Maryam Mirzakhani: 1977–2017

Communicated by Hélène Barcelo and Stephen Kennedy

Maryam Mirzakhani's Harvard PhD dissertation under Curt McMullen was widely acclaimed and contained already the seeds of what would become her first three major papers. All three of these results—a new proof of Witten's conjecture, a computation of the volume of the moduli space of curves, and an asymptotic count of the number of simple closed geodesics on a hyperbolic surface—were deep, beautiful, and unexpected. Subsequent papers revealed the geometry and dynamics of the moduli space of hyperbolic surfaces, culminating in the rigidity result achieved jointly with Alex Eskin. Her work exhibits great depth of insight, wide-ranging technical mastery, and creativity and imagination of a very high order. It earned her a Fields Medal in 2014.

Mirzakhani was born on May 12, 1977, in Tehran and grew up in postrevolutionary Iran during the Iran–Iraq War. One of four children, her father was an engineer, her mother a homemaker. All three of her siblings became engineers. It was in middle school that she discovered her passion and talent for mathematics. In high school she participated in the International Mathematical Olympiad, winning gold medals in 1994 and 1995. She earned a bachelor's degree in mathematics from Sharif University in Tehran and headed to Cambridge, Massachusetts, for graduate study at Harvard.

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Mirzakhani spoke at the 2015 Clay Research Conference at Oxford after receiving the Fields Medal in 2014.

When Mirzakhani was awarded the Fields Medal in 2014 she had already been diagnosed with the cancer that would eventually claim her life. Always a private and humble person, she did not welcome the attention and acclaim that accompanied the prize. Together with her husband, Jan Vondrák, she wanted to raise their daughter, live her life, and pursue mathematics.

Notices asked friends, collaborators, and students of Mirzakhani to write brief reminiscences of her life. We also sought expository work on her mathematical accomplishments from Ursula Hamenstädt, Scott Wolpert, and, writing jointly, Alex Wright and Anton Zorich. The portrait that emerges from these pieces is of a warm, generous, and modest friend, teacher, and colleague—a mathematician of great talent and imagination coupled with grit and persistence to a very unusual degree.

Maryam Mirzakhani died, aged forty, on July 14, 2017.

The following videos provide more information and testimonials on Maryam Mirzakhani's life and work. The Harvard and Stanford memorial services contain personal reflections and reminiscences from friends, colleagues, and family members who have asked that their contribution to this article be via this medium.

McMullen's Fields laudation: www.youtube.com watch?v=VC7MZv1JH8w&t=3s.

The Stanford memorial service: www.youtube.com /watch?v=IUfB2HadIBw.

The Harvard memorial service: www.youtube.com /watch?v=HUBnzTTQ5jk.

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Opening photo courtesy of Alex Wright and Anton Zorich.

Photo of Mirzakhani lecturing courtesy of Roya Beheshti.

Roya Beheshti

Maryam Mirzakhani in Iran

When we were teenagers, Maryam and I would often talk about where in life we wanted to be at forty. In our minds, forty was the peak of everything in life. What I could not imagine was that one of us would be writing in memory of the other at forty.

I met Maryam in 1988 when we were both eleven and had just started middle school. We became close friends almost instantly, and for the next seven years at school we sat next to each other at a shared desk. My early memories of her are that she was well read, had a passion for writing, could easily get into heated debates over social or political issues, and would get very bitter about any kind of prejudice against women. Our friendship did not have its genesis in math. In fact, math was the only subject at which Maryam was not among the top students in the sixth grade. A memory that I distinctly recall is when our math teacher was returning our tests to us toward the end of the academic year. Maryam had received a score of sixteen out of twenty, and although the test had been difficult, there were students who had done better than that. Maryam was so frustrated by her score that before putting the test in her bag, she tore it apart, and announced that that was it for her and she was not going to even try to do better at math! That did not last long. After the summer break, Maryam came back with her confidence regained and started to do very well. Soon after that math became a shared passion between us, and we started to spend a lot of time thinking and talking about math.

The middle/high school that we went to was a special school for gifted female students in Tehran. Each year around one hundred students were admitted to that school through a competitive entrance exam. The principal of the school, Ms. Haerizadeh, was a strong woman with a vision who would always do anything to make sure that the students in our school had the same opportunities as the students in the equivalent boys' school. We had a strong team of teachers, and the overall environment was very encouraging for students who were interested in math or science.

When we entered middle school the Iran–Iraq War had been over for a month, and the country was becoming more stable.[1] Around that time a few programs for high school students were initiated that I think played a big role in developing interest in mathematics for several Iranian mathematicians of my generation. One of them was the Math Olympiad competitions. Another was a summer workshop run by Sharif University for high school

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[1]*After the Islamic Revolution in 1979, all the universities in Iran were closed for almost three years as part of the Cultural Revolution, the Islamization of education. When the colleges resumed classes in 1983, the country was in the middle of a costly war that made it difficult for the educational system to move forward.*

As eleventh-grade students in 1994 Mirzakhani and Behesti became the first female students on the Iranian Mathematical Olympiad team. Maryam Mirzakhani (gold), Roya Beheshti (silver), Ali Norumohammadi (bronze), Omid Naghshineh (bronze), Maziyar Raminrad (gold), and Reza Sadeghi (silver).

students to introduce them to college-level mathematics. We participated in that workshop the summer after ninth grade. That workshop had a major impact on Maryam's growing interest in math and resulted in her first publication (joint with Professor E. Mahmoodian). We participated in the Math Olympiad competitions in the eleventh grade and made it to the national team as the first female students on the team (see above).

From 1995 to 1999 we attended Sharif University, the leading technical university in Iran, where we got a lot of support from our professors and the chair of the mathematics department, Y. Tabesh. By then Maryam had become very ambitious about her future in math, and her life was not imaginable without it. Although the universities, unlike the high schools, are coeducational in Iran, many forms of gender segregation were imposed in the 1990s. The atmosphere within the math department, however, was relaxed and friendly, and there were various activities from workshops to math competitions to reading seminars that kept us motivated and stimulated an even stronger passion for mathematics. There were six of us in our class who decided to apply to graduate schools in the US during our junior year. To increase the chances of all of us getting admitted to top-tier schools, we decided to each choose a school to which the others would not apply. Harvard and MIT were natural choices for Maryam and me because we wanted to stay close to each other. We all moved to universities in the northeastern US in 1999.

Maryam and I continued to see each other very often until 2003, when I left Boston. She was very generous with her time when it came to discussing math and was always happy to meet and talk about any problem I was thinking about. I remember in fall 1999 I took a course that Maryam was auditing. The final was a take-home exam, which I had planned to complete overnight. Maryam came to my office for moral support in case I had to stay up late to finish the exam. The last problem looked difficult, so I discussed it with Maryam. She got interested in it, and we thought on it for a while without success. After a few hours I gave up, as I was too sleepy and decided that I had done enough for the exam. But, as would always be her pattern, Maryam persisted. She stayed up all night working on it, and when I woke up early in the morning, she had figured out how to do the problem and was checking the last steps of her solution.

Maryam's work was always driven by a certain pure joy that did not change over the years by experience or wisdom. She drew pleasure from doing advanced mathematics in the same way as she did from solving high school-level Math Olympiad problems. She was driven by this joy and certainly not by a desire for fame or influence, and she continued to be driven by it even when fame and influence came within her reach. She avoided public attention religiously and that helped her stay focused on her research in the midst of the celebrity she attained winning the Fields Medal while also dealing with cancer.

I was at a workshop at MSRI in May 2013 when I received a message from Maryam informing me of her diagnosis. She had recently come back from a trip to Europe and Iran, and I had planned to meet with her after the workshop. When I saw her a couple of days later at Stanford, she was worried about the effect of the illness on her daughter and her career but was determined to stay hopeful and

Mirzakhani (right) and Beheshti (left) on Cape Cod in 2000.

upbeat. In the four years that followed, she stubbornly focused on the positive and refrained from complaining about the problems she was dealing with in the same way that she always avoided talking about her achievements. Even after she was diagnosed with recurrent cancer and the odds were not in her favor, she maintained hope and handled the situation with grit and grace.

Maryam Mirzakhani's parents heard the news of her receiving the Fields Medal through public media. She explained later that she did not think it was such a big deal! This accords with what I remember from my interactions with her when she was a student at Harvard. While explaining her beautiful work in math, Maryam made it sound as if what she was doing was simple.

Her humanity was very pronounced and she was kindhearted. I recall her helping disabled students at Harvard in getting around. After receiving the Fields Medal, the Iranian students in the Boston area were hoping to celebrate her accomplishments by organizing a big party for her. Maryam refused this, instead opting for attending, unannounced, their small monthly lunch gathering.

Maryam beautifully exemplified that the pursuit of knowledge is a timeless, borderless, and, yes, genderless adventure. Her untimely passing has deprived the world of math of a brilliant, yet humble talent at her peak. Her legacy will live forever.

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Mirzakhani with her daughter.

In her final month, the last time I met Maryam at her home, I told her that I had just read how difficult and painful one of her treatments was and expressed surprise that she had never complained about it. She dismissed that as something not worth focusing on and changed the topic. We then talked about various audiobooks that she planned to listen to. She was, as always, warm and engaging and got radiant and happy when her daughter interrupted us, jumping around Maryam and telling her that she loved her new bike. This is how I am going to always remember Maryam.

Photo Credits

Photo of Mirzakhani with daughter courtesy of Jan Vondrák.

All other section photos courtesy of Roya Beheshti.

Izzet Coskun, Laura DeMarco, and David Dumas

Maryam Mirzakhani in Graduate School

We met in David's office to talk about our memories of Maryam in graduate school and to write some words that might convey what she was like. But how do we even begin to write about Maryam, one of our closest friends in those years and an integral part of our development as mathematicians, when the pain of her absence is so fresh? We started with the present, when we first learned about her illness, when each of us saw her last, the recent interactions we had, whether talking with her about math or talking about our children, and little by little the memories and stories from graduate school started to flow.

Some themes quickly emerged: Maryam's determination to figure things out, her intense and insightful questioning, her excitement about new ideas, and the gleam in those brilliant blue eyes after making some absolutely hilarious comment. We recalled her seminar lectures and her struggles and also her jokes and her cooking, and we recalled how easy it was to spend time with her—and the countless hours we all spent together—though we never truly knew her most personal or private thoughts and feelings.

We were part of a tight-knit group of students at Harvard, and we were all actively involved in Curt McMullen's informal geometry and dynamics seminar. Every semester, each of us had to give a lecture in this seminar. In our first couple of years, we gave expository talks about research articles we were reading, and one of Maryam's first talks was on McShane's identity for the punctured torus. All three of us were there. Before the talk, she was anxious about her presentation, which was partly her apprehension about speaking in English, though, in fact, she was perfectly fluent.

Maryam's lecture style that day was unforgettable. She would race ahead of herself at times, unable to contain her excitement, and the barrage of words was accompanied by large circular hand gestures. At the time, we were all struggling with new mathematical ideas; we often felt

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we were swimming through a mathematical soup as we tried to find our thesis problems. Even then, however, Maryam seemed to have a direction and purpose, and the mathematical content of her lecture was potent and clear, even if the presentation itself was amusing.

I met Maryam in 1993. I had an appointment to meet Ebad Mahmoodian at IPM, the Institute for Research in Fundamental Sciences in Tehran. When I got there, Mahmoodian told me that he wanted to check some math that a high school student had handed to him and he asked if I'd be willing to help him. So, I spent the morning with Mahmoodian going over the arguments with him. Mahmoodian was teaching a summer course on graph theory for gifted school children at Sharif University. One of the topics he had talked about was decomposing graphs into disjoint unions of cycles, including the rather curious example of decomposing a tripartite graph into a union of 5-cycles. This was considered the first difficult case of the general problem. Mahmoodian had asked the students to find examples of tripartite graphs that were decomposable as unions of 5-cycles, offering one dollar for each new example. By the time of the next lecture, Maryam had found an infinite family of examples. She had also found a number of necessary and sufficient conditions for the decomposability. These were the results that Mahmoodian and I checked that day. Checking everything carefully took the whole morning. At the time we joked that Maryam was obviously smart, but not that smart; otherwise she could have milked Mahmoodian for all he was worth by revealing one example a day.

Those of us who knew Maryam in person probably have a hard time thinking of her as "the genius" that she has been portrayed as in the media. She didn't have any of the pretensions of the stereotypical genius of children's books. She had the same qualms and worries as the rest of us: she too had wondered at some point whom to work with, whether she would finish her thesis, whether she would find a decent academic job, whether she could balance the demands of motherhood with being a professor, whether she'd be tenured at some point in this century. And she was a lovely person. We loved her for who she was, and we would have loved her just the same even without her honors and awards.

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In later years, her lectures became much more polished, and she gave talks on enumerating simple closed geodesics on surfaces and on the earthquake flow on Teichmüller space. We still laugh about our own talks in the McMullen seminar sometimes being all show and no substance in contrast with hers that were all substance and no show. She really understood the material deeply. Looking back, it is remarkable that each of her talks in that informal setting was later developed into a substantial contribution to mathematics.

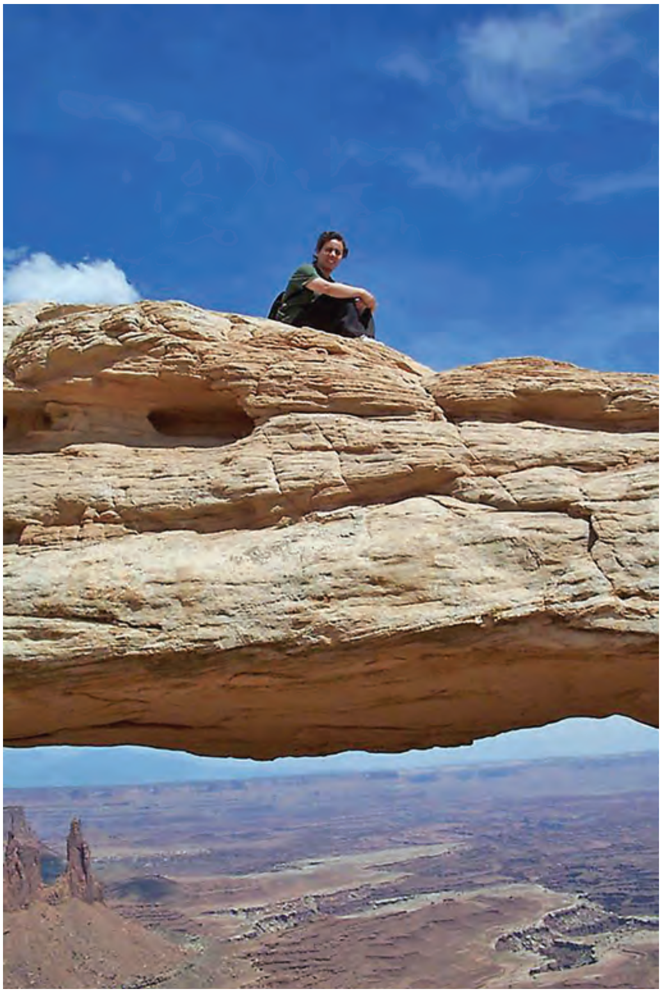
Maryam's perseverance stands out in our minds. In the early spring of her last year in graduate school, she rushed into one of our offices close to tears. She was holding what must have been the twentieth draft of her thesis, covered in McMullen's famous red ink. "I am not sure I will ever graduate," she moaned, and she joked that the last ten pages did not have as much red only because McMullen's pen must have run out of ink. After commiserating for half an hour, Maryam steeled herself and with her characteristic determination marched back to her office for yet another round of editing.

