

MULTIPLE SADDLE CONNECTIONS ON FLAT SURFACES AND THE PRINCIPAL BOUNDARY OF THE MODULI SPACES OF QUADRATIC DIFFERENTIALS

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Abstract. We describe typical degenerations of quadratic differentials thus describing “generic cusps” of the moduli space of meromorphic quadratic differentials with at most simple poles. The part of the boundary of the moduli space which does not arise from “generic” degenerations is often negligible in problems involving information on compactification of the moduli space.

However, even for a typical degeneration one may have several short loops on the Riemann surface which shrink simultaneously. We explain this phenomenon, describe all rigid configurations of short loops, present a detailed description of analogs of desingularized stable curves arising here, and show how one can reconstruct a Riemann surface endowed with a quadratic differential which is close to a “cusp” from the corresponding point at the principal boundary.

Introduction

Saddle connections on flat surfaces. We study flat metrics on a closed orientable surface of genus g , which have isolated conical singularities and linear holonomy restricted to $\{Id, -Id\}$. If the linear holonomy group is trivial, then the surface is referred to as a *translation surface*, such a flat surface corresponds to an Abelian differential ω on a Riemann surface. If the

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holonomy group is nontrivial, then such a flat surface arises from a meromorphic quadratic differential q with at most simple poles on a Riemann surface. In this paper, unless otherwise stated, a quadratic differential is *not* the square of an Abelian differential and a *flat surface* is the Riemann surface with the flat metric corresponding to an Abelian or to a quadratic differential.

It is natural to consider families of flat surfaces sharing the same combinatorial geometry: the genus, the number of singularities and the cone angles at singularities. Such families correspond to the *strata* $\mathcal{Q}(d_1, \dots, d_m)$ in the moduli space of quadratic differentials (correspondingly to the strata $\mathcal{H}(n_1, \dots, n_m)$ in the moduli space of Abelian differentials when the flat structure has trivial linear holonomy). Here $d_i \in \{-1, 0, 1, 2, 3, \dots\}$ stand for the orders of singularities (simple poles, marked points, zeroes) of quadratic differentials (correspondingly $n_i \in \{1, 2, 3, \dots\}$ stand for the degrees of zeroes of Abelian differentials). The collection $\alpha = \{d_1, \dots, d_m\}$ (correspondingly $\beta = \{n_1, \dots, n_m\}$) is called the *singularity data* of the stratum.

A *saddle connection* is a geodesic segment joining a pair of conical singularities or a conical singularity to itself without any singularities in its interior. For the flat metrics as described above, regular closed geodesics always appear in families; any such family fills a maximal cylinder bounded on each side by a closed saddle connection or by a chain of parallel saddle connections. Thus, when some regular closed geodesic becomes short the corresponding saddle connection(s) become short as well. More generally, a degeneration of an Abelian or of a quadratic differential corresponds to collapse of some saddle connections.

Any saddle connection on a flat surface $S \in \mathcal{Q}(\alpha)$ persists under small deformations of S inside $\mathcal{Q}(\alpha)$. It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections. We say that a collection $\{\gamma_1, \dots, \gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_n|$ of the lengths of all saddle connections in the collection.

Degeneration of Abelian differentials. In the case of Abelian differentials ω , rigid collections of saddle connections were studied in the paper [EMZ]. It was shown that all saddle connections in any rigid collection are *homologous*. In particular, they are all parallel and have equal length and

either all of them join the same pair of distinct singular points, or they are all closed.

This implies that when the saddle connections in a rigid collection are contracted by a continuous deformation, the limiting flat surface generically decomposes into several components represented by nondegenerate flat surfaces S'_1, \dots, S'_k , where k might vary from one to the genus of the initial surface. Let $\mathcal{H}(\beta'_j)$ be the stratum ambient for S'_j . The stratum $\mathcal{H}(\beta') = \mathcal{H}(\beta'_1) \sqcup \dots \sqcup \mathcal{H}(\beta'_k)$ of disconnected flat surfaces $S'_1 \sqcup \dots \sqcup S'_k$ is referred to as a *principal boundary stratum* of the stratum $\mathcal{H}(\beta)$. For any connected component of any stratum $\mathcal{H}(\beta)$ the paper [EMZ] describes all principal boundary strata; their union is called the *principal boundary* of the corresponding connected component of $\mathcal{H}(\beta)$.

The paper [EMZ] also presents the inverse construction. Consider any flat surface $S'_1 \sqcup \dots \sqcup S'_k \in \mathcal{H}(\beta')$ in the *principal boundary* of $\mathcal{H}(\beta)$; consider a sufficiently small value of a complex parameter $\delta \in \mathbb{C}$. One can reconstruct the flat surface $S \in \mathcal{H}(\beta)$ endowed with a collection of homologous saddle connections $\gamma_1, \dots, \gamma_n$ such that $\int_{\gamma_i} \omega = \delta$, and such that the degeneration of S that consists of contracting the saddle connections γ_i in the collection gives the surface $S'_1 \sqcup \dots \sqcup S'_k$. This inverse construction involves several *basic surgeries* of the flat structure. Given a disconnected flat surface $S'_1 \sqcup \dots \sqcup S'_k$ one applies an appropriate surgery to each S'_j producing a surface S_j with boundary. The surgery depends on the parameter δ : the boundary of each S_j is composed of two geodesic segments of lengths $|\delta|$; moreover, the boundary components of S_j and S_{j+1} are compatible, which allows one to glue the compound surface S from the collection of surfaces with boundary.

A collection $\gamma = \{\gamma_1, \dots, \gamma_n\}$ of homologous saddle connections determines the following data on combinatorial geometry of the decomposition $S \setminus \gamma$: the number of components, their boundary structure, the singularity data for each component, the cyclic order in which the components are glued to each other. These data are referred to as *configuration* of homologous saddle connections. A configuration \mathcal{C} uniquely determines the corresponding boundary stratum $\mathcal{H}(\beta'_{\mathcal{C}})$.

The constructions above explain how configurations of homologous saddle connections on flat surfaces $S \in \mathcal{H}(\beta)$ determine the “cusps” of the stratum $\mathcal{H}(\beta)$. Consider a subset $\mathcal{H}_1^\varepsilon(\beta) \subset \mathcal{H}(\beta)$ of surfaces of area one having a saddle connection shorter than ε . Up to a subset $\mathcal{H}_1^{\varepsilon, thin}(\beta)$ of negligibly small measure the set $\mathcal{H}_1^\varepsilon(\beta)$ can be represented as a disjoint

union over all admissible configurations \mathcal{C} (i.e. as a union over different “cusps”) of neighborhoods of the corresponding “cusps”. When a configuration \mathcal{C} is composed from homologous saddle connections joining distinct zeroes, the neighborhood of the corresponding cusp has the structure of a fiber bundle over the corresponding boundary stratum $\mathcal{H}(\beta'_{\mathcal{C}})$ with the fiber represented by an appropriate ramified cover over the Euclidean ε -disc. Moreover, the canonical measure in the corresponding connected component of $\mathcal{H}_1^{\varepsilon, thick}(\beta) = \mathcal{H}_1^{\varepsilon}(\beta) \setminus \mathcal{H}_1^{\varepsilon, thin}(\beta)$ decomposes as a product measure of the canonical measure in the boundary stratum and the Euclidean measure in the fiber, see [EMZ].

REMARK. We warn the reader that the correspondence between the compactification of the moduli space of Abelian differentials and the Deligne–Mumford compactification of the underlying moduli space of curves is not straightforward. In particular, the desingularized stable curve corresponding to the limiting flat surface generically *is not* represented as the union of corresponding Riemann surfaces S'_1, \dots, S'_k : the stable curve might contain more components.

1 Structure of the Paper and Statements of Theorems

This paper concerns the study of similar phenomena in the case of quadratic differentials that are not squares of Abelian differentials.

1.1 $\hat{\mathbf{H}}$ omologous saddle connections. A meromorphic quadratic differential q with at most simple poles on a Riemann surface S defines a canonical (ramified) double cover $p: \hat{S} \rightarrow S$ such that $p^*q = \omega^2$ is a square of an Abelian differential ω on \hat{S} . Let $P = \{P_1, \dots, P_m\} \subset S$ be the collection of singularities (zeroes and simple poles) of q ; let $\hat{P} = p^{-1}(P)$ be the set of their preimages under the projection $p: \hat{S} \rightarrow S$.

Given a saddle connection γ on S choose an orientation of γ and let γ', γ'' be its lifts to the double cover endowed with the orientation inherited from γ . If $[\gamma'] = -[\gamma'']$ as cycles in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$ we let $[\hat{\gamma}] := [\gamma']$, otherwise we define $[\hat{\gamma}]$ as $[\hat{\gamma}] := [\gamma'] - [\gamma'']$.

REMARK 1. It immediately follows from the above definition that the cycle $[\hat{\gamma}]$ defined by a saddle connection γ is always *primitive* in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$.

DEFINITION 1. The saddle connections γ_1, γ_2 on a flat surface S defined by a quadratic differential q are $\hat{\mathbf{h}}$ omologous if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$ under an appropriate choice of orientations of γ_1, γ_2 . (The notion “homologous

in the relative homology with local coefficients defined by the canonical double cover induced by a quadratic differential” is unbearably bulky, so we introduced an abbreviation “ $\hat{\text{homologous}}$ ”. We stress that the circumflex over the “h” is quite meaningful: as is indicated in the definition, the corresponding cycles are homologous *on the double cover*.)

REMARK 2. Note that since there is no canonical way to enumerate the preimages γ', γ'' of a saddle connection γ on the double cover, the cycle $\hat{\gamma}$ is defined only up to a sign, even when we fix the orientation of γ . Thus, $\hat{\gamma}_1$ is $\hat{\text{homologous}}$ to $\hat{\gamma}_2$ if and only if $\hat{\gamma}_1 = \pm \hat{\gamma}_2$.

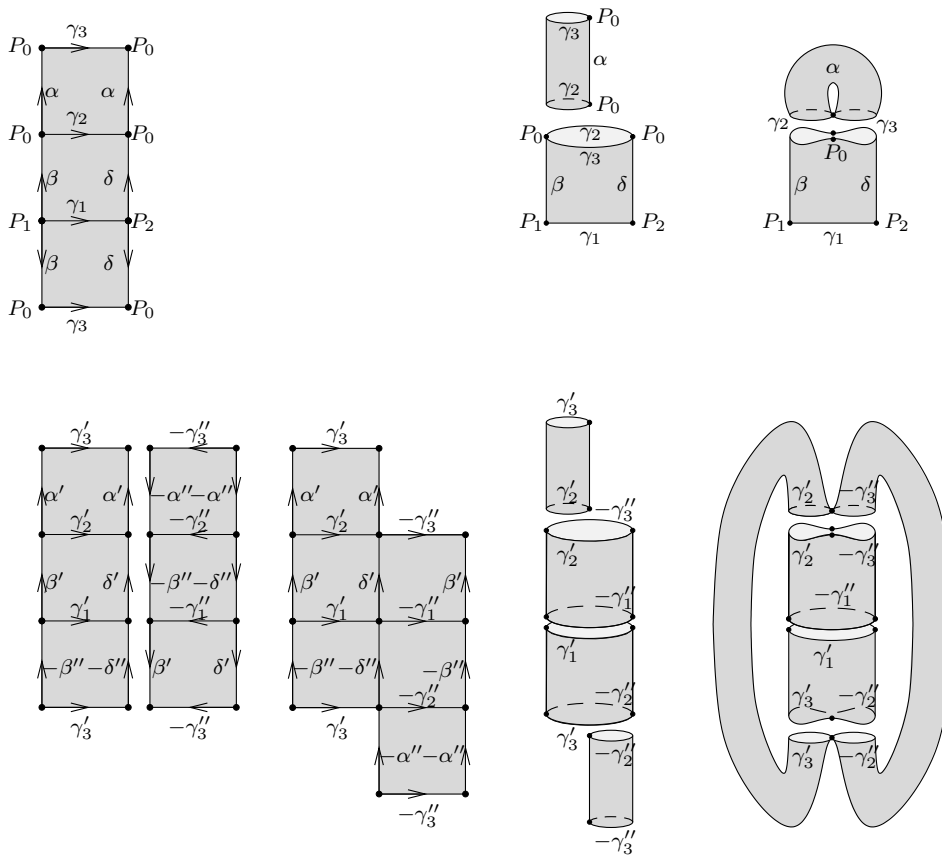


Figure 1: Saddle connections $\gamma_1, \gamma_2, \gamma_3$ on the torus (above picture) are $\hat{\text{homologous}}$, though γ_1 is a segment joining distinct points and γ_2 and γ_3 are closed loops.

We begin with the following example which illustrates many of the main ideas.

EXAMPLE 1. Consider three unit squares, or rather a rectangle 1×3 and glue a torus from it as indicated at the top left corner of Figure 1. Identifying the three corresponding sides β , γ_1 and δ of the two bottom squares we obtain a “pocket” with two “corners” P_1 and P_2 at the bottom and with two “corners” P_0 at the boundary on top. Identifying the points P_0 we obtain a “pocket” with a “figure-eight” boundary (the bottom fragment of the top right picture at Figure 1). Identifying the sides α of the remaining square we obtain a cylinder which we glue to the previous fragment. Topologically the surface thus obtained is a torus. Metrically this torus has three conical singularities. Two of them (“the corners P_1, P_2 of the pocket”) have cone angle π ; the third conical singularity P_0 has cone angle 4π . Such a flat torus gives us a point in the stratum $\mathcal{Q}(2, -1, -1)$.

The bottom picture illustrates the canonical double covering over the above torus. The cycle γ'_2 is homologous to γ'_3 on the double cover and the cycle γ''_2 is homologous to γ''_3 . This implies that the cycles $\hat{\gamma}_1, \hat{\gamma}_2$ and $\hat{\gamma}_3$ on the double cover are homologous to the waist curve of the thick cylinder fragment of the right bottom picture. Thus, the saddle connections γ_1, γ_2 and γ_3 are $\hat{\text{homologous}}$, though γ_1 is a segment joining distinct points P_1 and P_2 , and γ_2, γ_3 are the closed loops with the base point P_0 .

It essentially follows from the definition that $\hat{\text{homologous}}$ saddle connections are parallel on S and that their lengths either coincide or differ by a factor of two. The following simple statement proved in Appendix A characterizes rigid collections of saddle connections on a flat surface with nontrivial linear holonomy.

PROPOSITION 1. *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are $\hat{\text{homologous}}$.*

There is an obvious geometric test for deciding when saddle connections γ_1, γ_2 on a translation surface S are homologous: it is sufficient to check whether $S \setminus (\gamma_1 \cup \gamma_2)$ is connected or not (provided $S \setminus \gamma_1$ and $S \setminus \gamma_2$ are connected). It is slightly less obvious to check whether saddle connections γ_1, γ_2 on a flat surface S with nontrivial linear holonomy are $\hat{\text{homologous}}$ or not. In particular, a pair of closed saddle connections might be homologous in the usual sense, but not $\hat{\text{homologous}}$; a pair of closed saddle connections might be $\hat{\text{homologous}}$ even if one of them represents a loop homologous to

zero, and the other does not; finally, a saddle connection joining a pair of *distinct* singularities might be \hat{h} omologous to a saddle connection joining a singularity to itself, or joining another pair of distinct singularities.

Section 2 describes geometric criteria for deciding when two saddle connections are \hat{h} omologous and what is the structure of the complement $S \setminus (\gamma_1 \cup \gamma_2)$. These criteria are intensively used in the remaining part of the paper. In particular, we prove in section 2 the following statement.

Theorem 1. *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. Two saddle connections γ_1, γ_2 on S are \hat{h} omologous if and only if they have no interior intersections and one of the connected components of the complement $S \setminus (\gamma_1 \cup \gamma_2)$ has trivial linear holonomy. Moreover, if such a component exists, it is unique.*

1.2 Graph of connected components. A collection γ of \hat{h} omologous saddle connections $\gamma = \{\gamma_1, \dots, \gamma_n\}$ divides S into simpler surfaces S_j with boundary. We associate to any such decomposition a graph $\Gamma(S, \gamma)$. The vertices of the graph correspond to the connected components S_j of $S \setminus (\gamma_1 \cup \dots \cup \gamma_n)$. We denote the vertices corresponding to cylinders (if any) by small circles “o”. The remaining vertices are labelled with a “+” sign if the corresponding surface S_j has trivial linear holonomy and with a “-” sign if it does not. We do not label the vertices of “o”-type: it is easy to see that the cylinders always have trivial linear holonomy.

The edges of the graph are in the one-to-one correspondence with the saddle connections γ_i . Each saddle connection γ_i is on the boundary of either one or two surfaces. If γ_i is on the boundary of pair of surfaces, it corresponds to an edge joining the corresponding vertices. If γ_i is on the boundary of only one surface, then it corresponds to an edge of the graph which joins the vertex to itself; such an edge contributes 2 to the valence of the vertex.

REMARK 3. The union $\gamma = \gamma_1 \cup \dots \cup \gamma_n$ of saddle connections can be considered as a graph γ embedded into the surface S . By definition $\Gamma(S, \gamma)$ is a graph *dual* to γ . Namely, $\Gamma(S, \gamma)$ can be realized as graph embedded into the surface S in the following way. A vertex of $\Gamma(S, \gamma)$ corresponding to a connected component S_j of $S \setminus \gamma$ is mapped to a point v_j located in the interior of the corresponding surface with boundary S_j . The line representing the image of an edge of $\Gamma(S, \gamma)$ corresponding to a saddle connection γ_i has a single transversal intersection with γ_i in some interior point; it does not intersect any other saddle connection $\gamma_{i'}$, where $i' \neq i$. Moreover, this line does not intersect either itself or any other such line in an interior point.

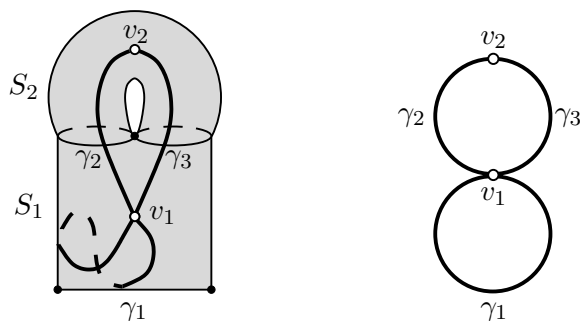


Figure 2: Graph $\Gamma(S, \gamma)$ of connected components.

EXAMPLE 2. Consider the surface S and the collection γ of homologous saddle connections $\{\gamma_1, \gamma_2, \gamma_3\}$ as in Example 1 above (see Figure 1). The complement $S \setminus \gamma$ has two connected components; both represented by flat cylinders. The graph $\Gamma(S, \gamma)$ contains two vertices, both of the “o”-type, and three edges. The graph $\Gamma(S, \gamma) \subset S$ is dual to the graph $\gamma \subset S$, see Figure 2.

It follows from the definition of homologous saddle connections that their lengths are either the same or differ by a factor of two. Having a collection γ of homologous saddle connections $\gamma_1, \dots, \gamma_n$ we can normalize the length of the shortest one to 1. Then the other saddle connections have lengths either 1 or 2, which endows the edges of the graph Γ with the weights 1 or 2.

The theorem below classifies all possible graphs corresponding to non-empty collections of homologous saddle connections.

Theorem 2. *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles; let γ be a collection of homologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$, and let $\Gamma(S, \gamma)$ be the graph of connected components encoding the decomposition $S \setminus (\gamma_1 \cup \dots \cup \gamma_n)$.*

The graph $\Gamma(S, \gamma)$ either has one of the basic types listed below or can be obtained from one of these graphs by placing additional “o”-vertices of valence two at any subcollection of edges subject to the following restrictions. At most one “o”-vertex may be placed at the same edge; a “o”-vertex cannot be placed at an edge adjacent to a “o”-vertex of valence 3 if this is the edge separating the graph.

The graphs of basic types, presented in Figure 3, are given by the following list:

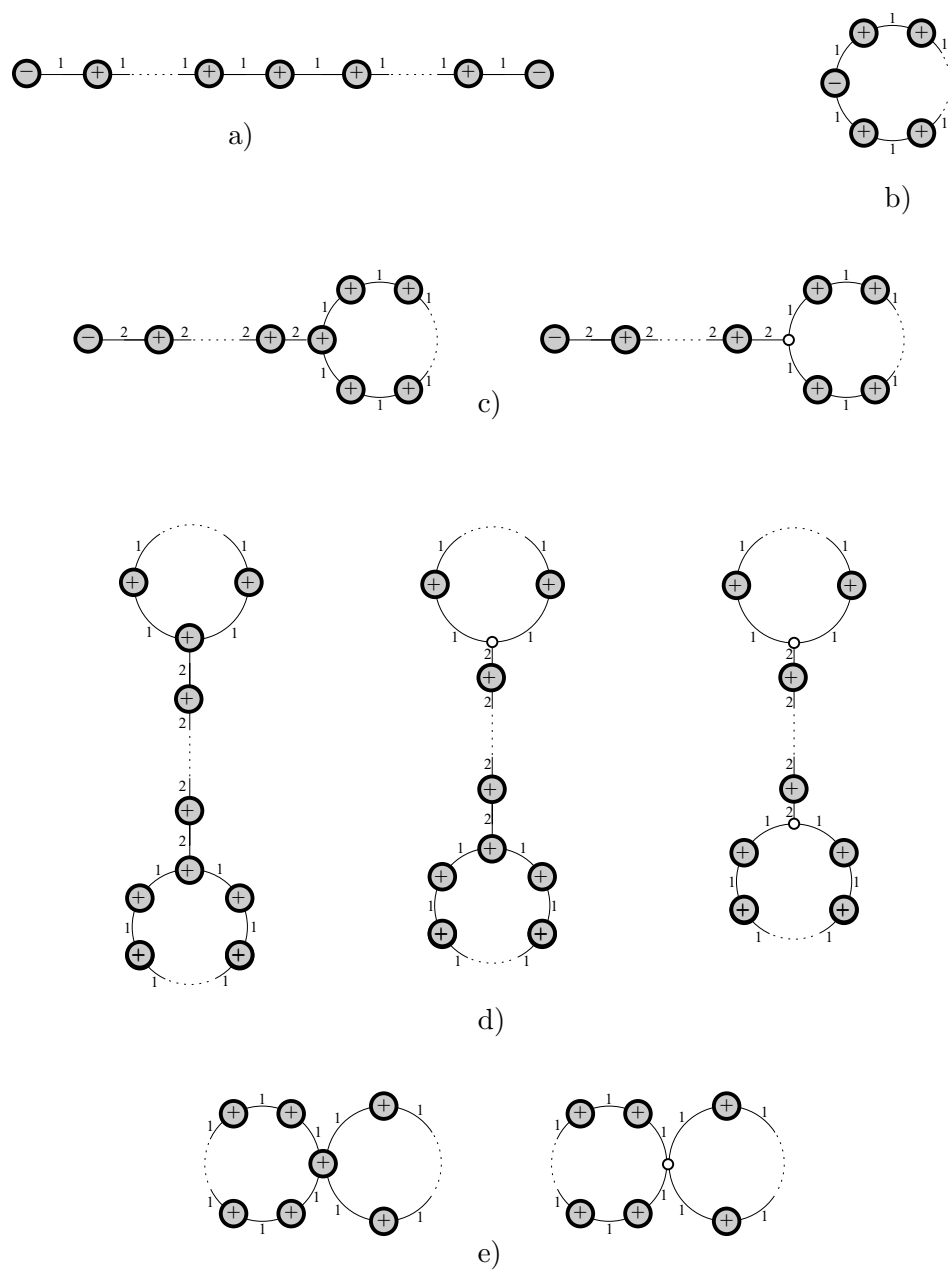


Figure 3: Classification of admissible graphs.

