}\min\big(t_1, 1-t_1, \ldots, t_4, 1-t_4\big) &= t_3+\min\big(0, 1-(t_1+t_3)\big)\\ &\quad+\min\big(0, 1-(t_2+t_3)\big)+\min\big(0, 1-(t_4+t_3)\big).\end{aligned} \tag{3.13}$$

Proof. First note that expression (3.12) is symmetric with respect to permutations of indices 1,2,3,4, just because the initial geometric setting is symmetric. Since the corresponding quadruples of numbers $t_i(k)$, taken for different data $N, a_1, \ldots, a_4$, form a dense set in $[0,1]^4$, we conclude that the continuous function in the right-hand side of (3.13) is symmetric with respect to permutations of indices 1,2,3,4. Hence, without loss of generality we may assume that

$t_1 \le t_2 \le t_3 \le t_4$. Under this assumption the left-hand side expression in (3.13) takes the value $\min(t_1, 1-t_4)$.

Consider the expression in the right-hand side. Note that $t_2+t_4 \ge t_1+t_3$. Since $t_2+t_4+t_1+t_3 = 2$, this implies that $t_1+t_3 \le 1$. Hence,

$$\min\big(0,\, 1-(t_1+t_3)\big) = 0.$$

Similarly, since $t_3+t_4 \ge t_1+t_2$, we conclude that $t_3+t_4 \ge 1$. Hence,

$$\min\big(0,\, 1-(t_4+t_3)\big) = 1-(t_4+t_3).$$

The value of the middle term depends on comparison of t_1 with $1-t_4$. If $t_1 \le 1-t_4$, then $t_1+t_4 \le 1$ and hence $t_2+t_3 \ge 1$. In this case, we get the value t_1 for the left-hand side expression in (3.13) and

$$t_3 + 0 + (1-(t_2+t_3)) + (1-(t_4+t_3)) = 2-(t_2+t_3+t_4) = t_1$$

as the value of the right-hand side expression.

If $t_1 > 1-t_4$, then $t_1+t_4 > 1$ and hence $t_2+t_3 < 1$. In this case, we get the value $1-t_4$ for the left-hand side expression in (3.13) and

$$t_3 + 0 + 0 + (1-(t_4+t_3)) = 1-t_4$$

as the value of the right-hand side expression. The desired identity is proved. □

To complete the proof of Theorem 2.3, it remains to prove the alternative concerning positivity of $d(k)$ claimed in the statement of the theorem.

By assumptions of Theorem 2.3, we have $t(N-k) = 2$. By formula (2.4) from Theorem 2.1, one gets $\dim_{\mathbb{C}} V^{1,0}(k) = 1$. Hence, $\dim_{\mathbb{C}} V(k) \ge 1$. Thus, formula (2.5) from Theorem 2.1 implies that $t(k)$ is equal either to 2 or to 1.

If $t(k) = 2$, then by formula (2.5), one gets $\dim_{\mathbb{C}} V(k) = 2$. By Lemma 2.2, this implies that $t_i(k) > 0$ for $i = 1,2,3,4$. Since by definition (2.2) of $t_i(k)$ one always has $t_i(k) < 1$, formula (2.6) implies that $d(k) > 0$ in this case.

If $t(k) = 1$, then by formula (2.5), one gets $\dim_{\mathbb{C}} V(k) = 1$. By Lemma 2.2, this implies that there is an index $i \in \{1,2,3,4\}$, such that $t_i(k) = 0$. Thus, in this case formula (2.6) implies that $d(k) = 0$.

The proof of Theorem 2.3 is now complete. □

APPENDIX A. HODGE BUNDLES ASSOCIATED TO QUADRATIC DIFFERENTIALS

Consider a cyclic cover $p\colon M_N(a_1, a_2, a_3, a_4) \to \mathbb{C}\mathrm{P}^1$ and a meromorphic quadratic differential $p^* q_0$ on $M_N(a_1, a_2, a_3, a_4)$ that defines the flat structure on $M_N(a_1, a_2, a_3, a_4)$. Here, a canonical quadratic differential q_0 on $\mathbb{C}\mathrm{P}^1$ is defined by equation (1.5). The quadratic differential q is a global square of an Abelian differential if and only if N is even and all a_i are odd, see [11].

Suppose that at least one of the following conditions is valid: N is odd, or one of a_i is even. Then q is not a global square of an Abelian differential. There exists a canonical (possibly ramified) double cover $p_2\colon \hat{X} \to M_N(a_1, a_2, a_3, a_4)$ such that $p_2^* q = \omega^2$, where ω is already a holomorphic 1-form on $\hat{X}$.

Let $\hat{g}$ be the genus of the cover $\hat{X}$. By *effective genus*, we call the positive integer

$$g_{\text{eff}} := \hat{g} - g.$$

The cohomology space $H^1(\hat{X},\mathbb{R})$ splits into a direct sum $H^1(\hat{X},\mathbb{R}) = H^1_+(\hat{X},\mathbb{R}) \oplus H^1_-(\hat{X},\mathbb{R})$ of invariant and anti-invariant subspaces with respect to the induced action of the involution $p_2^*\colon H^1(\hat{X},\mathbb{R}) \to H^1(\hat{X},\mathbb{R})$. Note that the invariant part is canonically isomorphic to the cohomology of the underlying surface, $H^1_+(\hat{X},\mathbb{R}) \simeq H^1(X,\mathbb{R})$. We consider subspaces $H^1_+(\hat{X},\mathbb{R})$ and $H^1_-(\hat{X},\mathbb{R})$ as fibers of natural vector bundles H^1_+ and H^1_- over the Teichmüller curve $\mathcal{C}$. The bundle H^1_+ is canonically isomorphic to the Hodge bundle H^1 considered above.