In some ways, Maryam was like all the other students, struggling with writing and expressing herself clearly. All of McMullen's students went through the ritual of countless rewrites. But Maryam had a rare dogged tenacity always to do better. She revised and revised tirelessly. The red on the page subsided, and she completed her thesis. Later she would adapt this thesis into three beautiful papers tying together Witten's conjecture, the volume of the moduli space of curves, and the counting of simple closed geodesics on hyperbolic surfaces.

Maryam had a whimsical side that was not often seen. We had many dinner parties together, hosted in our homes. Once when she was cooking us an Iranian meal, in the middle of preparing a rice dish she put on some music and started to dance. Soon we were all dancing around her kitchen, bumping hips and laughing as she stirred the pot. Another time, we were helping Maryam move into a new apartment, and we had to travel twenty-five miles to return the moving truck. While the rest of us were cranky and exhausted, Maryam spontaneously broke into song. She taught us some Iranian children's songs, and we sang the rest of the way, like school kids on a field trip.

Occasionally Maryam's lighter side would come out in mathematical conversations, too. We all have memories of standing in the hallways of the department with her, laughing raucously about something mathematical, probably disturbing all those seemingly serious professors in their offices. If she was really excited about an idea, she would start explaining it with energetic gestures. She would laughingly say "it's sooo complicated," though in truth the complexity never stopped her.

Maryam had an incredibly warm and infectious smile, but she eschewed the spotlight. While her warmth was apparent among friends, she was often quite reserved in her interactions with others. In 2002 David and Maryam attended a two-week summer program in complex hyperbolic geometry at the University of Utah. A group of students went on a weekend hiking trip together; Maryam came along and seemed to enjoy these quiet hikes in the mountains. When the group stopped to rest, she would often find a place to sit and reflect on the scenery in silence (see next page). Maryam was intensely private. Even after knowing her for so many years, we do not know her deepest religious, social, or political beliefs.



On a 2002 hike in Utah, Mirzakhani liked to reflect on the scenery in silence.

Maryam was extremely generous, both with her mathematical ideas and as a true friend. In her fourth year in graduate school, David had a serious fall and had to recover at home from several broken bones. Maryam organized a group of students to visit him and cheer him up during his recuperation. Maryam would not settle for half measures. When Laura married, Maryam teamed up with Izzet and another fellow graduate student, Alina Marian, to find something that would really please her. Laura had a cutlery set in her registry, but Maryam was not satisfied with ordering a few knives and forks—she insisted that only the entire set would do.

Maryam was equally gracious with her mathematical ideas. When David was working on a paper about Thurston's skinning maps, Maryam realized that part of his work had implications for some counting problems related to the mapping class group action on Teichmüller space that she and her collaborators had recently explored; she was happy to have David include this application in his paper.

Maryam was a brilliant mathematician, a Fields Medalist, an inspiration to many young mathematicians. To us, however, she was first a friend. We are lucky to have studied math with her, benefiting from her deep insights and her penetrating questions. We are equally lucky to have danced and cooked together, laughed together and struggled together through those hard but precious years of graduate school. It seems so wrong that she has died so young. We love and miss her dearly.

Photo Credit

Photo of Mirzakhani in Utah courtesy of David Dumas.

Elon Lindenstrauss

Maryam Mirzakhani at Princeton I

Maryam and I both came to Princeton in 2004. She came just after completing a stellar PhD thesis and had a very visible presence in the department—energetic, sharp, and interested in everything. Getting to know Maryam was to me an unexpected joy. Maryam and I were both going to a course given by Peter Sarnak that first year, and we probably started talking to each other then.

I was intrigued by one aspect of her remarkable thesis that was closest to me: Maryam's asymptotics for the number of simple closed geodesics of length at most T on a hyperbolic surface. Counting the number of all closed geodesics of length at most T is classical in dynamics—it is well known that for surfaces of constant curvature -1 the number of such geodesics grows like e^T/T. The simple closed geodesics—those that as curves on the surface do not cross themselves—are governed by a very different law, which is influenced by the topology of the surface, by the genus g, and the number of cusps n: their number is proportional to $T^{6g-6+2n}$. As sometimes happens when counting objects, the asymptotic formula for the number of simple closed geodesics is intimately related to an equidistribution statement, and as an ergodic theorist I was excited by the fact that the heart of the matter was a beautiful application of ergodicity but in a context I was not familiar with, namely, the action of the mapping class group on the space of measured laminations supported on the surface.

When I met her, Maryam was already dreaming about classifying the invariant measures and orbit closures for the action of $\mathrm{SL}(2,\mathbb{R})$ on the moduli space of quadratic differentials. This quest was a somewhat speculative one at the time, motivated by the fantastic results of Marina Ratner (who died unexpectedly only a week before Maryam's untimely death) on invariant measures and orbit closures for the action of Ad-unipotent subgroups on quotients of Lie groups. Alex Eskin was also obsessed with this problem, and some years later they would join forces (in part aided also by Amir Mohammadi) to fulfill this mathematical dream in a pair of long and exceedingly difficult papers, using what seems like almost every mathematical technology known to mankind.

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We started discussing mathematics, and in particular Maryam patiently explained to me some bits of her thesis. On the third floor of Fine Hall, the tower housing the Princeton mathematics department, there are some niches with blackboards, and they were the favorite places for us to sit and discuss. We decided to work together on an intermediate problem, motivated by the quest for an analogue of Ratner's theorems in the moduli space setting but quite closely connected to Maryam's thesis: the classification of locally finite measures on the space of measured laminations invariant under the action of the mapping class group.

This problem can be viewed as the analogue of the problem of classifying measures invariant under horospheric groups in the Lie group setting (horospheric groups are a special case of Ad-unipotent groups that are "big" in an appropriate sense), which in the Lie group setting is easier than the general case of Ratner's results and where rigidity was established earlier by S. G. Dani and H. Furstenberg. Working with Maryam on this project and learning from her all the multiple points of view on this problem was a pleasure. While working we also talked about life, politics,…. In some sense Iran and Israel are similar, with a tension between tradition/religion and modernity, and we both hoped a time would come when the conflicts between our nations would subside. Sadly, this does not seem to be the direction the world is heading.

When Maryam came to Princeton, she had to meet very high expectations due to her remarkable PhD thesis. Eventually she exceeded these expectations, but because she was not trying to reap low-hanging fruits, this took time, and I am sure it was probably not easy initially. When we worked together she was modest but sharp and persistent.

After Maryam moved to Stanford I had less frequent contact with her, but we had some email exchanges from time to time. In June 2013 she told me she would not be able to come to the Fields Symposium in Toronto due to "an unexpected medical issue that I will have to deal with," and in the ensuing email exchange she told me she had been diagnosed with breast cancer. In 2014–15 I was with my family on sabbatical in Berkeley. Just after we arrived at Berkeley I heard about her Fields Medal. I knew she was in the running, and she certainly very much deserved it, but since I had not heard anything I had assumed this did not work out. I was very pleased to be wrong, and we exchanged some emails about our experiences with the publicity of the medal (though of course her prize generated much greater public interest than mine).

During this year, and particularly in the second semester, when both of us participated in a special semester at MSRI, it was a pleasure to see her back with her full energy and insight, often asking prescient questions from the balcony in the MSRI main lecture auditorium. My family and I hosted Maryam and her family in the house we were renting, and her little girl enjoyed the trampoline. Everything seemed very idyllic. Unfortunately, this full recovery proved to be a mirage, and the cancer returned not long afterwards.

We all miss her terribly.

Peter Sarnak

Maryam Mirzakhani at Princeton II

The first I heard of Maryam was her spectacular thesis, written with Curt McMullen as her adviser. In it she resolved a long-standing question on the asymptotic count of the number of simple closed geodesics on a hyperbolic surface, as well as giving an ingenious new proof of Witten's conjecture relating certain intersection numbers on moduli spaces of surfaces of genus g with n punctures and a KdV hierarchy. These results won her immediate international acclaim, and until her untimely passing, she continued to produce one striking breakthrough after another, reshaping and molding the theory of hyperbolic surfaces and especially the dynamics connected with their moduli spaces. Her contributions will be celebrated and studied for years to come.

Maryam's first faculty position was at Princeton University, and we cherish the period that she spent with us. I met with Maryam many times over the years to learn about her latest work and other developments and to discuss problems of mutual interest. While she was very modest in both private and public exchanges, her deep mathematical intuition and brilliance shone right through in any discussion; one always felt elevated in her presence. On many occasions I would find myself standing at the back of the lecture hall together with Maryam, as we both preferred to stand while listening to a lecture. The bonus for me was that I would get her private clarifications and insights into what the speaker was saying.

Maryam's most recent visit to Princeton was with her husband, Jan, and daughter, Anahita. They spent the fall of 2015 at the Institute for Advanced Study as part of a core activity centered around her far-reaching works with Alex Eskin. This was a good time; Maryam was in remission and doing very well in all respects. She delivered to a packed audience the Minerva Lectures at Princeton University.[2] Her and Jan's research was moving forward handsomely, and there were many exciting opportunities for them going forward. Sadly, a few months after they got back to Stanford I got an email from Maryam saying that she had had a big setback healthwise. She confronted her cancer with the same courage and boldness that she attacked all the obstacles that her very rich and too short life presented. She is missed by so many of us—those whose lives were enriched by knowing her as well as those who are inspired by her story and her mathematics.

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[2] https://www.math.princeton.edu/events/seminars/minerva-lectures/archive/13.

Maryam Mirzakhani, Roya Beheshti, and others at the Math Olympiad preparation camp in summer 1994.

Maryam was a member of the Iranian Math Olympiad teams in 1994 and 1995, with whom she went on to receive gold medals at IMO with a perfect score on the later one. I was director of the Math Olympiad in Iran during that period, and it was exciting to see with what ease she could solve difficult problems. Later she joined the training camp of Math Olympiad to train the next generation of Olympians. It was so amazing for me to work with her to design new challenging problems! She also developed resources and wrote a number theory book with her lifelong friend, Roya Beheshti, as a reference for Olympiad camps. The book still is in use after a twelfth printing!

Maryam joined the math department of Sharif University for her undergraduate studies in 1995. I was the chairman of the math department during that period, and with a few colleagues we did our best to generate a center of excellence. We attracted many talented students, but Maryam was truly exceptional in all her courses.

Saeed Akbari, a young faculty member at that time, brought a twenty-four-year-old problem by Paul Erdős to the attention of his algorithmic graph theory students and offered a ten-dollar prize. The problem was to prove the existence of a planar 3-colorable graph that is not 4-choosable. Maryam solved it and received the prize. Her solution appeared in the BICA journal in 1996.

Then she went to Harvard and continued to be among the greatest mathematicians in her generation, but she always remembered the good old days of her undergraduate studies at Sharif. Her heart was always open to concerns about how we can help young and talented teenagers to have the opportunity to flourish and develop their own mindset and talent.

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Photo Credit

Photo courtesy of Roya Beheshti.

Alex Eskin

Maryam Mirzakhani as Collaborator

Maryam and I first met in person at Princeton. She was a postdoc, and I was a visitor at the Institute for Advanced Study. We started collaborating right away. For me it was an amazing experience; she was obviously brilliant, but she was also the nicest and most positive person I have ever met. We spent a lot of time at the IAS talking about math and other things, and every week we would walk to the university to sit in on Elon Lindenstrauss's course about homogeneous dynamics, which included a description of his "low-entropy method."

After we both left Princeton, we continued working together. Maryam's philosophy was to never go after the "low-hanging fruit," and our projects got increasingly ambitious. But the biggest open problem in our subfield, since Curt McMullen's pioneering work ten years before, was trying to find analogues of Marina Ratner's celebrated theorems in moduli space by proving that the $SL(2, \mathbb{R})$ orbit closures were always manifolds. For a few years we never talked about it, but we both could sense that the other person was working on it. Eventually we decided to join forces. For me it was an easy decision, since working with her was so incredibly fun.

At first we just had some vague dreams and no concrete plan. At one point we read a breakthrough paper by two French mathematicians, Yves Benoist and Jean-François Quint. The paper had already been available for a few years, and reading it was on our to-do list, but it was written in French, which neither of us knew. At some point I was visiting IHES, and there was a group of people there who wanted to read the paper, and that provided the needed push. It was ostensibly a result in a different area of mathematics, but we could tell immediately that it was relevant to our problem. At this point we both dropped all of our other projects. After a few months of struggling, we plotted a potential route to the solution based on a variant of their method.

In my senior year at Stanford I had an independent study with Professor Mirzakhani in topology. I learned from her the art of thinking through problems—her diligence and calm methodology of thinking through problems that we would formulate was like watching an artist construct a masterpiece. She would brush on the blackboard the basic fundamentals that she would ensure I had a strong grasp of and then slowly layer on various colors of detailing that would give a more vivid illustration of the problem we were tackling and, more importantly, a profound excitement for our work

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and potential as mathematicians. She would teach in a way to instill in her students a deep curiosity coupled with a heavy thirst for a greater mastery.

During a few of our meetings, I told her about my interests in utilizing structures for grasping insight into various biological phenomena, and we discussed a few examples I had been thinking about. She asked me about my career goals post graduation, and I told her that I was unsure, but that I loved what we were doing. She discussed with me her motivations that fuel her work. She propelled me to apply to PhD programs, which I ended up doing later that year.

Her impact on my life is unforgettable, and I would not be where I am now if not for her support and encouragement.

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We worked on this for two years while getting more and more excited. But at some point, seemingly near the end, we realized that there was a major problem and that our path was blocked. Our method needed a fact called the semisimplicity of the algebraic hull, which we could not prove. We could understand the $SL(2,\mathbb{R})$-invariant measures, but, because of the nonamenability of $SL(2,\mathbb{R})$, in order to understand the $SL(2,\mathbb{R})$ orbit closures we needed to understand the P-invariant measures, where P is the upper-triangular subgroup of $SL(2,\mathbb{R})$. We looked for ways to fix the argument but could not find any. We were still talking once a week on Skype, but we were making no progress. During this period, Maryam's daughter was born, but she continued working; she had this amazing inner strength.

After more than a year of being stuck, we realized that there was a different approach to the problem that was based on a distant relative of the "low-entropy method" that we had learned from Lindenstrauss. That eventually worked, but not all the way. However, it was enough to show that P-invariant measures are $SL(2,\mathbb{R})$-invariant, and for a long time we had an argument (based on the work of Giovanni Forni) that $SL(2,\mathbb{R})$-invariance implies the semisimplicity of the algebraic hull. Then we could fall back on our original argument based on the Benoist-Quint method to complete the proof. It took us another two years to work out all the details. During this time Maryam was first diagnosed with cancer.

Later, the cancer was in remission, and I remember a lot of happy times at MSRI, at IAS, and at Stanford. But eventually the cancer came back, and even Maryam could not beat it. I miss her both personally and professionally.

Mirzakhani working at home.

Photo Credits

Photos courtesy of Jan Vondrák.

Eugenia Sapir

Maryam Mirzakhani as Thesis Advisor

I think back often to my time in grad school as a student of Maryam Mirzakhani. The memories of her are still very much alive in my head. These memories are so full of life that sometimes I feel they will pop out of my mind and materialize in front of me. I will attempt to draw a picture here of the vibrant person who lives on in my mind.

