Maryam Mirzakhani (1977–2017)

Anton Zorich (Université Paris-Diderot, Paris, France)

> *... je dirai quelques mots sur toi, mais je ne te gênerai point en insistant avec lourdeur sur ton courage ou sur ta valeur professionnelle. C'est autre chose que je voudrais décrire ... Il est une qualité qui n'a point de nom. Peut-être est-ce la "gravité", mais le mot ne satisfait pas. Car cette qualité peut s'accompagner de la gaieté la plus souriante ...*
>
> Antoine de Saint-Exupéry

> *You have to ignore low-hanging fruit, which is a little tricky. I am not sure if it is the best way of doing things, actually – you are torturing yourself along the way. But life is not supposed to be easy.*
>
> Maryam Mirzakhani

On 14 July 2017, Maryam Mirzakhani died. Less than three years earlier, she had received the Fields Medal *"for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces"*, becoming the first woman to win the Fields Medal. She was often the first. For example, together with her friend Roya Beheshti, she was the first Iranian girl to participate in the International Mathematical Olympiad. She won two gold medals: in 1994 and in 1995. Despite all the glory, Maryam always remained extremely nice, friendly, modest and not the least bit standoffish. Meeting her at a conference, you would, at first glance, take her for a young postdoc rather than a celebrated star. She worked hard, mostly "keeping low profile" (using her own words). Kasra Rafi, Maryam's friend since school years, said about her: "*Everything she touched she made better*". This concerned things much broader than just mathematics.

Maryam was born and grew up in Tehran with a sister and two brothers. In one of her rare interviews (given on the demand of the Clay Mathematics Institute at the end of her Clay Research Fellowship), she said: *"My parents were always very supportive and encouraging. It was important for them that we have meaningful and satisfying professions but they did not care as much about success and achievement."* After passing a severe entrance test, Maryam entered the Farzanegan school for girls in Tehran. Having completed her undergraduate studies in Sharif University in Tehran in 1999, she came to Harvard University for graduate studies and received her PhD degree in 2004. The results of her thesis were astonishing for everybody, including Maryam's doctoral advisor C. McMullen: Maryam had discovered beautiful ties between seemingly very different geometric counting problems. In particular, she had discovered how the count of closed non-self-intersecting geodesics on hyperbolic surfaces is related to the Weil–Petersson volumes of the moduli spaces of bordered hyperbolic surfaces. As an application, Maryam had found an alternative proof of Witten's celebrated conjecture first proved by M. Kontsevich.

Maryam in CIRM, Luminy, 2008
(Picture by F. Labourie)

Amazing thesis

I cannot adequately describe the full depth of Maryam's amazing thesis. Maryam's perceptive insight on how to use dynamics in geometric problems would, unfortunately, remain invisible. Hopefully, this brief description will give an idea of the key actors and of the interplay between them.

Simple closed geodesics. Moduli spaces.

A closed curve on a surface is called *simple* if it does not have self-intersections. Closed geodesics on a hyperbolic surface usually do have self-intersections. Indeed, since the classical works of Delsarte, Hubert and Selberg, it is known that the number of closed geodesics of length at most L on a hyperbolic surface grows with the rate e^L/L when the bound L grows. However, Mary Rees and Igor Rivin showed that the number $N(X, L)$ of *simple* closed geodesics of length at most L grows only polynomially in L. Maryam went further and obtained striking results on this more subtle count of simple closed hyperbolic geodesics.

Let us start with a concrete example of a family of hyperbolic surfaces. Consider a configuration of six distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using an appropriate holomorphic automorphism of the Riemann sphere, we can send three out of six points to, say, 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of the Riemann sphere fixing 0, 1 and ∞ is already the identity transformation. Hence, the three remaining points serve as three independent complex parameters in the space of configurations $\mathcal{M}_{0,6}$ of six points on the Riemann sphere, considered up to a holomorphic diffeomorphism.

