THE WORK OF MARYAM MIRZAKHANI

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On August 13, 2014 (the opening day of ICM at Seoul) Maryam Mirzakhani received the Fields Medal *"for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces"* becoming the first woman to win the Fields Prize. We try to present two groups of results out of numerous results of Maryam. These two groups are quite different in the subject of study (hyperbolic world versus flat one); in the time when they were obtained (Ph.D. thesis versus most recent work); and also in the style (a firework of extremely elegant results and unexpected ties between them obtained individually versus a gigantic work in which took a decade of collaborative efforts aiming to prove one very concrete conjecture).

Courtesy of Maryam Mirzakhani

For a more personal biographical note we recommend the online paper of E. Klarreich [Kl]. We also recommend a short mathematical presentation [McM2] of works of Maryam written by C. McMullen for the ICM Proceedings.

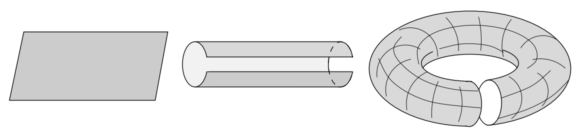
Introduction

Moduli spaces.

We are used to the fact that geometric objects might form continuous families endowed with rich and interesting topology. For example, the family of all straight lines in the plane passing through the origin forms a circle; the family of all m-dimensional vector subspaces V in the vector space $\mathbb{C}^n$ forms the Grassmann manifold $\mathbb{G}_m(\mathbb{C}^n)$.

Since the Grassmann manifold is mentioned anyway, we introduce the notions of tautological bundle and of Chern characteristic classes which we shall need later. The Grassmann manifold is endowed with a natural vector bundle called the *tautological bundle*: its fiber over a "point" $[V] \in \mathbb{G}_m(\mathbb{C}^n)$ is V considered as a vector space. Any m-dimensional complex vector bundle ξ over a compact manifold M can be induced from the tautological bundle by an appropriate map $f_\xi : M \to \mathbb{G}_m(\mathbb{C}^n)$ (for a sufficiently large n depending on M). The cohomology of the Grassmann manifold contains m distinguished elements $c_1, \dots, c_m$. The induced elements $f_\xi^* c_i \in H^*(M;\mathbb{C})$ are called the *Chern characteristic classes* of the vector bundle ξ. They provide extremely important invariants of vector bundles: the maps $f_{\xi'}, f_{\xi''} : M \to \mathbb{G}_m(\mathbb{C}^n)$ corresponding to isomorphic vector bundles $\xi' \sim \xi''$ are homotopic, so the Chern classes of isomorphic vector bundles $\xi' \sim \xi''$ coincide: $f_{\xi'}^* c_i = f_{\xi''}^* c_i$.

Coming back to families of geometric objects, one can consider continuous families of certain geometric structures on a fixed manifold. As an illustration we describe the family of flat metrics on a two-dimensional torus $\mathbb{T}^2$. One can glue a flat torus from a parallelogram identifying the opposite sides of the parallelogram as in the picture. Deforming continuously the parallelogram we get a continuous family of flat tori.



Actually, *any* flat torus can be glued from a parallelogram. Chose a basis of cycles a, b on the torus. For any flat metric on the torus we can find a pair of closed oriented flat geodesics in this metric such that their homology classes would be exactly a and b. Since the two closed geodesics represent a basis (a, b) of cycles, their topological intersection number is equal to one, $a \circ b = 1$. Since our two closed curves are *flat geodesics*, this implies that they have a single intersection. Thus, cutting the torus by this pair of geodesics we unwrap it in into a parallelogram. Note also that for a fixed basis of cycles (a, b) on the torus this parallelogram is defined in a unique way.

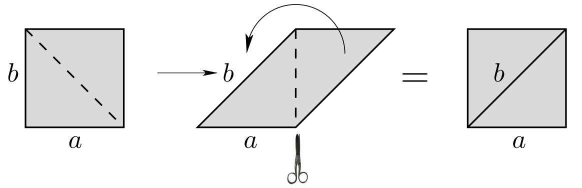
If we are interested in the flat metric on the torus only up to a uniform proportional rescaling of the metric (i.e. if we are interested only in the *conformall class* of the flat metric), the resulting parallelogram is defined up to homothety. The space of all such parallelograms defined up to rescaling can be identified with the upper half-plane: a point (x, y), such that $y > 0$, defines a parallelogram spanned by vectors $\vec{a} = (1, 0)$ and $\vec{b} = (x, y)$, and any parallelogram with pairs of opposite sides marked by letters a and b can be homothetically rescaled to a unique parallelogram like this. The condition $y > 0$ comes from the convention $a \circ b = +1$ for the intersection number of the cycles a and b. Choosing $a \circ b = -1$ we would get $y < 0$.

We have constructed the *Teichmüller space* $\mathcal{T}_1$ of conformal classes of flat metrics on a torus $\mathbb{T}^2$, where the subscript "1" stands for the genus $g = 1$ of the torus. In this very special situation, the Teichmüller space homogeneous: our construction shows that the upper half-plane which we obtained as a model of $\mathcal{T}_1$ can be identified with the left quotient $\mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R}) = \mathbb{H}^2$. Indeed, instead of rescaling proportionally the parallelogram to the unique representative having the a-edge of unit length, we could rescale it to a unique parallelogram of *unit area*. The space of parallelograms of unit area with pairs of opposite edges labeled by symbols a and b is isomorphic to the space of oriented frames of two vectors in $\mathbb{R}^2$ spanning a parallelogram of unit area, where the frames are considered up to a rotation. The latter space is isomorphic to

$$\mathcal{T}_1 \simeq \mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R}) = \mathbb{H}^2\,.$$

Recall now, that our flat tori in the Teichmüller space $\mathcal{T}_1$ carry an extra structure, namely, the choice of the basis of cycles (a, b). The space obtained by forgetting this extra structure is the *moduli space* $\mathcal{M}_1$ of conformal classes of flat metrics on the torus $\mathbb{T}^2$. Thus, by definition, the moduli space is the quotient of the Teichmüller space.

To illustrate the relation between the two spaces consider the torus T_0 defined in our coordinates by the point $(0, 1)$ in the upper half-plane. Let us work in the model of $\mathcal{T}_1$, where the a-cycle is normalized to have the unit length. Our initial torus T_0 is glued from the unit square. Applying a family of liner transformations $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$, where $t \in [0, 1]$, to the unit square, we get a one-parameter family of parallelograms; the corresponding family T_t of flat tori defines a curve in $\mathcal{T}_1$, and thus, in its projection $\mathcal{M}_1$. The picture below shows that the flat tori T_0 and T_1 are



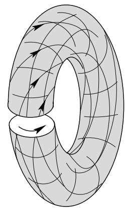
isometric, which means that they represent the same point of the moduli space $\mathcal{M}_1$. Note however, that the b-cycle on the torus T_0 is transformed to the diagonal of the leftmost square under our continuous family of deformations. The isometry between T_0 and T_1 does not send the b-cycle of T_0 to the b-cycle of T_1; the tori T_0 and T_1 endowed with distinguished bases of (a, b) cycles define two distinct points of the moduli space $\mathcal{T}_1$.