Maryam was full of boundless energy and enthusiasm for her work. This was evident when watching her do math or give a talk or when working with her on a problem. This excitement toward her subject was part of her driving force. When giving talks, she would bounce slightly while laying out all the disparate ideas that came together into the picture. When she got to the part when all the ideas came together, she'd talk faster, excited about getting to the punchline.

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Maryam would paint beautiful, detailed landscapes in her lectures. If she were giving a talk about concepts A, B, and C, she would not just explain that A implies B implies C. Rather, she would paint a mathematical landscape where A, B, and C lived together and interacted with one another in various complicated ways. More than that, she made it seem like the rules of the universe were working harmoniously together to make A, B, and C come about. I was often amazed by what I imagined her inner world to be like. In my imagination it contained difficult concepts from disparate fields of mathematics all living together and influencing one another. Watching them interact, Maryam would learn the essential truths of her mathematical universe.

Maryam's vision occasionally made for cryptic conversations. When, for example, we would talk about my thesis problem, there was a set of concepts that Maryam would repeatedly bring up, rather like a recurrent cast of characters in a television show. At first, it was unclear to me why these characters were relevant. Only after working on my problem for several months could I look back and see that the hints that Maryam dropped formed the backbone of the best, most illuminating way of understanding what was going on.

I met Maryam when she was a graduate student at Harvard. At that time I already saw her passion for mathematics. A couple of years later she did her fabulous dissertation. Subsequently, I saw her at conferences and the couple of visits she made to Chicago to work with Alex Eskin. Her work had a profound influence on me and many others. It was also always a joy to see her. She loved talking mathematics with everybody, and she was also just a wonderful person to be around. I got to know her a little better personally in the spring of 2015 when she and I were both at the Mathematical Sciences Research Institute. She had just won the Fields Medal. I remember how gracious and generous she was with everyone. I saw her for the last time at a seminar at Stanford in November 2016. I knew at that time that her health was deteriorating, so I was both a little surprised but also very happy to see her. She listened to the talk and then participated in a discussion for almost an hour afterwards. In spite of all she was going through, she exhibited the same enthusiasm and passion for mathematics. I think it was important to Maryam not to be identified with her illness. I miss her as a mathematician and as a friend.

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On the other hand, Maryam was completely down to earth. I've heard many of my peers say that they also found her completely approachable even though she didn't know them very well. It was easy for me to talk to her about my ideas. I would come into her office each week while I was in graduate school. I'd tell her what I'd been thinking about, and we'd just talk. We'd throw ideas back and forth and get excited when things worked. We would make math jokes and laugh. Sometimes I'd bring her an annoying problem I couldn't get around. Then her favorite phrase was, "How can that be?" She would repeat it several times, until she found the fissure we could use to crack the problem open. Our meetings would skip along happily until we came to a problem where we didn't see an immediate solution. We'd become silent and pensive, each of us retreating to our paper and pencil until we came up with something new to try. Maryam became engrossed in her drawing pad, sketching small drawings related to what we were discussing. When the silence dragged on for long enough and it had become clear that it would take too long for the next idea to come, the meeting was over. I knew that we'd discussed everything possible that week and that it was time to table the discussion for next time. I'd leave feeling energized and excited for the week to come.

I saw Maryam as my math coconspirator and also as a bottomless well of inspiration. She was an unpretentious person who held great depths. She was the embodiment of someone who studied math for the joy of the pursuit of knowledge. She is someone we have lost much too soon and who has left much undone.

Mirzakhani with her husband, Jan Vondrák, and daughter, Anahita.

Photo Credits

Photo courtesy of Jan Vondrák.

Hélène Barcelo and David Eisenbud

Maryam Mirzakhani and MSRI

Everyone in the mathematical community knows that Maryam Mirzakhani was an outstanding mathematician, a winner of the Fields Medal in 2014. Many have seen a video[3] with her on the floor drawing figures that reflect her deep mathematical imagination and suggest some of the charm of her fresh and vibrant personality. Still others knew Maryam from her other distinctions—for example, as one of *Nature Magazine*'s "10 People Who Matter" in 2014. But only a few know how deeply Maryam was engaged with the Mathematical Sciences Research Institute (MSRI) located in Berkeley, California.

Maryam was a key member of the MSRI Scientific Advisory Committee from 2012 to 2016, the committee charged with selecting the scientific programs and their members. During part of that time she was very ill with what turned out to be her terminal cancer, but she never missed a meeting—even when she couldn't travel from Stanford to Berkeley she attended by video between harsh medical treatments—and she faithfully did the substantial committee homework. It is fair to say that she was an inspiration to the entire committee.

She played other important roles at MSRI too. She was one of the main organizers of the semester-long program on Teichmüller theory and Kleinian groups in 2007 and a research professor for the spring of 2015. She was even a member of the complementary program when her husband, Jan Vondrák, came to MSRI for a computational program in the spring of 2005.

Finally, in the spring before her death, Maryam had agreed to be nominated as a trustee of MSRI, although she warned David Eisenbud, MSRI's director, in a long and deeply sad meeting that she might well not live to take office.

We believe that Maryam's engagement with MSRI came from a close alignment of priorities: first, for mathematics of the highest quality, done in a playful collaborative style; second, for the cultivation of talent also in young women, and the support of women with children; and, finally, for welcoming mathematicians from abroad, understanding that they enormously enrich the US mathematical scene.

It was an honor to know Maryam: brilliant, charming, and courageous to the end. We miss her deeply.

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[3] https://www.simonsfoundation.org/2017/07/20/fields-medalist-maryam-mirzakhani-dies-at-40/.

Ursula Hamenstädt

Zooming In and Out: The Work of Maryam Mirzakhani through the Eyes of a Geometer

Prelude

Let us imagine that we intend to embark on an attempt to understand the geometry of the universe. How would we proceed?

A perhaps naive idea is as follows. Identify some small region in the universe we are reasonably familiar with. What this means can be argued about, but the amount of information used to gain the level of understanding we feel comfortable with should be controlled. Then we send a spacecraft from our region in some arbitrary direction to collect information on the change of shape with respect to the fixed reference region. As the journey of the spacecraft continues and we collect more information, we may have to acquire more knowledge about the reference region to detect hidden similarities with the pictures taken along the journey. The vague hope is that the geometric interaction of nearby regions on the small scale reveals the laws of nature which control the large-scale shape of the universe as recorded in the random sampling.

To develop an understanding of the shape of the universe along such lines requires an intimate understanding of what "shapes" are supposed to represent, how we can describe their interaction, how simultaneously zooming in and out can be done in such a way that pattern formation can be correctly observed.

In my mind, Maryam Mirzakhani successfully implemented this program for the universe formed by Riemann surfaces and their canonical bundles, i.e., for the moduli space of Riemann surfaces and the moduli space of abelian differentials. These moduli spaces are well-studied mathematical objects which naturally emerge in almost any mathematical context. Building on the work of many distinguished researchers, Mirzakhani developed a new geometric way to study these spaces that puts earlier, seemingly unrelated, results into complete harmony and, at the same time, creates many new research directions which will be explored by future generations of mathematicians.

In the sequel I will discuss three of Mirzakhani's theorems which were obtained using the principle of zooming in and zooming out. Geometry should be understood in the broad sense of measurement, which includes counting of shapes sorted by sizes, etc.

Riemann Surfaces

An oriented *surface* is a two-dimensional oriented manifold and as such a quite simple object, in particular, if it is closed (i.e., compact, without boundary). Viewed as topological or smooth manifolds, surfaces can easily be classified. The most basic closed surface is the two-sphere, and any other closed surface arises from the two-sphere

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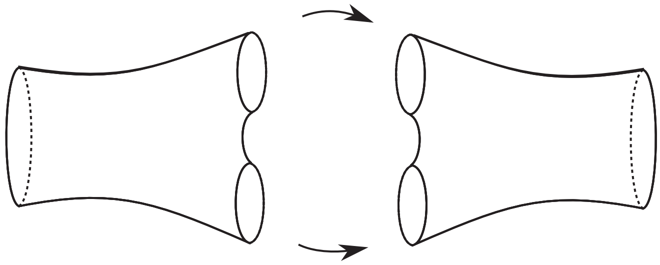

Figure 1. Mirzakhani (Theorem 1 here) proved estimates on the number of pair of pants decompositions.

by attaching a nonnegative number g of so-called *handles*. The number of handles attached is the *genus* of the surface.

Once this classification is established, seemingly there is not much more to say, with the exception of understanding the different geometric shapes a fixed closed surface S admits. The perhaps best known classical result that restricts the geometric structures on S is the *Gauss-Bonnet theorem*. It says that if we denote by K the Gauss curvature of a smooth Riemannian metric on S, then the integral $\frac{1}{2\pi}\int_S K d\text{vol}$ equals the *Euler characteristic* $\chi(S) = 2 - 2g$ of S, where vol is the volume form of the metric. Furthermore, for $g \geq 2$ there exists a *hyperbolic metric* on S, i.e., a metric of constant Gauss curvature -1.

The space of all isometry classes of such metrics can be described as follows. First, the set of all *marked* hyperbolic structures on S is defined to be the set of equivalence classes of hyperbolic metrics where two such metrics are identified if they can be transformed into each other by a diffeomorphism of S isotopic to the identity. This space, *Teichmüller space* $\mathcal{T}(S)$, admits a natural topology that identifies it with an open cell homeomorphic to $\mathbb{R}^{6g-6}$. The group of isotopy classes of diffeomorphisms of S coincides with the quotient of the automorphism group of $\pi_1(S)$ by the subgroup of inner automorphisms. This *mapping class group* $\text{Mod}(S)$ of S acts from the left on $\mathcal{T}(S)$ by precomposition. The space of isometry classes of hyperbolic surfaces is the quotient $\text{Mod}(S)\backslash\mathcal{T}(S)$, called the *moduli space* of S.

Zooming In: Local from Global

For a fixed hyperbolic metric on a closed surface S of genus g, there exists a decomposition of S into geometric pieces, *hyperbolic pairs of pants* as in Figure 1. Topologically, such a hyperbolic pair of pants is a two-sphere from which the interiors of three disjoint compact disks have been removed. This decomposition is done by cutting S open along a collection of $3g - 3$ pairwise disjoint *simple* geodesics. Here a geodesic is simple if it does not have self-intersections.

Clearly, the geometric description of the hyperbolic surface obtained from such a decomposition into hyperbolic pairs of pants depends on the choice of the boundary geodesics. A natural question is how many different such decompositions exist for a fixed hyperbolic metric? The easy answer is: This number is infinite, as there are simple closed geodesics on S of arbitrary length, and each simple closed geodesic belongs to such a decomposition. But what if we restrict the geometry by bounding the length of these geodesics?

A result going back to Huber and Selberg states that the number $c(L)$ of singly-covered closed geodesics of length at most L on a hyperbolic surface S is asymptotic to e^L/L as $L \to \infty$. In particular, the growth rate of the number of such geodesics is not only independent of the specific hyperbolic metric on S but also independent of the genus of S.

However, most of these geodesics have self-intersections, and if we want to understand the geomety of a hyperbolic surface S by decomposing it into pairs of pants, we need to understand simple closed geodesics on S. And, as we vary the hyperbolic metric over moduli space, we may expect that these simple geodesics and their growth rate give information on the geometry of the surface.

In her thesis, Mirzakhani gave an affirmative answer to quite a few questions of this type. Denoting by $\mathcal{M}_g$ the moduli space of closed surfaces of genus g, she showed:

Theorem 1 (Mirzakhani [7, Thm. 1.1]). *For a closed hyperbolic surface X of genus $g \geq 2$ and $L > 0$ denote by $s_X(L)$ the number of distinct geodesic pair of pants decompositions of X of total length at most L. Then*

$$\lim_{L\to\infty} \frac{s_X(L)}{L^{6g-6+2n}} = n(X),$$

where $n : \mathcal{M}_g \to \mathbb{R}_+$ is a continuous proper function.

Thus as moduli space is noncompact, the growth rate of the number of pants decompositions of bounded length depends not only on the genus of the surface but also on the surface itself!

Mirakhani showed this result by first zooming out and then zooming in. The first step is to prove that for fixed L, the locally bounded function $X \to s_X(L)$ can be integrated over moduli space, where this moduli space is equipped with the so-called Weil-Petersson volume form, the volume form of an (incomplete) Kähler metric. She discovers that this integral, denoted here by $p(L)$, is a polynomial in L of degree $6g - 6$.

Some pointwise estimates on the growth rate $s_X(L)$ were known earlier. But it requires far-reaching insight and courage to believe that one can basically effectively compute this integral by finding a global relation between these integrals for all $L > 0$. Mirzakhani [7, p. 99] explains this for a fixed pants decomposition γ of S, "the crux of the matter is to understand the density of $\text{Mod}_g \cdot \gamma$ in $\mathcal{ML}_g$." Here $\mathcal{ML}_g$ is the space of so-called *measured geodesic laminations* which is homeomorphic to a euclidean space of dimension $6g - 6$ with the point 0 removed. It contains *weighted simple multicurves*, i.e., collections of disjoint closed geodesics equipped with some positive weights as a dense subset, and it admits an action of the multiplicative group $(0, \infty)$ by scaling.

Each *marked* hyperbolic metric on S defines a continuous length function on $\mathcal{ML}_g$. This function associates to a multicurve the length of its geodesic representative for this metric.

The mapping class group $\mathrm{Mod}(S)$ acts transitively on the multicurves which define a pants decomposition of a fixed combinatorial type. If one equips each component of such a multicurve with the weight $T > 0$, then these weighted point masses define a $\mathrm{Mod}(S)$-invariant Radon measure μ_T on $\mathcal{ML}_g$. Mirzakhani observes that dividing by T^{6g-6}, the total mass of the set of laminations of length at most one for a fixed marked hyperbolic metric determines the value of $s_X(T)$. Furthermore, the measures μ_T converge as $T \to \infty$ to a $\mathrm{Mod}(S)$-invariant Radon measure λ on $\mathcal{ML}_g$ which was earlier constructed by Thurston. Thus zooming in again to the unit length ball defined by a fixed hyperbolic metric, she obtains the asymptotics of the function $s_X(L)$ as $L \to \infty$.

Mirzakhani's result indicates that the action of the mapping class group on $\mathcal{ML}_g$ has properties which resemble a more familiar setting in homogeneous dynamics: The group $\mathrm{SL}(2,\mathbb{Z})$ acts by linear transformations on the puncture plane $\mathbb{R}^2 - \{0\}$. This action preserves the Radon measure, which is just the sum of the Dirac masses on the standard integral grid $\mathbb{Z}^2$ or on any dilation of this grid, and it also preserves the Lebesgue measure of $\mathbb{R}^2$. These measures are all ergodic under the action of $\mathrm{SL}(2,\mathbb{R})$, which means that there are no invariant Borel subsets B of $\mathbb{R}^2$ of positive measure so that the measure of $\mathbb{R}^2 - B$ is positive as well. In fact, they are the only ergodic $\mathrm{SL}(2,\mathbb{Z})$-invariant Radon measures up to scale, with λ being the only nonwandering measure.

It turns out that the dynamics of the action of $\mathrm{Mod}(S)$ on $\mathcal{ML}_g$ has very similar properties: In joint work [6] with Lindenstrauss, Mirzakhani proved that the Thurston measure λ is the only nonwandering invariant Radon measure on $\mathcal{ML}_g$ up to scale.