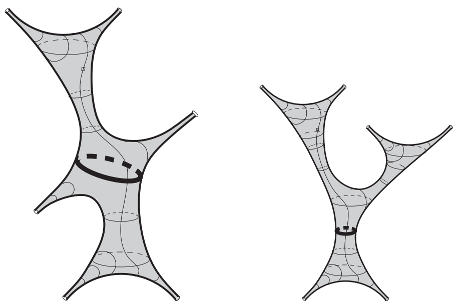

Figure 1. Schematic picture of hyperbolic spheres with cusps

By the uniformisation theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points, so the moduli space $\mathcal{M}_{0,6}$ can also be seen as the family of hyperbolic spheres with six cusps.

Deforming the configuration of points on $\mathbb{C}\mathrm{P}^1$, we can drastically change the shape of the corresponding hyperbolic surface, making it quite symmetric or, on the contrary, creating very long and very narrow bottlenecks between parts of the surface.

The space $\mathcal{M}_{g,n}$ of configurations of n distinct points on a smooth closed orientable Riemann surface of genus $g \geq 2$ is even richer. While the sphere admits only one complex structure, a surface of genus $g \geq 2$ admits a complex $(3g-3)$-dimensional family of complex structures. As in the case of the Riemann sphere, complex structures on a smooth surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points. For genus $g \geq 2$, one can let $n = 0$ and consider the space $\mathcal{M}_g = \mathcal{M}_{g,0}$ of hyperbolic surfaces without cusps.

Theorem 1 (Mirzakhani, 2008)**.** For any hyperbolic surface X in the family $\mathcal{M}_{g,n}$, the number of simple closed geodesics has exact polynomial asymptotics:

$$\lim_{L\to+\infty} \frac{N(X,L)}{L^{6g-6+2n}} = const(X)\,,$$

where the constant $const(X)$ admits explicit geometric interpretation, and the power of the bound L in the denominator is the dimension $\dim_{\mathbb{R}} \mathcal{M}_{g,n} = 6g-6+2n$ of the corresponding family of hyperbolic surfaces.

Now, I would like to describe a discovery of Maryam that I find particularly beautiful and, at first glance, even difficult to believe. Let us return to hyperbolic spheres with six cusps, as in Figure 1. A simple closed geodesic on a hyperbolic sphere separates the sphere into two components. We either get three cusps on each of these components (as in the left picture in Figure 1) or two cusps on one component and four cusps on the complementary component (as in the right picture in Figure 1). Hyperbolic geometry excludes other partitions. Denote the numbers of such specialised simple closed geodesics by $N_{3+3}(X,L)$ and by $N_{2+4}(X,L)$ respectively. We have $N_{3+3}(X,L) + N_{2+4}(X,L) = N(X,L)$.

Maryam proved that the asymptotic frequency of simple closed geodesics of each topological type is well defined for every hyperbolic surface and computed it. In our example, Maryam's computation gives the following proportions:

$$\lim_{L\to+\infty} \left(N_{3+3}(X,L) : N_{2+4}(X,L)\right) = 4:3\,.$$

Isn't it astonishing: the asymptotic frequency of simple closed geodesics of a given topological type is one and the same for *any* hyperbolic surface $X \in \mathcal{M}_{0,6}$ no matter how exotic the shape of the particular hyperbolic surface is!

The result of M. Mirzakhani is, of course, much more general than this particular example. There is a finite number of equivalence classes of simple closed curves on a topological surface of genus g with n punctures, considered up to a homeomorphism of the surface. M. Mirzakhani proved that the asymptotic frequency of simple closed geodesics of each type on any hyperbolic surface X in $\mathcal{M}_{g,n}$ is well defined and is one and the same for all X in $\mathcal{M}_{g,n}$. Maryam expressed any such asymptotic frequency in terms of the *intersection numbers* of moduli spaces. In this way, Maryam described geometric properties of individual hyperbolic surfaces in terms of geometry and topology of the ambient moduli spaces.

We shall come back to intersection numbers when discussing Maryam's proof of Witten's conjecture.

Weil–Petersson volumes of moduli spaces

Now, consider several closed hyperbolic geodesics simultaneously. Assume that they have neither self-intersections nor intersections between each other. Cutting the initial hyperbolic surface by such a collection of simple closed geodesics, we get several *bordered* hyperbolic surfaces with geodesic boundary components.