Fix a flat metric on the topological torus $\mathbb{T}^2$. The fiber of the projection from the Teichmüller space $\mathcal{T}_1$ to the moduli space $\mathcal{M}_1$ over the chosen point of $\mathcal{M}_1$ corresponds to all possible ways to chose a basis (a, b) in homology $H_1(\mathbb{T}^2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ such that $a \circ b = 1$ (and not -1). A matrix of "change of coordinates" from some fixed basis to any other basis lives in $\mathrm{SL}(2,\mathbb{Z})$; different matrices define different bases, and any matrix defines some new bases. We conclude that the projection $\mathcal{T}_1 \to \mathcal{M}_1$ is an infinite cover, and the group of this cover (called the *mapping class group*) is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$.

Note that any homeomorphism of the topological torus $\mathbb{T}^2$ defines a transformation of any basis of cycles, and that homeomorphisms homotopic to identity keep a basis of cycles unchanged. The mapping class group $\mathrm{Mod}(\mathbb{T}^2)$ of the torus $\mathbb{T}^2$ can be seen as the quotient of the group of all homeomorphisms of the torus quotient over the normal subgroup of homeomorphisms homotopic to identity. This definition of the mapping class group might seem excessively abstract: we quotient one infinite-dimensional group over the other. Its advantage is that it is much more general and can be applied to define moduli spaces of very general geometric structures.

We have to confess that in the discussion above we used the fact that the fundamental group of the torus is commutative, and thus isomorphic to the first homology group, $\pi_1(\mathbb{T}^2) \simeq H_1(\mathbb{T}^2;\mathbb{Z})$. Defining the Teichmüller space of surfaces of genus greater than 1 one uses a collection of simple closed curves or a basis of loops in the fundamental group and not a basis of cycles in homology. Certain homeomorphism of a surfaces of higher genera which are not homotopic to the identity map, act nontrivially on the fundamental group of the surface, but act trivially on the fist homology. (The corresponding subgroup of the mapping class group of a surface is called the *Torelli group*. For the torus $\mathbb{T}^2$ the Torelli group is trivial.)

Consider the closed loop in the moduli space $\mathcal{M}_1$ defined by the one-parameter family of flat tori $\{T_t\}_{t\in[0,1]}$ constructed above. We have seen that its lift to the Teichmüller space is not closed: the basis of cycles (a, b) obtained after the deformation of the flat metric is different from the original basis of cycles. Thus, our lift corresponds to a nontrivial element of the mapping class group. As a homeomorphism of the torus T_0 representing the resulting element of the mapping class group we can choose the *Dehn twist* as in the picture below. We cut the initial torus at the "equator" and then twist progressively the "parallels" of the resulting cylinder; the parallel located at the height t is twisted by the length t. Recall that our torus T_0 is glued from the unit square. Thus, arriving to the top of the cylinder we twist the equator by a complete turn. Identifying the two copies of the equator we get a torus and a well-defined map of the torus T_0 to itself. We suggest to the reader to recognize the Dehn twist in the picture above as well. By construction the Dehn twist maps the basis of cycles (a, b) exactly as the element of the mapping class group corresponding to the closed path $\{T_t\}_{t\in[0,1]}$ in $\mathcal{M}_1$.



In the model of $\mathcal{T}_1$ where we identify the points of $\mathcal{T}_1$ with oriented frames in $\mathbb{R}^2$ defining a parallelogram of unit area (and where we consider the frame up to a rotation) the torus $T_0(a, b)$ endowed with the original basis of cycles (a, b) is represented by the standard orthonormal frame in this model, $a = (1, 0)$, $b = (0, 1)$. We have already seen that the Teichmüller space $\mathcal{T}_1$ can be identified with the upper half-plane consiered as the homogeneous space

$$\mathcal{T}_1 \simeq \mathrm{SO}(2,\mathbb{R})\backslash \mathrm{SL}(2,\mathbb{R}) = \mathbb{H}^2 .$$

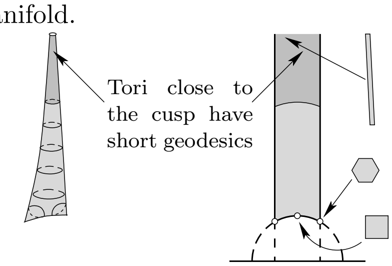
In this realization of $\mathcal{T}_1$ and for the point in the moduli space $\mathcal{M}_1$ represented by the square torus, the action of the mapping class group $\mathrm{Mod}(\mathbb{T}^2) \simeq \mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{T}_1$ is exactly the action of $\mathrm{SL}(2,\mathbb{Z})$ on the right on $\mathrm{SO}(2,\mathbb{R})\backslash \mathrm{SL}(2,\mathbb{R}) = \mathbb{H}^2$. Thus, the moduli space $\mathcal{M}_1$ is isomorphic to the modular surface:

$$\mathcal{M}_1 = \mathcal{T}_1 / \mathrm{Mod}(\mathbb{T}^2) = {}_{\mathrm{SO}(2,\mathbb{R})}\backslash\, \mathrm{SL}(2,\mathbb{R}) /_{\mathrm{SL}(2,\mathbb{Z})} .$$

We have also seen that the mapping class group $\mathrm{Mod}(\mathbb{T}^2)$ of the torus $\mathbb{T}^2$ is defined as the quotient of the group of all homeomorphisms of the torus over the subgroup of homeomorphisms homotopic to the identity map. Choosing a distinguished element in each class of homeomorphisms in an equivariant way we realize the mapping class group $\mathrm{Mod}(\mathbb{T}^2)$ as the group of certain particular transformation of $\mathbb{T}^2$. As such group of particular transformations one can choose the following group. Fix some flat metric on the torus, say the one of T_0. Consider all *linear automorphisms* of the torus T_0, that is the diffeomorphisms $f : T_0 \to T_0$ such that the differential Df is constant in flat coordinates of T_0. To exclude from consideration linear automorphisms isotopic to identity, represented by parallel translations of the torus, let us mark a point on the torus and consider only those linear automorphisms which leave the marked point fixed. The group of linear automorphisms of a flat torus with a marked point is isomorphic to the mapping class group $\mathrm{Mod}(\mathbb{T}^2) \simeq \mathrm{SL}(2,\mathbb{Z})$.

To fit a more general framework, it is convenient to consider the Teichmüller space $\mathcal{T}_1$ of the conformal classes of flat tori and the corresponding moduli space as the Teichmüller space $\mathcal{T}_{1,1}$ and the moduli space $\mathcal{M}_{1,1}$ of tori with a marked point. Now the group of automorphisms of the geometric structure (in our context — isometries of the flat torus preserving the marked point) is finite.