Hyperbolicity

Mirzakhani's thesis was built on the interplay between geometric information on individual hyperbolic surfaces and the geometry of moduli space, and it culminated in a counting result for simple closed multicurves on hyperbolic surfaces. But in which way does moduli space resemble the Riemann surfaces it is made of?

It is classical that moduli space admits a natural complete geometric structure in its own right, the so-called *Teichmüller metric*. Although this metric is not Riemannian, nor is it nonpositively curved in any reasonable sense, it shares many global features of a hyperbolic metric on a surface. In particular, any two points in Teichmüller space can be connected by a unique geodesic for the pullback of this metric, and periodic geodesics in moduli space are in bijection with conjugacy classes of so-called pseudo-Anosov mapping classes. These periodic geodesics can be sorted and counted according to their length.

In joint work with Alex Eskin [2] and with Eskin and Rafi [5], Mirzakhani uses random sampling and local information on surfaces to obtain precise counting results for periodic geodesics in moduli space. The starting point for this endeavor is the (classical) fact that the cotangent space of moduli space at a surface X can naturally be identified with the vector space of holomorphic quadratic differentials for X. More precisely, the hyperbolic metric of X determines a complex structure, and a holomorphic quadratic differential for this structure is a holomorphic section of the bundle $(K^*)^2$, where K^* is the cotangent bundle.

The geometric beauty of this observation lies in the fact that each nontrivial quadratic differential defines a new geometric structure on the surface S in the same conformal class as the hyperbolic structure. This geometric structure is a flat (Euclidean) metric with finitely many cone points of cone angle an integral multiple of π. As one varies over the fibre of the cotangent bundle at a given point, these flat metrics vary as well, and the space of all such flat metrics reflects the geometric shape of the hyperbolic metric.

Even more is true: Each such flat metric comes equipped with a pair of preferred orthogonal directions, the so-called horizontal and vertical directions, which are defined on the complement of the cone points Σ of the differentials and which are just the line bundles on $S - \Sigma$ on which the evaluation of the differential is positive or negative, respectively. These directions define orthogonal foliations of the surface away from the cone points. Furthermore, there is a natural one-parameter group Φ^t of transformations of such flat surfaces which consists of scaling the horizontal direction with the scaling factor $e^{t/2}$ and the vertical direction with the scaling factor $e^{-t/2}$. These transformations clearly preserve the surface area of the flat metric. Two such flat surfaces define the same cotangent vector in moduli space if they can be transformed into each other with finitely many cut-and-paste operations.

The flow thus defined by Φ^t on the unit area subbundle of the cotangent bundle of moduli space can be analyzed by zooming out: The local change of shape along a flow line can easily be described. A main result is the following:

Theorem 2 (Eskin and Mirzakhani [2])**.** *As $R \to \infty$, the number of periodic trajectories of the flow Φ^t of length at most R is asymptotic to $e^{(6g-6)R}/(6g-6)R$.*

This result was later extended in joint work [5] with Eskin and Rafi with a much more sophisticated argument to *strata* in the moduli space of abelian and quadratic differentials.

The idea of the proof consists of zooming out in random directions and averaging the random samples. It was noted earlier that the flow Φ^t has properties resembling the properties of a hyperbolic flow as long as the trajectories remain in a fixed compact set. What this means precisely was clarified by Mirzakhani in joint work [1] with Athreya, Bufetov, and Eskin. But the moduli space is very notably noncompact, and the metric near the cusp resembles a sup metric which can be thought of as a metric of infinite *positive* curvature. There are many recent results which make this idea precise.

Now moduli space is the quotient of Teichmüller space $\mathcal{T}(S)$ under the action of the mapping class group. Moving in Teichmüller space in a random direction can be thought of as looking at a randomly chosen finite set of points in some large ball in $\mathcal{T}(S)$ and trying to understand the change of shape of a hyperbolic or flat surface sampled on these finitely many points. A careful analysis of this sampling process reveals a bias of such random samples to return to the so-called thick part of Teichmüller space. This is defined to be the preimage of a fixed compact subset of moduli space. Understanding the hyperbolic behavior of the flow Φ^t on the cotangent bundle of the thick part allows Eskin and Mirzakhani to show that the growth rate of periodic orbits that are disjoint from some fixed compact subset of moduli space is strictly smaller than the growth rate of all orbits. That this is sufficient for the above counting result can be established by an adaptation of a classical strategy of Margulis.

Hidden Symmetries and Homogeneous Dynamics

The singular Euclidean metric on the surface S defined by a quadratic differential is given by a collection of charts on the complement of the finite number of singular points with values in the complex plane. Chart transitions are translations or compositions of translations with the reflection $z \to -z$. The differential is the square of a holomorphic one-form or, equivalently, a holomorphic section of the cotangent bundle K^* of S if all chart transitions can be chosen to be translations. The differential is, in this case, called *abelian*. Postcomposition of charts then defines a right action of the group $\mathrm{SL}(2,\mathbb{R})$ on the moduli space of abelian or quadratic differentials. The action of the diagonal group is just the Teichmüller flow Φ^t.

The $\mathrm{SL}(2,\mathbb{R})$-action preserves the *strata* which consists of differentials with the same number and order of singular points. Strata are complex suborbifolds in the moduli space of holomorphic one-forms on Riemann surfaces.

Let P be the set of singular points of an abelian differential ω. Then ω determines by integration over a smooth path a class in $H^1(S,P;\mathbb{C})$. In fact, this class determines ω locally uniquely in the following sense. Integration over a fixed basis of $H_1(S,P;\mathbb{Z})$ yields so-called period coordinates for the stratum. These period coordinates equip the stratum with an $\mathrm{SL}(2,\mathbb{R})$-invariant affine structure, i.e., local charts with affine chart transitions. These chart transitions are volume-preserving and, hence, define a Lebesgue measure on the stratum. This *Masur–Veech measure* is finite and invariant under the action of $\mathrm{SL}(2,\mathbb{R})$.

Orbits of the $\mathrm{SL}(2,\mathbb{R})$-action on the moduli space of area-one abelian differentials can be identified with the unit tangent bundle of *Teichmüller disks* which are totally geodesic immersed hyperbolic disks in moduli space. The analog of periodic Teichmüller geodesics are *closed* $\mathrm{SL}(2,\mathbb{R})$-orbits. Such orbits project to finite-volume immersed hyperbolic surfaces in moduli space. It is known that such closed orbits are dense in any stratum. But how can one globally understand the dynamics of the $\mathrm{SL}(2,\mathbb{R})$-action?

In homogenous dynamics, the celebrated solution of a conjecture of Raghunathan by Marina Ratner (who sadly passed away one week before the death of Maryam Mirzakhani) states for example the following.

Consider a simple Lie group G of noncompact type and a lattice Γ in G. Let $H < G$ be a closed subgroup that is generated by unipotent elements. The group H acts by right translation on $\Gamma\backslash G$. Then for every H-invariant Borel probability measure μ on $\Gamma\backslash G$ there exists a closed subgroup $L < G$ with the following properties:

(1) $\Gamma \cap L$ is a lattice in L.
(2) μ is the projection of the Haar measure on L to $\Gamma \cap L\backslash L$.

Furthermore, the closure of each H-orbit on $\Gamma\backslash G$ equals the closed orbit of a Lie subgroup $L < G$ which intersects Γ in a lattice.

Thus Ratner's theorem gives a complete classification of orbit closures on quotients $\Gamma\backslash G$ as well as of invariant measures.

Can one formulate a conjecture which translates Ratner's theorem into the framework of the action of $\mathrm{SL}(2,\mathbb{R})$ on a stratum of area-one abelian differentials? Remembering that an orbit closure in $\Gamma\backslash G$ is algebraic, an ad hoc guess could be that such an orbit closure is an affine suborbifold, i.e., cut out by polynomial equations in affine coordinate charts. But the affine structure on strata of moduli space is obtained by taking periods of holomorphic one-forms over relative homology classes, with an integral structure that arises from integral homology, and a priori this is not in harmony with algebraic structures obtained from global algebraic geometric information.

The amazing rigidity result of Eskin and Mirzakhani [3] states as a special case:

Theorem 3 (Eskin and Mirzakhani [3]). *Every $\mathrm{SL}(2,\mathbb{R})$-invariant Borel probability measure on the moduli space of area-one abelian differentials is an affine measure of an affine invariant manifold.*

Together with Mohammadi, they also show that every orbit closure is an affine invariant manifold. By a 2016 result of Filip, this result can be translated into an algebraic-geometric statement for moduli space: The quotient of an affine invariant manifold by the natural circle group of rotations which multiplies an abelian differential by a complex number of absolute value one is a projective subvariety of the projective bundle over moduli space whose fibre at a point X equals the projectivization of the vector space of holomorphic one-forms on X.

But how can one prove this result? By an amazing construction which simultaneously zooms in and out. Random sampling is used to reveal the global geometry. One of the main tools is random walks, and vast and sophisticated knowledge about such random walks which provides information on deviation, stationary measures, drifts, and structural insights related to the idea of proximality enters into the argument.

It requires the amazing insight, the unsurpassed optimism, and the enormous technical strength of Maryam Mirzakhani to carry out this program, using along the way tools from areas of mathematics as diverse as geometry, ergodic theory, homogeneous dynamics, algebraic groups, random walks, algebraic geometry, and mathematical physics. Her vision is the door to a wonderland where very concrete but seemingly unrelated mathematical structures combine to a composition of extreme complexity and breathtaking beauty, yet which is made from simple tunes and is orchestrated in perfect harmony. There is no coda, but there is an unspoken invitation to everyone to extract his or her favorite line and explore its variations and ramifications.

I met Maryam for the first time on the occasion of a workshop in Chicago (probably in 2002). Curtis McMullen introduced her to me during lunch, and, as mathematicians do, the small lunch group chatted about math. She fully participated in the discussion, making interesting comments, and answered a question that came up. Right after giving the answer, she realized that what she had said was not quite right, and she corrected herself with a laugh, mockingly embarrassed about her inaccuracy.

During our encounters in later years, I found many times again this laughter, reflecting her professional ambition, passion, and optimism which enabled her to adjust and fine-tune her fantastic vision until the next step had emerged in complete clarity and in an aesthetically pleasing way. Looking backwards, I am very grateful that I had the privilege to know her in person, early on.

With her creativity, persistence, and courage, Maryam Mirzakhani made a contribution to mathematics which I believe will be remembered and further developed by many future generations of mathematicians. Her memory should in particular serve as an inspiration and role model for all young women interested in pursuing a career in mathematics.

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Image Credit

Figure 1 courtesy of Ursula Hamenstädt.

Scott A. Wolpert

Maryam Mirzakhani and Hyperbolic Geometry

Reflection

In the spring of 2004, I set about reading a manuscript from one of Curt McMullen's graduating students. The manuscript included many of my favorite topics and concluded with a proof, beginning with hyperbolic geometry, of Witten's conjecture from $2d$-quantum gravity. In 1992 Maxim Kontsevich first proved the conjecture using a matrix Airy integral model and ribbon graphs. The conjecture posits that the intersection numbers on the moduli space $\mathcal{M}$ of Riemann surfaces of two tautological line bundles satisfy the recursion of the Korteweg–de Vries (KdV) hierarchy. I soon realized that the manuscript was closely interrelated with two other manuscripts, one on a recursive scheme in genus for integration over the moduli space $\mathcal{M}$ and one on the asymptotic count by length of simple geodesics on a hyperbolic surface. The constants in the asymptotic count formulas were given by the tautological intersection numbers and Thurston volumes. The graduate student was Maryam Mirzakhani, and so began my journey of studying and contemplating her beautiful results. Mirzakhani's number of publications is not so large, but the breadth and achievements are very special. Often her results are given by simple answers—broad statements without special hypotheses.

In my experience, studying Mirzakhani's works is somewhat akin to signing on for a hike with a young, freshly trained guide. You embark on familiar terrain but soon arrive at surprising new vistas. Also the pace of the hike can be quite varied—conceptual steps can be large, and detailed calculations with formulas go by quickly. I've thought that the latter might be the product of her Math Olympiad training. I always personally found Maryam to be full of energy, most gracious, and intellectually ambitious—always eager to unravel the next piece of the puzzle. For me, Maryam always embodied the mathematician persona that we aspire to fulfill.

Background

We describe Mirzakhani's thesis results from *Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces, Weil–Petersson volumes and intersection theory on the moduli space of curves*, and *Growth of the number of simple closed geodesics on hyperbolic surfaces*, as well as the main results from the joint work with Elon Lindenstrauss. An exposition of the results and necessary background material are provided in our CBMS lectures *Families of Riemann surfaces and Weil–Petersson geometry* and our PCMI lectures *Mirzakhani's volume recursion and approach for the Witten–Kontsevich theorem on moduli tautological intersection numbers.*

Mirzakhani considers Riemann surfaces R of finite topological type with hyperbolic metrics, possibly with

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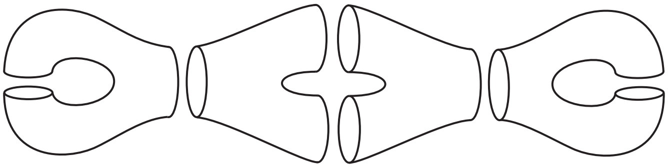

Figure 1. Assembling a genus 3 surface from pairs of pants.

punctures and geodesic boundaries. A surface is marked by choosing a homotopy class of a homeomorphism from a reference surface F. Teichmüller space $\mathcal{T}$ is the space of equivalence classes of marked Riemann surfaces. If the Riemann surfaces are compact or compact with punctures, then the Teichmüller space is a complex manifold. The mapping class group MCG consists of homotopy classes of orientation-preserving homeomorphisms of F and the group acts by precomposition on marking homeomorphisms. The quotient $\mathcal{T}/MCG$ is the classical moduli space $\mathcal{M}$ of Riemann surfaces. For compact genus one surfaces, Teichmüller space is the upper half-plane $\mathbb{H}$ and $\mathrm{PSL}(2 : \mathbb{Z})$ is the MCG. A point $\tau \in \mathbb{H}$ describes the lattice L_τ with generators 1 and τ and the Riemann surface $\mathbb{C}/L_\tau$. Surfaces of negative Euler characteristic are uniformized by the upper half-plane with hyperbolic metric. We consider Teichmüller spaces for negative Euler characteristic surfaces—compact surfaces, surfaces with a finite number of punctures, and surfaces with a finite number of geodesic boundaries. Denote the genus of a surface R by g and the number of punctures or boundaries by n.

A nontrivial, nonpuncture homotopic, free homotopy class α on the reference surface has a unique geodesic representative on a marked surface R—the geodesic length ℓ_α provides a natural function on $\mathcal{T}$. Collections of geodesic-length functions provide local coordinates and global immersions to Euclidean space for $\mathcal{T}$. A hyperbolic surface can be cut open on a simple geodesic. Since a neighborhood of a simple geodesic is an annulus with an S^1 symmetry, the cut boundaries can be reassembled with a relative rotation to form a new hyperbolic structure. For unit-speed hyperbolic relative displacement of reference points, the infinitesimal deformation is the Fenchel–Nielsen (FN) twist vector field t_α on $\mathcal{T}$.