The splitting $H^1 = H^1_+ \oplus H^1_-$ is covariantly constant with respect to the Gauss–Manin connection. The symplectic form restricted to each summand is nondegenerate. Thus, the monodromy of the Gauss–Manin connection on H^1_- is symplectic. The Lyapunov exponents of the bundle H^1_- with respect to the geodesic flow on $\mathcal{C}$ are denoted by $\lambda^-_1 \geq \cdots \geq \lambda^-_{2g_{\text{eff}}}$. As before we have $\lambda^-_k = -\lambda^-_{2g_{\text{eff}}-k+1}$. It is natural to study the Lyapunov spectrum ΛSpec_- of H^1_-.

When N is odd, the canonical double cover $\tilde{X}$ over $M_N(a_1, a_2, a_3, a_4)$ is again a cyclic cover. Namely, it is the cyclic cover $M_{2N}(a'_1,\dots,a'_4)$, where $a'_i := a_i$ when a_i is odd and $a'_i := a_i + N$ when a_i is even. We can apply Theorem 2.6 to compute the Lyapunov spectrum of the Hodge bundle over $M_{2N}(a'_1,\dots,a'_4)$. By construction, this Lyapunov spectrum is a union $\Lambda\text{Spec}_- \sqcup \Lambda\text{Spec}_+$ of Lyapunov spectra of the bundles H^- and H^+, respectively. Applying Theorem 2.6 once more, we compute the Lyapunov spectrum ΛSpec_+ of the Hodge bundle corresponding to $M_N(a_1, a_2, a_3, a_4)$. Taking the complement of two spectra, we obtain the Lyapunov spectrum ΛSpec_-.

When N is even and a pair of a_i is even, the canonical double cover $\tilde{X}$ is not a cyclic cover. However, in this case it is an *Abelian* cover, see [22].

APPENDIX B. SQUARE-TILED CYCLIC COVERS WITH SYMMETRIES

Following I. Bouw and M. Möller [3], we consider in this section a situation when a cyclic cover has an extra symmetry. Passing to a quotient over such an invariant holomorphic automorphism we get a new square-tiled surface and corresponding arithmetic Teichmüller curve for which we can explicitly compute the Lyapunov spectrum.

The fact that the Lyapunov spectrum corresponding to the "stairs" square-tiled surfaces discussed below forms an arithmetic progression (see Proposition B.5) was noticed by the authors in computer experiments about a decade ago. Recently M. Möller suggested that this property of the spectrum is a strong indication that the corresponding surface is a quotient of a cyclic cover over an automorphism (by analogy with examples of nonarithmetic Veech surfaces discovered in [3], having the same property of the spectrum).

The discussion with J.-C. Yoccoz on possible values of Lyapunov exponents of $\mathrm{SL}(2,\mathbb{R})$-cocycles "over continued fractions" was a strong motivation for us to prove complete integrability in this example. We show that the Hodge bundle $H^1_{\mathbb{R}}$ decomposes into a direct sum of two-dimensional covariantly constant subbundles and that such subbundles over a small arithmetic Teichmüller curve might have arbitrarily small Lyapunov exponents.

Consider a cyclic cover of the type $M_N(a, N-a, b, N-b)$. In this section, we always assume that N is even and all a_i are odd, so $p^*q_0 = \omega^2$. By Convention 1.2, the four ramification points P_1, P_2, P_3, P_4 of the Riemann sphere $\mathbb{C}\mathrm{P}^1$ are named, so even when $a = b$ we can distinguish preimages corresponding to P_1 and to P_3. There exists a unique holomorphic involution of $\mathbb{C}\mathrm{P}^1$ with interchanging the points in each of the two pairs P_1, P_2 and P_3, P_4 of the marked points. By construction it preserves the quadratic differential determining the flat structure on $\mathbb{C}\mathrm{P}^1$. This involution induces a holomorphic involution τ of the square-tiled cyclic cover $M_N(a, N-a, b, N-b)$ preserving the flat structure in the sense that

$$\tau^*\omega^2 = \omega^2.$$

By construction, the involution τ also preserves the structure of the cover. Moreover, it interwinds a generator T of the group of deck transformations with its inverse:

$$\tau T = T^{-1}\tau. \tag{B.1}$$

This relation implies that the holomorphic automorphism of the cyclic cover defined as $\tau_2 := \tau \circ T$ is also a holomorphic involution, and that it also preserves the flat structure in the sense that

$$\tau_2^*\omega^2 = \omega^2.$$

Note that this relation implies that

$$\tau_2^*\omega = \pm\omega.$$

Note also that

$$T^*\omega = -\omega.$$

Hence, if one of the two involutions does not preserve ω, then the other does. Thus, up to the interchange of notations for the involutions, we may assume that

$$\tau^*\omega = \omega.$$

PROPOSITION B.1. *Let N be even and let a and b be odd. A quotient of a square-tiled cyclic cover of the form $M_N(a, N-a, b, N-b)$ over the involution τ as above is a connected square-tiled surface. The Lyapunov spectrum* ΛSpec *of the Hodge bundle over an arithmetic Teichmüller curve of the initial square-tiled cyclic cover can be obtained by taking two copies of the Lyapunov spectrum* $\Lambda\mathrm{Spec}_+$ *of the Hodge bundle over the arithmetic Teichmüller curve of the quotient square-tiled surface and suppressing one of the two entries "1".*

By convention, by Lyapunov spectrum ΛSpec we call the top g Lyapunov exponents, where g is the genus of the flat surface under consideration.

The first statement of the proposition is a particular case of the following elementary lemma.