One cannot construct a *hyperbolic* metric on a compact torus. However, it is possible to construct a hyperbolic metric on a *punctured* torus; this hyperbolic metric has a cusp at the puncture. Moreover, for every flat metric one can choose the hyperbolic metric of constant curvature -1 in the same *conformal class* as the original flat metric: infinetesimal circles in one metric remain circles (not ellipses) in the other. Such hyperbolic representative is unique, and the construction can be reversed: any hyperbolic metric of curvature -1 on a once-punctured torus corresponds to some flat metric in the same conformal class. Thus, we can also view the moduli space $\mathcal{M}_{1,1}$ as the moduli space of hyperbolic metrics of constant negative curvature on a torus punctured at one point. More generally, one can consider the moduli space $\mathcal{M}_{g,n}$ of hyperbolic metrics of constant negative curvature on a surface of genus g punctured at n points, were the metric has cusps at the punctured points. These spaces have more dimensions than $\mathcal{M}_{1,1}$ and they are not homogeneous spaces anymore. However, they have certain common properties with $\mathcal{M}_{1,1}$. They are never compact since the metric on the surface might tend to a degenerate one. For example, the modular surface $\mathcal{M}_{1,1}$ has a cusp (see the picture): the tori might get arbitrary narrow and arbitrary long. The moduli spaces $\mathcal{M}_{g,n}$ are not manifolds but *orbifolds* (unless $g = 0$): they have some "conical" loci. For example, $\mathcal{M}_{1,1}$ has two conical points corresponding to the two flat tori which are more symmetric when all other tori (see the picture). However, passing to an appropriate finite cover ramified at the corresponding loci we get a true manifold.



During the last several decades various moduli spaces became one of the central object in mathematics and theoretical physics. Mathematicians are interested in them by various reasons. The moduli space is a *universal family* of the corresponding geometric structures (like the Grassmann manifold, which, in a sense, carries in its tautological bundle all possible vector bundles over all possible manifolds). Moduli spaces themselves are often extremely reach in beautiful geometric structures. And finally, whether you like them or not, you fatally run into the questions related with moduli spaces studying extremely naive objects like graphs or interval exchange transformations which at the first glance have nothing to do with moduli spaces of geometric structures.

Physicists have their own reasons. String theory suggests that at some microscopic level particles might, actually, resemble tiny circles. Then, a short trajectory of such particle-circle is rather a narrow tube than a curve. If in the process of evolution, the circle might break into two circles, then might break into two circles again, then might merge some other circle. A trajectory of such a circle-particle is not a tube anymore but a complicated surface of high genus. Variational approach suggests to compute a *statistical sum* of certain quantities as an integral over the space of all possible trajectories (weighted by their probabilities) which in our case, is the space of all surfaces. In certain important cases, the quantities depend only on the conformal class of the metric on the resulting surface, which allows to reduce the integral over the space of trajectories–surfaces to the integral over the corresponding moduli space.

Theoretical physics continues to develop its opinion on the nature of the relevant moduli space: strings leave place to d-branes, and the moduli spaces of Calabi–Yau manifolds step ahead moduli spaces of Riemann surfaces. Though it is not clear yet whether we live in the moduli space of Calabi–Yau manifolds on in the moduli space of G_2-structures, one can tell for sure that Maryam Mirzakhani has spend many years in the world of hyperbolic and flat surfaces.

1. Hyperbolic World

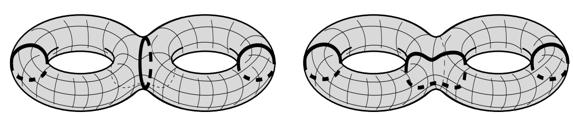
Hyperbolic surfaces.

Consider a smooth surface S of genus g with n holes. We assume that all holes (boundary components) are numbered once and forever. Considering diffeomorphisms of S we assume that the boundary component number i is mapped to the boundary component number i for $i = 1, \dots, n$. A closed curve α on S is called *simple* if it does not have self-intersections. Speaking about *simple closed curves* on a surface S we allways tacitly assume that they are not contractible neither to a point nor to one of the boundary components (if there are any: n is allowed to be zero).

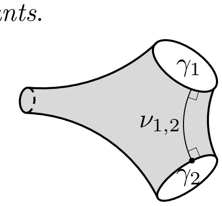
Suppose now that the surface S is endowed with a hyperbolic metric. By convention, we always assume that the boundary components of the resulting *hyperbolic surface* X are realized by geodesics β_i in the hyperbolic metric, where $i = 1, \dots, n$. The hyperbolic lengths of the geodesic boundary components β_i are denoted by $b_i(X)$ or by $L_i(X) = |\beta_i|_X$.

Imagine that a simple closed curve α on X is made from an elastic string, and that the string can contract sliding along X. It is not surprising that the contracted sting would take a form of a closed geodesic in the hyperbolic metric of X. What is not obvious, is that such closed geodesic is unique and does not have self-intersections neither, no matter what hyperbolic metric and what simple closed curve we chose on the original smooth surface S. The hyperbolic length $|\gamma|_X$ of the resulting unique *simple closed geodesic* $\gamma = \gamma(\alpha, X)$ in the free homotopy class of the simple closed curve α is denoted by $\ell_\alpha(X) := |\gamma(\alpha)|_X$.

A topological *pair of pants* is a two-dimensional sphere with three holes. One can cut any topological surface S along an appropriate collection of nonintersecting simple closed curves to get a decomposition of S into pairs of pants. It is clear from the picture that the pants decomposition of S is not unique (unless S is a pair of pants itself). However, the number of simple closed curves α_i in any pants decomposition of S is the same and equals $3g - 3 + n$ (here we do not count the boundary components of S).



Consider now a hyperbolic metric on S and the corresponding hyperbolic surface X. As before let all simple closed curves α_i defining the pants decomposition get contracted to unique simple closed geodesics $\gamma_i = \gamma(\alpha_i, X)$ in the free homotopy classes of the corresponding simple closed curves α_i. Recall that none of γ_i has self-intersections. Moreover, different γ_i, γ_j, $i \neq j$, never intersect either no matter what topological pants decomposition and what hyperbolic metric on S we choose. Thus, the hyperbolic surface X gets decomposed into *hyperbolic pairs of pants*.

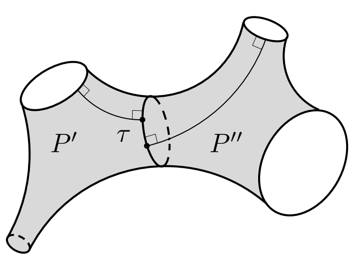


Consider one individual hyperbolic pair of pants $P \subset X$ (as in the picture). Denote by ℓ_k, $k = 1, 2, 3$, the lengths $\ell_k := |\gamma_k|_X$ of hyperbolic geodesics γ_k bounding P. It is known that for any triple of nonnegative numbers $(\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3_+$ there exists a hyperbolic pair of pants $P(\ell_1, \ell_2, \ell_3)$ with geodesic boundaries of given lengths, and that such hyperbolic pair of pants is unique (we always assume that the boundary components of P are numbered). If some of the lengths ℓ_k are equal to zero, the corresponding boundary components are represented by *hyperbolic cusps*.