- a) An arbitrary (possibly empty) chain of “+”-vertices of valence two bounded by a pair of “-”-vertices of valence one;
- b) A single loop of vertices of valence two having exactly one “-”-vertex and arbitrary number of “+”-vertices (possibly no “+”-vertices at all);
- c) A single chain and a single loop joined at a vertex of valence three. The graph has exactly one “-”-vertex of valence one; it is located at the end of the chain. The vertex of valence three is either a “+”-vertex, or a “o”-vertex (vertex of the cylinder type). Both the chain, and the cycle may have in addition an arbitrary number of “+”-vertices of valence two (possibly no “+”-vertices at all);
- d) Two nonintersecting cycles joined by a chain. The graph has no “-”-vertices. Each of the two cycles has a single vertex of valence three (the one where the chain is attached to the cycle); this vertex is either a “+”-vertex or a “o”-vertex. If both vertices of valence three are “o”-vertices, the chain joining two cycles is nonempty: it has at least one “+”-vertex. Otherwise, each of the cycles and the chain may have arbitrary number of “+”-vertices of valence two (possibly no “+”-vertices of valence two at all);
- e) “Figure-eight” graph: two cycles joined at a vertex of valence four, which is either a “+”-vertex or a “o”-vertex. All the other vertices (if any) are the “+”-vertices of valence two. Each of the two cycles may have arbitrary number of such “+”-vertices of valence two (possibly no “+”-vertices of valence two at all).

Every graph listed above corresponds to some flat surface S and to some collection of saddle connections γ .

Theorem 2 is proved in section 3 with exception of the final statement on realizability, which is proved in sections 5–6.

1.3 Parities of boundary singularities. In section 4 we give a detailed analysis of each connected component S_j of the decomposition $S \setminus \gamma$.

It is convenient to consider a closed surface with boundary S_j^{comp} canonically associated to S_j by taking the natural *compactification* of S_j . Note, that S_j^{comp} need not be the same as the *closure* of S_j in S . For example, if we cut a surface S along a single saddle connection γ joining a pair of distinct singularities we obtain a surface S_1 whose compactification is a surface with boundary composed of a pair of parallel distinct geodesics of the same length, while the closure of $S_1 = S \setminus \gamma_1$ in S coincides with S . The closure of S_j in S is obtained from the compactification S_j^{comp} of S_j by

identification of some boundary points (if necessary), or by identification of some boundary saddle connections (if necessary).

Ribbon graph. Given a vertex v of a finite graph Γ consider a tree Γ_v obtained as a small neighborhood of v in Γ in the natural topology of a one-dimensional cell complex. The tree Γ_v together with the canonical mapping of the graphs $\Gamma_v \rightarrow \Gamma$ will be referred to as the *boundary* of v . The number of edges of Γ_v is exactly the valence of v (and hence is at most 4 for the graphs from Figure 3).

Suppose that the boundary of S_j^{comp} has $r = r(j)$ connected components (called *boundary components*). Every boundary component is composed of a closed chain of saddle connections $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$, where $1 \leq i \leq r$. The case $p(i) = 1$ is not excluded: a boundary component might be composed of a single saddle connection. The canonical orientation of S_j^{comp} determines the orientation of every boundary component \mathcal{B}_i of ∂S_j^{comp} and hence determines the cyclic order

$$\rightarrow \gamma_{j_{i,1}} \rightarrow \dots \rightarrow \gamma_{j_{i,p(i)}} \rightarrow \tag{1}$$

on every such chain; by convention we let $j_{i,p(i)+1} := j_{i,1}$. Thus, we get a natural decomposition of the set of edges of Γ_{v_j} into a disjoint union of subsets, each endowed with a cyclic order,

$$\{\rightarrow \gamma_{j_{1,1}} \rightarrow \gamma_{j_{1,2}} \rightarrow \dots \rightarrow \gamma_{j_{1,p(1)}} \rightarrow\} \sqcup \dots \sqcup \{\rightarrow \gamma_{j_{r,1}} \rightarrow \dots \rightarrow \gamma_{j_{r,p(r)}} \rightarrow\}. \tag{2}$$

It is convenient to encode such a combinatorial structure by a *local ribbon graph* \mathbb{G}_{v_j} which is defined in the following way. We denote a saddle connection and the edge of the graph $\Gamma(S, \gamma)$ dual to it by the same symbol γ_i .

Consider a realization of $\Gamma(S, \gamma)$ by an embedded graph dual to the graph γ in S (see Remark 3 above). For every vertex v_j of $\Gamma(S, \gamma)$ we get an induced embedding $\Gamma_{v_j} \hookrightarrow S_j^{comp}$. Let a connected component \mathcal{B}_i of ∂S_j^{comp} be represented by a chain (1) of saddle connections. A tubular neighborhood in S_j^{comp} of the union of the corresponding edges $\{\gamma_{j_{i,1}} \cup \dots \cup \gamma_{j_{i,p(i)}}\}$ of $\Gamma_{v_j} \subset S_j^{comp}$ (as in the left picture of Figure 4) inherits the canonical orientation of S . This orientation induces a natural cyclic order on the edges $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$ of Γ_{v_j} . We choose the embedding $\Gamma_{v_j} \hookrightarrow S_j^{comp}$ in such way that turning counterclockwise around v_j (considered as a point of S_j^{comp}) we see the edges $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$ appear in the cyclic order (1).

When the boundary ∂S_j^{comp} contains several connected components, the ribbon graphs corresponding to different components overlap at v_j (as in the left picture of Figure 4). However, it is easy to make them disjoint by a small deformation, subject to an appropriate choice of the initial embedding

$\Gamma_{v_j} \hookrightarrow S_j^{comp}$. From now on we shall always assume that the embedding is chosen appropriately.

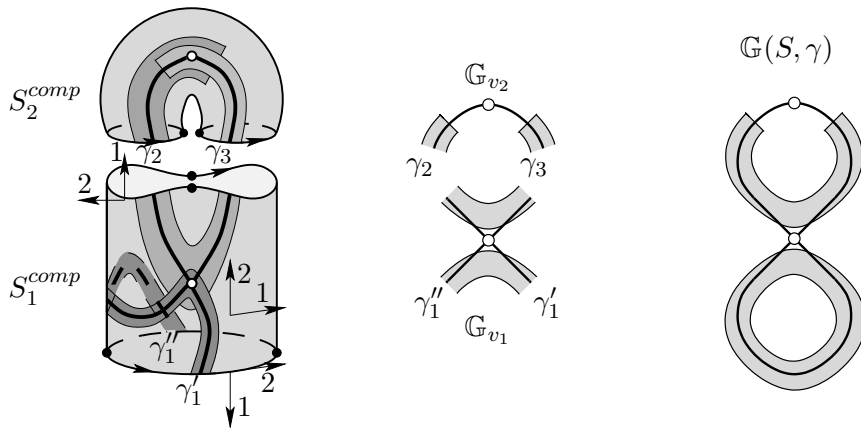


Figure 4: Compactifications S_1^{comp}, S_2^{comp} of connected components of $S \setminus \gamma$, the associated local ribbon graphs G_{v_1}, G_{v_2} and the global ribbon graph $G(S, \gamma)$.

EXAMPLE 3. Consider once again the surface S and the collection γ of $\hat{\text{homologous}}$ saddle connections $\{\gamma_1, \gamma_2, \gamma_3\}$ as in Example 1, see Figure 1. In Example 2 we have constructed the associated graph $\Gamma(S, \gamma)$, see Figure 2.

The complement $S \setminus \gamma$ has two connected components; their compactifications S_1^{comp}, S_2^{comp} are represented by a pair of flat cylinders. Each of the two connected components of the boundary of S_2^{comp} (the top cylinder in Figure 4) is formed by a single saddle connection, so we get $\partial S_2^{comp} = \{\gamma_2\} \sqcup \{\gamma_3\}$. Each of the two connected components of the boundary of S_1^{comp} (the bottom cylinder in Figure 4) is formed by a pair of saddle connections, so we get $\partial S_1^{comp} = \{\gamma_2 \rightarrow \gamma_3\} \sqcup \{\gamma_1' \rightarrow \gamma_1\}$. The orientation of the boundary components induced by the canonical orientation of S is indicated in the left picture.

The picture in the center of Figure 4 shows the corresponding local ribbon graphs and the picture on the right shows the global ribbon graph $G(S, \gamma)$ for this example.

For vertices v of valence 1, 2, 3, 4, Figure 5 gives a complete list of all possible partitions of the edges of Γ_v into a disjoint union of subsets endowed with a cyclic order and of the corresponding local ribbon graphs G_v .

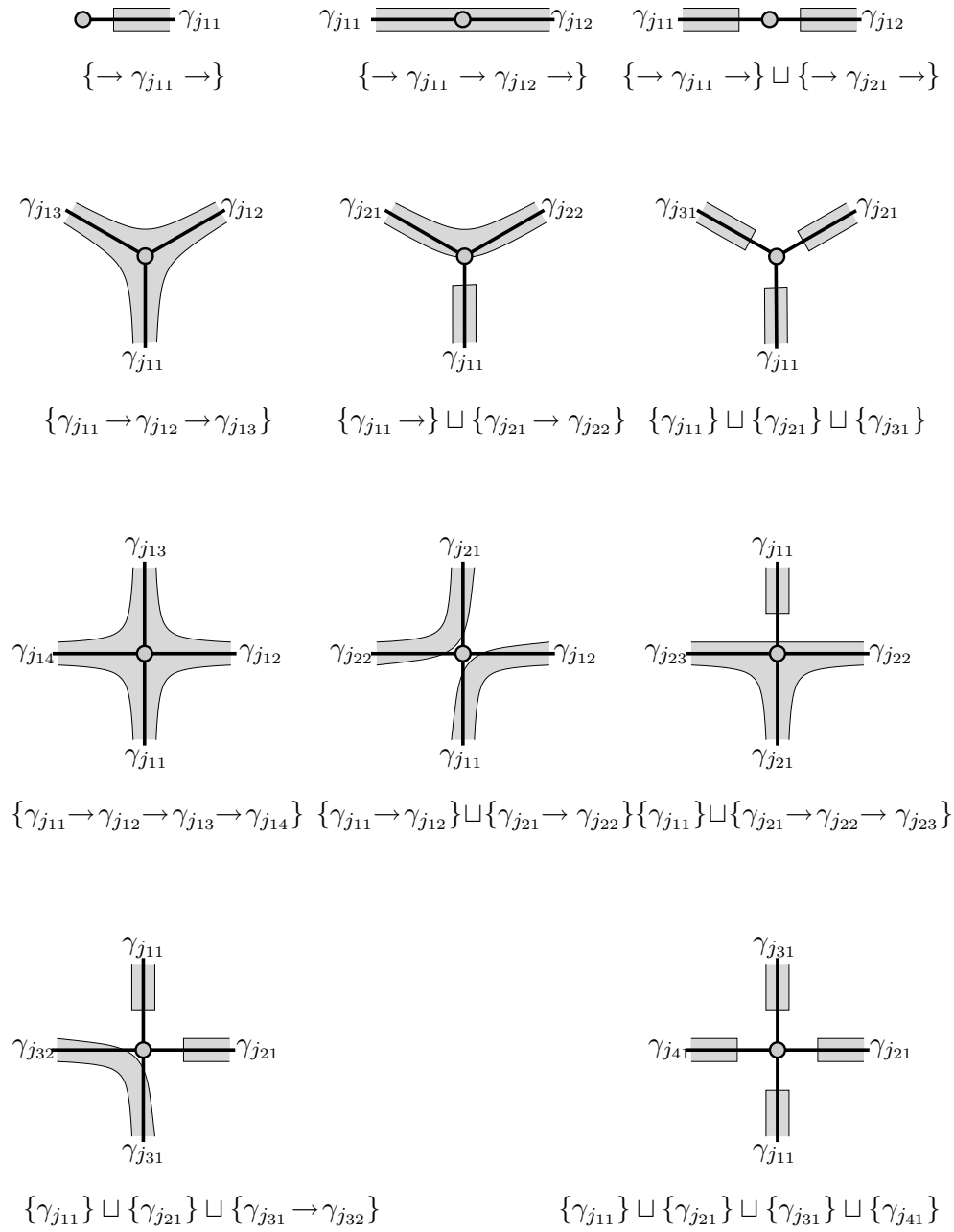


Figure 5: All local ribbon graphs G_v of valences from one to four.

Boundary singularities. Let S_j be a connected component of the decomposition $S \setminus (\gamma_1 \cup \dots \cup \gamma_n)$; let S_j^{comp} be its compactification, and let a connected component \mathcal{B}_i of the boundary ∂S_j^{comp} be represented by a chain (1) of saddle connections. The common endpoint of γ_{j_i} and $\gamma_{j_{i+1}}$ is called the *boundary singularity* of S_j^{comp} . Since all saddle connections $\gamma_1, \dots, \gamma_n$ are parallel, the corresponding angle between γ_{j_i} and $\gamma_{j_{i+1}}$ is an integer multiple of π . There might also be several conical singularities in the interior of S_j^{comp} ; they are called *interior singularities*.

DEFINITION 2. If the total angle at a boundary singularity is $(k+1)\pi$ the *order of the boundary singularity* is defined to be k , and the *parity of the boundary singularity* is defined to be the parity of k . If the total angle at an interior singularity is $(d+2)\pi$ the *order of the interior singularity* is defined to be d .

The order of the interior singularity coincides with the order of the zero (simple pole) of the corresponding germ of a quadratic differential. By convention, boundary singularities, and their orders will always refer to the compactification S_j^{comp} .

When S_j is represented by a “+”-vertex of the graph $\Gamma(S, \gamma)$, we include the parities of the boundary singularities in our combinatorial structure represented by the embedded local ribbon graph \mathbb{G}_{v_j} . Let \mathcal{B}_i be a connected component of the boundary ∂S_j^{comp} constituted by a chain (1) of saddle connections. The edges $\gamma_{j_{i,1}}, \dots, \gamma_{j_{i,p(i)}}$ of the embedded graph $\Gamma_{v_j} \hookrightarrow S_j^{comp}$ subdivide a neighborhood of v_j in S_j into $p(i)$ sectors. To each sector bounded by a pair of consecutive edges $\gamma_{j_{i,l}}$ and $\gamma_{j_{i,l+1}}$ we associate the parity of the order $k_{j_{i,l}}$ of the corresponding boundary singularity of S_j^{comp} : of the common endpoint of the consecutive saddle connections $\gamma_{j_{i,l}} \rightarrow \gamma_{j_{i,l+1}}$ in \mathcal{B}_i .

Any connected component S_j of the decomposition $S \setminus \{\gamma_1, \dots, \gamma_n\}$ determines the following combinatorial data which we refer to as the *boundary type* of S_j : the structure (2) of the local ribbon graph \mathbb{G}_{v_j} as in Figure 5; an embedding $\Gamma_{v_j} \hookrightarrow \Gamma(S, \gamma)$ and a collection of parities of boundary singularities of S_j .

Theorem 3. *Consider a decomposition of a flat surface S as in Theorem 2. Every connected component S_j of the decomposition has one of the boundary types presented in Figure 6 and all indicated boundary types are realizable.*

The dotted lines in Figure 6 indicate pairs of edges of a vertex $v \in \Gamma(S, \gamma)$ of valence 3 or 4, which are joined by a loop in the graph $\Gamma(S, \gamma)$ (see Figure 3) and encode in this way the embedding $\Gamma_{v_j} \hookrightarrow \Gamma(S, \gamma)$.

REMARK 4. We use the following convention on indexation of local ribbon graphs in Figure 6: the first symbol represents the type (“+”, “−”, or “o”) of the vertex v_j in the graph $\Gamma(S, \gamma)$; the second symbol is the valence of v_j ; the number after a dot is the number of boundary components of S_j . An extra letter “ a, b, c ” is employed when it is necessary to distinguish different embedded local ribbon graphs sharing the same vertex type, valence and number of boundary components.

The first part of the statement of Theorem 3 which claims that every connected component of the decomposition has one of the boundary types in Figure 6 is quite elementary; it is proved at the end of section 4. The statement about the realizability of all boundary types presented in Figure 6 is much less trivial; it follows from Theorem 4 which is proved in sections 5 and 6.

1.4 Configurations of homologous saddle connections. We formalize the data on combinatorial geometry of the decomposition $S \setminus \gamma$ in Definition 3 below.

DEFINITION 3. The following combinatorial structure is called a *configuration of homologous saddle connections*.

1. A finite graph Γ endowed with a labelling of each vertex by one of the symbols “+”, “−”, or “o”, of one of the types described in Theorem 2 (see Figure 3).
2. For any vertex v of the graph Γ an embedded ribbon graph \mathbb{G}_v (encoding the decomposition of Γ_v into a disjoint union of subsets, called *boundary components*, each endowed with a cyclic order; see equation (2)) of one of the types described in Theorem 3 (see Figure 6).
3. For every “+”-vertex v of Γ and for every pair of consecutive elements $\gamma_{i,l} \rightarrow \gamma_{i,l+1}$ of \mathbb{G}_v (called *boundary singularities*) an associated parity (even or odd) as in Figure 6.
4. For every vertex v of Γ and for every boundary singularity of \mathbb{G}_v a nonnegative integer $k_{i,l}$ (referred to as the *order of the boundary singularity*) satisfying the following conditions. The order of the boundary singularity respects the *parity* associated to the corresponding boundary singularity when v is of the “+”-type; the order of any boundary singularity of any vertex of the “o”-type is equal to zero. The sum $D_i + 2 = k_{i,1} + \dots + k_{i,p(i)}$ of orders of boundary singularities along any boundary component \mathcal{B}_i of v satisfies $D_i \geq 0$ for a vertex of “+”-type and $D_i \geq -1$ for a vertex of “−”-type.

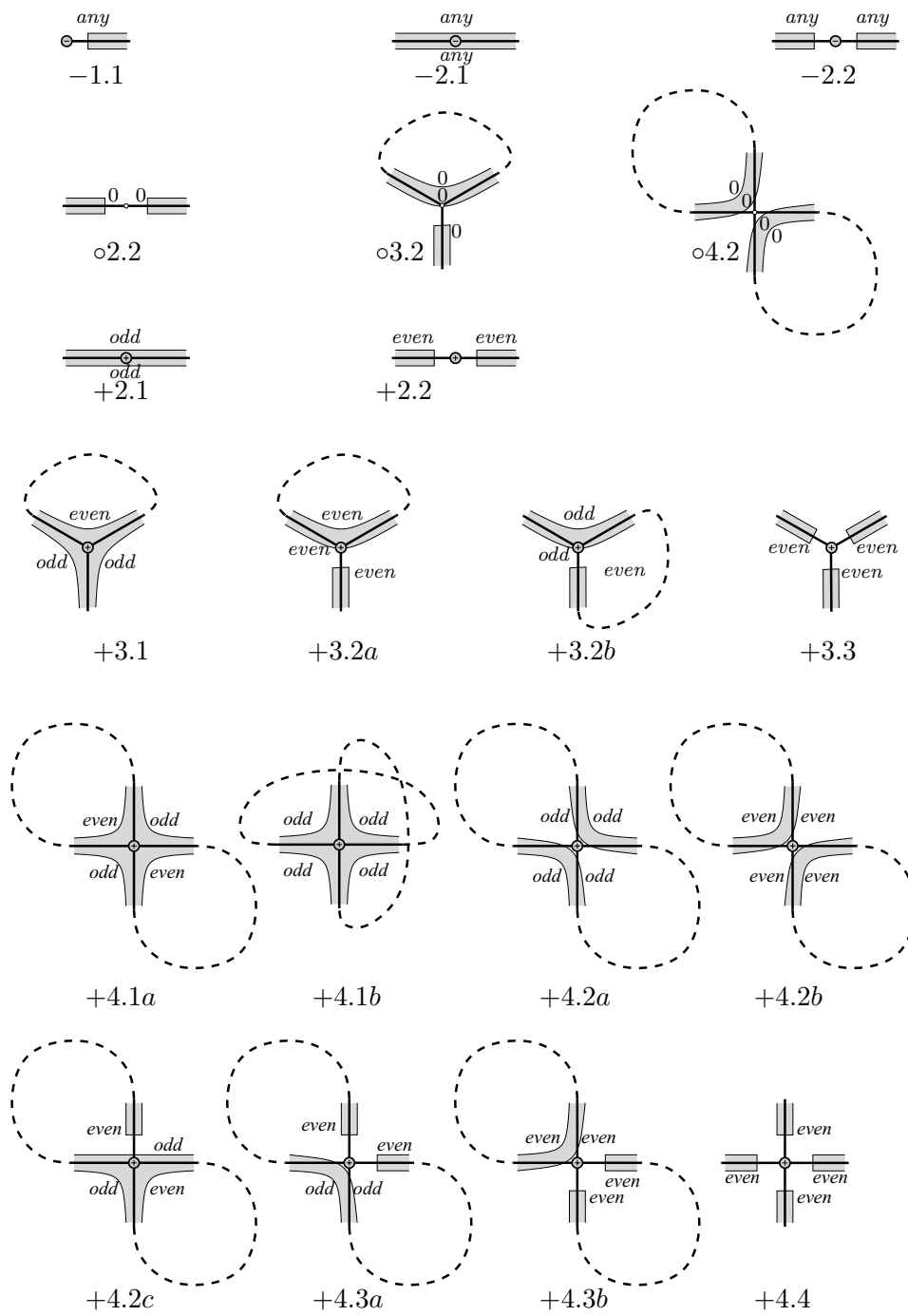


Figure 6: Classification of embedded local ribbon graphs.

5. For every vertex v of Γ of “-”-type an unordered (possibly empty) collection of integers $\{d_1, \dots, d_{s(v)}\}$, where $d_l \in \{-1, 1, 2, 3, \dots\}$; for every vertex v of Γ of “+”-type an unordered (possibly empty) collection of positive even integers $\{d_1, \dots, d_{s(v)}\}$, where $d_l \in \{2, 4, \dots\}$. In both cases these collections of integers (called *orders of interior singularities*) satisfy the following compatibility conditions with the collection of boundary singularities of \mathbb{G}_v :

$$-4 \leq \left(\sum d_l + \sum D_i \right) \equiv 0 \pmod{4},$$

where the first sum is taken over all interior singularities and the second sum is taken over all boundary components \mathcal{B}_i of \mathbb{G}_v .

6. When the vertex v is of the “-”-type the couple [unordered collection of interior singularities, unordered collection of boundary singularities] is in addition not allowed to belong to the following exceptional list:

	$\{\emptyset, \{2\}\}; \{-1, \{3\}\}; \{1, \{1\}\}; \{-1, 1, \{2\}\}$ $\{1, \{5\}\}; \{3, \{3\}\}; \{1, 3, \{2\}\}; \{\emptyset, \{6\}\}; \{4, \{2\}\}$
	$\{\emptyset, \{2, 0\}\}; \{\emptyset, \{1, 1\}\}; \{-1, \{0, 3\}\}; \{-1, \{1, 2\}\}$ $\{1, \{0, 1\}\}; \{1, -1, \{0, 2\}\}; \{1, -1, \{1, 1\}\}$ $\{3, 1, \{2, 0\}\}; \{3, 1, \{1, 1\}\}; \{3, \{3, 0\}\}; \{3, \{2, 1\}\}$ $\{1, \{5, 0\}\}; \{1, \{4, 1\}\}; \{1, \{3, 2\}\}; \{4, \{2, 0\}\}; \{4, \{1, 1\}\}$ $\{\emptyset, \{6, 0\}\}; \{\emptyset, \{5, 1\}\}; \{\emptyset, \{4, 2\}\}; \{\emptyset, \{3, 3\}\}$
	$\{\emptyset, \{2, 2\}\}; \{\emptyset, \{1, 3\}\}; \{-1, \{2, 3\}\}; \{1, \{1, 2\}\}; \{-1, 1, \{2, 2\}\}$ $\{\emptyset, \{3, 5\}\}; \{1, \{2, 5\}\}; \{3, \{2, 3\}\}; \{1, 3, \{2, 2\}\}$ $\{\emptyset, \{2, 6\}\}; \{4, \{2, 2\}\}$

The above definition might be viewed as an instruction for a “lego game”. Having an infinite stock of elementary “lego bricks” of twenty different kinds (we mean the embedded local ribbon graphs of twenty types presented at Figure 6) one constructs the entire building following the plan given by the global graph from Figure 3. After that one “decorates the building” with an arbitrary collection of integers matching the parities of the “lego bricks” and satisfying some elementary conditions.