Hyperbolic surfaces can be assembled from pairs of pants—from genus zero surfaces with three geodesic boundaries, as in Figure 1. For pants, boundary reference points are provided by the unique orthogonal geodesics between boundaries. At an assembly seam, the common boundary length ℓ and Fenchel–Nielsen twist displacement parameter τ are unrestricted. The length varies in $\mathbb{R}_{>0}$, and the twist varies in $\mathbb{R}$.

Theorem 1 (FN coordinates). *The parameters* $\prod_{j=1}^{3g-3+n}(\ell_j, \tau_j)$ *provide a real analytic equivalence of* $\mathcal{T}$ *to* $(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+n}$.

The symplectic geometry of $\mathcal{T}$ begins with the MCG-invariant Weil–Petersson Kähler metric $g_{WP\,K\ddot{a}hler}$ and its symplectic form $\omega = \omega_{WP\,K\ddot{a}hler}$. Symmetry reasoning shows that ℓ and τ provide action-angle coordinates.

Theorem 2 ([W]). *The WP symplectic form satisfies twist-length duality*

$$\omega(\ ,t_\alpha) = d\ell_\alpha$$

and has the expansion

$$\omega = \sum_{j=1}^{3g-3+n} d\ell_j \wedge d\tau_j.$$

In particular geodesic-length functions are Hamiltonian, and the symplectic structure is independent of pants decomposition. Also the quotient $\mathcal{T}/MCG$ is at least a real analytic manifold.

The Bers fiber space $\mathcal{B}$ is the complex disc holomorphic bundle over $\mathcal{T}$ with fiber over a marked surface R being the universal cover $\widetilde{R}$. A point on a fiber can be declared to be a puncture, and so $\mathcal{B}_g \approx \mathcal{T}_{g,1}$ and $\mathcal{B}_{g,n} \approx \mathcal{T}_{g,n+1}$. An extension $MCG_{\mathcal{B}}$ of MCG by the fundamental group of the fiber $\pi_1(F)$ acts properly discontinuously and holomorphically on $\mathcal{B}$. The resulting map $\pi : \mathcal{B}/MCG_{\mathcal{B}} \to \mathcal{T}/MCG$ describes an orbifold bundle, the *universal curve* over $\mathcal{M}$; the fibers are Riemann surfaces modulo their automorphism groups. A fiber of the vertical line bundle $(\mathrm{Ker}\, d\pi)$ is a tangent to a Riemann surface; the hyperbolic metrics of individual surfaces provide metrics for the fibers. The Chern form $\mathbf{c}_1(\mathrm{Ker}\, d\pi)$ on $\mathcal{B}/MCG_{\mathcal{B}}$ can be used to define forms/cohomology classes on $\mathcal{M}$ by integrating over fibers $\kappa_k = \int_{\pi^{-1}(R)} \mathbf{c}_1^{k+1}$.

Theorem 3 ([W]). *For the hyperbolic metrics on fibers,* $2\pi^2\kappa_1 = \omega$, *pointwise on* $\mathcal{M}$ *and in cohomology on the Deligne–Mumford compactification* $\overline{\mathcal{M}}$.

There is a unique way to fill in a puncture for a conformal structure. Accordingly a labeled puncture determines a section s of $\widetilde{\mathcal{B}}/MCG_{\mathcal{B}} \to \mathcal{M}$ (punctures on fibers are now filled in). Along a puncture section s we consider the family of tangent lines $(\mathrm{Ker}\, d\pi)|_s$ or the dual family of cotangent lines $(\mathrm{Ker}\, d\pi)^*|_s$—the pullback to $\mathcal{M}$ by the puncture section s of the dual family of cotangent lines is the tautological canonical class ψ.

Mirzakhani–McShane and the Volume Recursion

On a compact manifold, the number of closed geodesics of length less than a given bound is finite. In 2003 Mirzakhani wrote, “My work has been motivated by the problem of estimating $s_R(L)$, the number of primitive simple closed geodesics of hyperbolic length less than L on R.” A natural first step is to average the count over all Riemann surfaces—this requires an approach to computing integrals over the moduli space of Riemann surfaces. Fundamental domains in Teichmüller space cannot provide a workable approach. With an especially ingenious and useful insight, Mirzakhani considered Greg McShane’s remarkable identity to provide a *partition of unity* relative to the mapping class group. The Mirzakhani–McShane identity is based on studying the geodesics emanating orthogonally from a geodesic boundary of a Riemann surface. The behavior of emanating geodesics

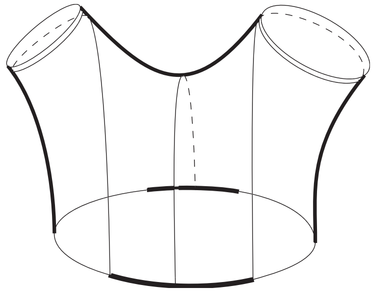

Figure 2. On the waist boundary: an ortho-boundary geodesic, a pair of distinguished intervals, and a pair of spirals.

prescribes a measure zero Cantor subset of the boundary. The remarkable identity is the formula that the sum of lengths of complementary intervals equals the boundary length.

A geodesic boundary β has an infinite number of *ortho-boundary geodesics*—simple geodesic segments orthogonal to the boundary at segment endpoints. Given an ortho-boundary geodesic γ, an ϵ-neighborhood of $\beta \cup \gamma$ is a topological pair of pants—the pants cuff boundaries are homotopic to simple geodesics α and λ. Along the boundary β there is a pair of distinguished intervals; each interval contains an endpoint of γ and has as endpoints spirals (one clockwise and one counterclockwise) about α and λ. A spiral is a simple infinite geodesic emanating orthogonally from β. The pair of pants bounded by β, α, and λ contains only the four spirals emanating orthogonally from β. The (equal) length of the two distinguished intervals is an elementary hyperbolic trigonometric expression in ℓ_β, ℓ_α, and ℓ_λ. The association of distinguished intervals to ortho-boundary geodesics can also be reversed. Given a pair of pants bounded by β, α', and λ', there is a corresponding ortho-boundary geodesic γ' and distinguished intervals bounded by spirals about α' and λ'. Distinguished intervals for distinct ortho-boundary geodesics are disjoint, and the complement of all distinguished intervals on a given boundary is a null Cantor set.

To state the resulting identity, let

$$\mathcal{D}(x, y, z) = 2 \log \frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}}.$$

Theorem 4 (The Mirzakhani–McShane identity)**.** *For a hyperbolic surface R with a single boundary β of length L,*

$$L = \sum_{\alpha,\lambda} \mathcal{D}(L, \ell_\alpha, \ell_\lambda),$$

where the sum is over unordered pairs of simple geodesics with β, α, and λ bounding an embedded pair of pants.

Mirzakhani applied the length identity to reduce the action of the mapping class group to actions of mapping class groups of subsurfaces. Computing the volume of the moduli space is reduced to an integral of products of lower-dimensional volumes—the desired recursion.

The integration approach is illustrated by computing the genus one, one boundary, volume. For such tori, the selection of an ortho-boundary geodesic is equivalent to selecting a simple closed geodesic. The length identity is

$$L = \sum_{\alpha \ simple} \mathcal{D}(L, \ell_\alpha, \ell_\alpha).$$

Introduce FN coordinates relative to a particular geodesic α and introduce $Stab(\alpha) \subset MCG$, the stabilizer for MCG acting on free homotopy classes. The stabilizer is generated by the Dehn twist about α.

The Dehn twist acts in FN coordinates by $(\ell, \tau) \to (\ell, \tau + \ell)$. The sector $\{0 \le \tau < \ell\}$ is a fundamental domain for the $Stab(\alpha)$ action. A mapping class $h \in MCG$ acts on a geodesic-length function by $\ell_\alpha \circ h^{-1} = \ell_{h(\alpha)}$.

Now we apply an unfolding argument and write the length identity as

$$L = \sum_\alpha \mathcal{D}(L, \ell_\alpha, \ell_\alpha) = \sum_{h \in MCG/Stab(\alpha)} \mathcal{D}(L, \ell_{h(\alpha)}, \ell_{h(\alpha)}),$$

use the MCG action on geodesic-length functions to find

$$LV(L) = \int_{\mathcal{T}(L)/MCG} \sum_{MCG/Stab(\alpha)} \mathcal{D}(L, \ell_\alpha \circ h^{-1}, \ell_\alpha \circ h^{-1})\, \omega,$$

change variables on $\mathcal{T}$ by $p = h(q)$ to find

$$= \sum_{h \in MCG/Stab(\alpha)} \int_{h(\mathcal{T}(L)/MCG)} \mathcal{D}(L, \ell_\alpha, \ell_\alpha)\, d\tau d\ell$$

$$= \int_{\mathcal{T}(L)/Stab(\alpha)} \mathcal{D}(L, \ell_\alpha, \ell_\alpha)\, d\tau d\ell,$$

and use the $Stab(\alpha)$ fundamental domain to obtain the integral

$$\int_0^\infty \int_0^\ell \mathcal{D}(L, \ell, \ell)\, d\tau d\ell.$$

The integral in τ gives a factor of ℓ. The derivative $\partial \mathcal{D}(x, y, z)/\partial x$ is a simpler function. Apply this observation and differentiate in L to obtain a formula for the derivative of $LV(L)$:

$$\frac{\partial}{\partial L} LV(L) = \int_0^\infty \frac{1}{1 + e^{\ell + \frac{L}{2}}} + \frac{1}{1 + e^{\ell - \frac{L}{2}}}\, \ell d\ell = \frac{\pi^2}{6} + \frac{L^2}{8}.$$

The genus one volume formula $V(L) = \frac{\pi^2}{6} + \frac{L^2}{24}$ results.

The general volume recursion is a sum over topological configurations for the complement of a pair of pants containing a surface boundary; see Figure 3 (see p. 1238). The possible configuration types for the complement are: connected with two internal cuffs, disconnected with two internal cuffs and one internal cuff. For the tuple of boundary lengths $L = (L_1, \dots, L_n)$ and $V_g(L)$ the volume of $\mathcal{T}_g(L)/MCG$, the recursion is

$$(1) \quad \frac{\partial}{\partial L_1} L_1 V_g(L) = \mathcal{A}_g^{connect}(L) + \mathcal{A}_g^{disconnect}(L) + \mathcal{B}_g(L),$$

where $\mathcal{A}_g^*(L)$ and $\mathcal{B}_g(L)$ are integral transforms of products of smaller topological-type volume functions. By direct calculation the volume functions are polynomials of lengths-squared with coefficients positive rational multiples of powers of π—products of factorials and Riemann zeta at nonnegative even integers. Mirzakhani very astutely recognized that symplectic geometry provides

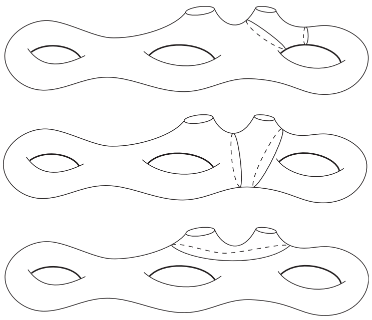

Figure 3. Pants complement configurations.

the setting for the volume formulas to be intersection numbers and for the recursion to give a solution for the Witten–Kontsevich conjecture. The relations between volumes of moduli spaces are the relations of $2d$-quantum gravity.

Symplectic Reduction

Mirzakhani considered equivalences of symplectic spaces. She introduced the space $\widehat{\mathcal{T}}_{g,n}$ of decorated Riemann surfaces with geodesic boundaries of varying lengths and a reference point on each boundary. The $\mathbb{R}$ dimension of $\widehat{\mathcal{T}}_{g,n}$ is $2n$ greater than that of $\mathcal{T}_g(L)$. To a Riemann surface with reference points on n boundaries is also associated a tailored surface with $2n$ cusps and no boundaries. For a boundary of length ℓ, an almost tight pair of pants with boundary lengths ℓ, 0, and 0 (zero length prescribes a puncture) can be attached to cap off the boundary. Almost tight pants are attached by aligning reference points. The space $\widehat{\mathcal{T}}_{g,n}$ is equivalent to $\mathcal{T}_{g,2n}$. The seams and variable points can be included in a pants decomposition of the resulting tailored surface. Theorems 1 and 2 can be applied to prescribe a symplectic form $\omega_{\widehat{\mathcal{T}}_{g,n}}$ on $\widehat{\mathcal{T}}_{g,n}$. Consider the resulting symplectic geometry. For a Riemann surface with boundaries $\beta_1, \dots, \beta_n$, let L_j be the length of β_j, τ_j describe the location of the reference point, and t_j be the infinitesimal FN twist of the jth almost tight pants. Introduce the moment map

$$\widehat{\mathcal{T}}_{g,n} \xrightarrow{\mu} \hat{L} = (L_1^2/2, \dots, L_n^2/2) \in \mathbb{R}^n_{\geq 0}.$$

The function $L_j^2/2$ is the Hamiltonian potential for the rescaled twist $\omega_{\widehat{\mathcal{T}}_{g,n}}(-L_j t_j, \ ) = d(\frac{1}{2}L_j^2)$, and $\omega_{\widehat{\mathcal{T}}_{g,n}}$ is twist flow invariant. In particular the vector field $-L_j t_j$ is the infinitesimal generator for an S^1 action on the jth reference point. The group $(S^1)^n$ acts on the level sets of $\mu : \widehat{\mathcal{T}}_{g,n} \to \mathbb{R}^n$ and the quotient of a level set of the moment map is naturally $\mathcal{T}_g(L)$. Theorem 2 leads to symplectic reduction: for small L, the level set quotient $\mu^{-1}(\hat{L})/(S^1)^n$ is diffeomorphic to $\mathcal{T}_g(L)$ with

$$\omega_{\widehat{\mathcal{T}}_{g,n}}|_{\mu^{-1}(\hat{L})/(S^1)^n} \approx \omega_{\mathcal{T}_g(L)},$$

and the level set quotients are mutually diffeomorphic. The volumes $V_g(L)$ are the volumes of $(S^1)^n$ quotients of μ level sets in $\widehat{\mathcal{T}}_{g,n}$.

Mirzakhani noted that the reference points on circle boundaries describe S^1 bundles. $\widehat{\mathcal{T}}_{g,n}$ is the total space of an $(S^1)^n$ bundle over $\mathcal{T}_g(L)$. Circle bundles can be endowed with S^1 principal connections with the associated curvature 2-forms giving the first Chern classes of the bundles in the cohomology of the compactification of $\mathcal{T}_g(L)/MCG$. What are the cohomology classes? Instead of capping off with almost tight pants, a boundary β can be capped off by a Euclidean disc of radius L with the boundary reference point prescribing a radius vector at the origin. The orientation for the twist parameter prescribes the orientation for rotating the radius vector. The fiber is equivalent to the cotangent plane at the origin—the description of the tautological class ψ.

An application of the Duistermaat–Heckman theorem now gives the desired cohomology relationship

$$\omega_{\mathcal{T}_g(L)} \equiv \omega_{\mathcal{T}_g(0)} + \sum_{j=1}^{n} \frac{L_j^2}{2} c_1(\hat{\beta}_j),$$

for $\hat{\beta}_j$ the S^1 principal bundle for β_j. The bundles $\hat{\beta}_j$ and ψ_j are smoothly equivalent. Combining with Theorem 3's formula $2\pi^2\kappa_1 = \omega$ gives Mirzakhani's beautiful relationship of volumes to intersection numbers.