It is also known that two geodesic boundary components γ_1, γ_2 of any hyperbolic pair of pants P can be joined by a single geodesic segment $\nu_{1,2}$ orthogonal to both γ_1 and γ_2 (see the picture). Thus, every geodesic boundary component γ of any hyperbolic pair of pants might be endowed with a canonical distinguished point. The construction can be extended to the situation, when both remaining boundary components of the pair of pants are represented by cusps.

Having two hyperbolic pairs of pants $P'(\ell_1', \ell_2', \ell)$ and $P''(\ell_1'', \ell_2'', \ell)$ sharing the same length $\ell > 0$ of one of the geodesic boundary components, we can glue such pairs of pants together (see the picture). The hyperbolic metric on the resulting hyperbolic surface Y is perfectly smooth and the common geodesic boundary of $P'(\ell_1, \ell_2, \ell)$ and $P''(\ell_1, \ell_2, \ell)$ becomes a simple closed geodesic γ in the hyperbolic surface Y.

Recall that each geodesic boundary component of any pair of pants is endowed with a distinguished point. These distinguished points record how the pairs of pants P' and P'' are twisted one with respect to another when we glue them together by a common boundary component (see the picture). Hyperbolic surfaces $Y(\tau)$ corresponding to different values of the twist parameter τ in the range $[0, \ell[$ are not isometric.



Let us return to the hyperbolic pants decomposition of the hyperbolic surface X corresponding to a collection of simple closed curves $\alpha_1, \ldots, \alpha_{3g-3+n}$ on the topological surface S. It is clear from what was said above that we can vary all $3g-3+n$ lengths $\ell_{\alpha_i}(X) = |\gamma_i|_X$ of the resulting simple closed geodesics γ_i on X and vary the twists $\tau_{\alpha_i}(X)$ along them to obtain a deformed hyperbolic metric. The resulting collection of $2 \cdot (3g-3+n)$ real parameters serve as *Fenchel–Nielsen coordinates* in the *Teichmüller space* $\mathcal{T}_{g,n}(b_1, \ldots, b_n)$ of boarded hyperbolic surfaces with n geodesic boundary components of lengths $b_1, \ldots, b_n$.

A hyperbolic surface X considered as a point of the Teichmüller space is endowed with a canonical collection of generators of the fundamental group of the underlying smooth surface S. Forgetting this choice of generators we identify many different points of the Teichmüller space into one, getting the *moduli space* $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ of boarded hyperbolic surfaces.

For example, consider a point of the Teichmüller space $\mathcal{T}_{0,4}(b_1, \ldots, b_4)$ represented by the hyperbolic surface $Y(0)$ as in the picture, and the point in the moduli space $\mathcal{M}_{0,4}(b_1, \ldots, b_4)$ obtained by forgetting the basis of cycles on $Y(0)$. Change continuously the twist parameter τ of the hyperbolic surface $Y(\tau)$ from 0 to ℓ, where $\ell = |\gamma|$ is the hyperbolic length of the geodesic γ along which we twist the pairs of pants. The family $\{Y(\tau)\}_{\tau \in [0;\ell]}$ defines a closed loop in the moduli space and a nonclosed path in the Teichmüller space. Indeed, the resulting *Dehn twist* along γ acts nontrivially on the fundamental group of the surface $Y(0)$.

The moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ of boarded hyperbolic surfaces can be viewed as the quotient of the simply-connected Teichmüller space $\mathcal{T}_{g,n}(b_1, \ldots, b_n)$ by the discrete *mapping class group*. The Dehn twist constructed above belongs to the mapping class group of $Y(0)$.

By the work of W. Goldman [G] each moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ carries a natural closed 2-form ω_{WP} called the *Weil–Peterson symplectic form*. S. Wolpert proved in [Wo1] that ω_{WP} has particularly simple expression in Fenchel–Nielsen coordinates. Namely, no matter what pants decomposition do we chose, we get

$$\omega_{WP} = \sum_{i=1}^{3g-3+n} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i} . \tag{1}$$

The wedge power ω^n of a symplectic form on a manifold M^{2n} of real dimension $2n$ defines a volume form on M^{2n}. The volume of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ with respect to the volume form ω_{WP}^{3g-3+n} is called the *Weil–Peterson* volume of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$; it is known to be finite.

Weil–Peterson volumes.
Citing Maryam "our point of departure for calculating the Weil–Peterson volumes of the moduli spaces is the following McShane identity [McS]:"

Theorem (McShane)**.** *Let $f(x) = (1+e^x)^{-1}$ and X be a hyperbolic once-punctured torus. Then we have*

$$\sum_{\gamma} f(\ell_\gamma(X)) = \frac{1}{2},$$

where the sum is over all simple closed geodesics γ on X.

This identity is in some sense a miracle: though the length spectrum of simple closed geodesics is different for different hyperbolic surfaces, the sum above is identically 1/2. When Greg McShane has discovered this identity, it had an effect of a sensational revelation. About ten years after McShane, Maryam Mirzakhani discovered a new identity valid already for any hyperbolic surface X of arbitrary genus with any number n of geodesic boundary components. This new identity (generalizing the McShane's one) has the following structure. Let $L_i = \ell_{\beta_i}(X)$ be the lengths of the hyperbolic boundary components $\beta_1, \dots, \beta_n$. Fix the boundary component β_1 and consider all possible simple closed curves α_1, α_2 such that $\beta_1, \alpha_1, \alpha_2$ bound a pair of pants. Take the sum over all such pairs α_1, α_2 of certain explicit function $\mathcal{D}$ in lengths of geodesics $\beta_1, \gamma(\alpha_1, X), \gamma(\alpha_2, X)$, where, as before, $\gamma(\alpha_i, X)$ is the unique simple closed geodesic in the hyperbolic metric X in the free homotopy class of the simple closed curve α_i. Now consider one by one all other boundary components β_i and all simple closed curves α such that β_1, β_i, α bound a pair of pants. Take a sum of another very explicit function $\mathcal{R}$ in hyperbolic lengths of $\beta_1, \beta_i, \gamma(\alpha, X)$ over all such triples, where $\gamma(\alpha, X)$ is the unique simple closed geodesic in the hyperbolic metric X in the free homotopy class of the simple closed curve α. The identity says that two sums together give the hyperbolic length $L_1 = \ell_{\beta_1}(X)$ of the initial geodesic boundary component β_1 no matter what hyperbolic surface X you take.

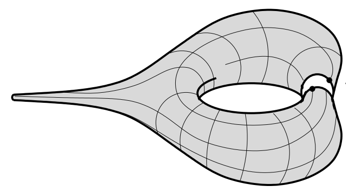
Theorem ([Mi2]). *For any hyperbolic surface X with n geodesic boundary components $\beta_1, \dots, \beta_n$ of lengths $L_1, \dots, L_n$ one has*

$$\sum_{\alpha_1, \alpha_2} \mathcal{D}\left(L_1, \ell_{\gamma_1}, \ell_{\gamma_2}\right) + \sum_{i=2}^{n} \sum_{\alpha} \mathcal{R}\left(L_1, L_2, \ell_\gamma\right) = L_1\,.$$

I have no idea how one might guess this kind of identities!