LEMMA B.2. *Let $\sigma: S \to S$ be an automorphism of a square-tiled surface preserving an Abelian differential ω, which defines the flat structure $\sigma^*\omega = \omega$. Then, the quotient surface S/σ is a connected square-tiled surface.*

Proof. First note that if a flat surface (S,ω) admits a tiling by unit squares, such tiling is unique. By convention, when S is a torus, it is endowed with a marked point serving as a "fake zero." Hence, an automorphism σ maps squares of the tiling to squares of the same tiling by parallel translations. This implies that the quotient surface is square-tiled.

Let us enumerate the squares by numbers $1,\dots,M$. Denote by π_h,π_v permutations indicating the squares adjacent to the right (respectively, atop) to every square of the tiling.

Consider some nonsingular component of the quotient surface. Let $A \subseteq \{1,\dots,M\}$ denote the subset of all squares of the initial surface S which project to this nonsingular component. The subset A is invariant under both π_h and π_v. Since the initial surface is nondegenerate, it implies that A coincides with the entire set $\{1,\dots,M\}$. □

REMARK B.3. We have, actually, shown in the proof of the lemma above that a square-tiled surface (S,ω) admits a nontrivial automorphism $\sigma: S \to S$ such that $\sigma^*\omega = \omega$ if and only if there exists a nontrivial permutation σ of squares of the tiling, which commutes with both π_h and π_v.

Proof of Proposition B.1. It remains to prove the statement concerning the Lyapunov spectrum. By construction of the involution τ, the induced involution τ^* on cohomology interchanges the eigenspaces corresponding to eigenvalues ζ^k and $\zeta^{-k} = \zeta^{N-k}$,

$$\tau^* V(k) = V(N-k), \qquad \tau^* V^{1,0}(k) = V^{1,0}(N-k), \tag{B.2}$$

and preserves the subspaces $W(k) \subset H^1(X,\mathbb{R})$ defined in the beginning of Section 2.4.

For any k such that $1 \le k < N/2$, we can represent the subspaces $V(k)\oplus V(N-k)$, $V^{1,0}(k)\oplus V^{1,0}(N-k)$, and $W(k)$ as direct sums of invariant and anti-invariant subspaces under the involution τ^* as follows:

$$V(k) \oplus V(N-k) = V_+(k) \oplus V_-(k)$$
$$V^{1,0}(k) \oplus V^{1,0}(N-k) = V_+^{1,0}(k) \oplus V_-^{1,0}(k)$$
$$W(k) = W_+ \oplus W_-.$$

Since for $k < N/2$ one has $V(k) \cap V(N-k) = \{0\}$, the relations (B.2) imply that for $k < N/2$ each of the subspaces $V_+(k)$ and $V_+^{1,0}(k)$ has half of a dimension of the corresponding ambient subspace $V(k) \oplus V(N-k)$ and $V^{1,0}(k) \oplus V^{1,0}(N-k)$. By construction, the subspace $V_+(k)$ is invariant under the complex conjugation. Hence, the real subspace $W_+(k)$ has half of a dimension of the ambient real subspace $W(k)$. Moreover, $W_+(k)$ is a symplectic $\mathrm{SL}(2,\mathbb{R})$-invariant and Hodge star-invariant subspace.

It follows from Theorem 2.6 that if the vector bundle $W(k)$, where $k < N/2$, has at least one nonzero Lyapunov exponent, the bundle $W(k)$ necessarily has dimension four and its Lyapunov spectrum has the form $\{\lambda, \lambda, -\lambda, -\lambda\}$. Hence, the Lyapunov spectrum of the subbundle $W_+(k)$ of such bundle has the form $\{\lambda, -\lambda\}$.

Note also that by construction the subspaces $V(N/2)$, $V^{1,0}(N/2)$ and, hence, $W(N/2)$ are invariant under the involution τ^* which acts on these subspaces as the identity map. Therefore, $V_+(N/2) = V(N/2)$, $V_+^{1,0}(N/2) = V^{1,0}(N/2)$, and $W_+(N/2) = W(N/2)$. Recall that the Lyapunov spectrum of $W(N/2)$ is $\{1, -1\}$.

Consider now a square-tiled surface S obtained as a quotient of the square-tiled cyclic cover $M_N(a, N-a, b, N-b)$ over τ. Clearly, we have canonical isomorphisms

$$\begin{aligned} H^1(S, \mathbb{C}) &\simeq \bigoplus_{k \le N/2} V_+(k) \\ H^{1,0}(S) &\simeq \bigoplus_{k \le N/2} V_+^{1,0}(k) \\ H^1(S, \mathbb{R}) &\simeq \bigoplus_{k \le N/2} W_+(k). \end{aligned}$$

Taking into consideration our observations concerning the Lyapunov spectrum of the subbundles $W(k)$, this implies the statement of Proposition B.1 concerning the spectrum of Lyapunov exponents of the quotient surface. □

Let us consider in more detail two particular cases.

B.1. **Lyapunov spectrum of "stairs" square-tiled surfaces.** Consider a following square-tiled surface $S(N)$. Take N squares and arrange them cyclically into a cylinder of width N and of hight 1. A permutation π_h, which indicates a right neighbor of a square number k, is given by a cycle $(1, \dots, N)$. Now identify by a parallel translation the top horizontal side of the square number k to the bottom horizontal side of the square number $N+1-k$. A permutation π_v, which indicates a neighbor above a square number k, is as follows:

$$\pi_v = \begin{pmatrix} N & N-1 & \dots & 2 & 1 \\ 1 & 2 & \dots & N-1 & N \end{pmatrix}.$$

The resulting square-tiled surface is presented in Figure 3.