Parts (1)–(2) of the definition describe the combinatorial geometry of the building; conditions (4)–(6) impose elementary restrictions on the collection of orders of singularities. Note that the *parities* of the boundary singularities are encoded in the “lego bricks”. Thus, condition (3) of the definition makes a bridge between the geometry (1)–(2) of the ribbon graph and the arithmetic (4)–(6) of the collection of integers representing the orders of singularities. See the Main Theorem for a formal statement and Appendix B for an explicit illustration of this approach.

Singularity data corresponding to a configuration. Any two flat surfaces realizing the same configuration \mathcal{C} of homologous saddle connections belong to the same stratum $\mathcal{Q}(\alpha)$ of quadratic differentials. The singularity data α are defined by the configuration \mathcal{C} as follows.

First note that any configuration \mathcal{C} determines a natural *global ribbon graph* \mathbb{G} in the following way. We have defined a structure of a local ribbon graph for a small neighborhood Γ_v of every vertex $v \in \Gamma$. For every vertex v of Γ we have a ribbon going along a germ of every edge of $\Gamma_v \subset \Gamma$ from v to the center of the edge. Note that all local ribbon graphs carry the canonical orientation induced from the canonical orientation of the embodying plane. For every edge of Γ we can extend the ribbons from the endpoints towards the center of the edge and glue them together respecting the canonical orientation. Applying this procedure to all edges of Γ we get a global ribbon graph endowed with the canonical orientation.

Consider the global ribbon graph \mathbb{G} as a surface with boundary. The boundary components of this surface are in a one-to-one correspondence with the subset of those conical points of S which serve as the endpoints of the saddle connections γ_i in the collection $\gamma_1, \dots, \gamma_n$. The orders of the corresponding singularities are calculated as follows. For any connected component $(\partial\mathbb{G})_m$ of its boundary define an integer b_m as

$$b_m = \sum_{\substack{\text{boundary singularities} \\ \text{which belong to } (\partial\mathbb{G})_m}} (k_{i,l} + 1) - 2. \tag{3}$$

The set with multiplicities α can be defined now as

$$\alpha = \left(\bigcup_{\substack{\pm\text{-vertices} \\ v_j \in \Gamma(\mathcal{C})}} \text{interior singularities of } v_j \right) \cup \left(\bigcup_{\substack{\text{components } (\partial\mathbb{G})_m \\ \text{of the boundary} \\ \text{of } \mathbb{G}(\mathcal{C})}} b_m \right). \tag{4}$$

EXAMPLE 4. The configurations \mathcal{C} presented in the left picture of Figure 7 has 8 saddle connections $\gamma = \{\gamma_1 \cup \dots \cup \gamma_8\}$; the surface $S \setminus \gamma$ decomposes into 7 connected components $S_1 \sqcup \dots \sqcup S_7$. Two components are represented by cylinders and thus have no interior singularities. Among the remaining five components three have no interior singularities and are denoted \emptyset , one has one interior singularity of order 2, and one has two interior singularities of order 4. Thus, we get

$$\bigcup_{\substack{\pm\text{-vertices} \\ v_j \in \Gamma(\mathcal{C})}} \text{interior singularities of } v_j = \{2, 4, 4\}.$$

The boundary of the global ribbon graph \mathbb{G} has two components $(\partial\mathbb{G})_1$ and $(\partial\mathbb{G})_2$ which correspond to two conical singularities P_1 and P_2 of S .

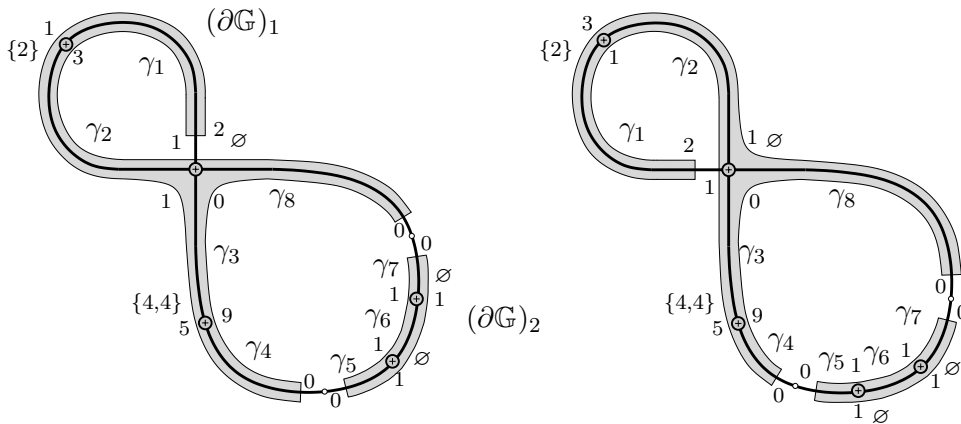


Figure 7: An example of a configuration.

The saddle connections $\gamma_5, \gamma_6, \gamma_7$ join P_2 to itself; the other saddle connections join P_1 to itself. Turning counterclockwise around the point P_l , $l = 1, 2$, we see geodesic rays parallel to γ_i appear in the same order as they appear when we follow the corresponding component $(\partial\mathbb{G})_l$ in the positive direction. Denoting by “ x ” the geodesic rays which do not belong to the configuration we get the following (cyclically ordered) list for the zero P_2 :

$$\gamma_5 x \gamma_6 x \gamma_7 \gamma_7 x \gamma_6 x \gamma_5 .$$

We have 10 geodesic rays; this corresponds to the cone angle 10π matching our formula for the order b_2 of the zero P_2 :

$$(0 + 1) + (1 + 1) + (1 + 1) + (0 + 1) + (1 + 1) + (1 + 1) - 2 = 8 ,$$

The analogous list for P_1 is as follows

$$x x \gamma_1 x \gamma_2 x \gamma_3 x x x x x \gamma_4 \gamma_4 x x x x x x x x x \gamma_3 \gamma_8 \gamma_8 x \gamma_2 x x x \gamma_1 .$$

The number of consecutive “ x ”s coincides with the order of the corresponding boundary singularity (see Definition 2). Thus, at P_1 we find 32 geodesic rays parallel to γ_i , which corresponds to the cone angle 32π , and the order b_1 of P_1 equals to

$$(2+1) + (1+1) + (1+1) + (5+1) + (0+1) + (9+1) + (0+1) + (0+1) + (1+1) + (3+1) - 2 = 30 .$$

Finally, we get the following set with multiplicities:

$$\alpha = (2, 4, 4, 8, 30) .$$

The surface S has genus $g = 13$; the configuration \mathcal{C} represents the stratum $\mathcal{Q}(2, 4, 4, 8, 30)$. Note, that the picture on the right represents the same configuration as the picture of the left.

REMARK 5. The example above gives an idea of how can one construct all configurations (in the sense of Definition 3) for a given stratum $\mathcal{Q}(\alpha)$ of meromorphic quadratic differentials with at most simple poles. This algorithm is discussed in more detail in Appendix B, where as an illustration we present a complete list of all configurations of homologous saddle connections for holomorphic quadratic differentials in genus two.

Principal boundary. Analogously to the case of Abelian differentials a configuration \mathcal{C} of homologous saddle connections determines the corresponding principal boundary stratum $\mathcal{Q}(\alpha'_\mathcal{C})$ or $\mathcal{H}(\beta'_\mathcal{C})$. Namely, to each boundary component \mathcal{B}_i

$$\{\rightarrow \gamma_{j_{i,1}} \rightarrow \cdots \rightarrow \gamma_{j_{i,p(i)}} \rightarrow\}$$

of every “+” or “-”-vertex v_j of the graph $\Gamma(\mathcal{C})$ (i.e. to each connected component of the corresponding local ribbon graph \mathbb{G}_j) we assign a number

$$D_{j_i} = k_{j_{i,1}} + \cdots + k_{j_{i,p(i)}} - 2, \tag{5}$$

where $k_{j_{i,1}}, \dots, k_{j_{i,p(i)}}$ are the orders of the boundary singularities corresponding to this boundary component. By Lemma 2.1 proved in the beginning of section 2 the number D_{j_i} is always a nonnegative even integer whenever v_j is a “+”-vertex. To every “+”-vertex v_j of the graph $\Gamma(\mathcal{C})$ we assign the stratum

$$\mathcal{H}(\beta'_j) = \mathcal{H}\left(\frac{d_1}{2}, \dots, \frac{d_{s(j)}}{2}, \frac{D_1}{2}, \dots, \frac{D_{r(j)}}{2}\right) \tag{6}$$

of holomorphic Abelian differentials, where $d_1, \dots, d_{s(j)}$ are the orders of interior singularities of v_j . Note that conditions (4) and (5) in Definition 3 of a configuration of homologous saddle connections imply that the entries of β'_j are integers and that their sum is even, so the stratum $\mathcal{H}(\beta'_j)$ is nonempty.

We assign to a “-”-vertex v_j the stratum

$$\mathcal{Q}(\alpha'_j) = \mathcal{Q}(d_1, \dots, d_{s(j)}, D_1, \dots, D_{r(j)}) \tag{7}$$

of meromorphic quadratic differentials with at most simple poles, where $d_1, \dots, d_{s(j)}$ are the orders of interior singularities of v_j . Note that condition (5) in Definition 3 of a configuration of homologous saddle connections guarantees that the sum of entries of α'_j defined above equals 0 modulo 4, while condition (6) guarantees that $\alpha'_j \notin \{(\emptyset, \{-1, 1\}), \{3, 1\}, \{4\}\}$, which implies that the stratum $\mathcal{Q}(\alpha'_j)$ is nonempty.

Given a configuration \mathcal{C} we assign to every “ \pm ”-vertex of the graph Γ the corresponding stratum. When Γ does not contain “-” vertices we get a stratum $\mathcal{H}(\beta'_\mathcal{C})$ of disconnected translation surfaces $S'_1 \sqcup \cdots \sqcup S'_k$,

where $S'_j \in \mathcal{H}(\beta'_j)$, $j = 1, \dots, k$. Otherwise we get a stratum $\mathcal{Q}(\alpha'_\mathcal{C})$ of disconnected flat surfaces $S'_1 \sqcup \dots \sqcup S'_k$, where $S'_j \in \mathcal{H}(\beta'_j)$ when S'_j is represented by a “+”-vertex and $S'_j \in \mathcal{Q}(\alpha'_j)$ when S'_j is represented by a “-”-vertex. The resulting stratum is called the *principal boundary stratum* corresponding to the admissible configuration \mathcal{C} .

EXAMPLE 5. Let us compute the principal boundary stratum corresponding to the configuration from Example 4, see Figure 7. The components represented by cylinders, encoded by \circ -vertices do not contribute to the principal boundary: they shrink and disappear. The vertex v_1 of valence four has type $+4.2c$, see Figure 6; the corresponding local ribbon graph \mathbb{G}_{v_1} has two connected components, $r(1) = 2$, which correspond to two connected components $\mathcal{B}_1, \mathcal{B}_2$ of the boundary ∂S_1^{comp} . The corresponding zeroes of the induced Abelian differential on S'_1 are calculated in terms of $D_1 = 2 - 2 = 0$ and $D_2 = 1 + 0 + 1 - 2 = 0$, see (5). Since S_1^{comp} does not have interior singularities, the corresponding closed flat surface S'_1 is a torus with two marked points, $S'_1 \in \mathcal{H}(0, 0)$, see (6).

The remaining four vertices of $\Gamma(S, \gamma)$ have type $+2.1$; the boundary of each of the corresponding components S_2, \dots, S_4 is connected. Applying formulae (5) and (6) we get the following list of surfaces S'_j :

$$S'_2 \in \mathcal{H}\left(\frac{2}{2}, \frac{3+1-2}{2}\right), \quad S'_3 \in \mathcal{H}\left(\frac{4}{2}, \frac{4}{2}, \frac{5+9-2}{2}\right), \quad S'_4, S'_5 \in \mathcal{H}\left(\frac{1+1-2}{2}\right).$$

The corresponding principal boundary stratum is

$$\mathcal{H}(0, 0) \sqcup \mathcal{H}(1, 1) \sqcup \mathcal{H}(6, 2, 2) \sqcup \mathcal{H}(0) \sqcup \mathcal{H}(0).$$

Main theorems. In sections 5 and 6 we describe some basic surgeries which depend continuously on a small complex parameter $\delta \in \mathbb{C}$ (responsible for the length and direction of the saddle connections which form the boundary) and on an additional discrete parameter having finitely many values. The theorem below makes a bridge between the formal combinatorial constructions discussed above and the geometry of the moduli spaces of quadratic differentials and is proved in those sections.

We denote by $\mathcal{Q}_1^\varepsilon(\alpha) \subset \mathcal{Q}_1(\alpha)$ the subset of those flat surfaces of area one, which have at least one saddle connection of length at most ε .

Theorem 4. *For each configuration \mathcal{C} of \hat{h} -homologous saddle connections as in Definition 3, let Γ be the graph of connected components corresponding to this configuration. Let $\mathcal{Q}(\alpha'_\mathcal{C})$ (or $\mathcal{H}(\beta'_\mathcal{C})$) be the boundary stratum corresponding to the configuration \mathcal{C} .*

For any flat surface $S' \in \mathcal{Q}(\alpha'_\mathcal{C})$ (correspondingly in $\mathcal{H}(\beta'_\mathcal{C})$), and any sufficiently small value of the complex parameter δ , if one applies the basic

surgeries to the connected components of S' and assembles a closed surface S from the resulting surfaces with boundary according to the structure of the graph $\Gamma(\mathcal{C})$, then the result is a surface in $\mathcal{Q}^\varepsilon(\alpha)$.

Similar to the case of Abelian differentials, we denote by $\mathcal{Q}_1^{\varepsilon, \text{thick}}(\alpha) \subset \mathcal{Q}_1(\alpha)$ the subset of those flat surfaces of area one, which have a collection of homologous saddle connections of length at most ε and no other short saddle connection. Here “short” means, of length less than $\lambda\varepsilon^r$ for some parameters $\lambda \geq 1$ and $0 < r \leq 1$, where the values of the parameters depend on the stratum. Then one can show that any surface in $\mathcal{Q}_1^{\varepsilon, \text{thick}}(\alpha)$ can be obtained by this construction. We will not prove this statement in order not to overload this paper.

We put Theorem 2, Theorem 3 and Theorem 4 together in one statement which may be considered as our main theorem.

We say that a collection γ of homologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$ on a flat surface $S \in \mathcal{Q}(\alpha)$ is *in general position* if there are no other saddle connections on S parallel to saddle connections in the collection γ . It follows from Proposition 4 stated in the end of section 2 that for almost all flat surfaces in any stratum any collection of homologous saddle connections is in general position. This implies, that we can always put a collection of homologous saddle connections in general position by an arbitrary small deformation of the flat surface inside the stratum.

Main Theorem. *Any collection γ of homologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$ in general position on a flat surface $S \in \mathcal{Q}(\alpha)$ naturally defines a corresponding configuration $\mathcal{C}(S, \gamma)$.*

Any “formal” configuration of homologous saddle connections as in Definition 3 corresponds to some actual collection of homologous saddle connections on an appropriate flat surface.

Proof. By Theorem 2 any collection γ of homologous saddle connections $\{\gamma_1, \dots, \gamma_n\}$ on a flat surface $S \in \mathcal{Q}(\alpha)$ naturally defines a graph of connected components $\Gamma(S, \gamma)$ (structure 1 of a configuration). According to Theorem 3, for every vertex v of $\Gamma(S, \gamma)$ the collection γ also defines a local ribbon graph (structure 2 of a configuration) as well as the orders d_i and $k_{i,l}$ of all interior and boundary singularities. By Theorem 3, for vertices of “+”-type, the orders $k_{i,l}$ of the boundary singularities are compatible with the corresponding parities (structures 3 and 4 of a configuration). The lower bounds for the sums D_i of orders of boundary singularities follow from Lemma 2.3. The necessary condition of the compatibility of the

orders of interior singularities with the orders of boundary singularities formalized as structure 5 is proved in Lemma 2.2. The list of nonrealizable singularity data for the vertices of the “—”-types presented in structure 6 of a configuration is justified in Lemma 6.2 at the end of section 6. This completes the proof of the first part of the statement.

The realizability of all formal configurations immediately follows from Theorem 4. \square

Appendices. Long saddle connections. In Appendix A we study collections of \hat{h} omologous saddle connections when they are not necessarily short.

The next proposition follows immediately from Definition 1 and the notion of configuration.

PROPOSITION. *Let $\gamma(S_0) = \{\gamma_1, \dots, \gamma_n\}$ be a collection of \hat{h} omologous saddle connections on a flat surface S_0 in $\mathcal{Q}(\alpha)$. Let a flat surface S be obtained by a sufficiently small continuous deformation of S_0 in $\mathcal{Q}(\alpha)$ and $\gamma(S)$ the corresponding collection of saddle connections. Then all saddle connections in the collection $\gamma(S)$ are \hat{h} omologous. The configuration $\mathcal{C}(S, \gamma(S))$ defined by the collection $\gamma(S)$ of \hat{h} omologous saddle connections on S coincides with the initial configuration $\mathcal{C}(S_0, \gamma(S_0))$.*

By definition, a configuration \mathcal{C} of \hat{h} omologous saddle connections is admissible for a given connected component $\mathcal{Q}^c(\alpha)$ of the stratum $\mathcal{Q}(\alpha)$ if there is *at least one* flat surface $S_0 \in \mathcal{Q}^c(\alpha)$ and at least one collection γ of \hat{h} omologous saddle connections $\gamma = \{\gamma_1, \dots, \gamma_n\}$ on S_0 realizing \mathcal{C} . Consider any surface S in the same connected component $\mathcal{Q}^c(\alpha)$. By $N_{\mathcal{C}}(S, L)$ denote the number of collections $\gamma = \{\gamma_1, \dots, \gamma_n\}$ of \hat{h} omologous saddle connections on S defining the same configuration $\mathcal{C}(S, \gamma) = \mathcal{C}$ and such that $\max_{1 \leq i \leq n} |\gamma_i| \leq L$. The results in [EM] imply the following statement proved in the appendix.

PROPOSITION 2. *For almost every flat surface S in the connected component $\mathcal{Q}^c(\alpha)$ containing S_0 the following limit exists:*

$$\lim_{L \rightarrow +\infty} \frac{N_{\mathcal{C}}(S, L)}{L^2} = c_{\mathcal{C}}(S),$$

and is strictly positive. Moreover, for almost all surfaces S in $\mathcal{Q}^c(\alpha)$ this limit is the same, $c_{\mathcal{C}}(S) = \text{const}_{\mathcal{C}}$. (This limit is called Siegel–Veech constant.)

In particular, any admissible configuration is presented on almost every flat surface in the corresponding connected component of the stratum by numerous collections of \hat{h} omologous saddle connections.

Final comments, open problems, applications. The thick part $\mathcal{Q}_1^{\varepsilon, \text{thick}}(\alpha)$ decomposes into a disjoint union

$$\mathcal{Q}_1^{\varepsilon, \text{thick}}(\alpha) = \bigsqcup_{\text{configurations } \mathcal{C}} \mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$$

of (not necessarily connected) components corresponding to admissible configurations; the surfaces in any such component of $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ share the same configuration \mathcal{C} of $\hat{\text{h}}$ omologous saddle connections. Following the lines of the paper [EMZ] one could extend Theorem 4 and prove that up to a defect of a very small measure, for every configuration \mathcal{C} there is an integer $M(\mathcal{C})$ such that $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ is a (ramified) covering of order $M(\mathcal{C})$ over the following space. The space is a fiber bundle over the boundary stratum $\mathcal{Q}_1(\alpha'_{\mathcal{C}})$ (correspondingly $\mathcal{H}_1(\beta'_{\mathcal{C}})$). It has a Euclidean ε -disc as a fiber when \mathcal{C} does not contain cylinders, and the space $\mathcal{H}_1^{\varepsilon}(0, \dots, 0)$ when \mathcal{C} contains cylinders (number of marked points on the torus equals the number of cylinders). In both cases it is easy to express the measure on $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ in terms of the product measure on the fiber bundle, and compute the volume of $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ in terms of volumes of the strata, and using the Siegel–Veech formula compute the constants $c_{\mathcal{C}}$.

However, the evaluation of the constants M (which depend on the configuration \mathcal{C}) requires some additional work. In particular, if the corresponding surgeries (see Theorem 4) are nonlocal (i.e. those, which use a path on a surface, see section 6) one needs to study the dependence of the resulting surface on the homotopy type of the path. These and related issues are partly discussed in the papers [B1] and [B2].

Another subject which we do not discuss in this paper is the individual study of the connected components of the strata of quadratic differentials: different connected components of the same stratum $\mathcal{Q}(\alpha)$ have their individual lists of admissible configurations, graphs, boundary strata, etc. In particular, one can use the lists of admissible configurations to determine the connected component to which a given flat surface belongs. For example, a saddle connection joining the zero and the simple pole on any flat surface from the component $\mathcal{Q}^{ir}(9, -1)$ has a $\hat{\text{h}}$ omologous saddle connection joining the zero to itself, while analogous saddle connections on surfaces from the complementary connected component $\mathcal{Q}^{reg}(9, -1)$ may have multiplicity one. The existing invariant called the *Rauzy class* used to distinguish these components is rather complicated, see [L], [BL], [Z]. Configurations of $\hat{\text{h}}$ omologous saddle connections for some nonconnected strata are studied in the paper [B1].

Given a billiard in a rational polygon Π , one can build a translation surface \hat{S} from an appropriate number $2N$ of copies of Π such that geodesics on S will project to the billiard trajectories in Π . Taking N copies instead of $2N$ one obtains a flat surface with $\mathbb{Z}/2\mathbb{Z}$ -holonomy with the same properties of geodesics. In some cases this latter construction is more advantageous. In the paper [AEZ] there is the study of billiards in polygons whose angles are multiples of $\pi/2$. Identifying two copies of such polygons by their boundaries one obtains a flat surface corresponding to a meromorphic quadratic differential on $\mathbb{C}P^1$ with at most simple poles. The results of this paper are used to classify closed billiard trajectories and generalized diagonals in the paper [AEZ], see also [B1].

Acknowledgements. Conceptually this paper is a continuation of the paper [EMZ]. We want to thank A. Eskin for his participation in the early stage of this project. We thank C. Boissy and E. Lanneau for several helpful conversations concerning nonconnected strata and G. Forni and Y. Minsky for valuable remarks. The first author thanks the University of Rennes for its support and hospitality during the preparation of this paper. The second author thanks the University of Chicago, UIC, Max-Planck-Institut für Mathematik at Bonn and IHES for hospitality and support during the preparation of this paper.

2 Preliminaries on Flat Surfaces and on $\hat{\text{Homologous}}$ Saddle Connections

In this section of preliminary results we describe geometric criteria for deciding when two saddle connections are $\hat{\text{homologous}}$ and describe the structure of the complement $S \setminus (\gamma_1 \cup \gamma_2)$. The key result in this section is Proposition 3.