Now let us discuss why such identities are relevant to the Weil–Peterson volumes of the moduli spaces $\mathcal{M}_{g,n}$. Integrating, the right-hand side of McShane's identity over the moduli space $\mathcal{M}_{1,1}$ with respect to the Weil–Peterson form, one, obviously, gets $\frac{1}{2}\operatorname{Vol}\mathcal{M}_{1,1}$. What is not tautology is that the integral of the sum on the left-hand side admits a geometric interpretation as the integral of f over the natural cover $\mathcal{M}^*_{1,1}$ of the initial moduli space $\mathcal{M}_{1,1}$. This cover is already much simpler than the original moduli space: it admits global coordinates in which the integral of f can be easily computed.

We present this instructive computation. The point of the cover $\mathcal{M}^*_{1,1}$ is a hyperbolic surface X endowed with a simple closed curve. The infinite cover $\mathcal{M}^*_{1,1}$ resembles the Teichmüller space $\mathcal{T}_{1,1}$. Though this cover is smaller than the simply connected universal cover $\mathcal{T}_{1,1}$ it admits global coordinates. Namely, given $(X, \alpha) \in \mathcal{M}^*_{1,1}$ cut X by the unique simple closed geodesic $\gamma(\alpha, X)$ in the free homotopy class of the simple closed curve α. A simple topological argument implies that we always get a hyperbolic pair of pants. By construction, two geodesic boundary components of this pair of pants have the same length $\ell = \ell_\alpha(X) = |\gamma(\alpha)|_X$ and the third boundary component is the cusp (see the picture). Reciprocally, from any hyperbolic pair of pants $P(\ell, \ell, 0)$ we can glue a hyperbolic surface X endowed with a distinguished simple closed curve α. Recall, that gluing such surface we have to take into account the twist parameter τ, see the picture below:



The hyperbolic surfaces X_τ corresponding to values of the twist parameter τ in the interval $[0; \ell[$ are pairwise non isometric, while X_0 and X_ℓ are isometric. Recall also, that X_0 and X_ℓ define the same point of the moduli space $\mathcal{M}_{1,1}$ but the lift of the path $\{X_\tau\}_{\tau \in [0;\ell]}$ is not closed in the Teichmüller space. Note, however, that this path lifts to a closed path in $\mathcal{M}^*_{1,1}$: by construction the Dehn twist sends the free homotopy class of the simple closed curve α to itself. This consideration proves that points in $\mathcal{M}^*_{1,1}$ are in the one-to-one correspondence with the points of the cone $\{\ell > 0; \tau \in \mathbb{R} \bmod \ell\}$. Integrating f in these global Fenchel–Nielsen coordinates on $\mathcal{M}^*_{1,1}$ we get

$$\frac{1}{2}\operatorname{Vol}_{WP}(\mathcal{M}_{1,1}) = \int_{\mathcal{M}_{1,1}} \sum_{\gamma} f(\ell_\gamma(X)) =$$

$$\int_{\mathcal{M}^*_{1,1}} f(\ell_\gamma(X)) = \int_0^\infty f(\ell) \int_0^\ell d\ell\, d\tau =$$

$$\int_0^\infty \ell\, f(\ell)\, d\ell = \int_0^\infty \frac{\ell\, d\ell}{1 + e^\ell} = \frac{\pi^2}{3}\,.$$

In the more general case of Mirzakhani's identity Maryam does not obtain the value of the volume right away. However, cutting the initial surface by simple closed geodesics γ_1, γ_2 for the first sum and cutting the surface by the simple closed geodesic γ in the second sum, and developping the idea of averaging over all possible hyperbolic surfaces, she gets a recursive relation for the volume $V_{g,n}(L_1, \dots, L_n) := \operatorname{Vol}\mathcal{M}_{g,n}(L)$ in terms of analogous volumes in smaller genera. These relations allow Maryam to prove the following statement and to compute the volumes explicitly for all sufficiently small values of g and n.

Theorem ([Mi2]). *The volume $V_{g,n}(L_1,\dots,L_n)$ is a polynomial in $L_1^2,\dots,L_n^2$; namely we have:*

$$V_{g,n}(L)=\sum_{|\alpha|\le 3g-3+n} C_\alpha\cdot L^{2\alpha}\,, \tag{2}$$

where $C_\alpha>0$ lies in $\pi^{6g-6+2n-2|\alpha|}\cdot\mathbb{Q}$.

A simple recursive formula for volumes in genus zero was found earlier by P. Zograf [Zog]. A very precise asymptotics of volumes for large genera was recently proved by M. Mirzakhani and P. Zograf [MiZ] (up to a universal multiplicative constant which is conjecturally equal to 1 and which still resists).

Counting simple closed geodesics.

Consider a hyperbolic surface X of finite area. In this section we do not allow any boundary components anymore, but we still allow to the hyperbolic metric of contant curvature -1 to have cusps. It is known since works of Delsarte, Hubert and Selberg that the growth rate of the number of closed geodesics of length at most L on any such surface X has the rate e^L/L when the bound L grows.

It is not surprising that most of long closed geodesics have self-intersections. A quantitative estimate of "most" is much more substle: the number $s_X(L)$ of *simple* (that is without any self-intersections) closed geodesics of length at most L, grows polynomially in L and not exponentially. More precisely, I. Rivin showed in [R] that for each X there are constants $c_1(X)\le c_2(X)$ such that

$$c_1L^d\le s_X(L)\le c_2L^d\,,$$

where $d=6g-6+2n=\dim_{\mathbb{R}}\mathcal{M}_{g,n}$ (see [R] for the historical perspective of this counting problem).

Counting simple closed geodesics on a surface of constant negative curvature X one can consider the geodesics of certain specific topological type only. For example, one can count separately those simple closed geodesics which separate X into two connected components, and those which do not. (In the picture with two pants decomposition of a surface of genus 2 five simple close curves are non-separating and one is separating.) More generally, two simple closed curves have the same *topological type* if one curve can be transformed to another by a homeomorphism of the punctured surface. For any surface of a fixed genus g with a fixed number n of punctures there is only a finite number of different topological types of simple closed curves. Maryam counted simple closed hyperbolic geodesics type by type.

Theorem ([Mi2]). *For any hyperbolic surface $X\in\mathcal{M}_{g,n}$ the number $s_X(L,type)$ of simple closed geodesics on X of length at most L and of a fixed topological type has exact polynomial asymptotics:*

$$\lim_{L\to+\infty}\frac{s_X(L,type)}{L^d}=n_{type}(X)\,,$$

where $d=6g-6+2n=\dim_{\mathbb{R}}\mathcal{M}_{g,n}$.

Moreover, n_{type} is a continuous proper function on $\mathcal{M}_{g,n}$ having the following structure:

$$n_{type}(X)=\frac{c(type)\cdot B(X)}{b_{g,n}}\,, \tag{3}$$

where $b_{g,n}=\int_{\mathcal{M}_{g,n}}B(X)\,dX<\infty$.