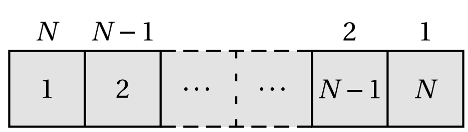


FIGURE 3. Square-tiled surface $S(N)$

CONVENTION B.4. Representing a square-tiled surface by a polygonal pattern, we usually try to respect a decomposition into horizontal cylinders. Thus, by convention, an unmarked vertical segment of the boundary is identified by a parallel translation with another unmarked vertical segment of the boundary located at the same horizontal level. To specify identification of the horizontal segments of the boundary we give a number of a square located atop of each square of the top boundary. When such square is not indicated, it means that we identify the two horizontal segments by a vertical parallel translation.

Note that a general square-tiled surface has neither distinguished polygonal pattern nor distinguished enumeration of the squares.

It is immediate to see that the square-tiled surface $S(N)$ from Figure 3 belongs to the stratum $\mathcal{H}(2g-2)$ for $N = 2g-1$, and to the stratum $\mathcal{H}(g-1, g-1)$ for $N = 2g$. Clearly, a central symmetry of the pattern from Figure 3 extends to an involution of the surface. This involution has two fixed points at the waist curve of the cylinder. It also has a fixed point at the middle of each of the N horizontal sides of squares. The involution fixes the single zero when $N = 2g-1$ and interchanges the two zeroes when $N = 2g$. Thus, the involution has $2g+2$ fixed points, so it is a hyperelliptic involution. We conclude that the surface $S(N)$ belongs to the hyperelliptic connected component $\mathcal{H}^{\mathrm{hyp}}(2g-2)$ when $N = 2g-1$, and to the hyperelliptic connected component $\mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$ when $N = 2g$.

PROPOSITION B.5. *The Hodge bundle $H^1_{\mathbb{R}}$ over the arithmetic Teichmüller curve of the surface $S(N)$ from Figure 3 decomposes into a direct sum of* $\mathrm{SL}(2,\mathbb{R})$*-invariant, Hodge star-invariant two-dimensional symplectic subbundles*

$$H^1_{\mathbb{R}} \simeq \bigoplus_{k \le N/2} W_+(k).$$

The Lyapunov spectrum $\Lambda\mathrm{Spec}$ *of the Hodge bundle $H^1_{\mathbb{R}}$ over the corresponding arithmetic Teichmüller curve is*

$$\Lambda\mathrm{Spec} = \begin{cases} 1/N,\ 3/N,\ 5/N, \dots,\ N/N & \text{when } N = 2g-1, \\ 2/N,\ 4/N,\ 6/N, \dots,\ N/N & \text{when } N = 2g. \end{cases}$$

Proof. We show in Lemma B.6 that for odd N the surface $S(N)$ in Figure 3 corresponds to a quotient of a cyclic cover $M_{2N}(2N-1, 1, N, N)$ over the involution τ defined by equation (B.1). Using Theorem 2.6 and Corollary 2.7, we compute in Lemma B.7 the Lyapunov spectrum for the Teichmüller curve of $M_{2N}(2N-1, 1, N, N)$ and apply Proposition B.1.

Similarly, we show in Lemma B.8 that for even N, the surface $S(N)$ in Figure 3 corresponds to a quotient of a square-tiled cyclic cover $M_N(N-1, 1, N-1, 1)$ over the involution τ as in (B.1). Using Theorem 2.6 and Corollary 2.7, we compute the Lyapunov spectrum for the arithmetic Teichmüller curve of $M_N(N-1, 1, N-1, 1)$ and apply Proposition B.1. □

LEMMA B.6. *For odd N, the surface $S(N)$ in Figure 3 corresponds to the quotient of the square-tiled cyclic cover $M_{2N}(2N-1,1,N,N)$ over the involution τ as in equation (B.1).*

Proof. Note that for any integer $a < 2N$ such that $\gcd(2N,a) = 1$, the square-tiled cyclic covers

$$M_{2N}(2N-1,1,N,N) \simeq M_{2N}(2N-a,a,N,N)$$

are isomorphic. To simplify combinatorics, it would be easier to work with $M_{2N}(N+2,N-2,N,N) \simeq M_{2N}(2N-1,1,N,N)$.

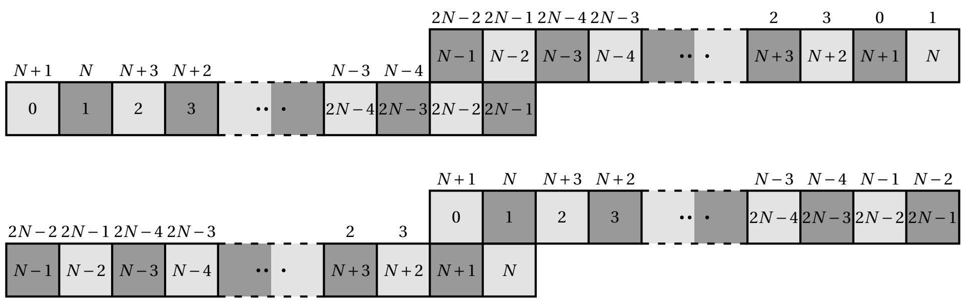
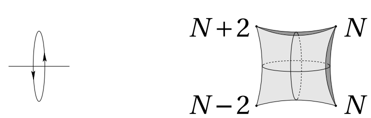


FIGURE 4. Square-tiled $M_{2N}(N+2,N-2,N,N)$, where N is odd. A square atop a white one is always black and vice versa

Suppose that the powers $N+2, N-2, N, N$ are distributed at the corners of a "square pillow" representing our flat $\mathbb{C}P^1$ as follows:



Let us color the "visible" square of the "square pillow" in white, and the complementary square in black. It is easy to see that Figure 4 represents two unfoldings (preserving enumeration of the squares) of the same square-tiled cyclic cover $M_{2N}(N+2,N-2,N,N)$, where we enumerate separately white and black squares by numbers from 0 to $2N-1$. Moreover, the natural generator of the group of deck transformations maps a square number k to a square of the same color having number $(k+1) \bmod 2N$.