Theorem 5 (Volume and intersections)**.** *For* $d = \dim_{\mathbb{C}} \mathcal{T}_{g,n}$,

$$V_g(L) = \frac{1}{d!} \int_{\mathcal{T}_{g,n}/MCG} \left(2\pi^2\kappa_1 + \sum_{j=1}^{n} \frac{L_j^2}{2}\psi_j\right)^d.$$

A very pleasing result: the coefficients of the volume polynomials are the κ_1-ψ tautological intersection numbers. The tautological classes are rational and nonnegative. The observations explain the positive coefficients and the pattern of powers of π^2 and L^2 in the volume polynomials.

Witten–Kontsevich and Counting Geodesics by Length

Witten posited that a generating function for the ψ intersection numbers should satisfy the KdV relations. To encode intersections, for any set of nonnegative integers $\{d_1, \dots, d_m\}$, write for top intersections

$$\langle \tau_{d_1}, \dots, \tau_{d_m} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{m} \psi_i^{d_i},$$

for the integral over the compactified moduli space. A product is defined for $\sum_{i=1}^m d_i = 3g-3+n$ and is otherwise set equal to zero. Introduce formal variables $t_i, i \geq 0$, and define F_g, the generating function for genus g, by

$$F_g(t_0, t_1, \dots) = \sum_{\{d_i\}} \langle \prod \tau_{d_i} \rangle_g \prod_{r>0} t_n^{n_r}/n_r!,$$

where the sum is over all finite sequences of nonnegative integers and $n_r = \#\{i \mid d_i = r\}$. The generating function

$$\mathbf{F} = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$$

is a partition function (probability of states) for a $2d$-quantum gravity model.

Known relations for the intersection numbers include the string equation for adding a puncture to a surface without a ψ factor and the dilaton equation for adding a puncture with a single ψ factor. For $n > 0$ and $\sum_i \alpha_i = 3g-2+n > 0$, the relations are

$$\text{string equation } \langle \tau_0 \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g = \sum_{\alpha_i \neq 0} \langle \tau_{\alpha_1} \cdots \tau_{\alpha_i - 1} \cdots \tau_{\alpha_n} \rangle_g,$$

and for $n \geq 0$ and $\sum_i \alpha_i = 3g-3+n \geq 0$, the dilaton equation

$$\langle \tau_1 \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g = (2g-2+n) \langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g.$$

Witten's posited relations are expressed in terms of the Virasoro Lie algebra generated by differential operators $\mathcal{L}_n = -z^{n+1}\frac{\partial}{\partial z}, n \geq -1$, with commutators $[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}$. The leading terms of the volume polynomials $V_g(L)$ are given by top ψ intersections. Mirzakhani was able to decipher the mystery—the leading terms of the volume recursion (1) give the desired relations $\mathcal{L}_*(e^{\mathrm{F}}) = 0$.

Theorem 6 (The Witten-Kontsevich conjecture)**.** *The moduli volume recursion gives the Virasoro constraints.*

Mirzakhani's prime simple geodesic theorem is based on studying the action of MCG on the space $\mathcal{MGL}$ of compactly supported measured geodesic laminations. A geodesic lamination is a foliation of a closed subset of a surface by geodesics. A transverse measure for a geodesic lamination $\mathcal{G}$ is an assignment for each smooth transverse arc τ with endpoints in the complement, a nonnegative measure which supports the intersection $\mathcal{G} \cap \tau$. Assigned measures coincide for smooth transversals that are homotopic through smooth arcs with endpoints in $\mathcal{G}^c$. Assigning masses to transverse arcs is a functional, and the assignment provides for a weak* topology—$\mathcal{MGL}$ is the space of compactly supported geodesic laminations. The space $\mathcal{MGL}$ is the closure of the set of positively weighted simple closed geodesics; $\mathbb{R}_+$ acts on $\mathcal{MGL}$ by multiplying measures. Typically a convergent sequence of simple closed geodesics limits to a disjoint union of infinite simple complete geodesics. The cross section of a measured geodesic lamination is the union of a finite set and a Cantor set. A multicurve is a finite positive weighted sum $\gamma = \sum_{j=1}^m a_j \gamma_j$ of disjoint, distinct simple closed geodesics with length $\ell_\gamma = \sum_{j=1}^m a_j \ell_{\gamma_j}$. For a general element ν of $\mathcal{MGL}$, the product of the transverse measure and hyperbolic length along leaves gives a measure whose total mass is ℓ_ν the length of ν. Multicurves with integer weights $\mathcal{MGL}(\mathbb{Z})$ form an MCG-invariant lattice-like subset. The space $\mathcal{MGL}$ has a natural volume element, the Thurston measure μ_{Th}. For a convex open set $U \subset \mathcal{MGL}$,

$$\mu_{Th}(U) = \lim_{T \to \infty} \frac{\#\{T \cdot U \cap \mathcal{MGL}(\mathbb{Z})\}}{T^{6g-6+2n}};$$

equivalently μ_{Th} can be computed from the masses of leaves intersecting a system of transverse arcs; μ_{Th} is the top exterior power of Thurston's train-track symplectic form. The volumes of geodesic-length balls $B(R) = \mu_{Th}(\{\nu \in \mathcal{MGL} \mid \ell_\nu(R) \leq 1\})$ and total volume $b(R) = \int_{\mathcal{M}_{g,n}} B(R)dV$ provide $\mathcal{MGL}(\mathbb{Z})$ normalizing factors in the following formulas.

Margulis's general counting result is that for a compact negatively curved closed manifold, the count of closed geodesics is asymptotic in length L to e^{hL}/hL, for h the topological entropy of the geodesic flow. The homeomorphisms of a surface act on the free homotopy classes of curves. Mirzakhani's focus is the length counting function for an MCG orbit

$$s_R(L,\gamma) = \#\{\alpha \in MCG\gamma \mid \ell_\alpha(R) \leq L\}$$

for a multicurve on a compact surface with possible cusps.

Theorem 7 (Prime simple geodesic theorem)**.** *For a rational multicurve γ, there is a positive constant $c(\gamma)$ such that*

$$\lim_{L \to \infty} \frac{s_R(L,\gamma)}{L^{6g-6+2n}} = \frac{c(\gamma)B(R)}{b(R)},$$

where the constant is computed from the weights for γ, the symmetries of γ, and the volume polynomial $V_{R(\gamma)}(L)$ for the cut open surface $R - \gamma$.

For a multicurve γ, introduce FN coordinates for $\mathcal{T}$, including the component curves of γ. Similar to the above unfolding argument for

$$\int_{\mathcal{T}/MCG} \sum_\alpha \mathcal{D}(L, \ell_\alpha, \ell_\alpha) dV = \int_0^\infty \mathcal{D}(L, \ell, \ell)\ell\, d\ell,$$

we have

$$\int_{\mathcal{T}/MCG} s_R(L,\gamma) dV = (\#Sym(\gamma))^{-1} \times \int_{\{0 < \sum a_j x_j \leq L, x_j > 0\}} V(R(\gamma); \mathbf{x})\, \mathbf{x} \cdot d\mathbf{x}$$

for the symmetry group of γ and $V(R(\gamma);\mathbf{x})$ the volume of the moduli space of the cut open surface $R(\gamma)$ with boundary lengths $\mathbf{x}$. The right-hand integral is calculated by the volume recursion.

Howard Masur showed that the MCG-invariant measure μ_{Th} is ergodic—an MCG-invariant subset of $\mathcal{MGL}$ either has full or zero μ_{Th} measure. Now if k_0 is a common denominator for the coefficients of γ, then the MCG orbit satisfies $MCG\gamma \subset 1/k_0 \mathcal{MGL}(\mathbb{Z})$ and counts for $s_R(L,\gamma)$ are bounded by counts for $1/k_0\mathcal{MGL}(\mathbb{Z})$. It follows for a convex open set U that a limit

$$\mu_\gamma(U) = \lim_{T_j \to \infty} \frac{\#\{T \cdot U \cap MCG\gamma\}}{T_j^{6g-6+2n}}$$

is bounded by $k_0^{-6g+6-2n}\mu_{Th}(U)$; the measure μ_γ is absolutely continuous with respect to μ_{Th}. By Masur ergodicity, we have that $\mu_\gamma = c'\mu_{Th}$. Now evaluate the measures on the unit balls $\{\nu \in \mathcal{MGL} \mid \ell_\nu \leq 1\}$ and integrate over the moduli space. The left-hand side is calculated above, and the right-hand side gives the normalizing factors $B(R)$ and $b(R)$, completing the argument.

Understanding all the invariant measures and all the possible closures of orbits for a group action is often very challenging. In the joint paper with Lindenstrauss, *Ergodic theory of the space of measured geodesic laminations,* the authors provide an understanding for the mapping class group MCG acting on $\mathcal{MGL}$. The description is in terms of *complete pairs* (S,γ), a multicurve γ, and a subsurface

S that is a union of components of the cut open $R(\gamma)$. For the natural embedding $\mathcal{I}_S$ of $\mathcal{MGL}(S)$ into $\mathcal{MGL}(R)$, define the subset

$$\mathcal{G}^{(S,\gamma)} = \{\gamma + \nu \mid \nu \in \mathcal{I}_S(\mathcal{MGL}(S))\},$$

sums of pairs: a multicurve and a measured geodesic lamination contained in the complement. By using either the lattice-like subset $\mathcal{MGL}(S,\mathbb{Z})$ or Thurston's train-track symplectic form for S, there is a natural measure $\mu_{Th}^{(S,\gamma)}$ on $\mathcal{G}^{(S,\gamma)}$. The measure can be distributed to a locally finite measure μ_{Th}^{S} on the orbit $MCG \cdot \mathcal{G}^{(S,\gamma)}$ in $\mathcal{MGL}(R)$. Mirzakhani and Lindenstrauss give an especially straightforward answer to the MCG orbit questions.

Theorem 8. *Let μ be a locally finite MCG-invariant measure on $\mathcal{MGL}$. Then μ is a constant multiple of a μ_{Th}^{S} for a complete pair (S,γ). Given $\nu \in \mathcal{MGL}$, there exists a complete pair (S,γ) such that $\overline{MCG \cdot \nu} = MCG \cdot \mathcal{G}^{(S,\gamma)}$.*

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Section figures 1–3 courtesy of Scott A. Wolpert.

Maryam Mirzakhani with Greg McShane on the terrace at Luminy.

Alex Wright and Anton Zorich

The Magic Wand Theorem

From Problems of Physics to Billiards in Polygons

In a sense, dynamical systems concern anything which moves. The things that move might be the planets in the solar system, or a system of particles in a chamber, or a billiard ball on an unusually shaped table, or currents in the ocean, or electrons in a metal, etc. One can observe certain common phenomena in large classes of dynamical systems; in particular, ideal billiards might be interpreted as toy models of a gas in a chamber. One can argue that one cannot model a gas with only one molecule. But if you let your molecule move and take a picture of it every second, then superposing millions of pictures you will get a simplified model of the gas, where we pretend that molecules do not interact between themselves.

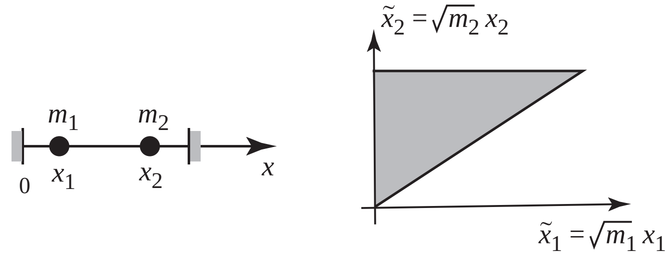

Figure 1. The configuration space of two molecules in a one-dimensional chamber is a triangle.

A very simple model of a gas where molecules do strongly interact also leads to a billiard. In this model there are only two molecules and the chamber is one-dimensional. Consider a system of two elastic beads confined to a rod placed between two walls,[1] as in Figure 1. (To the best of our knowledge this construction originates in lectures of Ya. G. Sinai.)

The beads have different masses m_1 and m_2; they collide between themselves and also with the walls. Assuming that the size of the beads is negligible, we can describe the configuration space of our system using coordinates $0 \le x_1 \le x_2 \le a$ of the beads, where a is the distance between the walls. Rescaling the coordinates as

$$\begin{cases} \tilde{x}_1 &= \sqrt{m_1}x_1, \\ \tilde{x}_2 &= \sqrt{m_2}x_2, \end{cases}$$

we see that the configuration space in the new coordinates is given by a right triangle Δ; see Figure 1. Consider now a trajectory of our dynamical system. It is not difficult to verify that in coordinates $(\tilde{x}_1, \tilde{x}_2)$ trajectories of the system of two beads on a rod correspond to billiard trajectories in the triangle Δ.

Dynamics of a triangular billiard is anything but trivial. For example, it is not known whether every acute triangle has at least one closed billiard trajectory different from the *Fagnano trajectory* presented in Figure 2. For a general obtuse triangle we do not know whether it has any closed billiard trajectories at all! (See papers of Richard Schwartz for recent results in this direction.)

From Billiards in Polygons to Flat Surfaces

Let us discuss now how billiards in rational polygons (polygons having angles rational multiples of π) give rise to surfaces and how billiard trajectories unfold to straight lines as in Figure 3 (see p. 1242).

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[1] *This paper uses some extracts from publications of the second author in* Gazette des Mathèmaticiens ***142** and **154**.*

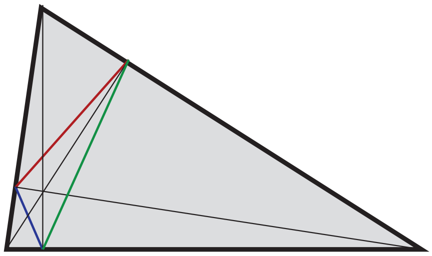

Figure 2. Every acute triangle has a closed Fagnano trajectory, joining the bases of the three heights. It is not known whether it always has another.

Consider a rectangular billiard and launch a billiard ball. When the ball hits the wall we can reflect the billiard table instead of letting the ball bounce from the wall. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. Note that at any moment the ball moves in one of the four directions defining the four types of copies of the billiard table. Copies of the same type are related by a parallel translation respecting the labeling of the corners.

Identifying the equivalent patterns by a parallel translation we obtain a torus; the billiard trajectory unfolds to a straight line on the corresponding torus, as in Figure 4 (see p. 1242).

One can apply a similar unfolding construction to any polygon with angles which are rational multiples of π to pass from a billiard to a flat surface.

Consider, for example, the triangle with angles $3\pi/8, 3\pi/8, \pi/4$, as in Figure 5 (see p. 1242). It is easy to check that a generic trajectory of such billiard moves at any time in one of 8 directions (compared to 4 for a rectangle). We can unfold the triangle to a regular octagon glued from 8 copies of the triangle. Identifying opposite sides of the octagon we get a flat surface. All straight lines on this surface project to the initial billiard trajectories.

The geometry of flat surfaces provides important information on billiards. For example, it was proved by H. Masur that rational billiards have plenty of closed trajectories. Closed trajectories appear in bands of parallel trajectories of the same length as in Figure 6 (see p. 1242). To quantify the number of closed billiards trajectories in a rational polygon one counts the number of such bands. It is natural to take into account how thick the band is by giving the band a weight equal to the ratio of the area of the band over the area of the billiard table. When some regions of the billiard table are covered by the band several times, we count their contribution to the area of the band with the corresponding multiplicity. In other words, the area of the band is measured on the flat surface obtained after unfolding the rational billiard as in Figure 10 (see p. 1245).