In the case of a translation surface S it is obvious that two saddle connections γ_1, γ_2 are homologous if and only if $S \setminus (\gamma_1 \cup \gamma_2)$ is disconnected (provided $S \setminus \gamma_1$ and $S \setminus \gamma_2$ are connected). It is less obvious to check whether saddle connections γ_1, γ_2 on a flat surface S with nontrivial linear holonomy are $\hat{\text{homologous}}$ or not. In particular, a pair of closed saddle connections might be homologous in the usual sense, but not $\hat{\text{homologous}}$; a pair of closed saddle connections might be $\hat{\text{homologous}}$ even if one of them represents a loop homologous to zero, and the other does not; finally, a saddle connection joining a pair of *distinct* singularities might be $\hat{\text{homologous}}$ to a saddle connection joining a singularity to itself, or joining another pair

of distinct singularities. The flat torus described in the introduction gives an example of these phenomena (see Example 1 and Figure 1).

We start this section with several lemmas establishing some restrictions on the orders of singularities of a flat surface with boundary. By convention we consider only those flat structures which have linear holonomy in $\{Id, -Id\}$. Throughout this paper we assume that the boundary components of any flat surface with boundary are made up of parallel saddle connections, unless otherwise noted. We also assume that a sufficiently small collar neighborhood of any boundary component is a topological annulus (or, in the other words, that the natural compactification of $S \setminus \partial S$ coincides with S).

LEMMA 2.1. *If a flat surface S_j with boundary has trivial linear holonomy, then the sum of the orders of the boundary singularities along each boundary component is even:*

$$k_{j_{i,1}} + \cdots + k_{j_{i,p(i)}} \equiv 0 \pmod{2}.$$

Proof. Take a loop following the i -th boundary component

$$\{\rightarrow \gamma_{j_{i,1}} \rightarrow \cdots \rightarrow \gamma_{j_{i,p(i)}} \rightarrow\}$$

at a sufficiently small constant distance. Recall that by Definition 2 of the order of a boundary singularity, the angle between the saddle connection $\gamma_{j_{i,l}}$ and the saddle connection $\gamma_{j_{i,l+1}}$ at the boundary singularity $\gamma_{j_{i,l}} \rightarrow \gamma_{j_{i,l+1}}$ equals $(k_{j_{i,l}} + 1)\pi$. Thus, the linear holonomy around the loop is trivial if and only if the total sum of the angles $k_{j_{i,1}}\pi + \cdots + k_{j_{i,p(i)}}\pi$ is an integer multiple of 2π , or, equivalently, if and only if the sum $k_{j_{i,1}} + \cdots + k_{j_{i,p(i)}}$ of the orders of the boundary singularities is even. \square

LEMMA 2.2. *Let $d_{j_l}, k_{j_{i,l}}$ denote the orders of correspondingly interior singularities and boundary singularities of a flat surface with boundary S_j . Then*

$$2r(S_j) - 4 \leq \sum d_{j_l} + \sum k_{j_{i,l}} \equiv 2r(S_j) \pmod{4},$$

where $r(S_j)$ is the number of boundary components, the first sum is taken over all interior singularities and the second sum is taken over all boundary singularities.

Proof. Consider one more copy of the surface S_j taken with the opposite orientation. We can naturally identify these two copies along the common boundary. It follows from our assumptions on S_j that the resulting surface S is a nonsingular oriented closed flat surface without boundary. In other words, the closed flat surface S corresponds to a meromorphic quadratic differential on a Riemann surface.

Each interior singularity of S_j of order d_{j_l} produces two distinct singularities of S of order d_i . Each boundary singularity of S_j of order $k_{j_{i,l}}$ gives rise to an interior singularity of S of order $2k_{j_{i,l}}$. The surface S has genus $\hat{g} = 2g + r(S_j) - 1$. Now recall that for any quadratic differential on a closed Riemann surface S of genus \hat{g} the sum of orders of singularities equals $4\hat{g} - 4$. Hence,

$$2\left(\sum_{\substack{\text{interior} \\ \text{singularities} \\ \text{of } S_j}} d_l + \sum_{\substack{\text{boundary} \\ \text{singularities} \\ \text{of } S_j}} k_{j_{i,l}}\right) = 4\hat{g} - 4 = 4(2g + r(S_j) - 1) - 4 = 8(g - 1) + 4r(S_j),$$

which implies the desired relation. □

LEMMA 2.3. *The sum $D_i + 2 = k_{i,1} + \dots + k_{i,p(i)}$ of orders of boundary singularities along some boundary component \mathcal{B}_i of a flat surface S is equal to zero if and only if a sufficiently narrow collar neighborhood of \mathcal{B}_i in S is isometric to a flat cylinder.*

When S has trivial linear holonomy and the sum $D_i + 2$ of orders of boundary singularities along a boundary component \mathcal{B}_i is strictly positive, D_i satisfies the inequality $D_i \geq 0$.

Proof. By Definition 2 the order $k_{i,l}$ of any boundary singularity is nonnegative. Thus, $D_i + 2$ is equal to zero if and only if the orders of all boundary singularities along the boundary component \mathcal{B}_i are equal to zero, which implies the first part of the statement.

The second statement is an obvious corollary of the first one combined with Lemma 2.1. □

LEMMA 2.4. *Let β denote the total boundary of a translation surface defined by a holomorphic 1-form ω . Then*

1. $\int_{\beta} \omega = 0$.
2. β cannot consist of a single saddle connection.
3. If β is composed of exactly two saddle connections γ_1, γ_2 then γ_1, γ_2 are parallel and have equal length. Moreover, the oriented surface obtained by isometric identification of γ_1 and γ_2 is a translation surface (i.e. it is a closed flat surface with trivial linear holonomy).

Proof. Note that the canonical orientation of the surface induces a canonical orientation of the boundary β . Thus, the first statement is an immediate consequence of Stokes' formula. The second statement follows from the first since the holonomy $\int_{\gamma} \omega$ along a saddle connection γ cannot be 0.

For the third let $\beta = \gamma_1 - \gamma_2$. Then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$. This implies that γ_1, γ_2 are parallel, have equal length and that their directions defined by

the chosen orientations are compatible with linear holonomy. We can isometrically identify γ_1 either with γ_2 or with $-\gamma_2$. However, the second identification produces a nonorientable surface, so γ_1 must be identified with γ_2 which implies that the resulting surface is a translation surface. \square

REMARK 6. Whenever it is possible we follow the following convention concerning unions of curves: by $\gamma_1 \cup \gamma_2$ we denote a set-theoretical union, when the curves γ_1, γ_2 are not oriented or when the orientation is irrelevant. We use the additive notation (say, $\gamma_1 - \gamma_2$ in the proof above) for the union of *oriented* curves endowed with the orientation defined by the choice of signs of the components.

Let S be a flat surface with boundary; let γ_1, γ_2 be a pair of nonoriented parallel saddle connections γ_1, γ_2 of equal length at the boundary of S . The canonical orientation of the surface induces a canonical orientation of γ_1, γ_2 . In this paper we often construct a new flat surface from the surface S by gluing part of the boundary of S represented by γ_1 to the part of the boundary represented by γ_2 . By convention such a surgery is always performed by an isometric identification of γ_1 and $-\gamma_2$, i.e. we always identify γ_1 and γ_2 in such way that the resulting flat surface is orientable.

Suppose that, moreover, S has trivial linear holonomy.

DEFINITION 4. We say that γ_1 and γ_2 are identified by a *translation* if the resulting flat surface has trivial linear holonomy; otherwise we say that γ_1 and γ_2 are identified by a *flip*.

LEMMA 2.5. *Assume that a flat surface S with nontrivial linear holonomy is divided by a pair of parallel saddle connections γ_1, γ_2 into two connected components S_1, S_2 . Then at least one of the components must have nontrivial linear holonomy.*

Proof. If one of the γ_1, γ_2 is a closed curve homologous to zero, say γ_1 , then γ_2 lies in one of the two components of the complement $S \setminus \gamma_1$. Then, the boundary of the other component, say, S_1 consists solely of γ_1 , $\partial S_1 = \gamma_1$. By property 2 of Lemma 2.4 the component S_1 has nontrivial linear holonomy.

Therefore, we may assume that γ_1 and γ_2 are not homologous to zero so they are homologous to each other. Choosing appropriate orientations of γ_1 and γ_2 we get

$$\partial S_1 = \gamma_1 - \gamma_2, \quad \partial S_2 = -\gamma_1 + \gamma_2.$$

where the orientations of S_1, S_2 are induced from the canonical orientation of S . If both S_1 and S_2 have trivial linear holonomy we can choose the defining holomorphic 1-forms ω_1, ω_2 on S_1 and S_2 in such way that

$$\int_{\gamma_1} \omega_1 = \int_{\gamma_1} \omega_2 = \int_{\gamma_2} \omega_1 = \int_{\gamma_2} \omega_2.$$

The latter relations imply the compatibility of ω_1 and ω_2 on S . Thus, the flat structure on S can be defined by a holomorphic 1-form ω such that $\omega|_{S_1} = \omega_1, \omega|_{S_2} = \omega_2$, and S has trivial linear holonomy contrary to the initial assumption. \square

LEMMA 2.6. *Any two homologous saddle connections γ_1, γ_2 on a flat surface S are parallel. When both relations $[\gamma'_1] = -[\gamma''_1]$ and $[\gamma'_2] = -[\gamma''_2]$ are simultaneously valid or simultaneously not valid the saddle connections γ_1, γ_2 have the same length, $|\gamma_1| = |\gamma_2|$. When one of the relations, say, $[\gamma'_1] = -[\gamma''_1]$, is valid while the other one is not, $[\gamma'_2] \neq -[\gamma''_2]$, the lengths differ by the factor of two, $|\gamma_1| = 2|\gamma_2|$.*

Proof. The proof is a straightforward corollary of Definition 1 and the fact that the length of a saddle connection δ on the translation surface \hat{S} is defined as $|\delta| = |\int_{\delta} \omega|$ and its direction is defined by the argument of $\int_{\delta} \omega$. \square

LEMMA 2.7. *Let γ be a saddle connection on a flat surface S having nontrivial linear holonomy. The following properties are equivalent:*

- (a) $[\gamma'] = -[\gamma'']$ in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$;
- (b) $\hat{S} \setminus (\gamma' \cup \gamma'')$ contains exactly two connected components;
- (c) $[\gamma] = 0$ in $H_1(S, P; \mathbb{Z})$.

Proof. (a) \Rightarrow (c). Consider the map $p_* : H_1(\hat{S}, \hat{P}; \mathbb{Z}) \rightarrow H_1(S, P; \mathbb{Z})$ induced by the covering p . By definition of γ', γ'' we have $[\gamma] = p_*[\gamma'] = p_*[\gamma'']$. Thus, when $[\gamma'] = -[\gamma'']$ we get $[\gamma] = -[\gamma]$, so $[\gamma] = 0$.

(c) \Rightarrow (b). Since $[\gamma] = 0$, $S \setminus \gamma$ contains two connected components S_1, S_2 , such that $\partial S_1 = \gamma, \partial S_2 = -\gamma$. By property 2 of Lemma 2.4 both S_1 and S_2 have nontrivial linear holonomy, which implies that both $\hat{S}_1 = p^{-1}(S_1), \hat{S}_2 = p^{-1}(S_2)$, are connected. Thus, $\hat{S} \setminus (\gamma' \cup \gamma'') = \hat{S}_1 \sqcup \hat{S}_2$ contains exactly two connected components.

(b) \Rightarrow (a). Since γ' and γ'' are symmetric, the two connected components \hat{S}', \hat{S}'' of $\hat{S} \setminus (\gamma' \cup \gamma'')$ are also symmetric with respect to the involution. This restricts the possible situations to the following three (up to an interchange of the superscripts of \hat{S}'_1, \hat{S}''_1 if necessary):

- either $\partial \hat{S}'$ is composed of two copies of γ' and $\partial \hat{S}''$ of two copies of γ'' ;
- or $\partial \hat{S}' = \gamma' - \gamma''$ and $\partial \hat{S}'' = \gamma'' - \gamma'$;
- or $\partial \hat{S}' = \gamma' + \gamma''$ and $\partial \hat{S}'' = -\gamma'' - \gamma'$.

The first situation implies that \hat{S} contains two connected components which contradicts the assumptions that S has nontrivial linear holonomy. Hence, the first situation is excluded. The second situation implies that isometrically identifying the boundary components γ' and γ'' of S' we obtain a flat surface isometric to S . By property 3 of Lemma 2.4 this again implies that S has trivial linear holonomy which contradicts the assumptions. This case is also excluded. In the only remaining case we have $\partial\hat{S}' = \gamma' + \gamma''$ which implies $[\gamma'] = -[\gamma'']$. \square

The next proposition is the key to the proofs of Theorems 1 and 2. We do not assume that the saddle connections in the proposition below are parallel.

PROPOSITION 3. *Let S be a flat surface having nontrivial linear holonomy. Two saddle connections γ_1, γ_2 on S are $\hat{\text{homologous}}$ if and only if they have no interior intersections and one of the following holds*

1. *The union $\gamma_1 \cup \gamma_2$ does not separate the surface S and the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has trivial linear holonomy. (In this case $|\gamma_1| = |\gamma_2|$; all combinations: loop–loop, loop–segment, segment–segment are possible.)*
2. *The union $\gamma_1 \cup \gamma_2$ separates S ; neither γ_1 nor γ_2 by itself separates; the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has two connected components, one of them has trivial linear holonomy, the other nontrivial. (In this case $|\gamma_1| = |\gamma_2|$; the saddle connections are either two segments or two loops.)*
3. *One of γ_1, γ_2 , say, γ_1 separates S , the other one does not; the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has two connected components, one of them has trivial linear holonomy, the other one, whose boundary is γ_1 , has nontrivial holonomy. (In this case $|\gamma_1| = 2|\gamma_2|$; the separating saddle connection γ_1 is a loop, γ_2 might be a segment or a loop.)*
4. *Both γ_1 and γ_2 separate S ; the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has three connected components; two of which have nontrivial linear holonomy, while the remaining one, whose boundary is $\gamma_1 \cup \gamma_2$, has trivial linear holonomy. (In this case $|\gamma_1| = |\gamma_2|$; both γ_1 and γ_2 are loops.)*

Proof. According to Lemma 2.6 $\hat{\text{homologous}}$ saddle connections are parallel. If two $\hat{\text{homologous}}$ saddle connections γ_1 and γ_2 have a common point, this point is an endpoint for both γ_1 and γ_2 . Thus, from now on we can assume that γ_1, γ_2 have no interior intersections.

Two saddle connections γ_1, γ_2 without interior intersections subdivide a flat surface S in one of the following ways:

- (i) The union $\gamma_1 \cup \gamma_2$ does not separate the surface S .
- (ii) The union $\gamma_1 \cup \gamma_2$ separates S ; neither γ_1 nor γ_2 by itself separates.

- (iii) One of γ_1, γ_2 , say, γ_1 separates S , the other one does not.
- (iv) Both γ_1 and γ_2 separate S .

For each of these cases we prove that the additional assumption that γ_1 and γ_2 are $\hat{\text{homologous}}$ is equivalent to the corresponding additional assumptions (1)–(4) on triviality of linear holonomy of the corresponding components. In each case we use Lemmas 2.6 and 2.7 to determine the corresponding relation between the lengths $|\gamma_1|$ and $|\gamma_2|$. We combine this information with Lemma 2.7 (when appropriate) to prove that one of γ_1, γ_2 (or both γ_1 and γ_2) is a closed cycle. The proof of realizability of combinations loop–loop, loop–segment, segment–segment indicated in Proposition 3 is left to the reader as an exercise. Note that Example 1 already proves realizability of combinations loop–segment in (1) and loop–loop in (2). The remaining combinations can be found in sections 5 and 6.

Let $X \subseteq S$ be a subset of S . By \hat{X} we denote the preimage $\hat{X} = p^{-1}(X)$. Let S_j be a connected component of $S \setminus (\gamma_1 \cup \gamma_2)$. We use the following obvious criterion: S_j has nontrivial linear holonomy if and only if the preimage \hat{S}_j is connected. Now let us pass to consideration of cases (i)–(iv).

(i) In this case $S \setminus (\gamma_1 \cup \gamma_2)$ is connected; denote it by S_1 . Neither of γ_1, γ_2 is homologous to zero, so $[\hat{\gamma}_1] = [\gamma'_1] - [\gamma''_1]$, and $[\hat{\gamma}_2] = [\gamma'_2] - [\gamma''_2]$ (see Lemma 2.7). By Lemma 2.6 when such γ_1 and γ_2 are homologous, we have $|\gamma_1| = |\gamma_2|$.

If the saddle connections γ_1 and γ_2 are $\hat{\text{homologous}}$ then the cycle $[\gamma'_1] - [\gamma''_1]$ is homologous (in the usual sense) to one of the $\pm([\gamma'_2] - [\gamma''_2])$ which means that $\hat{S}_1 = \hat{S} \setminus (\gamma'_1 \cup \gamma''_1 \cup \gamma'_2 \cup \gamma''_2)$ is not connected. Hence, by the above criterion S_1 has trivial linear holonomy.

Suppose now that $S_1 = S \setminus (\gamma_1 \cup \gamma_2)$ has trivial linear holonomy. Then \hat{S}_1 has two connected components \hat{S}'_1 and \hat{S}''_1 . By assumption the flat surface S has nontrivial linear holonomy. Hence, it follows from property 3 in Lemma 2.4 that both $S \setminus \gamma_1$ and $S \setminus \gamma_2$ have nontrivial linear holonomy. This implies that there exist a pair of loops ρ_1, ρ_2 on S such that ρ_i and γ_i have a single transversal intersection, $i = 1, 2$; such that $\rho_1 \cap \gamma_2 = \emptyset$, $\rho_2 \cap \gamma_1 = \emptyset$; and such that holonomy along each $\rho_i, i = 1, 2$, is nontrivial. Interchanging the superscripts of \hat{S}'_1, \hat{S}''_1 if necessary, we may assume that γ'_1 is on the boundary of \hat{S}'_1 . Since ρ_1 has nontrivial linear holonomy, the lift $\rho'_1 \subset \hat{S}'_1$ of ρ starting at γ'_1 is not closed and hence it ends on $-\gamma''_1$. This implies that both γ'_1 and $-\gamma''_1$ belong to the boundary of \hat{S}'_1 . Since $S_1 = S \setminus (\gamma_1 \cup \gamma_2)$ is connected, at least one of both $\pm\gamma'_2$ and $\pm\gamma''_2$

belongs to the boundary of \hat{S}'_1 . Applying the same argument as above and using the obvious symmetry between \hat{S}'_1 and \hat{S}''_1 we conclude that under an appropriate choice of orientations of γ_1 and γ_2 one has

$$\partial\hat{S}'_1 = \gamma'_1 - \gamma''_1 - \gamma'_2 + \gamma''_2,$$

which is equivalent to

$$[\gamma'_1] - [\gamma''_1] = [\gamma'_2] - [\gamma''_2],$$

and hence, γ_1 and γ_2 are \hat{h} omologous.

(ii) In this case γ_1 and γ_2 are homologous in the usual sense, and not homologous to zero; the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has two connected components S_1, S_2 . This implies that either both of γ_1 and γ_2 are segments, or both are closed cycles. Since neither of γ_1, γ_2 is homologous to zero, Lemma 2.7 implies that $[\gamma'_i] \neq -[\gamma''_i]$ for $i = 1, 2$. Thus, if such γ_1, γ_2 are \hat{h} omologous we get $|\gamma_1| = |\gamma_2|$ by Lemma 2.6.

We will now prove that in case (ii) the two curves are \hat{h} omologous if and only if exactly one of the two components has nontrivial linear holonomy.

By Lemma 2.5 at least one of two components, say, S_1 has nontrivial linear holonomy. Under an appropriate choice of orientations of γ_1, γ_2 we have $\partial S_1 = \gamma_1 - \gamma_2$, which implies

$$\partial\hat{S}_1 = (\gamma'_1 + \gamma''_1) - (\gamma'_2 + \gamma''_2).$$

Since $[\gamma'_i] \neq -[\gamma''_i]$, for $i = 1, 2$, the condition that γ_1 and γ_2 are \hat{h} omologous is equivalent to one of the relations $([\gamma'_1] - [\gamma''_1]) = \pm([\gamma'_2] - [\gamma''_2])$. Together with the above equation on $\partial\hat{S}_1$ it is equivalent to one of the following systems

$$\begin{cases} [\gamma'_1] = [\gamma'_2] \\ [\gamma''_1] = [\gamma''_2] \end{cases} \quad \begin{cases} [\gamma'_1] = [\gamma''_2] \\ [\gamma''_1] = [\gamma'_2] \end{cases}$$

The systems might be identified by interchange of superscripts of, say, γ'_2 and γ''_2 , thus we can consider just the first one.

Since by the second property of Lemma 2.4 neither of $[\gamma'_i], [\gamma''_i], i = 1, 2$, is homologous to zero, the latter system is valid if and only if cutting $\partial\hat{S}$ by any of two pairs $[\gamma'_1], [\gamma'_2]$ or $[\gamma''_1], [\gamma''_2]$ of saddle connections we get two connected components. Since \hat{S}_1 is connected the latter is true if and only if \hat{S}_2 contains two connected components. By the criterion formulated above this is true if and only if S_2 has trivial linear holonomy. The equivalence is proved.

(iii) In this case γ_1 is a closed cycle homologous to zero, while γ_2 is not homologous to zero. This implies that the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has two connected components S_1, S_2 . Combining Lemma 2.7 with Lemma 2.6 we see that if such γ_1 and γ_2 are \hat{h} omologous, we have $|\gamma_1| = 2|\gamma_2|$.

Choose the orientation of γ_1 and enumeration of the components in such way that

$$\partial S_1 = \gamma_1 \quad \partial S_2 = -\gamma_1 + \gamma_2 - \gamma_2.$$

Then

$$\partial \hat{S}_1 = \gamma'_1 + \gamma''_1, \quad \partial \hat{S}_2 = -\gamma'_1 - \gamma''_1 + \gamma'_2 - \gamma''_2 + \gamma''_2 - \gamma'_2.$$

Note that S_1 has nontrivial linear holonomy (see property 2 of Lemma 2.4) so \hat{S}_1 is connected. If γ_1 and γ_2 are $\hat{\text{homologous}}$, then $[\gamma'_1] = \pm([\gamma'_2] - [\gamma''_2])$. This implies that $\hat{S} \setminus (\gamma'_1 \cup \gamma'_2 \cup \gamma''_2)$ contains at least two connected components. Since $\hat{S} \setminus (\gamma'_1 \cup \gamma'_2 \cup \gamma''_2) = \hat{S}_1 \cup \gamma''_1 \cup \hat{S}_2$ where \hat{S}_1 is connected and γ''_1 connects \hat{S}_1 and \hat{S}_2 , this implies that \hat{S}_2 is not connected. Hence, S_2 has trivial linear holonomy.

Conversely, consider the connected component of $S \setminus \gamma_1$ containing γ_2 ; denote it by \tilde{S}_2 . Property 2 of Lemma 2.4 implies that \tilde{S}_2 has nontrivial linear holonomy. Note that $S_2 = \tilde{S}_2 \setminus \gamma_2$. Thus, when S_2 has trivial linear holonomy, there exists a closed path ρ on \tilde{S}_2 transversally intersecting γ_2 such that holonomy along ρ is nontrivial. Since S_2 has trivial linear holonomy, \hat{S}_2 has two connected components \hat{S}'_2, \hat{S}''_2 . Changing if necessary the superscripts of \hat{S}'_2, \hat{S}''_2 we may assume that γ'_2 is on the boundary of \hat{S}'_2 . Since the holonomy along ρ is nontrivial, following the lift of ρ which starts at γ'_2 and goes inside \hat{S}'_2 the path ρ ends at $-\gamma''_2$, which shows that γ'_2 and $-\gamma''_2$ make part of the boundary of the same component \hat{S}'_2 . Taking into consideration the symmetry between components \hat{S}'_2, \hat{S}''_2 and choosing an appropriate orientation of γ_1 we get

$$\partial \hat{S}'_2 = -\gamma'_1 + \gamma'_2 - \gamma''_2, \quad \partial \hat{S}''_2 = -\gamma''_1 - \gamma'_2 + \gamma''_2,$$

which implies that γ_1 and γ_2 are $\hat{\text{homologous}}$.