For geometers we add that the quantity $B(X)$ in (3) is the volume of the "unit ball" in the space of *measured laminations* $\mathcal{ML}_{g,n}$ where the "unit ball" is defined in terms of the hyperbolic metric on X and its volume is computed in terms of the Thurston measure on $\mathcal{ML}_{g,n}$.

The coefficient $n_{type}(X)$ in the polynomial asymptotics of $s_X(L,type)$ depends on the surface X. Note, however, that if we compute the proportions of simple closed geodesics of fixed topological types among all simple closed geodesics of bounded length on a fixed hyperbolic surface X, asymptotically such proportions do not depend on X anymore. For example, for a surface of genus two without cusps, consider simple closed geodesics which separate the surface into two pieces (we call such simple closed geodesics "of $type_2$") and those which do not separate the surface (we call them "of $type_1$"). The Theorem implies that asymptotically the ratio below does not depend on the hyperbolic surface X in $\mathcal{M}_2$ anymore:

$$\lim_{L\to+\infty}\frac{s_X(L,type_1)}{s_X(L,type_2)}=\frac{c(type_1)}{c(type_2)}\,. \tag{4}$$

The proof of the Theorem combines methods from two domains. On the one hand, technology elaborated by Maryam in [Mi2] discussed in the previous section allows to compute averages over $\mathcal{M}_{g,n}$ of all kind of counting functions of simple closed geodesics and not only Weil–Peterson volumes of the moduli spaces. Namely, consider a function $f:\mathbb{R}_+\to\mathbb{R}_+$ with compact support, or a function $f:\mathbb{R}_+\to\mathbb{R}_+$ decreasing sufficiently fast when the argument goes to infinity. Now define a function $\hat f$ on $\mathcal{M}_{g,n}$ as $\hat f(X):=\sum_\alpha f(\ell_\alpha(X))$, where the sum is taken over all simple closed curves α on X (or over all simple closed curves of some fixed topological type on X), and $\ell_\alpha(X)=|\gamma(\alpha)|_X$ is the length of the simple closed geodesics $\gamma(\alpha,X)$ measured in the hyperbolic metric of X. In particular, choosing the indicator function of the interval $[0;L]$ as f Maryam computes the normalized averages

$$\lim_{L\to+\infty}\frac{1}{L^d}\int_{\mathcal{M}_{g,n}} s_X(L,type)\,dX= \\ =\int_{\mathcal{M}_{g,n}} n_{type}(X)\,dX=c(type)\,.$$

The resulting values of $c(type)$ are quite effective. For example, Maryam proves that for surfaces in

$\mathcal{M}_2$ the ratio (4) is equal to 6, that is non separating simple closed curves (as in the picture with different pairs of pants on a surface of genus two) are six times more frequent than separating ones.

To prove asymptotical formulae for individual hyperbolic surfaces, Maryam elegantly relates the counting problems to the Thurston measure on the space of *measured laminations* $\mathcal{ML}_{g,n}$ and deduces the desired results from ergodicity of the action of the mapping class group on $\mathcal{ML}_{g,n}$ with respect to the Thurston measure proved by H. Masur in [Ma2].

2. Symplectic World

Witten's Conjecture.

A hyperbolic metric on a surface defines a *conformal structure*. Conformal structures on a surface are in the natural one-to-one correspondence with *complex structures*. This correspondence allows to identify the moduli space $\mathcal{M}_{g,n}$ of hyperbolic metrics of constant negative curvature -1 with n cusps on a closed surface of genus g with the moduli space of closed Riemann surfaces (complex curves) C of genus g with n distinct marked points. In this section we use this latter interpretation of $\mathcal{M}_{g,n}$. As before, we assume that the marked points $x_1, \dots, x_n \in C$ are numbered.

For a "point" $(C, x_1, \dots, x_n) \in \mathcal{M}_{g,n}$ consider the corresponding complex curve C and consider the (co)tangent space to C at x_i. Fixing the index i and varying $(C, x_1, \dots, x_n)$ in $\mathcal{M}_{g,n}$ we get a family of complex lines parameterized by "points" $(C, x_1, \dots, x_n)$ of the moduli space $\mathcal{M}_{g,n}$. This family carries a natural structure of a complex line bundle over $\mathcal{M}_{g,n}$. The resulting *tautological bundle* is denoted by $\mathcal{L}_i$. (It has certain ressemblance with the "tautological bundle" over the Grassmann manifold.)

Allowing to Riemann surfaces get "pinched" along short hyperbolic geodesics (in the language of complex curves we allow them now to have simple self-intersections) we get the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the initial moduli space. The marked points $x_1, \dots, x_n$ on degenerate Riemann surfaces (i.e. on nodal curves) are required to stay distinct from the singular points. Thus, the line bundles $\mathcal{L}_i$ initially defined over $\mathcal{M}_{g,n}$ extend naturally to the compactified moduli space $\overline{\mathcal{M}}_{g,n}$. The space $\overline{\mathcal{M}}_{g,n}$ is a nice complex orbifold, so for any $i = 1, \dots, n$ one can define the first Chern class $\psi_i := c_1(\mathcal{L}_i)$. Recall that the cohomology forms a ring, so taking a product of k cohomology classes of dimension 2 (as the first Chern class) we can integrate the resulting cohomology class over a compact complex manifold of complex dimension k. In particular, for any partition $d_1 + \dots + d_n = 3g-3+n$ of $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g-3+n$ into the sum of nonnegative integers, one can integrate the product $\psi_1^{d_1} \cdot \dots \cdot \psi_n^{d_n}$ over the orbifold $\overline{\mathcal{M}}_{g,n}$. Formally speaking, we should first pass to an appropriate finite cover of $\overline{\mathcal{M}}_{g,n}$, which is already a manifold (and not just an orbifold), and work there. By convention, the "*intersection number*" (or the "correlator" in the physical context)

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \tag{5}$$

is defined as the integral over the corresponding cover (which is already an integer number) divided by the degree of the cover. Thus, any such integral is a rational number. It is convenient to extend the definition to any collection of nonnegative integers $(d_1, \dots, d_n)$ defining the quantity (5) as zero when $d_1 + \dots + d_n \neq 3g-3+n$. Also by convention $\int_{\mathcal{M}_{0,3}} \psi_1^0 \psi_2^0 \psi_3^0 = 1$ meaning that the "integral" of a scalar 1 over $\mathcal{M}_{0,3}$ (which is a single point) equals 1.