In enumeration of Figure 4, the involution τ defined in equation (B.1) acts as

$$\tau(k_{\text{white}}) = (N-1-k)_{\text{black}},$$

which corresponds to a superposition of the two patterns of our square-tiled surface indicated at Figure 4. Passing to a quotient over τ we obtain a square-tiled surface represented at Figure 5. Note that the top horizontal sides of each pair of adjacent squares having numbers $2k$ and $2k+1$ is glued to bottom horizontal sides of a pair of consecutive squares number $2N-2-2k$ and $2N-1-2k$. Hence, we can tile the surface at Figure 5 with rectangles of size 2×1. Rescaling the resulting surface by a contraction in the horizontal direction and by an

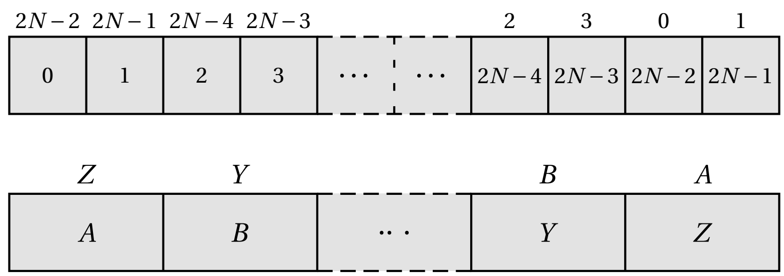


FIGURE 5. Quotient of a square-tiled cyclic cover $M_{2N}(N+2, N-2, N, N)$ over the involution τ

expansion in a vertical direction, which transforms rectangles of size 2×1 into squares, we obtain a square-tiled surface $S(N)$ as at Figure 3. □

LEMMA B.7. *The spectrum of nonnegative Lyapunov exponents of the Hodge bundle $H^1_{\mathbb{R}}$ over the Teichmüller curve of a square-tiled cyclic cover $M_{2N}(2N-1, 1, N, N)$ has the following form for odd $N > 1$:*

$$\Lambda\mathrm{Spec} = \left\{\frac{1}{N}, \frac{1}{N}, \frac{3}{N}, \frac{3}{N}, \dots, \frac{N-2}{N}, \frac{N-2}{N}, 1\right\}.$$

Proof. Since $\gcd(2N, 2N-1) = \gcd(2N, 1) = 1$, we conclude that

$$t_1(k) > 0 \quad \text{and} \quad t_2(k) > 0 \qquad \text{for } k = 1, \dots, N-1,$$

where

$$t_1(k) = \left\{\frac{2N-1}{2N} \cdot k\right\} \qquad \text{and} \qquad t_2(k) = \left\{\frac{1}{2N} \cdot k\right\},$$

see (2.2). Since

$$\frac{2N-1}{2N} \cdot k + \frac{1}{2N} \cdot k = k \in \mathbb{Z},$$

we obtain

$$t_1(k) + t_2(k) = 1 \qquad \text{for } k = 1, \dots, N-1.$$

Note that

$$t_3(k) = t_4(k) = \left\{\frac{N}{2N} \cdot k\right\} = \left\{\frac{k}{2}\right\}.$$

Hence, by definition (2.3) of $t(k)$ and by formula (2.4), we get

$$\dim_{\mathbb{C}} V^{1,0}(k) = \dim_{\mathbb{C}} V^{1,0}(N-k) = \begin{cases} 1 & \text{when } k \text{ is odd,} \\ 0 & \text{when } k \text{ is even.} \end{cases}$$

Finally, our computation shows that

$$2d(k) = 2\min(t_1(k), 1-t_1(k), \dots, t_4(k), 1-t_4(k)) = \frac{k}{N}$$

for $k = 1, 3, \dots, 2j+1, \dots, N$. The statement of the lemma follows now from Corollary 2.7. □

LEMMA B.8. *For any even N the surface $S(N)$ as in Figure 3 corresponds to a quotient of a square-tiled cyclic cover $M_N(N-1,1,N-1,1)$ over an involution τ as in equation* (B.1).

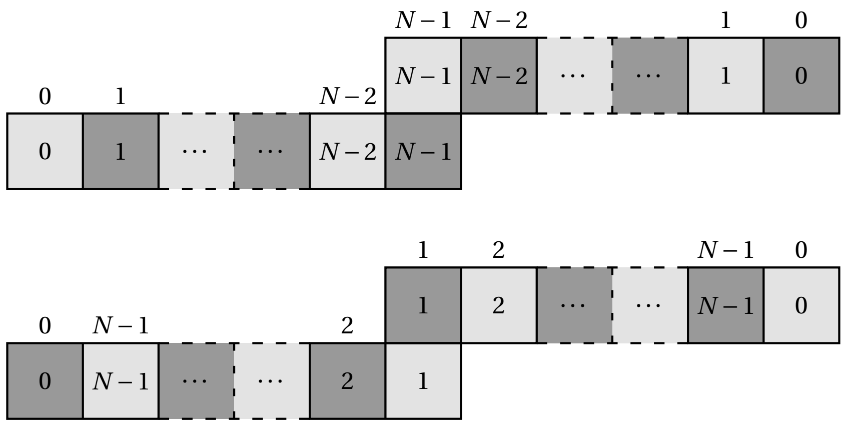


FIGURE 6. Square-tiled $M_N(N-1,1,N-1,1)$ for even N. A square atop a white one is always black and vice versa