(iv) In this case the complement $S \setminus \{\gamma_1 \cup \gamma_2\}$ has three connected components. Both γ_1 and γ_2 are homologous to zero, so they are represented by closed cycles. This also implies that $[\hat{\gamma}_i] = \gamma'_i, i = 1, 2$. If such γ_1 and γ_2 are $\hat{\text{homologous}}$, we have $|\gamma_1| = |\gamma_2|$ (see Lemma 2.6).

Denote the connected components of $S \setminus (\gamma_1 \cup \gamma_2)$ in such way that under an appropriate choice of orientations of γ_1, γ_2 one gets

$$\partial S_1 = \gamma_1, \quad \partial S_2 = -\gamma_2, \quad \partial S_3 = -\gamma_1 + \gamma_2.$$

By property 2 of Lemma 2.4 the components S_1 and S_2 have nontrivial linear holonomy, so \hat{S}_1 and \hat{S}_2 are connected. We get

$$\partial \hat{S}_1 = \gamma'_1 + \gamma''_1, \quad \partial \hat{S}_2 = -\gamma'_2 - \gamma''_2, \quad \partial \hat{S}_3 = -\gamma'_1 - \gamma''_1 + \gamma'_2 + \gamma''_2.$$

If γ_1 and γ_2 are $\hat{\text{homologous}}$ then $[\gamma'_1] = \pm[\gamma'_2]$ which implies that cutting \hat{S} by γ'_1, γ'_2 we get two connected components, which means that \hat{S}_3 is not connected and hence S_3 has trivial linear holonomy.

Conversely, if S_3 has trivial linear holonomy then \hat{S}_3 contains two connected components \hat{S}'_3, \hat{S}''_3 which (under appropriate enumeration) have boundaries

$$\partial\hat{S}'_3 = -\gamma'_1 + \gamma'_2, \quad \partial\hat{S}''_3 = -\gamma''_1 + \gamma''_2,$$

which implies that γ_1 and γ_2 are homologous. Proposition 3 is proved. \square

Theorem 1 follows from Proposition 3.

Proof of Theorem 1. Cutting S by γ_1, γ_2 we get one of the decompositions (i)–(iv). According to Proposition 3, the additional assumptions (1)–(4) on the triviality of the linear holonomy of the corresponding component are necessary and sufficient conditions for γ_1, γ_2 to be \hat{h} omologous. It remains to note that in each of the cases (1)–(4) there is a unique component with trivial linear holonomy. \square

The following criterion is analogous to the corresponding statement in [EMZ]. It is proved in Appendix A, where the notion of a measure on each stratum is discussed.

PROPOSITION 4. *For almost every flat surface in any stratum, two saddle connections are parallel if and only if they are \hat{h} omologous.*

3 Graph of Connected Components

In this section we give the proof that every graph is given by the list in Theorem 2. Denote by \dot{S} the surface S punctured at the singularities. Any closed loop ρ on \dot{S} can be homotoped to have a finite number of transverse intersections with the saddle connections from the collection $\gamma = \{\gamma_1 \dots \gamma_n\}$. It naturally induces a path ρ_* on the graph $\Gamma(S, \gamma)$ by recording the surfaces S_j intersected by ρ . It is easy to see that paths $\rho \sim \rho'$ homotopic on the punctured surface \dot{S} define paths $\rho_* \sim \rho'_*$ homotopic on the graph. Mark a point $x \in \dot{S} \setminus \{\gamma_i\}$; let $v(x)$ be the corresponding vertex of the graph $\Gamma(S, \gamma)$. We get a natural homomorphism $\pi_1(\dot{S}, x) \rightarrow \pi_1(\Gamma(S, \gamma), v(x))$.

Any finite connected graph can be retracted by a deformation to a bouquet of circles (possibly to a point). We can choose the retraction in such way that $v(x)$ retracts to the base point of the bouquet of circles. We can consider the bouquet of circles B as a graph obtained from the graph $\Gamma(S, \gamma)$ by identifying some subtree of $\Gamma(S, \gamma)$ to a single vertex of B . Thus, some edges of $\Gamma(S, \gamma)$ remain nondegenerate under the retraction, and some edges collapse to a point.

Now we can prove the lemma which is the main technical tool in the proof of Theorem 2.

LEMMA 3.1. *Let $\alpha \subset \Gamma(S, \gamma)$ be a closed path on $\Gamma(S, \gamma)$ realized as a subgraph of $\Gamma(S, \gamma)$. If under some retraction of $\Gamma(S, \gamma)$ to a bouquet of circles, α retracts to one of the circles, then there exists a closed path ρ on the punctured surface \dot{S} such that $\rho_* = \alpha$ and the linear holonomy along ρ is nontrivial.*

Proof. We suppose that a retraction of $\Gamma(S, \gamma)$ to a bouquet of circles is fixed. We start with consideration of the general case, when the bouquet of circles contains at least two circles.

Let γ_1 and γ_2 be a pair of edges of $\Gamma(S, \gamma)$, which remain nondegenerate under retraction, such that $\gamma_1 \in \alpha$ and $\gamma_2 \notin \alpha$ (since the bouquet contains at least two circles, such γ_2 exists). Cutting $\Gamma(S, \gamma)$ by these edges we get a connected graph. Equivalently, cutting the surface S by a pair of homologous saddle connections γ_1, γ_2 we get a connected surface $S_{(1,2)} = S \setminus (\gamma_1 \cup \gamma_2)$ which, by Proposition 3, has trivial linear holonomy. By construction $\partial S_{(1,2)} = \gamma_1 \cup -\gamma_1 \cup \gamma_2 \cup -\gamma_2$. Gluing back the boundary components γ_1 and $-\gamma_1$ of $S_{(1,2)}$ we get a surface $S_{(2)} = S \setminus \gamma_2$ which has nontrivial linear holonomy by Lemma 2.4. Thus, the boundary components γ_1 and $-\gamma_1$ of the *translation* surface $S_{(1,2)} = S \setminus (\gamma_1 \cup \gamma_2)$ are identified by a flip (see Definition 4 in the previous section).

Consider any path ρ in S such that $\rho_* = \alpha$ and such that ρ has unique transversal intersection with γ_1 . By construction ρ gives a nonclosed connected path ρ' on $S_{(1,2)}$ joining a pair of points on $P_+ \in \gamma_1$ and $P_- \in -\gamma_1$ corresponding to the same point $P \in \gamma_1$ on S upon gluing of γ_1 with $-\gamma_1$. Since γ_1 and $-\gamma_1$ are identified by a flip, we see that the linear holonomy along ρ is nontrivial.

It remains to consider the case, when the bouquet of circles corresponding to the graph $\Gamma(S, \gamma)$ has a single circle. It follows from Proposition 3 that the graph cannot be just a single loop composed of “+”-vertices of valence two and of “cylinder vertices” of valence two. Thus, either $\Gamma(S, \gamma)$ is a loop composed of vertices of valence two with some “-”-vertices, or there is at least one nontrivial subtree with a vertex on the base loop.

In the first case choose any path ρ' on S such that $\rho'_* = \alpha$. If the linear holonomy along the path ρ' is nontrivial, we choose $\rho := \rho'$ and the lemma is proved. If the holonomy is trivial, we can compose ρ' with a closed path ρ'' , such that ρ'' is contained entirely inside some S_j^- , and such that the linear holonomy along ρ'' is nontrivial. Since $\rho'' \subset S_j^-$ the projection

ρ''_* is a trivial path. Thus, $(\rho' \cdot \rho'')_* = \rho'_* = \alpha$, and by construction the linear holonomy along $(\rho' \cdot \rho'')$ is nontrivial. The required path ρ is given by $\rho' \cdot \rho''$.

In the second case the subtree necessarily has a vertex S_j of valence one, which by Lemma 2.4 is a “-”-vertex. Denote by γ_1 the edge adjacent to this vertex of valence one; we denote by the same symbol γ_1 the corresponding saddle connection in S . Consider any path ρ' on S such that $\rho'_* = \alpha$. If $\alpha \subset \Gamma(S, \gamma)$ passes through S_j^- , we apply the same construction as in the previous case. If α does not pass through S_j^- then $\gamma_1 \cap \alpha = \emptyset$, and any path $\rho \in S$ such that $\rho_* = \alpha$ has trivial intersection with the saddle connection γ_1 . Choose some edge $\gamma_2 \in \alpha$ which is nondegenerate under retraction. Cutting S by the pair of homologous saddle connections γ_1, γ_2 we get two connected components: a connected surface $S_{(1,2)}$ and a surface S_j^- (corresponding to the vertex of valence one). The closed path ρ on S becomes a nonclosed connected path on $S_{(1,2)}$ joining the boundary components γ'_2 and γ''_2 . By Proposition 3 the surface $S_{(1,2)}$ has trivial linear holonomy. By construction $\partial S_{(1,2)} = \gamma_1 \cup \gamma_2 \cup -\gamma_2$. Glue back the boundary components γ_2 and $-\gamma_2$ of $S_{(1,2)}$. We get a surface $S_{(1)}$ which coincides with one of the two components of the initial surface S cut by a single saddle connection γ_1 . Since $\partial S_{(1)} = \gamma_1$, by Lemma 2.4 the surface $S_{(1)}$ has nontrivial linear holonomy. Thus, the boundary components γ_2 and $-\gamma_2$ of the translation surface $S_{(1,2)}$ were identified by a flip which implies that the linear holonomy along ρ is nontrivial. □

LEMMA 3.2. *Consider a connected subgraph Υ of the initial graph $\Gamma(S, \gamma)$. If Υ has a vertex labelled with “-” or if it is not a tree, the surface with boundary S_Υ corresponding to this subgraph has nontrivial linear holonomy.*

Proof. If the subgraph has some vertex labelled with “-”, the corresponding surface S_j^- has a closed path with nontrivial linear holonomy. The bigger surface S_Υ has the same path, so it also has nontrivial linear holonomy. If the subgraph is not a tree, then it has a loop which is not homotopically trivial. By Lemma 3.1 there is a closed path ρ on S_Υ corresponding to this loop such that ρ has nontrivial linear holonomy. □

Proof of Theorem 2 (Necessity). First we note that a cylinder has trivial linear holonomy, so by Lemma 2.4 a “o”-vertex cannot have valence 1.

If the valence of a “o”-vertex is two, then each of the two boundary components of the corresponding cylinder represents a single saddle connection.

Hence, two “o”-vertices of valence two cannot have a common edge, otherwise the pair of corresponding cylinders would be identified along a boundary component of each which would result in a longer cylinder contradicting the assumption that each cylinder is maximal.

Now note that the bouquet of circles to which the graph $\Gamma(S, \gamma)$ is retracted contains at most two circles. Otherwise there would be edges γ_1 and γ_2 such that $\Gamma(S, \gamma) \setminus (\gamma_1 \cup \gamma_2)$ would be connected but not simply connected. Thus, according to Lemma 3.2 the surface $S_{(1,2)} = S \setminus (\gamma_1 \cup \gamma_2)$ would have nontrivial linear holonomy, which contradicts Proposition 3.

Two loops. Suppose that the bouquet of circles contains exactly two circles. Cut them by some edges γ_1 and γ_2 which correspond to different circles of the bouquet. By Proposition 3 the resulting surface has trivial linear holonomy. Lemma 3.2 implies that the surface and therefore the graph does not have any “-”-vertices, in particular, no vertices of valence 1. Since the Euler characteristic of $S^1 \vee S^1$ equals to -1 we get

$$\begin{aligned}
 -1 = \chi(S^1 \vee S^1) = \chi(\Gamma(S, \gamma)) &= -\frac{1}{2} \cdot (\text{number of vertices of valence 3}) \\
 &\quad - \frac{2}{2} \cdot (\text{number of vertices of valence 4}) \\
 &\quad - \frac{3}{2} \cdot (\text{number of vertices of valence 5}) - \dots,
 \end{aligned}$$

which means that either $\Gamma(S, \gamma)$ has two vertices of valence 3 while all the other vertices have valence 2, or $\Gamma(S, \gamma)$ has a single vertex of valence 4 while all the other vertices have valence 2. All graphs of this type except one are in the list of Theorem 2, see types d and e. The type which we have to rule out is schematically presented in Figure 8.

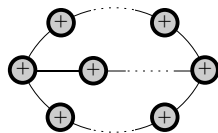


Figure 8: A graph of this type is not realizable as $\Gamma(S, \gamma)$.

We prove by contradiction that this graph is not realizable as $\Gamma(S, \gamma)$. Let S_n^+ be a vertex of valence three; let $\gamma_1, \gamma_2, \gamma_3$ be the edges adjacent to it. Cutting $\Gamma(S, \gamma)$ by any pair of distinct edges $\gamma_i, \gamma_j, i = 1, 2, 3$, we still get a connected graph. This means that no pair of homologous saddle connections $\gamma_i \cup \gamma_j, i = 1, 2, 3$, separates S . Hence, by Proposition 3 the lengths $|\gamma_i|, i = 1, 2, 3$ are equal and all γ_i are parallel.

Let ω be the holomorphic 1-form representing the translation structure on S_n^+ . Under an appropriate choice of orientations of $\gamma_1, \gamma_2, \gamma_3$ we get $\partial S_n^+ = \gamma_1 \cup \gamma_2 \cup \gamma_3$, and hence $\int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega = 0$. On the other hand the fact that all the lengths $|\gamma_i|, i = 1, 2, 3$ are equal and all γ_i are parallel implies that $\int_{\gamma_1} \omega = \pm \int_{\gamma_2} \omega = \pm \int_{\gamma_3} \omega$. These two relations are incompatible, which is a contradiction.

One loop. If $\Gamma(S, \gamma)$ is a loop of vertices of valence two then by Proposition 3 it has at least one “-”-vertex S_i^- . Let γ_1 and γ_2 be the edges of $\Gamma(S, \gamma)$ adjacent to S_i^- . The complement $S \setminus (\gamma_1 \cup \gamma_2)$ has two connected components: S_i^- and $S \setminus S_i^-$. Since S_i^- has nontrivial linear holonomy, by Proposition 3 the flat surface $S \setminus S_i^-$ has trivial linear holonomy. It follows now from Lemma 3.2 that $S \setminus S_i^-$ is a chain of “+”-vertices of valence two and of “o”-vertices of valence two. Thus, in this case the graph $\Gamma(S, \gamma)$ is of the type b, see Theorem 2 and Figure 3.

If the graph has a nontrivial subtree attached to the base loop then any such subtree necessarily has a vertex of valence one, which by Lemma 2.4 is a “-”-vertex. Let us show that $\Gamma(S, \gamma)$ can have only one “-”-vertex of valence one. Suppose that there are two vertices S_i^- and S_j^- of valence one; denote by γ_1 and γ_2 the edges of $\Gamma(S, \gamma)$ adjacent to these vertices. Cutting S by γ_1 and γ_2 we obtain three connected components: S_i^-, S_j^- and $S_{(1,2)} := S \setminus (S_i^- \cup S_j^-)$. Since the first two surfaces have nontrivial linear holonomy, it follows from Proposition 3 that $S_{(1,2)}$ has trivial linear holonomy. But by assumption the graph Υ corresponding to $S_{(1,2)}$ has a nontrivial loop, so by Lemma 3.1 the flat surface $S_{(1,2)}$ has nontrivial linear homology, which is a contradiction.

Thus, the graph has the structure of a union of a circle with a segment attached to a circle. The graph has a single vertex of valence three, a single vertex of valence one and an arbitrary number of vertices of valence two. Choosing an appropriate pair of edges γ_1, γ_2 and combining Proposition 3 with Lemma 3.1 and Lemma 3.2 we see that the only “-”-vertex of the graph is the vertex of valence one located at the free end of the segment. This is the graph of the type c in the list of graphs in Theorem 2.

A tree. In this case $\Gamma(S, \gamma)$ has at least two vertices of valence one which are therefore of “-”-type. Let γ_1 and γ_2 be the edges of $\Gamma(S, \gamma)$ adjacent to this pair of vertices S_i^-, S_j^- . Cutting the surface S by γ_1, γ_2 we get three connected components S_i^-, S_j^- , and $S \setminus (S_i^- \cup S_j^-)$. By Proposition 3 the component $S \setminus (S_i^- \cup S_j^-)$ has trivial linear holonomy.

Thus, by Lemma 3.2 it does not have any “-”-vertices. Since $\Gamma(S, \gamma)$ is a tree it means that $\Gamma(S, \gamma)$ is a chain of “+”-vertices of valence two bounded at the ends by a pair of “-”-vertices of valence one. This is the graph a from Theorem 2 (see also Figure 3).

Two “o”-vertices of valence two cannot be neighbors. It remains to prove that a “o”-vertex of valence 3 cannot be joined by a separating edge to a “o”-vertex of valence 2. If that were the case then on one boundary component of the cylinder there would be a marked point. If this boundary component were joined to a “o”-vertex of valence 2 it would produce a “fake singularity” on S .

We have proved that all graphs must be of the type in Theorem 2. The fact that the weights are as described follows from the next lemma.

LEMMA 3.3. *The type of the graph $\Gamma(\mathcal{C})$ uniquely determines the distribution of unsigned weights 1 and 2 on the edges of the graph; the corresponding weights are presented in Figure 3.*

Proof. For every vertex representing a component with trivial linear holonomy we can choose signs for the weights 1 and 2 on the edges adjacent to the vertex. The sum of these signed weights is zero. This immediately implies that the globally defined unsigned weights on both edges adjacent to a valence two “+”-vertex or to a valence two “o”-vertex are the same. This in turn implies that all the weights on the graphs of types a) and b) coincide, and hence are marked by 1.

The remaining graphs do not have “-”-vertices. The edges of any vertex of valence three are weighted by 1, 1 and 2. This implies that the weights of the graphs of types c) and d) are as in Figure 3.

Let γ_1, γ_2 be a pair of edges adjacent to a valence four vertex, and belonging to two different loops. The surface cut along these saddle connections is connected. By Proposition 3, $|\gamma_1| = |\gamma_2|$, and hence the corresponding edges have the same weight. Since all the edges in a chain of “+”-vertices or “o”-vertices of valence 2 have the same weight, we see that all edges of a graph of type e) are weighted by 1. \square

This completes the proof of the necessity part of Theorem 2. \square

4 Parities of Boundary Singularities

In this section we prove the necessity part of Theorem 3 which says that for any decomposition of a flat surface S as in Theorem 2 every connected component S_j has one of the boundary types presented in Figure 6.

Theorem 2 and Figure 3 give the types of graphs Γ ; Figure 5 gives the list of all abstract local ribbon graphs of valences from one to four. Basically, what remains to check is that for any “+”-vertex v of Γ an embedding $\mathbb{G}_v \hookrightarrow \Gamma$ of the local ribbon graph \mathbb{G}_v into the graph Γ uniquely determines the parities of the boundary singularities, and that these parities are exactly as in Figure 6.

Signs of the weights. Given a collection γ of $\hat{\gamma}$ homologous saddle connections $\gamma_1, \dots, \gamma_n$ on a flat surface S we have assigned weights 1 and 2 to saddle connections γ_i (see the paragraph preceding Theorem 2 for the definition of the weights and Figure 3 for the distribution of the weights in Γ). If a connected component S_j of $S \setminus \gamma$ has trivial linear holonomy (i.e. if it is represented by a “+” or by a “o”-vertex of Γ) we may assign signs \pm to the weights of the saddle connections on the boundary of S_j . The canonical orientation of S_j induces the canonical orientation of the boundary ∂S_j . Let ω be a holomorphic 1-form representing the flat structure on S_j normalized so that

$$\int_{\gamma_i} \omega = \text{weight of } \gamma_i,$$

for some saddle connection γ_i on the boundary of S_j . Then for the other saddle connections on ∂S_j we get $\int_{\gamma_{i'}} \omega = \pm 1$ or $\int_{\gamma_{i'}} \omega = \pm 2$ (see also Figures 10–13).

There is an ambiguity in the choice of signs: we may simultaneously change the signs of all weights to the opposite ones. This corresponds to choosing $-\omega$ instead of ω .

LEMMA 4.1. *Consider two consecutive saddle connections $\gamma_{j_i,l} \rightarrow \gamma_{j_i,l+1}$ on the same boundary component \mathcal{B}_i of ∂S_j . The parity of the corresponding boundary singularity is even if the weights of $\gamma_{j_i,l}$ and $\gamma_{j_i,l+1}$ have the same signs, and odd if the weights of $\gamma_{j_i,l}$ and $\gamma_{j_i,l+1}$ have opposite signs.*

Proof. The holomorphic 1-form ω chosen above defines an oriented horizontal foliation on S_j : the kernel foliation of $\text{Im}(\omega)$. The above normalization of ω implies that any saddle connection at the boundary ∂S_j is horizontal. The weight of a saddle connection $\gamma_{j_i,l}$ on the boundary of S_j is positive if the orientation of $\gamma_{j_i,l}$ induced from the orientation of the boundary matches the orientation of the foliation and negative if it does not.

The cone angle between two incoming or two outgoing separatrix rays (in the sense of the orientation of the foliation) is an even multiple of π and the cone angle between an incoming and an outgoing separatrix ray (in the sense of the orientation of the foliation) is an odd multiple of π .

The statement of the lemma now follows from Definition 2 of the order of a boundary singularity. \square

Consider now a particular case when S_j is represented by a vertex v_j of valence four of the graph $\Gamma(S, \gamma)$. Four edges of Γ_{v_j} are distributed into two pairs: each pair bounds one of the two loops of the graph $\Gamma(S, \gamma)$, see Figure 3.

LEMMA 4.2. *The weights of saddle connections on the boundary of a component S_j represented by a vertex of valence four have same signs if they bound the same loop in $\Gamma(S, \gamma)$ and opposite signs otherwise.*

Proof. From Lemma 3.3 we know that the absolute values of weights of all edges of Γ_{v_j} for a vertex v_j of valence four are equal to one (see also Figure 3). Hence, it follows from Stokes' theorem that we have two edges of weight $+1$ and two edges of weight -1 in Γ_{v_j} . We want to show that the weights of a pair of edges of Γ_{v_j} bounding the same loop in Γ have the same signs.

Let $\gamma_1, \gamma_2 \in \Gamma_{v_j}$ bound *different* loops in Γ . Cutting S by γ_1, γ_2 we get a connected flat surface S_{12} . Using the same notation as in the proof of Proposition 3 we get

$$\partial S_{12} = \gamma'_1 \cup -\gamma''_1 \cup \gamma'_2 \cup -\gamma''_2.$$

By Theorem 1 the surface S_{12} has trivial linear holonomy. Hence, we can extend the form ω to S_{12} which enables us to assign signs to the weights of saddle connections $\gamma'_1, \gamma''_1, \gamma'_2, \gamma''_2$ on the boundary ∂S_{12} of S_{12} .

The last statement of Lemma 2.4 implies that gluing the initial closed surface S from S_{12} the boundary component γ'_1 is glued to $-\gamma''_1$ by a flip (see Definition 4 in section 2). Similarly γ'_2 is glued to $-\gamma''_2$ by a flip. Hence, the weights of γ'_1 and of γ''_1 have the same signs, and the weights of γ'_2 and of γ''_2 have the same signs.

This completes the proof of the lemma in the case when the corresponding loop contains no vertices at all. An induction on the number of vertices in the loop completes the proof in general case. \square

LEMMA 4.3. *For any “+”-vertex or “o”-vertex v of the graph $\Gamma(S, \gamma)$ the type of the graph uniquely determines the distribution of signed weights ± 1 and ± 2 on the edges of Γ_v (up to simultaneous interchange of all signs to the opposite ones).*

Proof. By Stokes' theorem the sum of weights of all saddle connections of Γ_v is equal to zero. Taking into consideration Lemma 3.3 (see also Figure 3) this implies that when the vertex v has valence 2, the weights of the edges

of Γ_v are $+1, -1$; when v has valence 3, the weights are $+1, +1, -2$; when v has valence 4, the weights are $+1, +1, -1, -1$. Moreover, according to Lemma 4.2 the weights of edges of Γ_v which bound the same loop in $\Gamma(S, \gamma)$ coincide. \square

Now we are ready to prove the following proposition, which corresponds to the necessity part of Theorem 3. (The sufficiency part of Theorem 3 immediately follows from Theorem 4 proved in the next section).