Since the roles of the different marked points are completely symmetric, the intersection numbers (5) are invariant under any permutations of entries $(d_1, \dots, d_n)$. Hence, they depend only on the number n_0 of entries d_i equal to 0; number n_1 of d_i equal to 1, etc. For example,

$$\langle \tau_1 \tau_0 \tau_0 \tau_0 \rangle_0 = \langle \tau_0 \tau_1 \tau_0 \tau_0 \rangle_0 = \langle \tau_0 \tau_0 \tau_1 \tau_0 \rangle_0 =$$
$$= \langle \tau_0 \tau_0 \tau_0 \tau_1 \rangle_0 = \langle \tau_0^3 \tau_1 \rangle_0 = \int_{\mathcal{M}_{0,4}} \psi_i \quad \text{for } i = 1,2,3,4\,.$$

As always, when there are plenty of rational numbers indexed by partitions or such, it is useful to wrap them into a single generating function. Namely, let us introduce formal parameters $t_1, t_2, \dots$ and define the following generating function

$$F_g(t_0, t_1, \dots) := \sum_n \sum_{d_1, \dots, d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \frac{t_{d_1} \cdots t_{d_n}}{n!}\,.$$

To get rid of any fear of F_g we suggest to the reader to write the terms of F_0 corresponding to $n = 3, 4, 5$ to obtain after simplification:

$$\langle \tau_0^3 \rangle_0 \frac{t_0^3}{3!} + \langle \tau_0^3 \tau^1 \rangle_0 \frac{t_0^3 t_1}{3!1!} + \langle \tau_0^4 \tau_2 \rangle_0 \frac{t_0^4 t_2}{4!1!} + \langle \tau_0^3 \tau_1^2 \rangle_0 \frac{t_0^3 t_1^2}{3!2!}.$$

To somebody who has never seen generating functions, the above manipulation might seem quite bizarre. However, the language of generating functions proves to be extremely convenient, especially in combinatorics, and especially, in encoding the recurrence relations and identities.

Now let us introduce one more formal variable λ to wrap all these generating functions into one:

$$F := \sum_{g=0}^{+\infty} \lambda^{2g-2} F_g\,.$$

The resulting generating function is really famous. For physicists it is a *partition function in two-dimensional quantum gravity*. In mathematical terms, E. Witten conjectured in 1991 certain recursive formula for the numbers (5) and interpreted this recursion in the form of KdV differential equations satisfied by F. The conjecture attracted an explosion of interest in mathematical community: a single formula interlaced quantum gravity, algebraic geometry, enumerative geometry, combinatorics, topology, and integrable systems.

By now we have several proofs of Witten's conjecture based on different ideas. One of them is due to Maryam Mirzakhani [Mi1]. She has very elegantly applied the technique of symplectic geometry and topology to moduli spaces of bordered Riemann surfaces $\mathcal{M}_{g,n}(L_1,\dots,L_n)$ discussed in the previous section and recognized in coefficients C_α in formula (2) for the volumes $V_{g,n}(L)$ the intersection numbers (5) (up to a routine normalization factor). This allowed Maryam to reduce the recurrence relations for the intersections numbers contained in Witten's formula to recurrence relations for the volumes $V_{g,n}(L)$ discussed above and thus prove Witten's conjecture.

Historically, the first proof of Witten's conjecture is due to M. Kontsevich, who used ribbon graphs as a "combinatorial model" of the moduli space to express the intersection numbers (5) as a sum over 3-valent ribbon graphs. He deduced that after an appropriate substitution the Witten's generating function transforms into the matrix Airy function. Another proof was suggested by A. Okounkov and R. Pandharipande [OP]; they used the Gromov–Witten theory of $\mathbb{P}^1$ and Hurwitz numbers (the numbers of different ramified covers of a two-dimensional sphere of a given ramification type). Hurwitz numbers are related to intersection numbers (5) by the *ELSV formula* [ELSV]. An alternative proof based on the ELSV-formula is due to M. Kazarian and S. Lando [KaL].

To present the idea of Maryam's proof of Witten's conjecture we have to outline briefly the relevant constructions from symplectic geometry and topology.

Momentum map and symplectic reduction.

A *symplectic manifold* (M,ω) is a manifold M endowed with a closed non-degenerate 2-form ω. Here non-degeneracy means that for any point x of M and for any nonzero tangent vector $\vec{v}\in T_xM$ one can find a tangent vector $\vec{w}\in T_xM$ such that $\omega(\vec{v},\vec{w})\neq 0$. Linear algebra tells us that any symplectic manifold is necessarily even-dimensional, $\dim_{\mathbb{R}} M = 2k$, and that the top exterior power ω^k of the symplectic form is a non-degenerate volume form on M^{2k}.

Suppose that the group of rotations $\mathbb{S}^1$ acts on a symplectic manifold (M,ω), and that this action preserves the symplectic form ω. For example, rotations around the origine in $\mathbb{R}^2$ preserve the symplectic form $dx\wedge dy$. The circle action g_t, where $t\in\mathbb{S}^1$, defines a vector field $\vec{v} := \dot{g}_t$ on M. For example, the group of rotations around the origine in $\mathbb{R}^2$ defines the vector field $y\partial_x - x\partial_y$. The symplectic form ω has two vectors as its arguments. Plugging the vector field $\vec{v}$ as the first argument of ω we get already a 1-form $\theta(\cdot) := \omega(\vec{v},\cdot)$. It is immediate to check that when the action of the group $\mathbb{S}^1$ preserves ω, the 1-form θ is closed, $d\theta = 0$. It might happen that θ is not only closed, but exact, $\theta = dH$. From now on we consider only this particular situation. In our example with the circle action on the symplectic $\mathbb{R}^2$ we get $H = (x^2+y^2)/2$ as such *Hamiltonian*. It is immediate to check that the circle action g_t preserves the function H. The orbits of the action follow the level hypersurfaces of H (in our example with symplectic $\mathbb{R}^2$ they follow the level curves $(x^2+y^2)/2 = const$ of the Hamiltonian H).

Consider a symplectic manifold (M,ω) endowed with a circle action as above. Suppose that the map $H : M \to \mathbb{R}$ is proper and that some value, say, value 0 is a regular value of H. (In the way we present this construction, the Hamiltonian H is defined up to an additive constant anyway.) Then all values a of H sufficiently close to 0 are also regular, and, hence, all sets $H^{-1}(a)$ for $|a|<\epsilon$ carry the structure of diffomorphic submanifolds.

In this sense, our example with the circle action on the symplectic $\mathbb{R}^2$ does not fit: the value 0 is the minimum of the Hamiltonian $H = (x^2+y^2)/2$, so it is the only critical value of H. All other values are regular, and all other submanifolds $H^{-1}(a)$ are pairwise diffeomomorphic: they are represented by circles in $\mathbb{R}^2$.

Since the group $\mathbb{S}^1$ acts along the level hypersurfaces, we can consider the *reduced spaces* $M_a := H^{-1}(a)/\mathbb{S}^1$. We claim without proof that under certain natural assumptions we get a family of (canonically) diffeomorphic compact manifolds parameterized by the real parameter $a\in]-\epsilon,+\epsilon[$ such that each of these manifolds is endowed with its own natural symplectic form ω_a. Though the manifolds M_a for different a are diffeomorphic, the *symplectic manifolds* (M_a,ω_a) are not *symplectomorphic*: the symplectic structure varies when we vary a. It is known that it varies *linearly* in the following sense.