We denote by $\mathcal{M}_{g,n}(b_1,\ldots,b_n)$ the moduli space of hyperbolic surfaces of genus g with n geodesic boundary components of lengths $b_1,\ldots,b_n$. By convention, the zero value $b_i = 0$ corresponds to a cusp of the hyperbolic metric, so the moduli space $\mathcal{M}_{g,n}$ considered in the previous section corresponds to $\mathcal{M}_{g,n}(0,\ldots,0)$ in this more general setting.

A hyperbolic *pair of pants* (as in Figure 2) is by far the most famous bordered hyperbolic surface. Topologically, a pair of pants is a sphere with three holes. It is known that for any triple of nonnegative numbers $(b_1,b_2,b_3) \in \mathbb{R}^3_+$, there exists a hyperbolic pair of pants $P(b_1,b_2,b_3)$ with geodesic boundaries of given lengths, and that such a hyperbolic pair of pants is unique (we always assume that the boundary components of P are numbered). It is also known that two geodesic boundary components γ_1,γ_2 of any hyperbolic pair of pants P can be joined by a single geodesic segment $\nu_{1,2}$ orthogonal to both γ_1 and γ_2 (see Figure 2). Thus, every geodesic boundary component γ of any hyperbolic pair of pants might be endowed with a canonical distinguished point. The construction can be extended to the situation when both remaining bound-

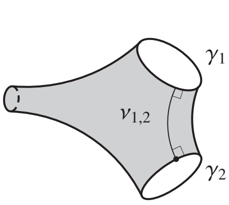

Figure 2. Hyperbolic pair of pants

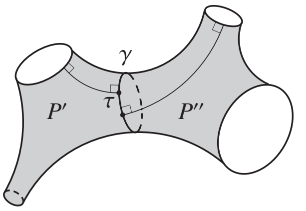

Figure 3. Twist parameter τ responsible for gluing together two hyperbolic pairs of pants

ary components of the pair of pants are represented by cusps.

Two hyperbolic pairs of pants $P'(b_1', b_2', \ell)$ and $P''(b_1'', b_2'', \ell)$ sharing the same length $\ell > 0$ of one of the geodesic boundary components can be glued together (see Figure 3). The hyperbolic metric on the resulting hyperbolic surface Y is perfectly smooth and the common geodesic boundary of P' and P'' becomes a simple closed geodesic γ on Y.

Recall that each geodesic boundary component of any pair of pants is endowed with a distinguished point. These distinguished points record how the pairs of pants P' and P'' are twisted, with respect to each other, when we glue them together by a common boundary component (see Figure 3). Hyperbolic surfaces $Y(\tau)$ corresponding to different values of the twist parameter τ in the range $[0, \ell[$ are generically not isometric.

In a similar way, any hyperbolic surface X of genus g with n geodesic boundary components admits a decomposition in hyperbolic pairs of pants glued along simple closed geodesics $\gamma_1, \ldots, \gamma_{3g-3+n}$. It is clear from what was said above that we can vary all $3g - 3 + n$ lengths $\ell_{\gamma_i}(X)$ of the resulting simple closed geodesics γ_i on X and vary the twists $\tau_{\gamma_i}(X)$ along them to obtain a deformed hyperbolic metric. The resulting collection of $2 \cdot (3g - 3 + n)$ real parameters serve as local *Fenchel–Nielsen coordinates* in the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$.

By the work of W. Goldman, each space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ carries a natural closed non-degenerate 2-form ω_{WP} called the *Weil–Peterson symplectic form*. S. Wolpert proved that ω_{WP} has a particularly simple expression in Fenchel–Nielsen coordinates, that is, no matter what pants decomposition we choose, we get

$$\omega_{WP} = \sum_{i=1}^{3g-3+n} d\ell_{\gamma_i} \wedge d\tau_{\gamma_i} .$$

The wedge power ω^n of a symplectic form on a manifold M^{2n} of real dimension $2n$ defines a volume form on M^{2n}. The volume $V_{g,n}(b_1, \ldots, b_n)$ of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ with respect to the volume form $\frac{1}{(3g-3+n)!}\,\omega_{WP}^{3g-3+n}$ is called the *Weil–Petersson* volume of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$; it is known to be finite.