PROPOSITION 5. *For any decomposition of a flat surface S as in Theorem 2 every connected component S_j has one of the boundary types presented in Figure 6.*

Proof. The necessity part of Theorem 2 proved in section 3 claims that the graph $\Gamma(S, \gamma)$ of the decomposition has one of the types presented in Figure 3. Note that for “+” and “-”-vertices, Figure 6 describes *all* possible embeddings of abstract local ribbon graphs \mathbb{G}_v that are given in Figure 5 into graphs Γ as in Figure 3. We use dotted lines to indicate the pairs of edges bounding cycles in the graphs in Figure 6; dotted lines are not indicated in symmetric situations. Since there are no restrictions on the parities of boundary singularities of “-”-vertices this completes the proof for “-”-vertices.

Any “o”-vertex S_j^{comp} corresponds to a flat cylinder. Hence, it has exactly two distinct boundary components. The boundary singularities on each of the components correspond to marked points, so the order of any boundary singularity of a “o”-vertex is zero. By Lemma 4.1 this implies that all edges of Γ_{v_j} which correspond to the same boundary component of the cylinder S_j have weights of the same sign. Taking into consideration Lemma 4.3 these two conditions restrict the possible structures of an embedded local ribbon graph $\mathbb{G}_v \hookrightarrow \Gamma(S, \gamma)$ for “o”-vertices to structures $\circ 2.2$, $\circ 3.2$ and $\circ 4.2$ in Figure 6.

By Lemma 4.3 for any “+”-vertex of $\Gamma(S, \gamma)$ we know the signed weights of the edges of Γ_v (up to simultaneous interchange of all signs to the opposite ones). For “+”-vertices of valence two and three this distribution follows immediately from Figure 3 and from Stokes’ theorem; for “+”-vertices of valence four this distribution is described by Lemma 4.2. Hence, using Lemma 4.1 we can determine the parities of all boundary singularities for any embedded local ribbon graph $\mathbb{G}_v \hookrightarrow \Gamma$. It remains to check that for all possible embeddings listed in Figure 6 the parities are the ones listed. This is an easy exercise. \square

COROLLARY 1. Given any abstract graph Γ as in Theorem 2 (see Figure 3), any “+” or “-”-vertex v_j of Γ , any choice of the structure of a local ribbon graph \mathbb{G}_{v_j} on Γ_{v_j} and any embedding $\mathbb{G}_{v_j} \hookrightarrow \Gamma$, one can find a flat surface S and a collection γ of homologous saddle connections on it such that $\Gamma(S, \gamma) = \Gamma$ and such that the boundary type of the component S_j is represented by the chosen embedded ribbon graph. Moreover, if v_j is represented by a “+”-vertex of Γ , then the parities of boundary singularities of S_j are completely determined by the choice of the embedded ribbon graph.

Conversely, given an abstract graph Γ as in Theorem 2 (see Figure 3), a “+”-vertex v_j of Γ , an abstract local ribbon graph \mathbb{G}_{v_j} , and a choice of the parities of boundary singularities as given in Figure 6 there is a unique way (up to a symmetry of the ribbon graph \mathbb{G}_{v_j}) to embed the local ribbon graph with marked parities into the graph Γ . This unique way is expressed by the dotted lines in Figure 6.

Proof. For “+” and “-”-vertices v_j all possible embeddings of local ribbon graphs as in Figure 5 into the graphs Γ as in Figure 3 are represented in Figure 6. Thus, the first statement follows from Theorem 3. The second statement immediately follows from Theorem 3 combined with Lemmas 4.1 and 4.3. \square

5 Neighborhood of the Principal Boundary: Local Constructions

In this section and in the next one we construct surfaces with boundaries representing all boundary types listed in Figure 6. We first prove the key proposition below. Combining it with some elementary extra arguments we prove Theorem 4 (and, hence, the missing realizability parts of Theorems 2 and 3).

PROPOSITION 6. Consider any configuration \mathcal{C} as in Definition 3, and any vertex v_j of the graph $\Gamma(\mathcal{C})$. Let S'_j be any flat surface from the component $\mathcal{Q}(\alpha'_j)$ (or $\mathcal{H}(\beta'_j)$) of the principal boundary stratum $\mathcal{Q}(\alpha')$ (or $\mathcal{H}(\beta')$) corresponding to v_j . Choose any sufficiently small value of a complex parameter δ (depending on S'_j).

Applying to S'_j an appropriate basic surgery (depending on δ) as described below one gets a surface S_j with boundary, such that the boundary type of S_j and the collections of interior singularities and of boundary singularities of S_j are represented by the local ribbon graph \mathbb{G}_{v_j} and by the corresponding structures $\{d_l\}_j, \{k_{i,l}\}_j$ of the configuration \mathcal{C} .

Recall that the principal boundary stratum corresponding to a “+”-vertex is of type $\mathcal{H}(\beta'_j)$; the principal boundary stratum corresponding to a “-”-vertex is of type $\mathcal{Q}(\alpha'_j)$. The singularity data β'_j, α'_j are defined by equations (6) and (7) correspondingly. Unlike the initial singularity data α the collections β'_j and α'_j might contain entries “0” representing marked points of the surface S'_j .

Though the principal boundary stratum corresponding to a “o”-vertex is empty, Proposition 6 is not meaningless (though very simple) even for such vertices. We leave the construction of surfaces S_j with boundary realizing each of o2.2, o3.2, o4.2-boundary types to the reader as an elementary exercise; see Figure 10.

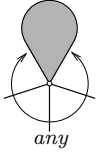
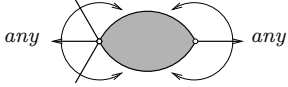
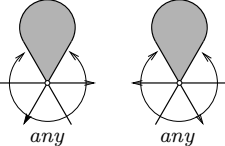
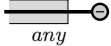
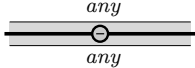
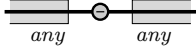
-1.1	-2.1	-2.2
		
		

Figure 9: Surfaces with boundary versus local ribbon graphs: “-”-vertices.

We split Proposition 6 into a collection of Propositions 7, 8 and 9. To avoid excessive repetitions we abbreviate the statements of the corresponding propositions; they should be read as the statement of Proposition 6 applied to vertices of specified types.

Part of the surgeries (namely, “breaking up a zero by a local construction” and a “parallelogram construction”) are taken from the paper [EMZ]. For the sake of completeness we present their outline in the current paper. For more details we address the reader to the original paper [EMZ].

5.1 Surfaces with boundary versus local ribbon graphs. Figure 6, Theorem 3 and Proposition 6 are formulated in terms of local ribbon

graphs. By definition they correspond to flat surfaces with boundary. Figures 9–13 explicitly describe this correspondence.

By convention we orient the boundary of a surface as follows. Choose an orthogonal frame $(\vec{n}, \vec{\tau})$ equivalent to a canonical frame in such way that \vec{n} is the external normal, and $\vec{\tau}$ is a vector tangent to the boundary. The vector $\vec{\tau}$ defines the orientation of the boundary (see Figure 4).


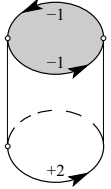
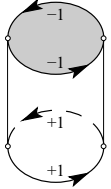
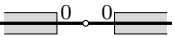
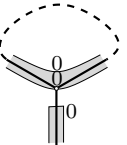
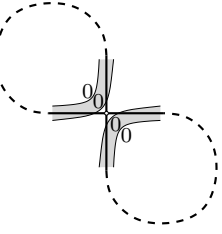
o2.2	o3.2	o4.2
		
		

Figure 10: Surfaces with boundary versus local ribbon graphs: “o”-vertices.

In those pictures in Figures 9–13, which represent surfaces with boundary (and not the ribbon graphs), the shadowed regions represent small holes inside a flat surface. The same remark concerns Figures 15–21.

Our convention on orientation implies that the boundaries of the small holes are oriented *clockwise*. The same convention implies that the edges of the graphs \mathbb{G}_v are oriented *counterclockwise*.

Choosing a line element in the tangent space to some point of a flat surface one can transport this line element to the tangent space at any other point. The resulting distribution defines a foliation. For the surfaces with boundary under consideration the foliation can be chosen parallel to all boundary saddle connections.

+2.1	+2.2

Figure 11: Surfaces with boundary versus local ribbon graphs: “+”-vertices of valence two.

When the flat surface has trivial linear holonomy, the corresponding foliation is orientable. The arrows on the saddle connections in Figures 10–13 represent the orientation of the foliation and *not* the canonical orientation of the saddle connections induced from the canonical orientation of the boundary. We also use this convention for Figures 14–19.

5.2 Local constructions. We reserve the word “degree” for the zeroes of *Abelian* differentials. A zero of degree l has cone angle $\pi(2l + 2)$. We reserve the word “order” for the zeroes of *quadratic* differentials. A zero of order m has cone angle $\pi(m + 2)$. Recall that a boundary singularity of order k has cone angle $\pi(k + 1)$.

We distinguish two kinds of surgeries. The surgeries of the first type are purely local: they do not change the flat metric on S'_j outside a small neighborhood of one or two points on S'_j . The surgeries of the second type depend on a nonlocal construction. In the remaining part of this section we describe local surgeries.

PROPOSITION 7. *Every surface with boundary type +2.1, +3.1, +4.1a, +4.1b, +4.2a is realizable by a local construction.*

We use the indexation of the boundary types as in Figure 6 and in Remark 4 in section 1.3.

+3.1	+3.2a	+3.2b	+3.3

Figure 12: Surfaces with boundary versus local ribbon graphs: “+”-vertices of valence three.

Proof. The principle boundary stratum corresponding to a vertex v_j of a “+”-type has type $\mathcal{H}(\beta'_j)$. The singularity data β'_j is given by equation (6), namely

$$\beta'_j = \{d_1/2, \dots, d_{s(j)}/2, D_1/2, \dots, D_{r(j)}/2\},$$

where $d_1, \dots, d_{s(j)}$ are the orders of interior singularities, and $D_1, \dots, D_{r(j)}$ are expressed in terms of the orders of boundary singularities by formula (5). Conditions 4 and 5 in Definition 3 of a configuration guarantee that all the entries of β'_j are nonnegative integer numbers, and that the total sum of these numbers is even. According to [MS2] this implies that the stratum $\mathcal{H}(\beta'_j)$ is nonempty.

Consider any surface S'_j in $\mathcal{H}(\beta'_j)$. Denote the length of the shortest saddle connection on S'_j by 4ε . We shall apply a surgery to S'_j , which would continuously depend on a small complex parameter considered as a vector \vec{v} in $\mathbb{R}^2 \simeq \mathbb{C}$. It is convenient to change slightly the notation and to denote by δ the *norm* of \vec{v} . We always assume that $\delta < \varepsilon$. Our surgery would not affect interior singularities of S'_j .

We provide all the details of the proof in the case of the boundary type +2.1 and we point out the differences in the other cases.

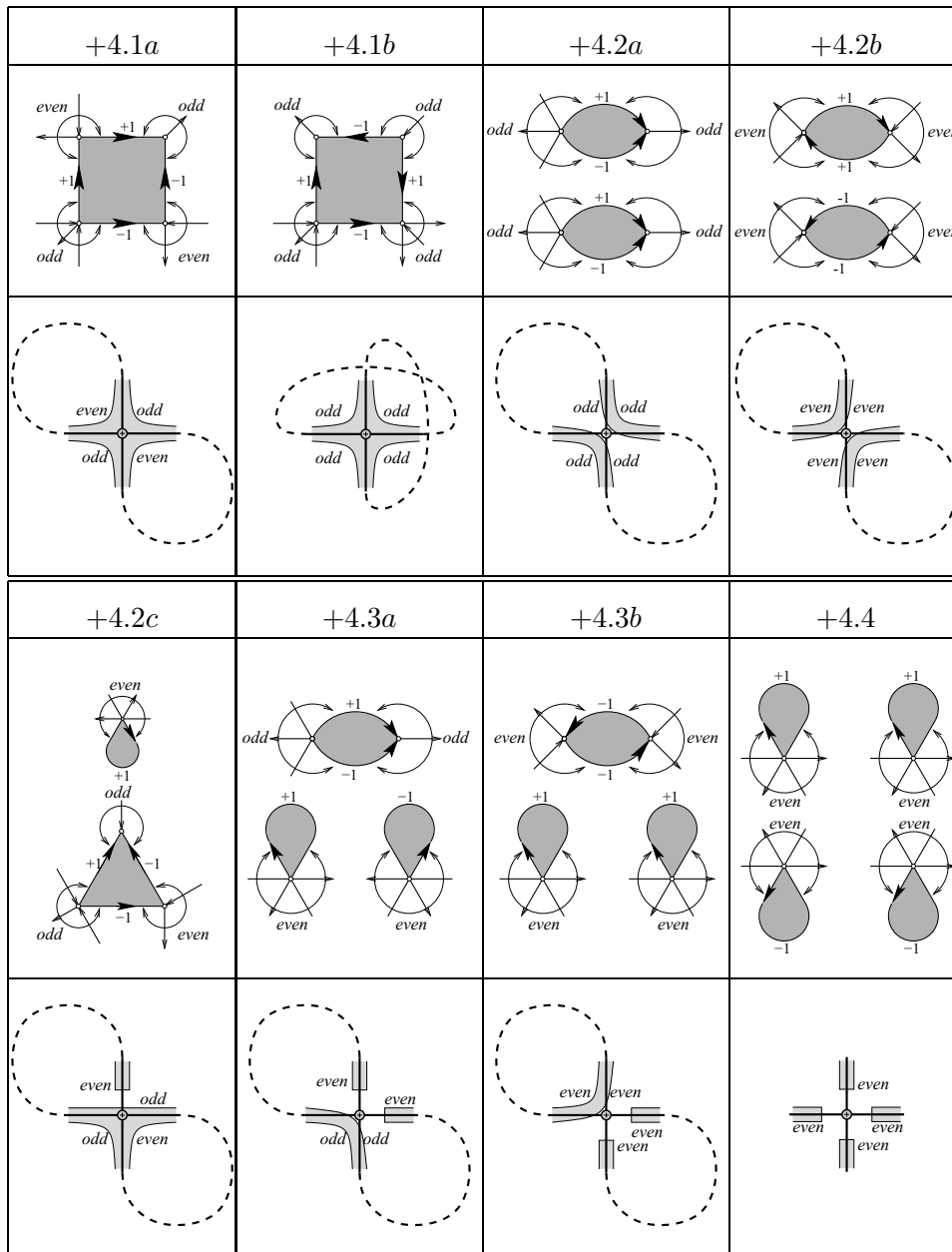


Figure 13: Surfaces with boundary versus local ribbon graphs: “+”-vertices of valence four.

Boundary type +2.1. In this case (see Figure 11) the boundary has single component, $r(j) = 1$, and $D_1 = k_{1,1} + k_{1,2} - 2$, where $k_{1,1}, k_{1,2}$ are the orders of the two boundary singularities of \mathbb{G}_{v_j} . Both $k_{1,1}, k_{1,2}$ are odd positive integers, see Figure 6.

Let P be the zero of S'_j of degree m , where $m = D_1/2$. We can represent m as the sum $m = m' + m''$, where $m' = (k_{1,1} - 1)/2$ and $m'' = (k_{1,2} - 1)/2$. Consider a metric disc of radius ε centered at P . By the choice of ε the disc does not contain any other singularities and is isometrically embedded into S'_j . It can be glued from $2(m + 1)$ copies of standard metric half-discs of radius ε ; see the picture at the top of Figure 14. Let $\vec{v} \in \mathbb{R}^2$ be a vector of length $\delta < \varepsilon$. Following [EMZ] we may *break up* the zero P of degree m into a pair of zeroes of degrees m' and m'' joined by a single saddle connection with affine holonomy \vec{v} .

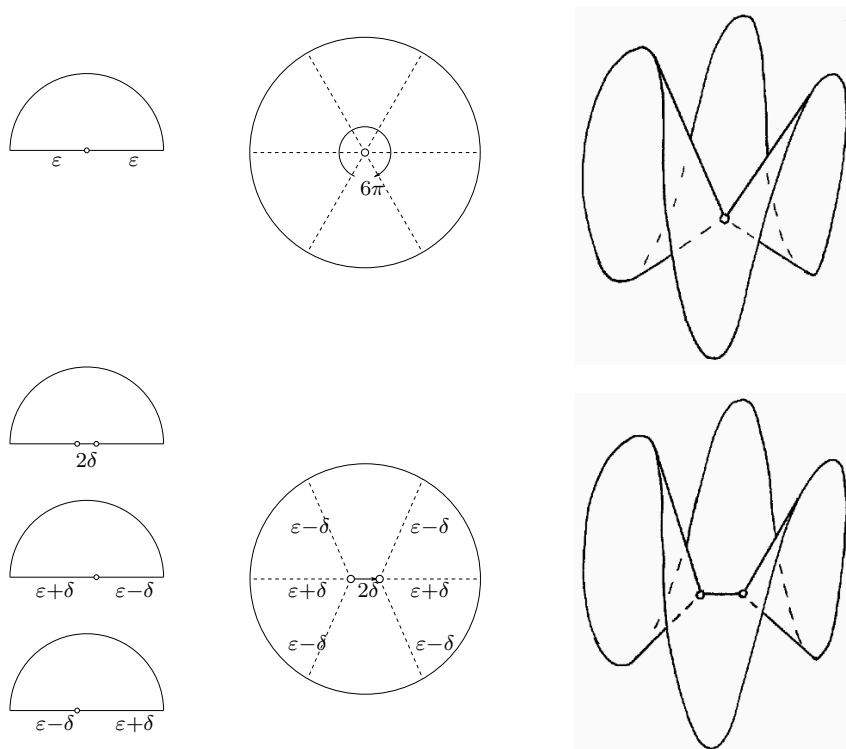


Figure 14: Breaking up a zero into two zeroes (after [EMZ]).

We do this by changing the way of gluing the half-discs as indicated on the bottom picture of Figure 14. As patterns we still use the standard

metric half-discs, but move the marked points on their diameters. Two special half-discs have two marked points on the diameter at distance δ from the center. Each of the remaining $2m$ half-discs has a single marked point at distance δ from the center. We alternate the half-discs with the marked point moved to the right and to the left of the center. The picture shows that all the lengths along identifications match; gluing the half-discs we obtain a topological disc with a flat metric. Now the flat metric has two cone-type singularities with cone angles $2\pi(m' + 1)$ and $2\pi(m'' + 1)$. Here $2m'$ and $2m''$ are the numbers of half-discs with one marked point glued in between the distinguished pair of half-discs with two marked points.

The case when one of m', m'' (or both of them) is equal to zero is not excluded, in this case the corresponding “newborn” singularity is just a marked point.

Note that a small annular neighborhood of the boundary of the initial disc is isometric to the corresponding annular neighborhood of the boundary of the deformed disc. Thus, we can glue the deformed disc back into the surface. Gluing back we can turn it by any angle φ , where $0 \leq \varphi < 2\pi(m + 1)$ in such way that the newborn saddle connection will have the prescribed affine holonomy \vec{v} .

Making a slit along the resulting saddle connection we get a surface S_j with boundary having prescribed boundary type +2.1, a pair of boundary singularities of prescribed orders $k_{1,1}, k_{1,2}$, and a collection of interior singularities of prescribed orders $d_1, \dots, d_{s(j)}$ (see Figure 11). We have completed the proof of Proposition 7 for the boundary type +2.1.

Boundary types +3.1 and +4.1a. Boundary type +3.1 can be considered as a particular case of boundary type +4.1a. To see this compare the surfaces with boundary representing the corresponding ribbon graphs (see the appropriate entries in Figures 12 and 13). Marking a point in the middle of the saddle connection labelled by “+2” on the boundary of the surface of type +3.1 we get a surface with boundary type +4.1a, where the boundary singularity joining the pair of edges labelled by “+1” has order 0.

Consider a local ribbon graph of type +4.1a, a collection $\{2m_1, \dots, 2m_n\}$ of orders of interior singularities and a collection $\{2a_1 + 1, 2a_2, 2a_3 + 1, 2a_4\}$ of orders of four boundary singularities (see Figure 6 for their parities). The singularity data β'_j of the corresponding component $\mathcal{H}(\beta'_j)$ of the principal boundary stratum has the form $\beta' = \{m_1, \dots, m_n, a_1 + a_2 + a_3 + a_4\}$, see equations (5) and (6).

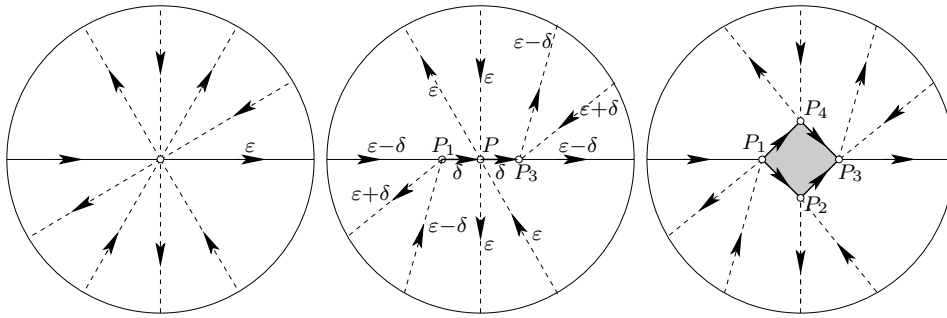


Figure 15: Breaking up a zero into three ones and performing a surgery we get a surface of type “+4.1a”.

Choose an Abelian differential $S'_j \in \mathcal{H}(\beta')$. As before denote the length of the shortest saddle connection on S'_j by 4ε . This time we split the distinguished zero P of degree $a_1 + a_2 + a_3 + a_4$ into *three* zeroes P_1, P, P_3 such that the zero P_1 of degree a_1 is joined to the zero P of degree $a_2 + a_4$ by a saddle connection, and the zero P of degree $a_2 + a_4$ is joined to the zero P_3 of degree a_3 by a saddle connection, see Figure 15. The two saddle connections have the same holonomy vector \vec{v} . We assume as before that $\|\vec{v}\| = \delta < \varepsilon$. We then cut along both saddle connections and detach the zero P into two boundary singularities P_2, P_4 of orders $2a_2$ and $2a_4$ correspondingly, getting a surface S_j with boundary of desired geometric combinatorial type (see Figures 12 and 13).

Boundary type +4.1b. Consider a local ribbon graphs of type +4.1b (see Figure 13) and a corresponding collection $\{2m_1, \dots, 2m_n\}$ of orders of interior singularities and a collection $\{2a_1 + 1, 2a_2 + 1, 2a_3 + 1, 2a_4 + 1\}$ of orders of four boundary singularities.

The singularity data β'_j of the corresponding component $\mathcal{H}(\beta'_j)$ of the principal boundary stratum has the form $\beta' = \{m_1, \dots, m_n, a_1 + a_2 + a_3 + a_4 + 1\}$, see equations (5) and (6).

Choose an Abelian differential $S'_j \in \mathcal{H}(\beta'_j)$; let P be a zero of S'_j of degree $(a_1 + a_2 + a_3 + a_4 + 1)$. We split P into three zeroes P_1, P, P_3 such that the zero P_1 of degree a_1 is joined to the zero P of degree $a_2 + a_4 + 1$ by a saddle connection with a holonomy vector \vec{v} and the zero P_3 of degree a_3 is also joined to P by a saddle connection with the same holonomy vector \vec{v} , see Figure 16. Note that the new saddle connections are oriented differently than in the previous case. We then cut along both saddle connections and

detach the zero P into two boundary singularities P_2, P_4 of orders $2a_2 + 1$ and $2a_4 + 1$ correspondingly.