Proof. The proof is analogous to the one of Lemma B.6. Suppose that the powers $N-1,1,N-1,1$ are distributed at the corners of a "square pillow" representing our flat $\mathbb{C}P^1$ as follows:

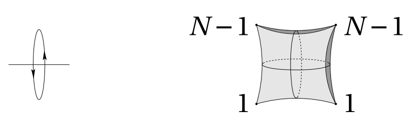


Figure 6 represents two unfoldings (preserving an enumeration of squares) of the square-tiled cyclic cover $M_N(N-1,1,N-1,1)$, where we enumerate separately white and black squares by numbers from 0 to $N-1$. The natural generator of the group of deck transformations maps a square number k to a square of the same color having number $(k+1) \bmod N$.

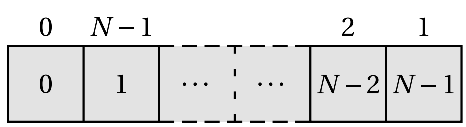


FIGURE 7. Quotient of a square-tiled cyclic cover $M_N(N-1,1,N-1,1)$ over an involution

In enumeration of squares as in Figure 6, the involution τ defined in equation (B.1) acts as

$$\tau(k_{\text{white}}) = (N-k)_{\text{black}}$$

which corresponds to a superposition of the two patterns of our square-tiled surface indicated at Figure 6. Passing to a quotient over τ we obtain a square-tiled surface represented at Figure 7. Clearly, it coincides with the square-tiled surface $S(N)$ as in Figure 3. □

LEMMA B.9. *The spectrum of nonnegative Lyapunov exponents of the Hodge bundle $H^1_{\mathbb{R}}$ over the Teichmüller curve of a square-tiled cyclic cover $M_N(N-1,1,N-1,1)$ has the following form for even $N>2$:*

$$\Lambda\mathrm{Spec}=\left\{\frac{2}{N},\,\frac{2}{N},\,\frac{4}{N},\,\frac{4}{N},\,\dots,\,\frac{N-2}{N},\,\frac{N-2}{N},\,1\right\}.$$

Proof. The proof is completely analogous to the proof of Lemma B.7 and is left to the reader. □

Proposition B.5 is proved.

REMARK B.10. Note that the surface $S(2N)$ as in Figure 3 admits an automorphism $\sigma\colon S\to S$ such that $\sigma^*\omega=\omega$. In enumeration of Figure 7, the corresponding permutation σ is defined as

$$\sigma(k)=k+N \mod 2N.$$

One easily checks that $S(2N)/\sigma = S(N)$. For even N, the induced double cover $S(2N)\to S(N)$ has two ramification points at the two zeroes of $S(2N)$. For odd N, the cover $S(2N)\to S(N)$ is unramified.

This cover extends to an involution of the corresponding universal curve and, thus, induces a map of the Teichmüller curve of $S(2N)$ to the Teichmüller curve of $S(N)$. The latter map is an isomorphism for even N and a double cover for odd N, see Section B.2 below.

REMARK B.11. Note that the quotient τ of a square-tiled cyclic cover $M_{2N}(2N-1,1,2N-1,1)$ over $S(2N)$ *does not* extends to an automorphism of the universal curve. In particular, the Teichmüller curve of the square-tiled $M_{2N}(2N-1,1,2N-1,1)$ is a double *quotient* of the one of $S(2N)$, see Section B.2 below. There is no contradiction with Proposition B.1: if we *give names* to the four zeroes of $M_{2N}(2N-1,1,2N-1,1)$ breaking part of the symmetry (as it was done in Proposition B.1), the Teichmüller curve of the square-tiled $M_{2N}(2N-1,1,2N-1,1)$ will become isomorphic to the one of the $S(2N)$.

A square-tiled cyclic cover $M_{2N}(2N-1,1,2N-1,1)$ admits another involution τ_2 preserving the flat structure, $\tau_2\omega=\omega$. It is induced from a unique involution of the underlying $\mathbb{C}\mathrm{P}^1$ which interchanges the points in each of the two pairs P_1,P_3, and P_2,P_4 (the points in each pair correspond to the *same* power $2N-1$ or 1, respectively). This automorphism commutes with the deck transformations, $\tau_2 T = T\tau_2$. The quotient of a square-tiled cyclic cover $M_{2N}(2N-1,1,2N-1,1)$ over τ_2 is isomorphic to a cyclic cover $M_{2N}(2N-1,1,N,N)$ tiled with rectangles $1\times\frac{1}{2}$.

REMARK B.12. A square-tiled cyclic cover $M_{2N}(2N-1,1,N,N)$ is obviously a double cover over $M_N(N-1,1,N,N)$. Clearly, a quadratic differential p^*q_0 induced from q_0 on $\mathbb{C}\mathrm{P}^1$ by the projection

$$p\colon M_N(N-1,1,N,N) \to \mathbb{C}\mathrm{P}^1$$

has two zeroes of degrees $N-2$ and $2N$ simple poles. Hence, the cyclic cover $M_N(N-1,1,N,N)$ is a Riemann sphere, which implies that $M_{2N}(2N-1,1,N,N)$ is a hyperelliptic Riemann surface. Notice that the square-tiled cyclic cover $M_{2N}(2N-1,1,N,N)$ belongs to the stratum $\mathcal{H}(N-1,N-1)$. However, the hyperelliptic involution of this flat surface fixes the zeroes, so the square-tiled cyclic cover $M_{2N}(2N-1,1,N,N)$ belongs to the component $\mathcal{H}^{\text{odd}}(N-1,N-1)$ and not to the hyperelliptic component (see Proposition 7 in [14]).