To give an account of Mirzakhani's work on Weil–Petersson volumes, we start with the identity of G. McShane.

Theorem (G. McShane, 1998)**.** Let $f(x) = (1 + e^x)^{-1}$ and let X be a hyperbolic torus with a cusp. Then,

$$\sum_{\gamma} f(\ell_\gamma(X)) = \frac{1}{2} ,$$

where the sum is taken over all simple closed geodesics γ on X, and $\ell_\gamma(X)$ is the length of the geodesic γ.

This identity is, in some sense, a miracle: though the length spectrum of simple closed geodesics is different for different hyperbolic tori with a cusp, the sum above is identically $\frac{1}{2}$ for any X in $\mathcal{M}_{1,1}$. Ten years after McShane's breakthrough, M. Mirzakhani was asked to present his result at the seminar of her scientific advisor Curt McMullen. Preparing the talk, Maryam discovered a remarkable generalisation of McShane's identity to hyperbolic surfaces of any genus with any number of boundary components.

Let us discuss why such identities are relevant to the Weil–Petersson volumes of the moduli spaces. Integrating the right side of McShane's identity over the moduli space $\mathcal{M}_{1,1}$ with respect to the Weil–Petersson form, one obviously gets $\frac{1}{2}\,\mathrm{Vol}\,\mathcal{M}_{1,1}$. It is less obvious that the integral of the sum on the left side admits a geometric interpretation as the integral of f over a certain natural cover $\mathcal{M}^*_{1,1}$ of the initial moduli space $\mathcal{M}_{1,1}$. This cover is already much simpler than the original moduli space: it admits global coordinates in which the integral of f can be easily computed.

Mirzakhani's more general identity does not immediately yield the volume. However, cutting the initial surface by simple closed geodesics involved in her identity and developing the idea of averaging over all possible hyperbolic surfaces, Mirzakhani got a recursive relation for the volume $V_{g,n}(b_1, \ldots, b_n)$ in terms of volumes of simpler moduli spaces. These relations allowed Maryam to prove the following statement and to compute the volumes explicitly.

Theorem 2 (Mirzakhani, 2007)**.** The volume $V_{g,n}(b_1, \ldots, b_n)$ is a polynomial in $b_1^2, \ldots, b_n^2$; namely, we have:

$$V_{g,n}(b_1, \ldots, b_n) = \sum_{d_1+\cdots+d_n \le 3g-3+n} C_{d_1,\ldots,d_n} \cdot b_1^{2d_1} \ldots b_n^{2d_n}, \quad (1)$$

where $C_{d_1,\ldots,d_n} > 0$ lies in $\pi^{6g-6+2n-2(d_1+\cdots+d_n)} \cdot \mathbb{Q}$.

Simple recursive formulae for volumes in genera $0, 1, 2$ were found earlier by P. Zograf. Precise asymptotics of volumes for large genera were recently proved by M. Mirzakhani and P. Zograf up to a multiplicative constant conjecturally equal to $\frac{1}{\sqrt{\pi}}$, which still resists a rigorous evaluation.

Witten's conjecture

The family of all complex lines passing through the origin in $\mathbb{C}^{n+1}$ forms the complex projective space $\mathbb{C}\mathrm{P}^n$. This space carries the natural *tautological line bundle*: its fiber over a "point" $[\mathcal{L}] \in \mathbb{C}\mathrm{P}^n$ is the line $\mathcal{L}$ considered as a vector space. Any complex line bundle ξ over a compact manifold M can be induced from the tautological bundle by an appropriate map $f_\xi : M \to \mathbb{C}\mathrm{P}^n$ (for a sufficiently large n depending on M). The second cohomology of the complex projective space $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \simeq \mathbb{Z}$ has a distinguished generator c_1. The pullback $c_1(\xi) = f_\xi^* c_1 \in H^2(M; \mathbb{Z})$ is called the *first Chern class* of the line bundle ξ.