The results of Masur imply that the number $N(\Pi, L)$ of closed trajectories of length at most L in a rational polygon Π admits lower and upper bounds

$$\liminf_{L\to+\infty} \frac{N(\Pi,L)}{L^2} = c(\Pi) > 0, \limsup_{L\to+\infty} \frac{N(\Pi,L)}{L^2} = C(\Pi).$$

It is not known, however, whether the number of closed trajectories in every rational polygon Π admits exact quadratic asymptotics, i.e., with $c(\Pi) = C(\Pi)$. We are not even aware of an algorithm which produces realistic estimates of the constants $c(\Pi)$ and $C(\Pi)$ for a general rational polygon Π.

Translation Surfaces

We have seen that unfolding a billiard in a rectangle gives us a flat torus. Unfolding more complicated rational polygons gives us more complicated translation surfaces. We present now a construction of a general translation surface, not necessarily related to a billiard.

Consider a collection of vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^2$ and arrange these vectors into a broken line as in Figure 7 (see p. 1242). Construct another broken line starting at the same point as the first one arranging the same vectors in the order $\vec{v}_{\pi(1)}, \dots, \vec{v}_{\pi(n)}$, where π is some permutation of n elements. By construction the two broken lines share the same endpoints; suppose that they bound a polygon as in Figure 7. Identifying the pairs of sides corresponding to the same vectors $\vec{v}_j$, $j = 1, \dots, n$, by parallel translations we obtain a closed topological surface.

By construction, the surface is endowed with a flat metric. When $n = 2$ and $\pi = (2, 1)$ we get a usual flat torus glued from a parallelogram. For a larger number of elements we might get a surface of higher genus, where the genus is determined by the permutation π. It is convenient to impose from now on some simple restrictions on the permutation π which guarantee nondegeneracy of the surface; see the original *Annals of Mathematics* papers of H. Masur and W. Veech for details.

For example, a regular octagon gives rise to a surface of genus two as in Figure 8 (p. 1243). Indeed, identifying pairs of horizontal and vertical sides of a regular octagon we get a usual torus with a hole in the form of a square. We slightly cheat in the next frame, where we turn this hole by 45° and only then glue the next pair of sides. As a result we get a torus with two isolated holes as on the third frame. Identifying the remaining pair of sides (which represent the holes) we get a torus *with a handle* or, in other words, a surface of genus two.

Similar to the torus case, the surface glued from the regular octagon or from an octagon as in Figure 7 also inherits from the polygon a flat metric, but now the resulting flat metric has a singularity at the point obtained from identified vertices of the octagon.

Note that the flat metric thus constructed is very special: since we identify the sides of the polygon only by translations, the parallel transport of any tangent vector along a closed cycle (avoiding conical singularities) on the resulting surface brings the vector back to itself. In other words, our flat metric has *trivial holonomy*. In particular, since a parallel transport along a small loop around any conical singularity brings the vector to itself, the cone

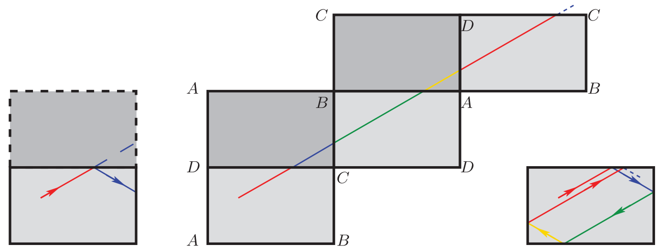

Figure 3. A billiard trajectory (right) can be unfolded to a straight line.

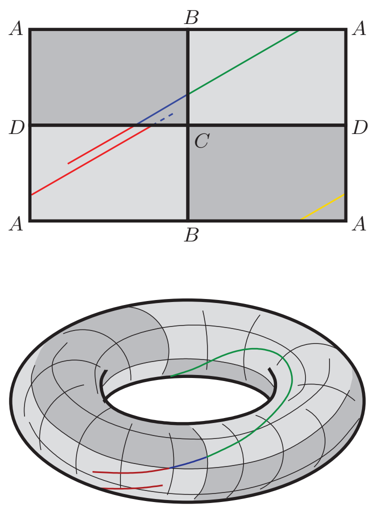

Figure 4. An unfolded billiard trajectory is a straight line on the flat torus.

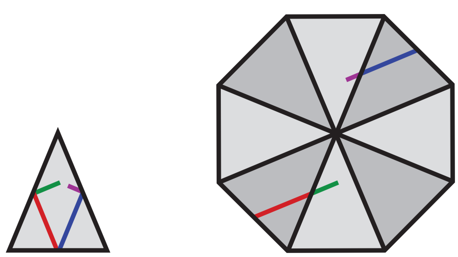

Figure 5. Billiard trajectories on the descriptor $\frac{3\pi}{8}$-$\frac{3\pi}{8}$-$\frac{\pi}{4}$ triangle unfold to straight lines on the octagon with opposite sides identified.

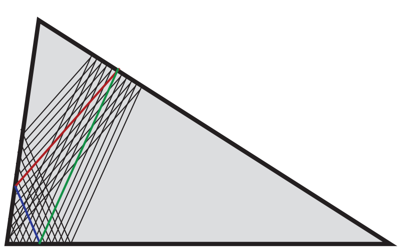

Figure 6. Closed trajectories on the triangle appear in parallel bands.

angle at any singularity is an integer multiple of 2π. In the most general situation the flat surface of genus g would have several conical singularities with cone angles $2\pi(d_1+1), \ldots, 2\pi(d_m+1)$, where $d_1+\cdots+d_m = 2g-2$.

It is convenient to consider the vertical direction as part of the structure. A surface endowed with a flat metric with trivial holonomy and with a choice of a vertical direction is called a *translation surface*. Two polygons in the plane obtained one from another by a parallel translation give rise to the same translation surface, while polygons obtained one from another by a nontrivial rotation (usually) give rise to distinct translation surfaces.

We can assume that the polygon defining our translation surface is embedded into the complex plane $\mathbb{C} \simeq \mathbb{R}^2$ with coordinate z. The translation surface obtained by identifying the corresponding sides of the polygon inherits the complex structure. Moreover, since the gluing rule for the sides can be expressed in local coordinates as $z = \tilde{z} + \text{const}$, the closed 1-form $dz = d(\tilde{z} + \text{const})$ is well-defined not only in the polygon but on the surface. An exercise in complex analysis shows that the complex structure extends to the points coming from the vertices of the polygon and that the 1-form $\omega = dz$ extends to the holomorphic 1-form on the resulting Riemann surface. This 1-form ω has zeroes of degrees $d_1, \ldots, d_m$ exactly at the points where the flat metric has conical singularities of angles $2\pi(d_1+1), \ldots, 2\pi(d_m+1)$.

Conversely, given a holomorphic 1-form ω on a Riemann surface one can always find a local coordinate z (in a simply-connected domain not containing zeroes of ω) such that $\omega = dz$. This coordinate is defined up to an

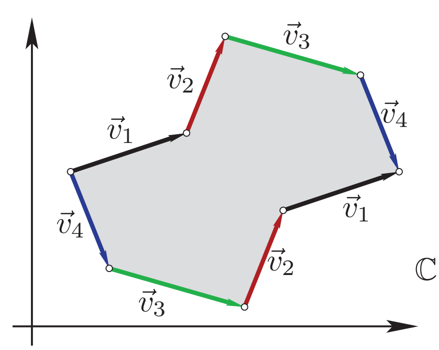

Figure 7. Identifying corresponding pairs of sides of this polygon we obtain a flat surface of genus two with a single conical singularity.

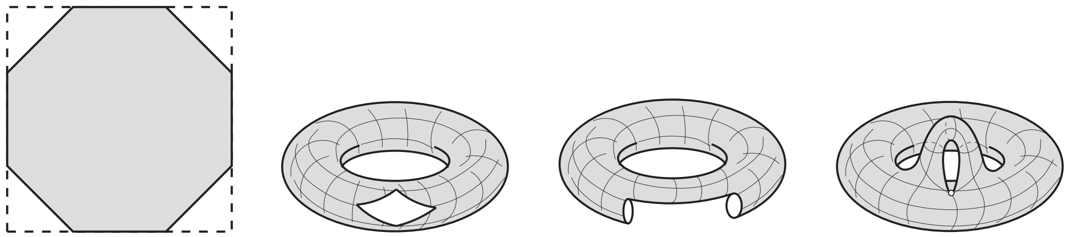

Figure 8. Identifying opposite sides of a regular octagon gives rise to a surface of genus two.

additive constant. It defines the translation structure on the surface. Cutting up the surface along an appropriate collection of straight segments joining conical singularities, we can unwrap the Riemann surface into a polygon as above. Polygons that differ by cut and paste correspond to the same ω; we will elaborate on this later.

This construction shows that the two structures are completely equivalent: the flat metric with trivial holonomy plus a choice of distinguished direction or a Riemann surface endowed with a nonzero holomorphic 1-form.

Families of Translation Surfaces and Dynamics in the Moduli Space

The polygon in our construction depends continuously on the vectors $\vec{v}_i$. This means that the topology of the resulting translation surface (its genus g, the number and the types of the resulting conical singularities) does not change under small deformations of the vectors $\vec{v}_i$. For every collection of cone angles $2\pi(d_1+1), \ldots, 2\pi(d_m+1)$ satisfying $d_1 + \cdots + d_m = 2g - 2$ with integer d_i for $i = 1, \ldots, n$, we get a *space* $\mathcal{H}(d_1, \ldots, d_m)$ *of translation surfaces*. Vectors $\vec{v}_1, \ldots, \vec{v}_n$ can be viewed as local complex coordinates on this space, called *period coordinates*. These coordinates define a structure of a *complex orbifold* (manifold with moderate singularities) on each space $\mathcal{H}(d_1, \ldots, d_m)$. The geometry and topology of spaces of translation surfaces is not yet sufficiently explored.

Readers preferring algebro-geometric language may view the space of translation surfaces with fixed conical singularities $2\pi(d_1+1), \ldots, 2\pi(d_m+1)$ as the stratum $\mathcal{H}(d_1, \ldots, d_m)$ in the moduli space $\mathcal{H}_g$ of pairs (Riemann surface C; holomorphic 1-form ω on C), where the stratum is specified by the degrees $d_1, \ldots, d_m$ of zeroes of ω, where $d_1 + \cdots + d_m = 2g - 2$. Note that while the moduli space $\mathcal{H}_g$ is a holomorphic $\mathbb{C}^g$-bundle over the moduli space $\mathcal{M}_g$ of Riemann surfaces, individual strata are not. For example, the minimal stratum $\mathcal{H}(2g-2)$ has complex dimension $2g$, while the moduli space $\mathcal{M}_g$ has complex dimension $3g-3$. The very existence of a holomorphic form with a single zero of degree $2g-2$ on a Riemann surface C is a strong condition on C.

To complete the description of the space of translation surfaces we need to present one more very important structure: the action of the group $\mathrm{GL}(2,\mathbb{R})$ on $\mathcal{H}_g$ minus the zero section. This action preserves strata. The description of this action is particularly simple in terms of our polygonal model Π of a translation surface S. A linear transformation $g \in \mathrm{GL}(2,\mathbb{R})$ of the plane maps the polygon Π to a polygon $g\Pi$. The new polygon again has all sides arranged into pairs, where the two sides in each pair are parallel and have equal length. We can glue a new translation surface and call it $g \cdot S$. It is easy to see that unwrapping the initial surface into different polygons would not affect the construction. Note also that we explicitly use the choice of the vertical direction: any polygon is endowed with an embedding into $\mathbb{R}^2$ defined up to a parallel translation.

The subgroup $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$ preserves the flat area. This implies that the action of $\mathrm{SL}(2,\mathbb{R})$ preserves the real hypersurface $\mathcal{H}_1(d_1, \ldots, d_m)$ of translation surfaces of area one in any stratum $\mathcal{H}(d_1, \ldots, d_m)$. The codimension-one subspace $\mathcal{H}_1(d_1, \ldots, d_m)$ can be compared to the unit hyperboloid in $\mathbb{R}^{2n}$.

Recall that under appropriate assumptions on the permutation π, the n vectors

$$\vec{v}_1 = \begin{pmatrix} v_{1,x} \\ v_{1,y} \end{pmatrix}, \ldots, \vec{v}_n = \begin{pmatrix} v_{n,x} \\ v_{n,y} \end{pmatrix}$$

as in Figure 7 define local coordinates in the space $\mathcal{H}(d_1, \ldots, d_m)$ of translation surfaces. Let $dv := dv_{1x}dv_{1y}\ldots dv_{nx}dv_{ny}$ be the associated volume element in the corresponding coordinate chart $U \subset \mathbb{R}^{2n}$. It is easy to verify that dv does not depend on the choice of coordinates, $\vec{v}_1, \ldots, \vec{v}_n$, so it is well-defined on $\mathcal{H}(d_1, \ldots, d_m)$. Similarly to the case of Euclidean volume element, we get a natural induced volume element dv_1 on the unit hyperboloid $\mathcal{H}_1(d_1, \ldots, d_m)$. It is easy to check that the action of the group $\mathrm{SL}(2,\mathbb{R})$ preserves the volume element dv_1.

The following theorem proved independently and simultaneously by Masur and Veech is the keystone of this area. (The two papers were published in the same volume of *Annals of Mathematics* in 1982).

Theorem (H. Masur, W. A. Veech)**.** *The total volume of every stratum* $\mathcal{H}_1(d_1, \ldots, d_m)$ *is finite.*

The group $\mathrm{SL}(2,\mathbb{R})$ *and its diagonal subgroup act ergodically on every connected component of every stratum* $\mathcal{H}_1(d_1, \ldots, d_m)$.

Here "ergodically" means that every measurable subset invariant under the action of the group has necessarily measure zero or full measure. The Birkhoff Ergodic Theorem then implies that the orbit of almost every point homogeneously fills the ambient connected component. In plain terms, a consequence of the ergodicity of the action

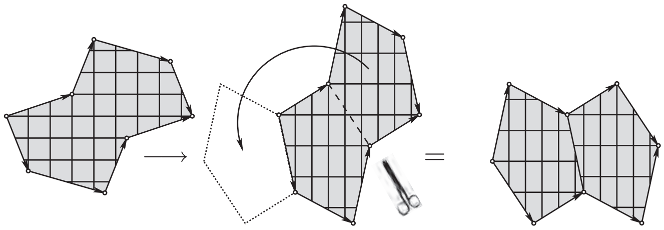

Figure 9. Expansion-contraction, combined with cut-and-paste transformations, approximates any other prescribed octagon.

of the diagonal subgroup can be interpreted as follows. Having almost any polygon as above, we can choose an appropriate sequence of times t_i such that contracting the polygon horizontally with a factor e^{t_i} and expanding it vertically with the same factor e^{t_i} and modifying the resulting polygonal pattern of the resulting translation surface by an appropriate sequence of cut-and-paste transformations (see Figure 9) we can get arbitrarily close to, say, a regular octagon rotated by any angle chosen in advance. Note that expansion-contraction (action of the diagonal group) changes the translation surface, while cut-and-paste transformations change only the polygonal pattern and do not change the flat surface.