B.2. **Orbits of several distinguished square-tiled surfaces.** For the sake of completeness, we describe the $\mathrm{SL}(2,\mathbb{Z})$-orbits of the square-tiled surfaces discussed above. Clearly, these data also describes the structure of the corresponding arithmetic Teichmüller curve.

All surfaces considered below are hyperelliptic, so their $\mathrm{SL}(2,\mathbb{Z})$-orbits coincide with $\mathrm{PSL}(2,\mathbb{Z})$-orbits. We denote generators of $\mathrm{SL}(2,\mathbb{Z})$ by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Veech groups of the "stairs" square-tiled surfaces which appear in the orbits of surfaces $S(N)$ were found by G. Schmithüsen in [18]; the generalized "stairs" square-tiled surfaces are studied by M. Schmoll in [20].

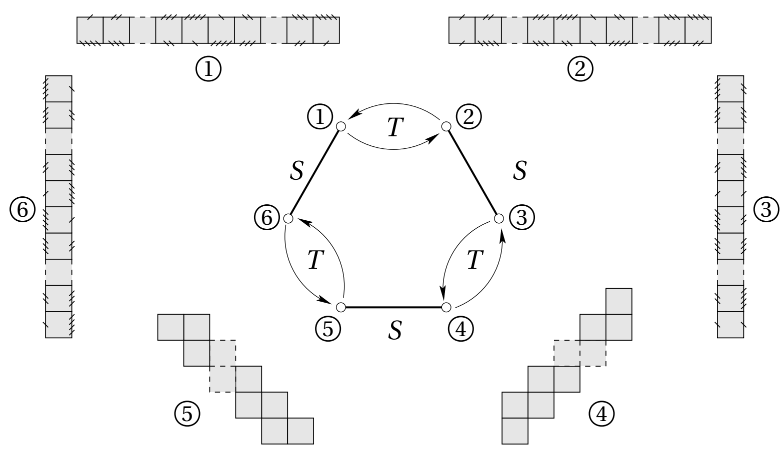


FIGURE 8.

Figures 8 and 9 represent the $\mathrm{SL}(2,\mathbb{Z})$-orbit of the surface $S(N)$ from Figure 3 when N is even and when N is odd, respectively.

Figure 9 also represents the $\mathrm{SL}(2,\mathbb{Z})$-orbit of the square-tiled cyclic cover $M_N(N-1,1,N-1,1)$ (the three square-tiled surfaces at the bottom of Figure 9).

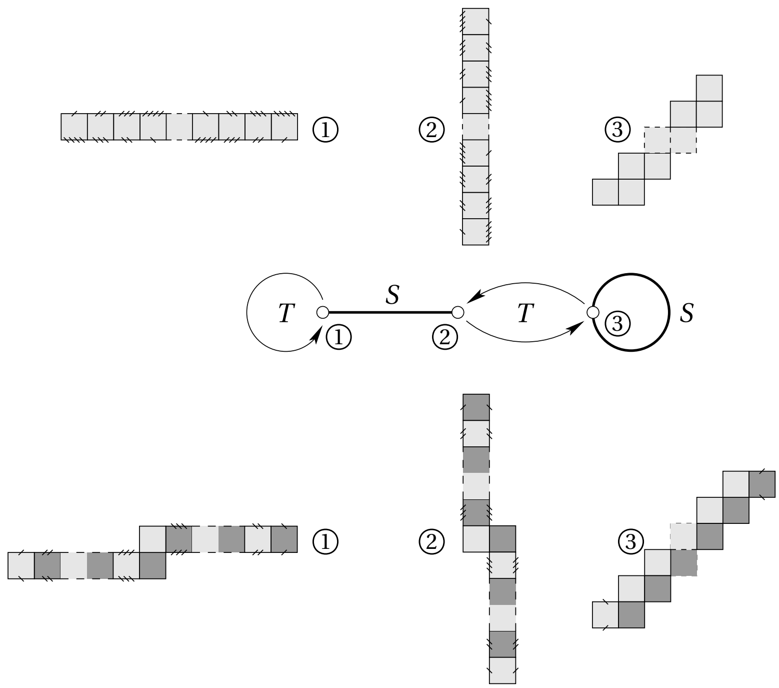


FIGURE 9.

APPENDIX C. HOMOLOGICAL DIMENSION OF THE SQUARE-TILED CYCLIC COVER $M_N(N-1,1,N-1,1)$

The following characteristic of a square-tiled surface is important for applications, see [9]. Given a square-tiled surface S_0, consider a collection of cycles $c_1,\dots,c_j$ representing the waist curves of its maximal horizontal cylinders, and let $d(S_0)$ be the dimension of the linear span of $c_1,\dots,c_j$ in $H_1(S_0,\mathbb{R})$. The *homological dimension* of an arithmetic Teichmüller curve corresponding to a square-tiled surface S_0 is defined as the maximum of $d(S)$ over S in the $\mathrm{PSL}(2,\mathbb{Z})$-orbit of S_0.

Answering the question of G. Forni [9], we show that the homological dimension of the arithmetic Teichmüller curve corresponding to the square-tiled cyclic cover $M_N(N-1,1,N-1,1)$ for even N is maximal possible: it is equal to the genus g of the corresponding Riemann surface.

The square-tiled cyclic cover $M_N(N-1,1,N-1,1)$ was discussed in the previous section. It has four conical singularities with cone angles $N\pi$. The corresponding Abelian differential has four zeroes of degrees $N/2-1$. Hence, the underlying Riemann surface has genus $g = N-1$.