We have already used a natural bijective correspondence between hyperbolic metrics of constant negative curvature with n cusps and complex structures endowed with n distinct marked points $x_1, \ldots, x_n$ on a closed smooth surface of genus g. In this section, we use this latter interpretation of the moduli space $\mathcal{M}_{g,n}$.

Consider the cotangent space $\mathcal{L}(C, x_i)$ to the Riemann surface C at the marked point x_i. Varying $(C, x_1, \ldots, x_n)$ in $\mathcal{M}_{g,n}$, we get a family of complex lines $\mathcal{L}(C, x_i)$ parameterised by the points of $\mathcal{M}_{g,n}$. This family forms a line bundle $\mathcal{L}_i$ over the moduli space $\mathcal{M}_{g,n}$. This *tautological line bundle* $\mathcal{L}_i$ extends to the natural *Deligne–Mumford compactification* $\overline{\mathcal{M}}_{g,n}$ of the initial moduli space. The space $\overline{\mathcal{M}}_{g,n}$ is a nice compact complex orbifold so, for any $i = 1, \dots, n$, one can define the first Chern class $\psi_i := c_1(\mathcal{L}_i)$. Recall that cohomology has a ring structure so, taking a product of k cohomology classes of dimension 2 (as the first Chern class), we can integrate the resulting cohomology class over a compact complex manifold of complex dimension k. In particular, for any partition $d_1 + \cdots + d_n = 3g - 3 + n$ of $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$ into the sum of nonnegative integers, one can integrate the product $\psi_1^{d_1} \cdot \cdots \cdot \psi_n^{d_n}$ over the orbifold $\overline{\mathcal{M}}_{g,n}$. By convention, the "*intersection number*" (or the "correlator" in a physical context) is defined as

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} . \tag{2}$$

As always, when there are plenty of rational numbers indexed by partitions or such, it is useful to wrap them into a single generating function. The resulting generating function for correlators (2) is really famous. For physicists, it is the *free energy of two-dimensional topological gravity*. In mathematical terms, E. Witten conjectured in 1991 a certain recursive formula for the numbers (2) and interpreted this recursion in the form of KdV differential equations satisfied by the generating function. The conjecture caused an explosion of interest in the mathematical community: a single formula interlaced quantum gravity, enumerative algebraic geometry, combinatorics, topology and integrable systems.

The first proof of Witten's conjecture was discovered by M. Kontsevich, who used metric ribbon graphs as a "combinatorial model" of the moduli space to express the intersection numbers (2) as a sum over 3-valent ribbon graphs. Maryam Mirzakhani suggested an alternative proof. She ingeniously applied techniques of symplectic geometry to the moduli spaces of bordered Riemann surfaces $\mathcal{M}_{g,n}(b_1, \dots, b_n)$ discussed in the previous section. Maryam recognised the intersection numbers (2) in the coefficients $C_{d_1,\dots,d_n}$ from formula (1) for the Weil–Petersson volumes $V_{g,n}(b)$ (up to a routine normalisation factor). This allowed Maryam to reduce the recurrence relations for the intersection numbers contained in Witten's formula to recurrence relations for the volumes $V_{g,n}(b)$ discussed above and thus prove Witten's conjecture.

Echoes

Geometer Kasra Rafi, Maryam's school friend, says the following about her studies at Harvard: "*She faced the same challenges as the rest of us but she moved through them much more quickly. And not that everything in her life was perfect. When she finished her amazing thesis, she asked herself: 'And what if all of this is wrong … ?' She had the fears that everybody has, she felt all the anxiety, but she managed to pass through it way more quickly than you can imagine …*"

Maryam's thesis was not wrong. It is a true masterpiece. The proofs are neither very technical nor particularly complicated. However, they insightfully put together tools and ideas from many different areas of contemporary dynamics and geometry. Reading the three relatively short papers describing this work gives the same euphoric feeling as listening to your favourite piece of music, reading your best-loved poet or gazing upon your preferred painting. Re-reading these papers might echo something you had been thinking about and reveal a simple and unexpected solution.