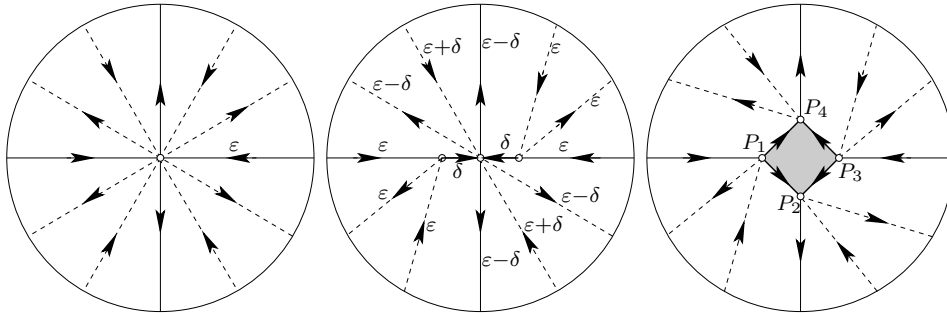


Figure 16: Breaking up a zero into three ones and performing a surgery we get a surface of type “+4.1b”.

By construction the resulting surface S_j with boundary has the desired boundary type “+4.1b” (see Figure 13), and collections of interior and boundary singularities of prescribed orders.

Boundary type +4.2a. Consider a local ribbon graphs of type +4.2a (see Figure 13) and corresponding collection $\{2m_1, \dots, 2m_n\}$ of orders of interior singularities and collections $\{2a' + 1, 2a'' + 1\}, \{2b' + 1, 2b'' + 1\}$ of orders of two pairs of boundary singularities (see Figure 6 for their parities). The singularity data β'_j of the corresponding component $\mathcal{H}(\beta'_j)$ of the principal boundary stratum has the form $\beta' = \{m_1, \dots, m_n, a' + a'', b' + b''\}$, see equations (5) and (6).

Choose an Abelian differential $S'_j \in \mathcal{H}(\beta'_j)$; let P_1 be a zero of S'_j of degree $a' + a''$; let P_2 be a zero of degree $b' + b''$. As in the case +2.1 we break each of the distinguished zeroes P_1, P_2 into a pair of zeroes of degrees a', a'' and b', b'' correspondingly. We apply the surgery in such way that each of the two corresponding pairs of zeroes is joined by a saddle connection with a holonomy vector \vec{v} . We then cut open the modified flat surface along the saddle connections. As a result we get a surface S_j with boundary of the desired boundary type “+4.2a” (see Figure 13) and with collections of interior and boundary singularities of prescribed orders.

Proposition 7 is proved. □

6 Neighborhood of the Principal Boundary: Nonlocal Constructions

Recall that a direction $\pm v \in \mathbb{R}^2 \setminus \{0\}$ determines a corresponding line field on the flat surface and a foliation in direction v . The foliation is orientable if and only if S has trivial linear holonomy. Such an auxiliary direction v is an element of all our constructions. We are creating surfaces with boundary from closed flat surfaces; the direction v is the direction of parallel geodesic segments which form the boundary components.

An interior singularity P of order d has $d+2$ adjacent separatrix rays (or just separatrices) of the foliation in direction v . They divide a disc of small radius ε centered at P into $d+2$ sectors, each with cone angle π (see the top part of Figure 14 which represents a singularity of order 4). When P is a regular point (a marked point) we still have two such adjacent sectors, each having cone angle π . When P is a simple pole, we have a single separatrix adjacent to P ; cutting an ε -neighborhood of P by this separatrix we get a single sector. When we speak about “sectors” adjacent to a singularity we always mean the sectors bounded by a pair of neighboring separatrices of the foliation in direction v .

When a flat surface has trivial linear holonomy, the foliation parallel to v is oriented by the choice of direction \vec{v} . The separatrix rays adjacent to any point P inherit the natural orientation: incoming and outgoing rays alternate with respect to the natural cyclic order on the collection of rays adjacent to P . The sectors adjacent to any singularity P are also naturally divided into two classes: the ones which are located to the right of the corresponding oriented separatrix rays and the ones which are located to the left. We shall refer to them as to the “right” and to the “left” sectors correspondingly.

In all nonlocal constructions we shall use a surgery along a smooth path without self-intersections joining a pair of singularities of a compact flat surface (sometimes joining a singularity to itself). This path ρ (two paths in some constructions) will be always chosen to be transverse to the direction v (and hence, transverse to the foliation in direction v); in particular, ρ never passes through singularities. We shall often call such path a “transversal”.

The following theorem from [HM] gives us a key instrument for all nonlocal constructions:

Theorem (Hubbard–Masur). *Consider a closed flat surface S with non-trivial linear holonomy, a pair of points P_1, P_2 on S , a direction $\pm v \in \mathbb{R}^2 \setminus \{0\}$ and a pair of sectors Σ_i adjacent to the corresponding points P_i , $i = 1, 2$.*

For any such data there exist a transversal ρ with the endpoints at P_1 and P_2 which leaves P_1 in Σ_1 and arrives at P_2 in Σ_2 . The case when P_1 and P_2 coincide, or even when Σ_1 and Σ_2 coincide is not excluded.

If S has trivial linear holonomy the statement above is valid under additional assumption that one of the sectors is a “right” sector, and the other one is a “left” sector.

6.1 Parallelogram construction. In this section we extend the “parallelogram construction” from [EMZ] to flat surfaces with nontrivial linear holonomy. For more details (including restrictions on the choice of parameter δ in terms of the length 4ε of the shortest saddle connection on S , and generalization of the “parallelogram construction” to piecewise-transverse paths) we address the reader to the original paper [EMZ] and to the forthcoming paper [B2].

Consider a transversal ρ as in the theorem above. In the construction below, we assume that if S has trivial linear holonomy, then $P_1 \neq P_2$. If S has nontrivial linear holonomy, then we allow $P_1 = P_2$ unless P_1 is a singularity of order -1 . If $P_1 = P_2$ we allow $\Sigma_1 = \Sigma_2$.

Fix the orientation of ρ from P_1 to P_2 . Since the path is smooth, it has well-defined tangent directions $\vec{u}_1 = \dot{\rho}|_{P_1}$ and $\vec{u}_2 = \dot{\rho}|_{P_2}$ at the endpoints. If the surface has trivial linear holonomy we assume that the frame $\{\vec{u}_i, \vec{v}\}$ represents the canonical orientation (upon interchanging, if necessary, the ordering of P_1, P_2).

For some interior point $P \in \rho$ let $\vec{u} = \dot{\rho}|_P$ be the vector tangent to ρ . Chose a vector $\vec{v} \in T_P(S)$ at P parallel to v such that the frame \vec{u}, \vec{v} represents the canonical orientation of the surface. Perform a parallel transport of \vec{v} along ρ to all points of ρ .

For a sufficiently small $\delta > 0$ and any positive $s \leq \delta$ we can construct a *parallel shift* ρ_s of ρ in direction \vec{v} at the distance s . Suppose that Σ_1, Σ_2 do not coincide nor are adjacent. Then for any $0 \leq s_1 < s_2 \leq \delta$ the corresponding shifts ρ_{s_1} and ρ_{s_2} do not intersect and do not have self-intersections. If Σ_1 and Σ_2 coincide or are adjacent, the same is true upon an appropriate choice of orientation of ρ .

Let γ_i be a segment of the separatrix ray in direction \vec{v} at P_i of length δ . Even when $P_1 = P_2$ (in particular, when $\Sigma_1 = \Sigma_2$) the segments $\gamma_1 \neq \gamma_2$ are well defined. The interior of the domain Ω bounded by $\rho, \rho', \gamma_1, \gamma_2$ is homeomorphic to an open disc and can be thought of as a “curvilinear parallelogram”, see Figure 17.

Remove Ω from S and identify ρ and ρ' by a parallel translation. When $P_1 \neq P_2$, as a result of this surgery we get a surface with two boundary

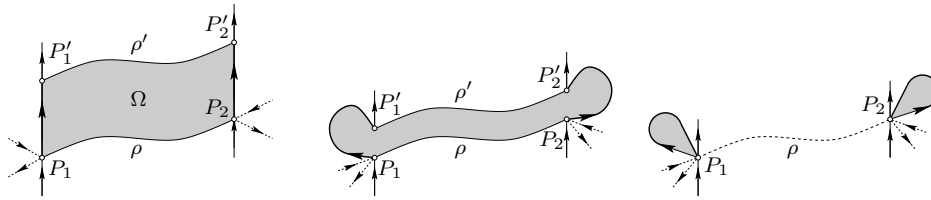


Figure 17: “Parallelogram construction”.

components each with a single singular point, see Figure 17. When $P_1 = P_2$ we detach the resulting boundary singularity into two getting a surface with a single boundary component with two boundary singularities, see Figure 21. We refer to this surgery as to the *parallelogram construction*.

If the parallelogram construction is applied to a pair of distinct points $P_1 \neq P_2$, let D_i be the order of the corresponding singularity $P_i \in S$, $i = 1, 2$. In the case when $P_1 = P_2$, let $\pi(a_1 + 1)$ be the angle between γ_1 and γ_2 counted in the positive direction, and let $\pi(a_2 + 1)$ be the angle between γ_1 and γ_2 counted in the negative direction. By construction $a_i \geq 0$, $i = 1, 2$. The order of the singularity P in this case is $a_1 + a_2$.

In this notation, the orders of the boundary singularities of a surface obtained by a parallelogram construction are equal to $\{D_1 + 2\}, \{D_2 + 2\}$, when $P_1 \neq P_2$ and to $\{a_1, a_2 + 2\}$ when $P_1 = P_2$. To see this, when $P_1 \neq P_2$ it is sufficient to observe Figure 17; in the remaining case it is sufficient to observe Figure 21.

6.2 Nonlocal surgeries. The remaining constructions are a combination of one of the local constructions described in the previous section with a parallelogram construction. The parameters ε, δ are chosen as before.

PROPOSITION 8. *Every surface of any of boundary types +2.2, +3.2a, +4.2b, +3.2b, +4.2c, +4.3a, +3.3, +4.3b, +4.4 is realizable by a combination of a local construction with a parallelogram construction.*

Proof. Applying the same arguments as in the beginning of the proof of Proposition 7 we check that the singularity data β' defined by equation (6) from formal combinatorial data $(\mathbb{G}_{v_j}, \{d_1, \dots, d_s\}, \{k_{1,1}, \dots, k_{r,p(r)}\})$ as in Proposition 8 represents a nonempty stratum $\mathcal{H}(\beta'_j)$. Having a closed flat surface $S'_j \in \mathcal{H}(\beta'_j)$ we now need to construct a surface S_j with boundary realizing the initial combinatorial data $(\mathbb{G}_{v_j}, \{d_1, \dots, d_s\}, \{k_{1,1}, \dots, k_{r,p(r)}\})$.

Boundary type +2.2. We begin with boundary type +2.2 (see Figure 11). All interior singularities $\{2m_1, \dots, 2m_s\}$ have positive even orders;

each of the two boundary components contains a single boundary singularity. The boundary singularities also have positive even orders $2m', 2m''$ (see Figure 6 and condition (4) of Definition 3 of a configuration), so in this case $\beta' = \{m_1, \dots, m_s, m' - 1, m'' - 1\}$.

Choosing an Abelian differential $S'_j \in \mathcal{H}(\beta')$ and performing the parallelogram construction at the zeroes of degrees $m' - 1, m'' - 1$ (see Figure 17) we get a flat surface S_j with boundary of type “+2.2” (see Figure 11), having collections of interior and of boundary singularities of prescribed orders.

Boundary types +3.2a and +4.2b. Boundary type +3.2a can be considered as a particular case of +4.2b when one of the boundary singularities has order 0 (see the appropriate entries in Figures 12 and 13).

Consider a ribbon graph of type +4.2b. Let $\{2m_1, \dots, 2m_s\}$ be a collection of orders of interior singularities. According to Figure 6 the orders of all boundary singularities are even for boundary type +4.2b; denote by $2a_1, 2a_2 + 2$ the orders of boundary singularities corresponding to the first boundary component and by $2a_3, 2a_4 + 2$ the orders of boundary singularities corresponding to the second component. By condition (4) of Definition 3 of a configuration the numbers a_i are nonnegative integers for $i = 1, \dots, 4$. We get $\beta' = \{m_1, \dots, m_s, a_1 + a_2, a_3 + a_4\}$.

Choose a flat surface $S'_j \in \mathcal{H}(\beta'_j)$. Choose a pair of separatrices γ_1, γ_2 in direction \vec{v} adjacent to the first zero. Choose γ_1 to be an outgoing separatrix and γ_2 to be an incoming separatrix in such way that the angle from the separatrix ray γ_1 to the separatrix ray γ_2 in the clockwise direction is $(2a_1 + 1)\pi$. Let Σ_1 be the sector adjacent to γ_1 counterclockwise; let Σ_2 be the sector adjacent to γ_2 clockwise.

Similarly, choose a pair of separatrices γ_3, γ_4 in direction \vec{v} adjacent to the zero of degree $a_3 + a_4$ in such a way that γ_3 is outgoing, γ_4 is incoming; the counterclockwise angle from γ_3 to γ_4 is $(2a_3 + 1)\pi$. Let Σ_3 be the sector adjacent to γ_3 clockwise; let Σ_4 be the sector adjacent to γ_4 counterclockwise.

Join Σ_3 to Σ_1 by a transversal ρ_1 ; join Σ_2 to Σ_4 by a transversal ρ_2 . If ρ_1 intersects ρ_2 we can resolve the intersections to achieve nonintersecting transversals.

Suppose that in resolving the intersections (if any) we did not change the correspondence between the sectors and ρ_1 still joins Σ_3 to Σ_1 and ρ_2 joins Σ_2 to Σ_4 . Choosing some small δ we can apply parallelogram construction to the transversal ρ_1 and the direction \vec{v} and to the transversal ρ_2 and the direction $-\vec{v}$, see Figure 18.

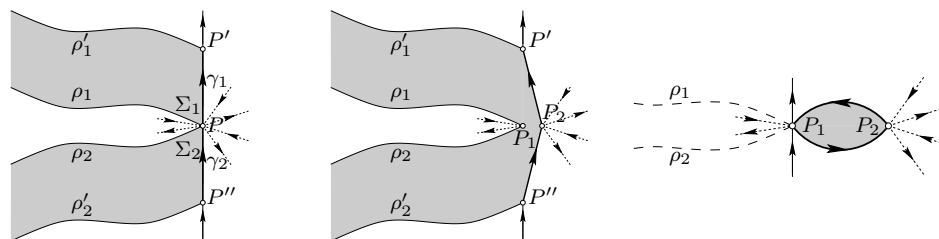


Figure 18: A pair of simultaneous parallelogram constructions.

In the remaining case after the resolution of intersections, the correspondence between the sectors changes and the transversal ρ_1 joins sector Σ_4 to Σ_1 while the transversal ρ_2 joins Σ_2 to Σ_3 . In this case we deform the transversals slightly in such way that they still do not intersect and ρ_2 lands on the ray γ_3 at a distance δ from the zero (in the same sector Σ_3) and ρ_1 starts at a point on the ray γ_4 at a distance δ from the zero (in the same sector Σ_4). We can construct two “curvilinear parallelograms” Ω_1, Ω_2 (see Figure 18) which do not intersect, so we can proceed as above.

Detaching each of the resulting singularities into pairs P_1, P_2 and P_3, P_4 (see Figure 18) we get the desired surface S_j with boundary of type “+4.2b” (see Figure 13) and with prescribe collections of interior and of boundary singularities.

REMARK 7. Recall that if we identify the opposite sides of each hole of a surface constructed above we obtain a closed surface with a pair of even order zeroes simultaneously broken up into a pair of odd order zeroes.

Boundary types +3.2b and +4.2c. Boundary type +3.2b can be considered as a particular case of boundary type +4.2c. To see this compare the surfaces with boundary representing the corresponding ribbon graphs (see the appropriate entries in Figures 12 and 13). Marking a point in the middle of the saddle connection labelled by “+2” on the boundary of the surface of type +3.2b we get a surface of boundary type +4.2c with the corresponding boundary singularity of order 0.

Consider a ribbon graph of type +4.2c. Let $\{2m_1, \dots, 2m_s\}$ be a collection of orders of interior singularities. Let $2a_1 + 1, 2a_2, 2a_3 + 1$ be the orders of the boundary singularities on the boundary component composed from three saddle connections; let $2a_4 + 2$ be the order of the single boundary singularity on the complementary boundary component, see Figure 6. By

condition (4) of Definition 3 of a configuration the numbers a_i are nonnegative integers for $i = 1, \dots, 4$. We get $\beta' = \{m_1, \dots, m_s, a_1 + a_2 + a_3, a_4\}$.

Choose a flat surface $S'_j \in \mathcal{H}(\beta'_j)$. Let P be the zero of degree $a_1 + a_2 + a_3$, and P_4 be the zero of degree a_4 . Choose a separatrix γ_1 in direction \vec{v} adjacent to P_4 and a separatrix γ_2 in direction \vec{v} adjacent to P . Let Σ_1 be the “right” sector adjacent to γ_1 ; let Σ_2 be the “left” sector adjacent to γ_2 , see Figure 19. Join Σ_1 to Σ_2 by a separatrix ρ . Choose δ small enough, so that the intersection of ρ with an δ -neighborhood of P contains a single connected component contained in Σ_2 .

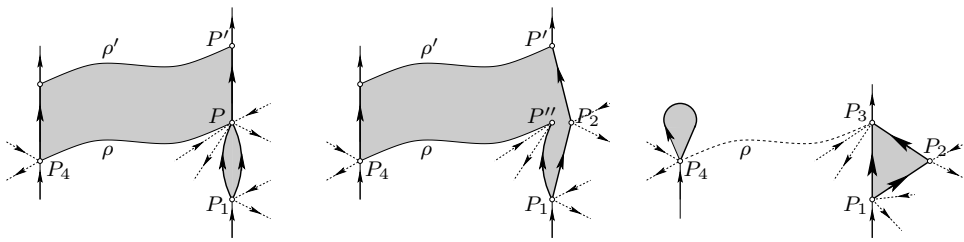


Figure 19: Parallelogram construction combined with breaking up a zero creates a surface of type $+4.2c$.

Choose a separatrix γ_3 at P such that the angle from γ_2 to γ_3 (in the counterclockwise direction) equals $\pi(2a_3 + 1)$. Break the zero P along γ_3 into two zeroes P and P_1 of degrees $a_2 + a_3$ and a_1 correspondingly joined by a saddle connection in direction \vec{v} of length δ and perform the parallelogram construction along ρ (strictly speaking to a transversal naturally corresponding to ρ), see Figure 19. Detaching P into two points we obtain a surface of type “ $+4.2c$ ” with the desired singularity data (see Figure 13).

Boundary type $+4.3a$. Let $\{2a_1 + 2\}$, $\{2a_2 + 2\}$, $\{2a_3 + 1, 2a_4 + 1\}$ be the orders of the boundary singularities naturally distributed into the corresponding boundary components, see Figures 6 and 13. By condition (4) of Definition 3 of a configuration the numbers a_i are nonnegative integers for $i = 1, \dots, 4$. We get $\beta' = \{m_1, \dots, m_s, a_1, a_2, a_3 + a_4\}$.

Surfaces with boundary of this type are obtained by a trivial combination of a parallelogram construction applied to a pair of distinct zeroes of degrees a_1 and a_2 and by breaking up a zero of degree $a_3 + a_4$ into two zeroes of degrees a_3, a_4 with a subsequent slit along the resulting saddle connection.

Boundary types +3.3 and +4.3b. The boundary type +3.3 (see Figure 12) can be considered as a particular case of the boundary type +4.3b when the appropriate boundary singularity has order 0 (see Figure 13).

Surfaces of type +4.3b can be constructed in complete analogy with surfaces of type +4.2b (see Figure 18) with the only difference that now we choose sectors Σ_3 and Σ_4 at two distinct points.

Boundary type +4.4. In this case the orders of boundary singularities have the form $\{2a_1 + 2, 2a_2 + 2, 2a_3 + 2, 2a_4 + 2\}$; according to condition (4) of Definition 3 of a configuration all numbers a_i are nonnegative integers. Thus, in this case we get $\beta' = \{m_1, \dots, m_s, a_1, a_2, a_3, a_4\}$. To construct a desired surface with boundary of type +4.4 (see Figure 13) it is sufficient to apply a pair of independent parallelogram constructions.

Proposition 8 is proved. \square

6.3 Surfaces with boundary of “–” type. To complete the proof of Proposition 6 it remains to construct surfaces with boundary realizing any combinatorial data $(\mathbb{G}_{v_j}, \{d_1, \dots, d_s\}, \{k_{1,1}, \dots, k_{r,p(r)}\})$ satisfying conditions 2–6 of Definition 3 for local ribbon graphs \mathbb{G}_{v_j} of “–”-types; see also Figure 9.

PROPOSITION 9. *Combinatorial data representing boundary types –2.2, –1.1 and –2.1 are realizable by appropriate surfaces with boundaries.*

(See initial Proposition 6 for the detailed formulation.)

Proof. The component of the principle boundary stratum corresponding to a vertex v_j of “–”-type has type $\mathcal{Q}(\alpha'_j)$. The singularity data α'_j is given by equation (7), namely

$$\alpha'_j = \{d_1, \dots, d_{s(j)}, D_1, \dots, D_{r(j)}\},$$

where $d_1, \dots, d_{s(j)}$ are the orders of interior singularities, and $D_1, \dots, D_{r(j)}$ are expressed in terms of the orders of boundary singularities by formula (5). Conditions 4 and 5 in Definition 3 of a configuration guarantee that all the entries of α'_j are from the set $\{-1, 0, 1, 2, \dots\}$, that the total sum of the entries of α'_j is divisible by 4 and that this sum is greater than or equal to -4 . Moreover, condition 6 in Definition (3) implies that α'_j neither belongs to the exceptional list given by equation (8) below, nor can be obtained from an entry of this list by adding additional elements “0” (see Lemma 6.1 in the next section). According to the results of the paper [MS2] this implies that the stratum $\mathcal{Q}(\alpha'_j)$ is a nonempty.

Consider any flat surface S'_j in $\mathcal{Q}(\alpha'_j)$. We use the same conventions on parameters δ, ε , and v as in the proof of Proposition 7. Applying an

appropriate surgery to the closed surface S'_j we are going to construct a surface S_j with boundary realizing the initial combinatorial data $(\mathbb{G}_{v_j}, \{d_1, \dots, d_s\}, \{k_{1,1}, \dots, k_{r,p(r)}\})$.

Boundary type -2.2 . Boundary type -2.2 (see Figure 9) is constructed in complete analogy to $+2.2$ by a parallelogram construction. Each of the two boundary components contains a single boundary singularity. The boundary singularities have strictly positive orders $k_{1,1}, k_{2,1}$ (see inequality on D_i in condition (4) of Definition 3 of a configuration), so in this case $\alpha' = \{d_1, \dots, d_s, k_{1,1} - 2, k_{2,1} - 2\}$.

Choosing a quadratic differential $S' \in \mathcal{Q}(\alpha')$ and performing the parallelogram construction at the zeroes of orders $k_{1,1} - 2, k_{2,1} - 2$ (see Figure 17) we get a flat surface S_j with boundary of type “ -2.2 ”, having collections of interior and of boundary singularities of prescribed orders.

Boundary types -1.1 and -2.1 . Note next that boundary type -1.1 can be considered as a particular case of boundary type -2.1 when one of the two boundary singularities has order 0 (see Figure 9).