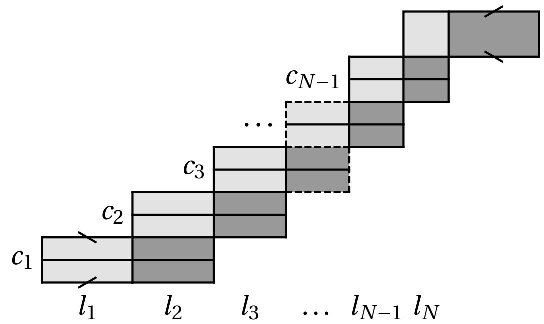


FIGURE 10. A horizontal deformation of the square-tiled cyclic cover $M_N(N-1,1,N-1,1)$

The PSL$(2,\mathbb{Z})$-orbit of the square-tiled $M_N(N-1,1,N-1,1)$ is represented at the bottom picture at the end of Section B.2. Consider a deformation of surface 3 from this orbit as in Figure 10. By $l_1,\dots,l_N$ we denote the lengths of the horizontal sides of the corresponding rectangles.

The integrals of the holomorphic form ω representing the flat structure over the cycles $c_1,\dots,c_{N-1}$ are equal to l_1+l_2, l_2+l_3, ..., $l_{N-1}+l_N$, respectively. Since $l_1,\dots,l_N$ are arbitrary positive real numbers, we can chose them in such a way that the integrals over $c_1,\dots,c_{N-1}$ become rationally independent. Hence, the integer cycles $c_1,\dots,c_{N-1}$ are linear independent over rationals and, thus, linear independent over reals. Since $g=N-1$, they span a g-dimensional Lagrangian subspace.

APPENDIX D. CYCLIC COVER $M_{30}(3,5,9,13)$

The table below (Table 1) evaluates the quantities discussed in this paper in the concrete case of the square-tiled cyclic cover $M_{30}(3,5,9,13)$. Here, $V^{1,0}$, $V^{0,1}$, and V denote $V^{1,0}(N-k)$, $V^{0,1}(N-k)$, and $V(N-k)$, respectively.

The Lyapunov exponents in the right column of the table are constructed following the algorithm from Corollary 2.7. The cyclic cover $M_{30}(3,5,9,13)$ has genus $g=25$. The table also illustrates Lemma 2.2: depending on how many of $t_i(k)$, where $i=1,2,3,4$, are null for a given k, the eigenspace $V(k)$ might have dimension 0, 1 or 2. The resulting spectrum $\{\lambda_1,\dots,\lambda_{25}\}$ is presented below:

$$\left\{1, \left(\frac{2}{5}\right)^4, \left(\frac{1}{3}\right)^2, \left(\frac{4}{15}\right)^2, \left(\frac{1}{5}\right)^6, 0^{10}\right\},$$

where powers denote multiplicities.

Acknowledgments. We are indebted to Giovanni Forni, Carlos Matheus, and Martin Möller for important discussions. We thank Alex Wright and the anonymous referees for careful reading the manuscript, indicating to us some typos and clarifying to us the paper [3]. Appendix B is motivated by a question of J.-C. Yoccoz, for which we would like to thank him.

k	$t_i(k) = \left\{\frac{a_i}{N}k\right\}$				$t(k)$	$\dim V^{1,0}$	$\dim V^{0,1}$	$\dim V$	λ
1	1/10	1/6	3/10	13/30	1	0	2	2	–
2	1/5	1/3	3/5	13/15	2	1	1	2	4/15
3	3/10	1/2	9/10	3/10	2	1	1	2	1/5
4	2/5	2/3	1/5	11/15	2	1	1	2	2/5
5	1/2	5/6	1/2	1/6	2	1	1	2	1/3
6	3/5	0	4/5	3/5	2	1	0	1	0
7	7/10	1/6	1/10	1/30	1	0	2	2	–
8	4/5	1/3	2/5	7/15	2	1	1	2	2/5
9	9/10	1/2	7/10	9/10	3	2	0	2	0, 0
10	0	2/3	0	1/3	1	0	0	0	–
11	1/10	5/6	3/10	23/30	2	1	1	2	1/5
12	1/5	0	3/5	1/5	1	0	1	1	–
13	3/10	1/6	9/10	19/30	2	1	1	2	1/5
14	2/5	1/3	1/5	1/15	1	0	2	2	–
15	1/2	1/2	1/2	1/2	2	1	1	2	1
16	3/5	2/3	4/5	14/15	3	2	0	2	0, 0
17	7/10	5/6	1/10	11/30	2	1	1	2	1/5
18	4/5	0	2/5	4/5	2	1	0	1	0
19	9/10	1/6	7/10	7/30	2	1	1	2	1/5
20	0	1/3	0	2/3	1	0	0	0	–
21	1/10	1/2	3/10	1/10	1	0	2	2	–
22	1/5	2/3	3/5	8/15	2	1	1	2	2/5
23	3/10	5/6	9/10	29/30	3	2	0	2	0, 0
24	2/5	0	1/5	2/5	1	0	1	1	–
25	1/2	1/6	1/2	5/6	2	1	1	2	1/3
26	3/5	1/3	4/5	4/15	2	1	1	2	2/5
27	7/10	1/2	1/10	7/10	2	1	1	2	1/5
28	4/5	2/3	2/5	2/15	2	1	1	2	4/15
29	9/10	5/6	7/10	17/30	3	2	0	2	0, 0

TABLE 1. Example: decomposition of the Hodge bundle for the square-tiled cyclic cover $M_{30}(3,5,9,13)$.

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