For my collaborators and me, it has already happened several times: Maryam's papers are full of beautiful ideas that are still at the stage of being absorbed by the mathematical community. For example, the proportion 4 : 3 for asymptotic frequencies of simple closed geodesics on a hyperbolic sphere with six cusps was very recently verified and confirmed by M. Bell and S. Schleimer, who used train tracks in their experiments. V. Delecroix, E. Goujard, P. Zograf and I recently proved that square-tiled surfaces of different combinatorial types have the same asymptotic frequencies as those discovered by Maryam for the corresponding simple closed hyperbolic geodesics.

Slow thinker

Having defended her PhD thesis, Maryam Mirzakhani got a prestigious Clay Mathematics Institute Research Fellowship. (Note that three out of four 2014 Fields Medallists are also former Clay Research Fellows.) In the same interview that I mentioned above, she said about this period of time: *"It was a great opportunity for me; I spent most of my time at Princeton, which was a great experience. The Clay Fellowship gave me the freedom to think about harder problems, travel freely and talk to other mathematicians. I am a slow thinker and have to spend a lot of time before I can clean up my ideas and make progress. So I really appreciate that I didn't have to write up my work in a rush."* What Maryam called "slowness" is actually "depth" or a kind of quality that Saint-Exupéry fails to describe in one word. In 2008, at the age of 31, Maryam Mirzakhani became a full professor at Stanford University, where she worked ever since.

To mention just one of Mirzakhani's numerous results of this period, I have to say a word about the *earthquake flow* introduced by Thurston. Given a non-self-intersecting closed geodesic on a hyperbolic surface, you can cut the surface by the geodesic, twist the two sides of the cut with respect to each other and reglue the cuts to get a new hyperbolic surface as in Figure 3. Having the imagination of Bill Thurston, you can twist a hyperbolic surface X along a closed subset of X, formed as a disjoint union of simple geodesics (such a subset is called a *hyperbolic lamination*). Moreover, Thurston defined a continuous family of simultaneous twists on the large space $\mathcal{ML}_{g,n}$ of all *measured geodesic laminations* on all hyperbolic surfaces. For many years, the properties of the resulting earthquake flow were completely enigmatic. In particular, it was not known whether it had any dense trajectories.

One more time Maryam Mirzakhani discovered unexpected ties between seemingly different objects. In some sense, she recognised in Thurston's earthquake flow the much more familiar and better understood horocycle flow on the moduli space of quadratic differentials. More precisely, she established a *measure isomorphism* between the two flows with respect to the corresponding natural invariant measures. Some important applications of this theorem were obtained instantly; some were recognised very recently – a decade later. I am sure that more will appear in the future. Mathematics is a slow science (in the same sense that Maryam was a "slow thinker").

If you ever saw Maryam attend a lecture in a large auditorium like in MSRI, she was always standing behind the back row of seats. It was neither impatience nor extravagance. I have never seen the slightest trace of a capricious "genius Olympiad kid" in Maryam: she simply had serious health problems with her back, which she never manifested otherwise. She would later mention with humour and irony that "serious" might become very relative.

Magic Wand theorem

In addition to having brilliant ideas, Maryam worked hard, as she worked on the Magic Wand theorem. From a dynamical point of view, the moduli space of holomorphic differentials can be viewed as a "homogeneous space with difficulties". I am citing Alex Eskin, who knows both facets very well: how the dynamics on the moduli space might mimic the homogeneous dynamics in some situations and how deep the difficulties might be.

The rigidity theorems, including and generalising the theorems proved by Marina Ratner at the beginning of the 1990s, show why homogeneous dynamics is so special. (Sadly, Marina Ratner also died just a week before Maryam.) General dynamical systems usually have some very peculiar trajectories living in very peculiar fractal subsets. Such trajectories are rare but there are still plenty of them. In particular, the question of identification of all (versus almost all) orbit closures or of all invariant measures has no sense for most dynamical systems: the jungle of exotic trajectories is too large. In certain situations, this diversity creates a major difficulty: even when you know plenty of fine properties of the trajectory launched from almost every starting point, you have no algorithm to check whether the particular initial condition you are interested in is generic or not. Ergodic theory is aimed at responding to statistical questions but might become completely powerless in the study of specific initial data.