Now everything is prepared to present the first marvel of this story. Suppose that we want to find out some fine properties of the vertical flow on an individual translation surface. Applying a homothety we can assume that the translation surface has area one. Masur and Veech suggest the following approach. Consider our translation surface (endowed with a vertical direction) as a point S in the ambient stratum $\mathcal{H}_1(d_1, \ldots, d_m)$. Consider the orbit $\mathrm{SL}(2,\mathbb{R}) \cdot S$ (or the orbit of S under the action of the diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ depending on the initial problem). Numerous important properties of the initial vertical flow are encoded in the geometrical properties of the closure of the corresponding orbit. This approach places the problem of finding the orbit closures under the action of $\mathrm{SL}(2,\mathbb{R})$ and studying their geometry at the center of the work in this area for the last three decades.

At first glance, we have just reduced the study of a rather simple dynamical system, namely, the vertical flow on a translation surface, to a really complicated one, the study of the action of the group $\mathrm{GL}(2,\mathbb{R})$ on the space $\mathcal{H}(d_1, \ldots, d_m)$. Nevertheless, this approach proved to be extremely fruitful, despite the fact that geometry and topology of spaces of translation surfaces is still under exploration. It is the Magic Wand Theorem of A. Eskin, M. Mirzakhani, and A. Mohammadi which made this approach particularly powerful.

"Almost All" versus "All"

From the dynamical point of view, the moduli space of holomorphic differentials can be viewed as a "homogeneous space with difficulties." We are citing Eskin, who knows both facets very well: how the dynamics on the moduli space might mimic the homogeneous dynamics in some situations and how deep the difficulties might be.

The rigidity theorems including and generalizing the theorems proved by Marina Ratner at the beginning of the 1990s show why homogeneous dynamics is so special. General dynamical systems usually have some very peculiar trajectories living in very peculiar fractal subsets. Such trajectories are rare, but there are still plenty of them. In particular, the question of identification of all (versus almost all) orbit closures or of all invariant measures makes no sense for most dynamical systems: the jungle of exotic trajectories is too large. In certain situations this diversity creates a major difficulty: even when you know plenty of fine properties of the trajectory launched from almost every starting point, you have no algorithm to check whether the particular initial condition you are interested in is generic or not. Ergodic theory answers statistical questions but says nothing about specific initial data.

Consider, for example, the geodesic flow on a compact Riemannian surface of constant negative curvature. We get a very nice dynamical system on the three-dimensional total space of the unit tangent bundle. It was observed long ago by H. Furstenberg and B. Weiss that the closures of individual geodesics might have almost arbitrary Hausdorff dimension from 1 (closed trajectories) to 3 (typical trajectories).

The situation in homogeneous dynamics is radically different. In certain favorable cases (such as certain unipotent flows) one manages to prove that *every* orbit closure is a nice homogeneous space, *every* invariant measure is the corresponding Haar measure, etc. This kind of rigidity gives rise to fantastic applications to number theory.

For several decades it was not clear to what extent the dynamics of the $\mathrm{SL}(2,\mathbb{R})$ action on the moduli space of Abelian and quadratic differentials resembles homogeneous dynamics. For Eskin, who came to the dynamics in the moduli space from homogeneous dynamics, it was, probably, the main challenge for fifteen years. Mirzakhani joined him in working on this problem about 2006. She was challenged by the result of her scientific advisor, C. McMullen, who had solved the problem in the particular case of genus two. After several years of collaboration, the first major part of the theorem, namely, the measure classification for $\mathrm{SL}(2,\mathbb{R})$-invariant measures, was proved in 2010.

To illustrate the importance of this theorem we can cite what Artur Avila has said about this result of Eskin and Mirzakhani to S. Roberts for the *New Yorker* article in memory of Mirzakhani:

> Upon hearing about this result, and knowing her earlier work, I was certain that she would be a front-runner for the Fields Medals to be given in 2014, so much so that I did not expect to have much of a chance.

We do not think that Maryam thought much about the Fields Medal at this time. Several years later she took the email message from I. Daubechies announcing that Maryam got the Fields Medal for a joke and just ignored it. But she certainly knew how important the theorem was. We'd say that most recent papers in our domain have used this Magic Wand Theorem in one way or another.

It took Eskin and Mirzakhani another several years of extremely hard work to extend their result from $\mathrm{SL}(2,\mathbb{R})$ to its subgroup of upper-triangular 2×2 matrices. The difference might seem insignificant, but exactly this difference is needed for the most powerful version of the Magic Wand Theorem.

Magic Wand Theorem

Now comes the Magic Wand Theorem of Eskin, Mirzakhani [EMi], and Mohammadi [EMiMh]. (A. Mohammadi joined the collaboration for the part of the theorem concerning orbit closures.)

As we have already mentioned, the moduli space is not a homogeneous space. Nevertheless, the orbit closures of $\mathrm{GL}(2,\mathbb{R})$ in the space of translation surfaces are as nice as one can only hope: they are complex manifolds, possibly with very moderate singularities (so-called "orbifolds"). In this sense the action of $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ on the space of translation surfaces mimics certain properties of the dynamical systems in homogeneous spaces.

Magic Wand Theorem (A. Eskin, M. Mirzakhani, and A. Mohammadi [EMiMh]). *The closure of any* $\mathrm{GL}(2,\mathbb{R})$*-orbit is a complex suborbifold (possibly with self-intersections). In period coordinates* $\vec{v}_1,\dots,\vec{v}_n$ *in the corresponding space* $\mathcal{H}(d_1,\dots,d_m)$ *of translation surfaces it is locally represented by a linear subspace.*

Every ergodic $\mathrm{SL}(2,\mathbb{R})$*-invariant measure is supported on a suborbifold. In coordinates* $\vec{v}_1,\dots,\vec{v}_n$ *this suborbifold is represented by the intersection of the hypersurface* $\mathcal{H}_1(d_1,\dots,d_m)$ *of translation surfaces of area one with a linear subspace, and the invariant measure is induced from the usual Lebesgue measure on this subspace.*

As a vague conjecture (or optimistic dream) this property was discussed long ago, and there was not the slightest hint of a general proof. The only exception is the case of surfaces of genus two, for which ten years ago C. McMullen proved a very precise statement, classifying all possible orbit closures. He used a special case of Ratner's results which are applicable here. And he, very ingeniously, used the special properties of surfaces of genus two.

The proof of the Magic Wand Theorem is a titanic work, which absorbed numerous fundamental recent developments in dynamical systems. Most of these developments do not have any direct relation to moduli spaces. It incorporates certain ideas of the low entropy method of M. Einsiedler, A. Katok, and E. Lindenstrauss; results of G. Forni and of M. Kontsevich on Lyapunov exponents of the Teichmüller geodesic flow; the ideas from the works of Y. Benoit and J.-F. Quint on stationary measures; iterative improvement of the properties of the invariant measure inspired by the approach of G. Margulis and G. Tomanov to the actions of unipotent flows on homogeneous spaces; some fine ergodic results due to Y. Guivarch and A. Raugi; and many others.

For many experts in the area it remains a miracle that Eskin and Mirzakhani managed to accomplish this project. Very serious technical difficulties appeared at every stage of the project. Not to mention that in the four years between 2010 and 2014 Maryam gave birth to a daughter and overcame the first attack of cancer. Since then we believed that Maryam could do anything.

The theorem itself really works like a Magic Wand, which allows one to touch a given billiard and obtain precious information describing its geometry and dynamics.

This Magic Wand works best if the corresponding orbit closure in the moduli space of translation surfaces can be computed. Unfortunately, until recently, the orbit closures associated to most billiards remained impossible to calculate. As the result of recent progress stemming from the Magic Wand Theorem, this situation is undergoing rapid improvement, some of which is witnessed by one of the last results of Mirzakhani [MiW]:

Theorem (M. Mirzakhani and A. Wright[MiW]). *There are infinitely many triangular billiards that unfold to translation surfaces whose* $\mathrm{GL}(2,\mathbb{R})$*-orbit is dense in the ambient stratum of flat surfaces.*

For example, the triangle Δ with angles $\left(\frac{2\pi}{11},\frac{4\pi}{11},\frac{5\pi}{11}\right)$ unfolds to a surface whose orbit is dense in $\mathcal{H}(1,3,4)$. A simple consequence for billiards is the existence of bands of parallel periodic trajectories which cover the

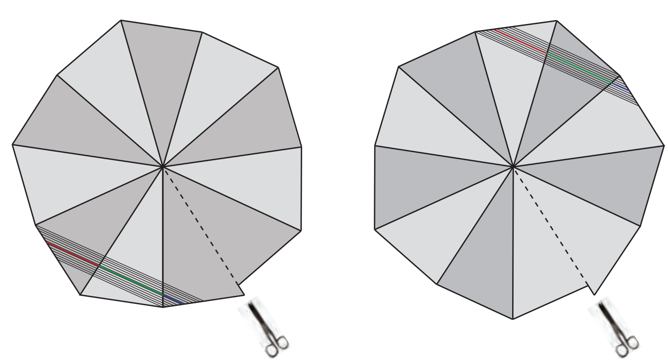

Figure 10. There are bands of closed trajectories which cover an arbitrarily large part of the 22-fold unfolding of the $\frac{2\pi}{11}$– $\frac{4\pi}{11}$– $\frac{5\pi}{11}$ triangle (obtained by sewing the above images along the dotted cut).

table almost 22 times. In other words, one can find bands of parallel closed geodesics as in Figure 10 covering an arbitrarily large part of the flat surface obtained by unfolding the triangle Δ. This follows directly from the density of the orbit, because there is an open set of translation surfaces in the ambient stratum with a band of parallel geodesics covering an arbitrarily large part of the surface.

This also gives a much more precise counting result for the number $N(\Delta, L)$ of bands of parallel periodic billiard trajectories of length less than L, weighted by the width of the band. Under some additional averaging one has

$$N(\Delta, L) \sim \frac{3350523}{760400\,\pi} \cdot \frac{L^2}{\text{area of the billiard table}}.$$

Here the constant in front of L^2 (the *Siegel–Veech constant*) is a function of the geometry of the orbit closure, so a billiard giving a different orbit closure would have a different constant.

There are only countably many $\mathrm{GL}(2,\mathbb{R})$-orbit closures. Each parameterizes surfaces with exceptional flat geometry, and by work of Filip, each is an algebraic variety that can be defined in terms of special properties of the Jacobian. Despite important progress, in general it remains an open problem to classify $\mathrm{GL}(2,\mathbb{R})$-orbit closures.

At first, the proof of the Magic Wand Theorem seemed to use the special properties of the $\mathrm{GL}(2,\mathbb{R})$ action on strata of translation surfaces. However, the techniques have already proved very useful far outside their original setting. For example, they have been applied to random dynamics on surfaces by Brown and Rodriguez Hertz and to homogeneous spaces by Eskin and Lindenstrauss.

In fact, we use not only the results but also the ideas of Mirzakhani all the time. Rereading her papers often echoes something which you were thinking about for a long time and all of a sudden it reveals a simple and unexpected solution. Mirzakhani foresaw many beautiful further results but did not have time to work on them. We often feel that we are following the paths and trails which she imagined and outlined.

Epilogue

We would feel insincere limiting ourselves to mathematical results of Maryam. We feel happy to have known Maryam as a person. Alex spent the last three years in Stanford working with Maryam, speaking with Maryam. We reproduce his recollection presented at the Stanford memorial for Maryam.

I remember vividly the first time I met Maryam in person. It was in her office at Stanford, and I was a graduate student, visiting at her invitation. I sat on the couch opposite the blackboard, which was layered with equations and mathematical doodles. Whenever she needed to write something, she would erase only the smallest portion of the board possible, and squeeze her writing into the small space. She spoke excitedly, sharing her ideas without reservation. Sometimes she repeated herself, sometimes she abandoned a sentence midway through, and her thoughts came out somewhat in a jumble. As her former student Jenya Sapir puts it, she would present different ideas like a cast of characters, all talking to each other. But as I started to grasp the depth of her understanding and vision, I felt awe.

Following the Prize Sessions at the 2009 Joint Mathematics Meetings, Mirzakhani stands with a group of Iranian-American mathematicians, including Ali Enayat, Mojtaba Moniri, Iraj Kalantari, Vali Siadat, and Bahman Kalantari.

Two years later, I was back in that same office, having just moved to Stanford to work with Maryam. She was so eager to get to work that she arranged for me to stay in a hotel for the week before my lease began, so that we could get started that much earlier.

We would discuss mathematics for hours at a time, standing at her blackboard, sometimes pacing in excitement or concentration. She was always optimistic. As her student Ben Dozier recalls, she would find some aspect of any question, even if naive, to be fascinating. Her enthusiasm was infectious.

I would leave our meetings determined, even if the mathematical challenges we faced were daunting. Maryam was not intimidated, and with her working with me, any fear of failure evaporated. I was energized to work on problems I previously would have judged to be too difficult to even attempt.

We had lunch together frequently, always at the business school cafeteria, right next to this auditorium. There our mathematical discussions would pause. Maryam spoke of her daughter, Anahita, and of the joys and challenges of being a parent. She spoke of teaching and complained that she was unable to stick with just one textbook, so that her house would be covered with different textbooks whenever she taught.

She rarely gave advice and spoke in mildly worded suggestions rather than instructions. On career issues, like the best journal to submit a paper to or how to get the best job, she was often silent. Perhaps, as a result of her own meteoric career, she had different firsthand experience with such struggles. But more than that, she radiated a sense of *perspective and focus, as if the adversities she had faced had taught her that which journal accepts a paper is not the most important thing and as if her intellectual curiosity dwarfed academic politics. She once told me: "Know what you want, and don't get distracted."*

She seemed to follow that advice herself. After she was awarded the Fields Medal, she wanted to get right back to thinking deeply. She tried to ignore reporters and the press coverage and seemed unchanged by her newfound recognition and celebrity.

But she did want to help fellow mathematicians, especially young mathematicians. One of the graduate students at Stanford, Pedram Safee, contacted her from Iran as a high school student. They had never met, but she still took the time to reply to him and give him advice on studying mathematics. Recently, Maryam became an editor of one of the most prestigious journals in mathematics. Most papers submitted to that journal get rejected, and the review process is very long. She told me repeatedly, "I just want to accept a paper, I just want to press the accept button."

When Maryam's cancer recurred, she wanted to keep working. Instead of her office, we met at the Coupa Cafe in front of Green Library, closer to her home. Between the meetings, now much less frequent, I was sad and worried, knowing that Maryam's medical situation was dire. But when I saw her in person, even when she spoke of her struggle, I felt almost elated. Maryam was still herself, still curious, brilliant, perseverant, and brave, and her optimism would make me feel, if only briefly, as if she would surely be ok.

Now, as we mourn her, I want to remember the beauty and power of her mathematics and the force of her personality, which could convince me, even as she was in pain, to smile.

For more on the the Magic Wand Theorem, see [W]. If you want to learn more about Maryam as a person, read the article by Erica Klarreich [K] in *Quanta*.

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Earlier this year, the American Mathematical Society governance developed and approved two ways by which to commemorate Maryam Mirzakhani and her mathematical legacy.

The Maryam Mirzakhani Lecture will be an annual AMS Invited Address at the Joint Mathematics Meetings (JMM). The lecture will allow leading scholars to present their research, while commemorating the exceptional accomplishments of Maryam Mirzakhani. *The first lecture will be at the 2020 JMM in Denver*.

The Maryam Mirzakhani Fund for The Next Generation is an endowment that exclusively supports programs for early career mathematicians; i.e., doctoral and postdoctoral scholars. It is part of a special initiative and aims to assist rising scholars each year at modest but impactful levels. A donation to the Maryam Mirzakhani Fund honors her memory by supporting emerging mathematicians now and in the future.

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