Consider a ribbon graph of type -2.1 . Let $\{d_1, \dots, d_s\}$ be the orders of interior singularities, let $\{k_{1,1}, k_{1,2}\}$ be the orders of boundary singularities. By condition (4) of Definition 3 we have $D_1 \geq -1$, where $D_1 = k_{1,1} + k_{1,2} - 2$, which implies that nonnegative integers $k_{1,1}, k_{1,2}$ cannot be simultaneously equal to zero. Thus, we may assume that $k_{1,1} \geq 1$. We get $\alpha'_j = \{d_1, \dots, d_s, k_{1,1} + k_{1,2} - 2\}$.

Consider a flat surface S'_j in $\mathcal{Q}(\alpha'_j)$. When both $k_{1,1}, k_{1,2}$ are odd we can break up the zero of order $k_{1,1} + k_{1,2} - 2$ into a pair of zeroes of orders $k_{1,1} - 1$ and $k_{1,2} - 1$ as in Figure 14.

When one of $k_{1,1}, k_{1,2}$ is odd and another one is even we can break up the zero of order $k_{1,1} + k_{1,2} - 2$ into a pair of zeroes of orders $k_{1,1} - 1$ and $k_{1,2} - 1$ by a similar construction, see Figure 20. (Recall that by convention a “zero of order -1 ” is a simple pole of the corresponding meromorphic quadratic differential.) Cutting along the saddle connection we obtain the desired surface of type -2.1 with prescribed orders of interior and boundary singularities (see Figure 9).

When both $k_{1,1}, k_{1,2}$ are even, in fact $k_{1,1} \geq 2$ and $k_{1,1} + k_{1,2} - 2 \geq 0$. Let P be the zero of order $k_{1,1} + k_{1,2} - 2$ of the quadratic differential representing the flat surface S'_j . Choose a pair of separatrices γ_1, γ_2 in such way that the angle from γ_1 to γ_2 counted counterclockwise is $\pi(k_{1,2} + 1)$. Let Σ_1 be the sector adjacent to γ_1 in the clockwise direction and Σ_2 be the sector adjacent to γ_2 in the counterclockwise direction. Apply the

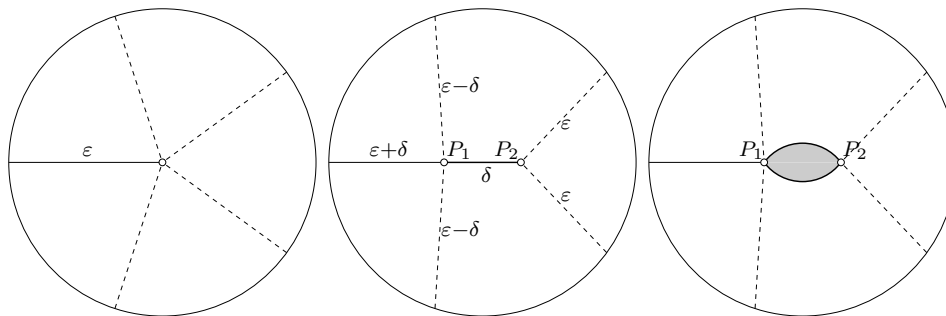


Figure 20: Breaking up a zero of odd order into two zeroes and splitting the saddle connection we get a surface of type -2.1 .

parallelogram construction to Σ_1, Σ_2 and detach P into two singularities P_1, P_2 (see Figure 21). The orders of the boundary singularities of the resulting surface S_j are $k_{1,1}$ and $k_{1,2}$ (see Figure 9).

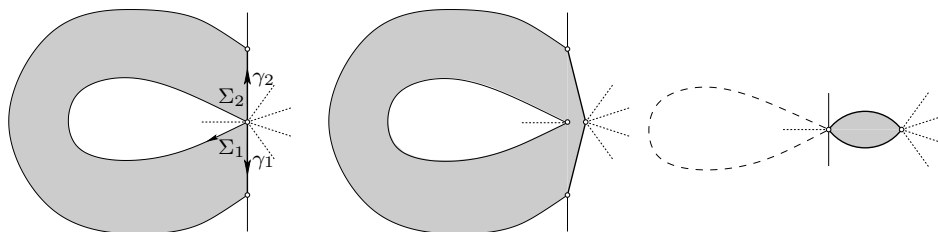


Figure 21: Applying the parallelogram construction to a pair of sectors of the same zero we get the missing surfaces of type -2.1 .

We have completed the proof of Proposition 9 and, thus, the proof of Proposition 6. □

Now we are ready to prove Theorem 4. Note that Theorem 4 immediately implies the missing realizability parts of Theorems 2 and 3.

Proof of Theorem 4. Consider a configuration \mathcal{C} (in the sense of the formal combinatorial Definition 3). Let $\mathcal{Q}(\alpha'_\mathcal{C})$ (resp. $\mathcal{H}(\beta'_\mathcal{C})$) be the principal boundary stratum corresponding to the configuration \mathcal{C} . Let S' be a (possibly nonconnected) flat surface in $\mathcal{Q}(\alpha'_\mathcal{C})$ (resp. $\mathcal{H}(\beta'_\mathcal{C})$). To every connected

component of S' apply the appropriate surgeries as in sections 5 and 6 realizing the corresponding local ribbon graphs. We apply the surgeries in such a way that the saddle connections on the boundary of each surface S_j are, say, horizontal, and have length proportional to their weight in Γ with coefficient δ . For every “o”-vertex of Γ consider an appropriate flat cylinder, with the same requirement for the boundary. Now we glue a compound surface from the components S_j as prescribed by the graph Γ . By construction the result is a closed surface S endowed with a flat metric with linear holonomy restricted to $\{Id, -Id\}$.

By construction each flat surface S_j with boundary is endowed with the canonical orientation. By definition the global ribbon graph $\mathbb{G}(\mathcal{C})$ is endowed with the canonical orientation compatible with the canonical orientation of the embedded local ribbon graphs. This implies that the resulting closed surface S inherits the canonical orientation. By construction S has a collection of saddle connections $\gamma_1, \dots, \gamma_n$ realizing the configuration \mathcal{C} .

It remains to prove that S is nonsingular, i.e. that it does not have any double (triple, ...) points. Suppose it does. Detaching them we get a nonsingular closed flat surface \tilde{S} . By construction \tilde{S} still has a collection of saddle connections $\gamma_1, \dots, \gamma_n$ realizing the configuration \mathcal{C} , which means that assembling the initial surface S we have performed some superfluous identifications of several points of \tilde{S} . \square

6.4 Nonrealizable collections of singularities. It was proved in [MS2] that, for the following exceptional list $\{\alpha''_1, \dots, \alpha''_4\}$ of singularity data

$$\{\emptyset, \{1, -1\}, \{3, 1\}, \{4\}\}, \quad (8)$$

the four corresponding strata $\mathcal{Q}(\alpha''_j)$ are empty. It is clear, that completing any of these lists with entries “0” (which stand for marked points) we also get an empty stratum. This gives rise to restriction 6 in Definition 3 of a configuration which we justify in this section.

Let \mathbb{G}_{v_j} be a local ribbon graph of one of types $-1.1, -2.1, -2.2$ and let $\{d_1, \dots, d_s\}, \{k_{1,1}, \dots, k_{r,p(r)}\}$ be a couple of unordered collections of integers satisfying conditions 4 and 5 of Definition 3 of a configuration. (In our formal combinatorial definition they represent orders of interior and of boundary singularities of a virtual flat surface with boundary.) Applying formally equations (5) and (7) (which evaluate the singularity data of the corresponding component of the virtual principal boundary stratum) to our combinatorial data we obtain an unordered collection α'_j of integers. Consider a collection α''_j obtained from α'_j by omitting all entries “0” (if any).

LEMMA 6.1. *The collection α''_j belongs to the exceptional list (8) if and only if the combinatorial data $(\mathbb{G}_{v_j}, \{d_1, \dots, d_s\}, \{k_{1,1}, \dots, k_{r,p(r)}\})$ as above belongs to the list (6) in Definition 3 of a configuration.*

Proof. The proof of the lemma is an exercise in elementary combinatorics. \square

Having justified the combinatorial obstructions we complete this section with the corresponding geometric lemma.

LEMMA 6.2. *Let S_j be a flat surface with boundary of one of types $-1.1, -2.1, -2.2$. Assume that S_j does not have any saddle connections parallel to the boundary different from those which belong to the boundary. Then the corresponding combinatorial data $(\mathbb{G}_{v_j}, \{\text{unordered collection of interior singularities}\}, \{\text{unordered collection of boundary singularities}\})$ does not belong to the exceptional list (6) in Definition 3 of a configuration.*

Proof. We use the following strategy to prove the lemma. If some surface S_j with boundary would define an entry from the list (6) in Definition 3 we would shrink the boundary of S_j to get as a limit a nondegenerate surface S'_j from the corresponding component $\mathcal{Q}(\alpha'_j)$ of the principal boundary stratum. However, Lemma 6.1 implies that such $\mathcal{Q}(\alpha'_j)$ is empty, which leads to a contradiction.

To complete the proof we need to describe how can one “shrink the boundary” of a flat surface. First note, that boundary type “ -1.1 ” can be considered as a particular case of boundary type “ -2.1 ” when the order of one of the boundary singularities is equal to zero (see the corresponding surfaces with boundary in Figure 9).

Having a surface S_j of type -2.1 we can isometrically identify the pair of boundary components to get a closed flat surface S . The corresponding singularity data α of $S \in \mathcal{Q}(\alpha)$ is expressed in terms of the singularity data of S_j as follows:

$$\alpha = \{d_1, \dots, d_{s(j)}, k_{1,1} - 1, k_{1,2} - 1\}.$$

This implies that the couples $[\{d_1, \dots, d_{s(j)}\}, \{k_{1,1}, k_{1,2}\}]$ of collections of orders of interior and of boundary singularities in the list below

$$\begin{array}{ccc} \emptyset, \{1, 1\} & \emptyset, \{2, 0\} & \\ \{1, -1\}, \{1, 1\}; & \{1\}, \{1, 0\}; & \{-1\}, \{2, 1\} \\ \{3, 1\}, \{1, 1\}; & \{3\}, \{2, 1\}; & \{1\}, \{4, 1\} \\ \{4\}, \{1, 1\}; & \emptyset, \{5, 1\}; & \emptyset, \{4, 2\}. \end{array}$$

are not realizable by any surface S_j with boundary of type -2.1 , for in these cases we would get a flat surface S from an empty stratum, see equation (8).

In the remaining cases we get a closed surface $S \in \mathcal{Q}(\alpha)$ with a distinguished pair of singularities P_0, P_1 joined by a distinguished saddle connection γ . By assumptions of the lemma this saddle connection is not parallel to any other saddle connection on S . This implies that deforming, if necessary, S and then applying an appropriate element of $SL(2, \mathbb{R})$ the surface S can be continuously deformed inside $\mathcal{Q}(\alpha)$ to a surface \tilde{S} with a single *short* saddle connection $\tilde{\gamma}$ and with no other short saddle connections. The deformation might be performed in such a way that the conical singularities \tilde{P}_0, \tilde{P}_1 serving as endpoints of $\tilde{\gamma}$ would have the same cone angles as P_0 and P_1 correspondingly. But then we would apply an appropriate surgery inverse to the one presented in Figures 14, 20 or 21 to coalesce the corresponding pair of zeroes into one. This would give a nondegenerate flat surface S' . Forgetting, if necessary, the resulting marked points on S' we get $S' \in \mathcal{Q}(\alpha'')$, where α'' is in the list (8) (see Lemma 6.1). The latter leads to a contradiction since these strata are empty.

The proof in the case of boundary type -2.2 is completely analogous. Lemma 6.2 is proved. □

Appendices

A Long Saddle Connections

We recall the definition of the natural $GL(2; \mathbb{R})$ -invariant measure in the stratum $\mathcal{Q}(\alpha)$. Let $\hat{P} = p^{-1}(P)$ be the collection of preimages of the singularities of a flat surface $S \in \mathcal{Q}(\alpha)$. Let $H_1^-(\hat{S}, \hat{P}; \mathbb{Z})$ be the subgroup in the relative homology group of \hat{S} , odd with respect to the involution τ . Similarly, let $H_-^1(\hat{S}, \hat{P}; \mathbb{C})$ be the subspace in the relative cohomology odd with respect to the involution τ (i.e. the invariant subspace corresponding to the eigenvalue -1 of the induced linear involution $\tau^* : H^1(\hat{S}, \hat{P}; \mathbb{C}) \rightarrow H^1(\hat{S}, \hat{P}; \mathbb{C})$). We can choose a basis in $H_1^-(\hat{S}, \hat{P}; \mathbb{Z})$ obtained as lifts $\hat{\gamma}_i$, $i = 1, \dots, \dim_{\mathbb{C}} \mathcal{Q}(\alpha)$, of a collection of saddle connections on S . For any surface near S the affine holonomy vectors $\int_{\hat{\gamma}} \omega$ serve as local coordinates for $\mathcal{Q}(\alpha)$. We define a measure $d\nu(S)$ on $\mathcal{Q}(\alpha)$ as Lebesgue measure defined by these coordinates, normalized so that the volume of a fundamental domain of the integer lattice in

$$H_-^1(\hat{S}, \hat{P}; \mathbb{Z} \oplus i\mathbb{Z}) \subset H_-^1(\hat{S}, \hat{P}; \mathbb{C})$$

is equal to one.

REMARK. Note that the Abelian differential ω on \hat{S} has a regular point at the preimage $P'_i \in p^{-1}(P_i)$ of a simple pole P_i of the quadratic differential q on S . Consider the set $\tilde{P} \subseteq \hat{P}$ obtained by removing these regular points. It is easy to see that the canonical homomorphism $H^1_-(\hat{S}, \hat{P}; \mathbb{C}) \rightarrow H^1_-(\hat{S}, \tilde{P}; \mathbb{C})$ induced by the inclusion $\tilde{P} \subseteq \hat{P}$ is actually an isomorphism. Thus, it does not matter which of two sets \hat{P}, \tilde{P} is used to define the coordinate charts.

Proof of Proposition 2. Let \mathcal{C} be an admissible configuration of \hat{h} omologous saddle connections. Let $\gamma = \{\gamma_1, \dots, \gamma_n\}$ be a collection of \hat{h} omologous saddle connections on the flat surface S_0 representing configuration \mathcal{C} . Choose some saddle connection γ_i corresponding to an edge of weight 1 of the graph $\Gamma(S, \gamma)$; such edge always exists, see Figure 3. We associate to the collection γ a pair of vectors $\pm \vec{v}(\gamma) \in \mathbb{R}^2$ setting $v = \int_{\gamma_i} \omega \in \mathbb{C} \cong \mathbb{R}^2$. For every surface S in the same connected component we consider the discrete subset $V_{\mathcal{C}}(S)$ by taking the union $V_{\mathcal{C}}(S) = \cup \pm v(\gamma)$ over all collections of \hat{h} omologous saddle connections γ realizing \mathcal{C} .

It is easy to see that the set $V_{\mathcal{C}}(S)$ satisfies axioms (A), (B), (C_μ) in [EM]. Proposition 2 now follows from the general results in [EM] and from Theorem 4 which implies that the Siegel–Veech constant $const_{\mathcal{C}}$ is nonzero. \square

Proof of Proposition 4. By Remark 1 the cycles $[\hat{\gamma}_1]$ and $[\hat{\gamma}_2]$ are primitive. This implies that if γ_1 and γ_2 are not \hat{h} omologous the homology classes of the lifts $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are independent in $H^1_-(\hat{S}, \hat{P}; \mathbb{Z})$.

If saddle connections γ_1 and γ_2 are parallel, then $\int_{\hat{\gamma}_1} \omega = r \int_{\hat{\gamma}_2} \omega$ for r real. Since the cycles $[\hat{\gamma}_1]$ and $[\hat{\gamma}_2]$ are independent, the above equation holds only for a set of measure zero in $H^1_-(\hat{S}, \hat{P}; \mathbb{C})$. Taking a countable union of sets of measure zero corresponding to possible pairs of integer cycles and different coordinate charts, we see that two non \hat{h} omologous saddle connections on S are parallel only for a set of S of measure zero. \square

Proof of Proposition 1. Suppose that there are two saddle connections γ_1, γ_2 in the collection which are not \hat{h} omologous. Then the corresponding periods $\int_{\hat{\gamma}_1} \omega$ and $\int_{\hat{\gamma}_2} \omega$ correspond to two independent coordinates in a small neighborhood of the initial flat surface, and hence they can be deformed independently. Since the length $|\gamma|$ equals $|\int_{\hat{\gamma}} \omega|$ or $1/2|\int_{\hat{\gamma}} \omega|$ (depending on whether γ is homologous to zero or not), we conclude that a collection containing two non \hat{h} omologous saddle connections cannot be rigid.

The necessity of the condition in Proposition 1 is proved. Sufficiency immediately follows from Lemma 2.6 which says that the lengths of homologous saddle connections are either the same or differ by a factor of two. \square

B List of Configurations in Genus 2

Using Definition 3, Theorem 4 and Corollary 1, and following Examples 4 and 5 in section 1 one can construct a complete list of configurations for any given stratum $\mathcal{Q}(\alpha)$. In this section we present an outline of the algorithm and list all configurations for holomorphic quadratic differentials in genus 2.

There are two natural parameters measuring “complexity” of singularity data $\alpha = \{d_1, \dots, d_m\}$: the genus g of a flat surface S in $\mathcal{Q}(\alpha)$ and the number N of simple poles on S (i.e. the number of conical points with the cone angle π). Having a configuration \mathcal{C} denote by N' the number of interior singularities of order -1 corresponding to this configuration and by g'_1, \dots, g'_k the genera of surfaces S'_1, \dots, S'_k corresponding to the principal boundary $\mathcal{Q}(\alpha'_\mathcal{C})$ (correspondingly $\mathcal{H}(\beta'_\mathcal{C})$ when \mathcal{C} does not have “ $-$ ”-vertices). It is easy to see that the number of simple poles on S (i.e. the number of entries “ -1 ” of α) might vary from N' to $N' + 4$, and that the genus g might vary from $\sum_{j=1}^k g'_j$ to $\sum_{j=1}^k g'_j + 2$ (see [B1] for an explicit expression of $g(S)$ in terms of genera g'_j of components and of a structure of the global ribbon graph). Thus, having fixed the upper bounds for g and N , we confine the list of corresponding configurations to a finite one.

A naive algorithm of enumeration of all configurations for a given stratum $\mathcal{Q}(\alpha)$ can be represented as follows. Let $g = g(\alpha)$ be the genus corresponding to the singularity data α ,

$$d_1 + \dots + d_m = 4g(\alpha) - 4.$$

Consider complete lists of (possibly disconnected) strata $\mathcal{H}(\beta')$ of genera $g-2, g-1, g$. These lists are finite and can be easily constructed. Consider complete lists of (possibly disconnected) strata $\mathcal{Q}(\alpha')$ of genera $g-2, g-1, g$ such that α' contains from $N-4$ to N entries “ -1 ” and at most two connected components α'_i, α'_j representing strata of quadratic differentials $\mathcal{Q}(\alpha'_i), \mathcal{Q}(\alpha'_j)$ (the remaining connected components are represented by strata of holomorphic differentials $\mathcal{H}(\alpha'_i)$). These lists are also finite and can be easily constructed. Add the empty set to these lists when $0 \leq g \leq 2$.

For every entry $\alpha' = \alpha'_1 \sqcup \dots \sqcup \alpha'_k$ (correspondingly β') as above consider all possible ways to organize the set $\{\alpha'_1, \dots, \alpha'_k\}$ into one of the graphs as in

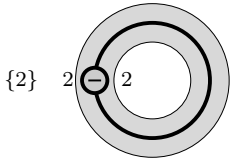
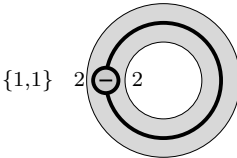
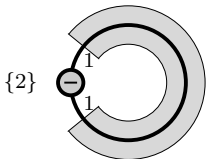
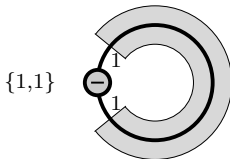
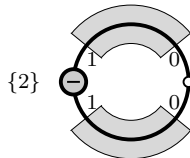
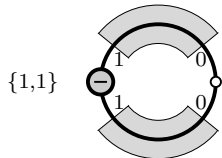
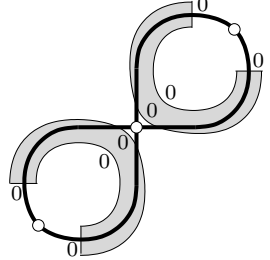
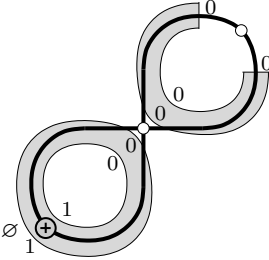
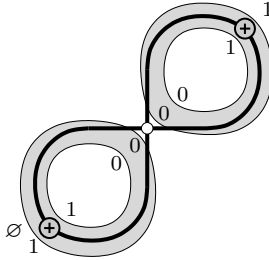
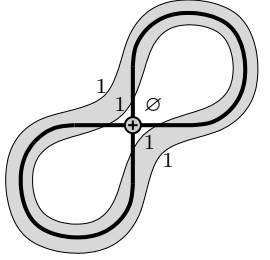
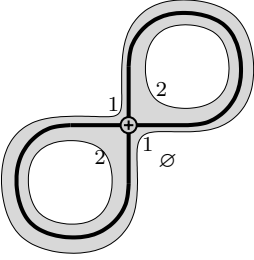
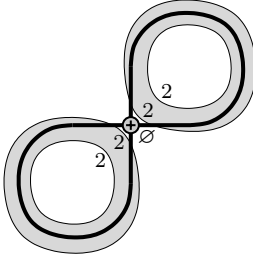
$\mathcal{Q}(2, 2)$	$\mathcal{Q}(2, 1, 1)$	$\mathcal{Q}(1, 1, 1, 1)$
		
		
		
		
		

Figure 22: Configurations of \hat{h} homologous saddle connections for holomorphic quadratic differentials in genus 2.

Figure 3, in such way that vertices corresponding to the strata $\mathcal{H}(\alpha'_j)$, $\mathcal{H}(\beta'_j)$ have “+”-type, and vertices corresponding to the strata $\mathcal{Q}(\alpha'_j)$ have “-”-type. Using these basic graphs, construct all possible “extended” graphs adding vertices of the “o”-type as described in Theorem 2.

For every vertex of every graph as above consider all possible structures of an embedded local ribbon graph as in Figure 6.

At the current stage we have already chosen $\alpha' = \{\alpha'_1, \dots, \alpha'_k\}$ (correspondingly β'), the graph Γ , the bijection of $\{\alpha'_1, \dots, \alpha'_k\}$ (correspondingly $\{\beta'_1, \dots, \beta'_k\}$) with the set of vertices of Γ compatible with the structure of “+” and “-”-vertices, and the structure of a local ribbon graph for every vertex of Γ . Now for every local ribbon graph \mathbb{G}_j representing a “+” or “-”-vertex S_j consider all possible ways to arrange orders of interior singularities and of boundary singularities of S_j in a way compatible with conditions (3)–(6) of Definition 3 and with equation (6) for the corresponding singularity data β'_j (correspondingly equation (7) for the singularity data α'_j).

By “compatibility” with equations (6)–(7) we mean that singularity data computed by these equations should produce β'_j (correspondingly α'_j) possibly completed with several (from 1 to r_j) entries “0” (where r_j is the number of connected components of the local ribbon graph \mathbb{G}_j).

From the resulting lists of configurations extract those which correspond to the required singularity data α .

Certainly this algorithm is not very efficient for large values of g or N . Nevertheless, for strata in small genera having reasonable number of simple poles, it works quite well (especially being slightly optimized using specific properties of given data α).

As an example we present a complete list of configurations of $\hat{\text{homologous}}$ saddle connections for holomorphic quadratic differentials in genus 2. We are grateful to Alex Eskin, who helped us to test completeness of this list.

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