The situation in homogeneous dynamics is radically different. In certain favourable cases, one can prove that *any* orbit closure is a nice homogeneous space, *any* invariant measure is the corresponding Haar measure, etc. This kind of rigidity allows one to obtain fantastic applications to number theory, developed, in particular, by J. Bourgain, E. Lindenstrauss, G. Margulis and T. Tao (to mention only Fields Medallists out of an extremely impressive list of celebrated scientists working in this area).

For several decades, it was not clear to what extent the dynamics of $SL(2,\mathbb{R})$-action on the moduli space of Abelian and quadratic differentials resembles homogeneous dynamics. For Alex Eskin, who came to dynamics in the moduli space from homogeneous dynamics, it was, probably, the main challenge for 15 years. Maryam Mirzakhani joined him in working on this problem around 2006. She was challenged by the result of her scientific advisor Curt McMullen, who had solved the problem in the particular case of genus two, ingeniously reducing it to the homogeneous case of genus one. After several years of collaboration, the first major part of the theorem, namely the measure classification for $SL(2,\mathbb{R})$-invariant measures, was proved. We forced Alex Eskin to announce it at the conference in Bonn in the Summer of 2010.

To illustrate the importance of this theorem, I cite what Artur Avila said about this result of Eskin and Mirzakhani to S. Roberts for the *New Yorker* article in memory of Maryam: *"Upon hearing about this result, and knowing her earlier work, I was certain that she would be a front-runner for the Fields Medals to be given in 2014, so much so that I did not expect to have much of a chance."* I do not think that Maryam thought much about the Fields Medal at this time (several years later she took the email message from Ingrid Daubechies announcing that she had received the Fields medal as a joke and just ignored it) but she certainly knew how important the theorem was. For the last few years, basically every paper in my domain has used the Magic Wand theorem in one way or another.

However, it took Alex Eskin and Maryam Mirzakhani several more years of extremely hard work to extend their result, proving the rigidity properties not only for the group $SL(2,\mathbb{R})$ of all 2×2 matrices with unit determinant but also for its subgroup of upper-triangular 2×2 matrices (which is already *amenable*). The difference might seem insignificant. However, exactly this difference is needed for the most powerful version of the Magic Wand theorem. Part of the theorem concerning orbit closures was proved in collaboration with Amir Mohammadi; an important complement was proved by Simion Filip.

Suppose, for example, that you have to study billiards in a rational polygon (that is, in a polygon with angles that are rational multiples of π). What mathematical object can be more simple-minded and down-to-earth than a rational triangle? However, the only known efficient approach to the study of billiards in rational polygons consists of the following. Applying symmetries over the sides of the polygon, unfold your billiard trajectory into a closed surface. The billiard trajectory gets unfolded into a non-self-intersecting winding line on this closed *translation surface*. This nice trick is called Katok–Zemlyakov construction.

Consider, for example, the triangle with angles $\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{4}$. It is easy to check that a generic billiard trajectory moves in one of eight directions at any time. We can unfold the triangle to a regular octagon glued from eight copies of the triangle. Identifying the opposite sides of the octagon, we get a closed surface of genus two endowed with a flat metric. There is no

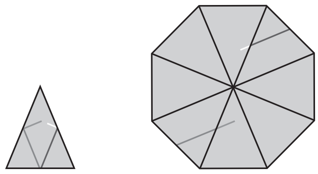

Figure 4. Billiard trajectories unfold into leaves of surface foliation

contradiction with Gauss–Bonnet theorem, since our flat metric has a conical singularity: all vertices of the octagon are glued into one point with the cone angle 6π. Geodesics on the resulting flat surface correspond to unfolded billiard trajectories.

It is convenient to incorporate the direction of the straight-line foliation into the structure of the translation surface and, turning the resulting polygon, place the distinguished direction into a vertical position. Acting on such "polarised" octagons with linear transformations of the plane, we get other octagons with sides distributed into pairs, where sides in each pair are parallel and have equal lengths. From any such polygon, we can glue a translation surface. Having a translation surface, we can unwrap it to a polygon in many ways (see Figure 5). This gives an idea of why the $GL(2,\mathbb{R})$-action on the space of translation surfaces is anything but easy to study.

Now, touch the resulting translation surface with the Magic Wand theorem to get the closure of its $GL(2,\mathbb{R})$-orbit in the moduli space of all translation surfaces sharing the same combinatorial geometry as the initial surface. According to the Magic Wand theorem, the orbit closure is a very nice orbifold that would provide you with plenty of fine information about the initial state. Is it not like getting a Cinderella Pumpkin Coach? One of the last works of Maryam Mirzakhani, performed in collaboration with Alex Wright, proves that despite the fact that the translation surface obtained after unfolding a rational triangle has plenty of symmetries, the corresponding orbit closure is often as large as it can be: it coincides with the entire ambient moduli space of translation surfaces. (The triangle considered above, however, gives a small orbit closure.)

The proof of the Magic Wand theorem is a titanic work, which absorbed numerous recent fundamental developments in dynamical systems; most of these developments do not have any direct relation to moduli spaces. I still do not understand how they managed to do it. Very serious technical difficulties appeared at every stage of the project, not to mention that in the four years between 2010 and 2014 Maryam gave birth to a daughter and managed to overcome the first attack of cancer. Since then, I believed that Maryam could do everything.

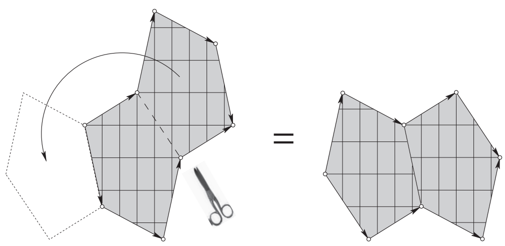

Figure 5. Polygonal patterns of the same translation surface

I cannot help telling a story that is symbolic to me. After M. Mirzakhani received the Fields Medal, I was asked by the "Gazette of the SMF" to write an article about the Magic Wand theorem and to ask Maryam for her picture to include in the article. The photograph that I received from Maryam was unexpected for a scientific paper: a three-year-old girl was holding two balloons of sophisticated shape (Riemann surfaces) almost as big as the girl herself.

However, the picture seemed to me absolutely appropriate. It perfectly represented my own image of Maryam; I was just surprised that she would suggest such a picture herself. Maryam carried through her entire life the curiosity and imagination that are so natural for children but which, unfortunately, are lost by most grown-ups.

Then came the next email: "Oops, sorry Anton, you got a picture of my daughter :-)" I had taken Anahita for Maryam.

Curt McMullen has a story related to Anahita that occurred during the ICM laudation. Presenting Maryam's work to thousands of people, Curt was nervous, asking himself how Maryam, sitting in the front row, might perceive his description of her accomplishments. During the talk, he realised that Maryam was spending most of the time tending to Anahita sitting on her knees.

In the Autumn of 2016, I learned that the illness had come back. But I also knew for sure that Maryam was doing her best to stay with her daughter and with her family as long as possible. I was not the only one to believe that Maryam could do what no other human can do. But by admiring someone's outstanding courage, we cannot expect that person to produce miracles.

If you want to learn more about Maryam as a personality, read the article "A Tenacious Explorer of Abstract Surfaces", written by Erica Klarreich for *Quanta* Magazine, and watch the Stanford Memorial recorded on October 2017 on YouTube. Maryam's husband, Jan Vondrák, her shield and support, said at the memorial: "*I want to say to the young people who are asking questions 'What would Maryam say?' that though she was a role model, it does not mean that you have to be exactly like her … You have to find your own path. You have to find what you love. You have to find what you are good at and what is meaningful to you. And if you do it well then you would have made Maryam happy.*"

"A light was turned off today," wrote Firouz Naderi, announcing Maryam's death. Both Maryam's work and her personality inspired and encouraged many people all over the world – women and men. Maryam's light will be kept inside us.

Anton Zorich (anton.zorich@gmail.com) works at University Paris Diderot and frequently visits the Center of Advanced Studies at Skoltech. His current research interests are mostly focused on interactions between geometry and dynamics in moduli spaces.

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