Similar to a complex line bundle, any circle bundle defines a characteristic class in the second integer cohomology of the base. Denote by $\phi\in H^2(M_0;\mathbb{Z})$ the characteristic class of the natural circle bundle $H^{-1}(0)\xrightarrow{\mathbb{S}^1} M_0$.

Using the natural diffeomorphism $M_0\to M_a$, where $a\in]-\epsilon,+\epsilon[$, induce the symplectic form ω_a from M_a to M_0 and compare its cohomology class

$[\omega_a]$ with those of ω_0. Symplectic topology tells us that

$$[\omega_a] = [\omega_0] + a \cdot \phi \,.$$

In a slightly more general situation when we have a Hamiltonian action of a torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ on the initial symplectic manifold (M,ω) we can repeat the above considerations line by line. The only difference is that instead of one Hamiltonian, we now get n independent Hamiltonians, which we consider as a single Hamiltonian H with values in $\mathbb{R}^n$. (The resulting map $H : M \to \mathbb{R}^n$ is a particular case of a *moment* or *momentum* map.) The *reduced space* is now defined as the quotient $M_a := H^{-1}(a)/\mathbb{T}^n$. Finally, the linear relation for the symplectic form ω_a on M_a, where $\|a\| \le \epsilon$, takes the form

$$[\omega_a] = [\omega_0] + \sum_{i=1}^{n} a_i \cdot [\phi_i] \,,$$

where $[\phi_1], \dots, [\phi_n]$ are characteristic classes on n circle bundles over M_0. An immediate corollary of the above formula is the expression for the volume of the compact symplectic manifold (M_a, ω_a) for the natural volume form $\omega_a \wedge \cdots \wedge \omega_a$. Let $2m := \dim M_a$. Then for $\|a\| \le \epsilon$ the volume $\mathrm{Vol}(M_a)$ is a polynomial of degree m in $a = (a_1, \dots, a_n)$, namely

$$\mathrm{Vol}(M_a) := \int_{M_a} (\omega_0 + \sum a_i \omega_i)^m = \qquad (6)$$

$$= \sum_{|\alpha| \le m} C(\alpha) \cdot a^\alpha \,,$$

where the coefficients $C(\alpha)$ of the polynomial are equal to the integrals

$$C(\alpha) = c_{comb} \cdot \int_{M_0} \phi_1^{\alpha_1} \cdots \phi_n^{\alpha_n} \cdot \omega_0^{m-|\alpha|} \qquad (7)$$

(up to a very explicit combinatorial coefficient c_{comb} equal to the product of the multinomial coefficient in the power ω_a^m and the combinatorial factor depending on a convention in defnition of the wedge power).

We admit that our presentation of the momentum map is criminal in numerous aspects. It neither provides exact references, nor gives relevant names. It hides the beautiful general construction of a Hamiltonian action of a Lie group G, and the important fact that the momentum map, actually, takes values in the dual $\mathfrak{g}^*$ of the corresponding Lie algebra $\mathfrak{g}$. Our only excuse is the lack of space and abundance of excellent expositions of this beautiful subject.

Back to the intersection theory.

Return now to the moduli space $\mathcal{M}_{g,n}(b)$ of hyperbolic metrics with n geodesic boundary components on a closed surface of genus g. To follow original notations of Maryam we denote in this section the hyperbolic lengths of the geodesic boundary components by $b_1, \dots, b_n$; they have exactly the same meaning as $L_i = b_i$ used in the previous section. Consider some decomposition $\mathcal{P}$ of the underlying topological surface into pairs of pants. We have seen, that each pants decomposition defines *Fenchel–Nielsen coordinates* $\{\ell_{\alpha_i}, \tau_{\alpha_i}\},\ i = 1, \dots, 3g-3+n$, on the Teichmüller space $\mathcal{T}_{g,n}(b)$, and that the Weil–Peterson symplectic form ω_{WP} on $\mathcal{T}_{g,n}(b)$ (which descends to $\mathcal{M}_{g,n}(b)$), has particularly simple form (1) in the Fenchel–Nielsen coordinates.

We have also seen that any simple closed curve α defines a natural one-parameter family of twist deformations of the hyperbolic metric (see the picture with the family Y_τ of hyperbolic surfaces glued from two pairs of pants after a twist τ). In Fenchel–Nielsen coordinates, the diffeomorphisms

$$\mathrm{tw}_{t,\alpha_i} : \mathcal{T}_{g,n}(b) \to \mathcal{T}_{g,n}(b)$$

defined by the simple closed curves α_i involved in the pants decomposition has particularly simple form: the coordinate τ_i is transformed to $\tau_i + t$, and all the other coordinates remain unchanged. It is clear from the coordinate representation of the Weil–Peterson symplectic form ω_{WP} that it is preserved under the twists $\mathrm{tw}_{t,\alpha_i}(X)$. This implies that the vector fields defined by any 1-parameter family of twists along a fixed simple closed curve is *Hamiltonian.*

Now everything is ready to present the chain of elegant observations which allow to Maryam Mirzakhani fit the framework of hamiltonian reduction. Maryam passes from the moduli space $\mathcal{M}_{g,n}(b)$ to a larger space $\widehat{\mathcal{M}_{g,n}}$. The $2n$ extra dimensions of this larger space correspond to the following extra parameters. Now she lets the hyperbolic lengths $b_i = \ell_{\beta_i}(X)$, of n boundary components vary for $i = 1, \dots, n$. She also marks a point on each of n boundary components, getting another n new parameters. The parameters b_i are now allowed to take any nonnegative values, where $b_i = 0$ means that the i-th boundary component is reduced to a cusp. Letting a marked point on the geodesic boundary turn around i-th boundary component Maryam gets a natural circle action g_t on this new space. The action is nontrivial even when the boundary component is a cusp: when the hyperbolic length of the boundary component β_i becomes too short, Maryam starts to draw a canonically chosen curve around it; this curve is endowed with a mark point shadowing the marked point on the boundary. When the boundary component degenerates to a cusp, the canonically chosen curve becomes a closed horocycle around the cusp; this horocycle is endowed with a marked point which thus survives under the degeneration of the boundary. Considering all n boundary components Maryam gets the torus action of $\mathbb{T}^n$ on $\widehat{\mathcal{M}_{g,n}}$.

The next observation is that the space $\widehat{\mathcal{M}_{g,n}}$ is endowed with a natural symplectic form generalizing the Weil–Peterson symplectic form (here Maryam uses the result of S. Wolpert, who showed that ω_{WP} admits a smooth extension to $\overline{\mathcal{M}}_{g,n}$). By construction this new symplectic form is $\mathbb{T}^n$-invariant. Moreover, the function

$$a_i(X) = \frac{b_i^2}{2} = \frac{\ell^2_{\beta_i}(X)}{2}\,, \tag{8}$$

where $b_i = \ell_{\beta_i}(X)$ is the hyperbolic length of the i-th geodesic boundary component, is the Hamiltonian function for the corresponding circle action. We are now in the framework of the previous section. Considering the n-tuple of Hamiltonians for n boundary components, Maryam obtains the Hamiltonian (momentum map) H on $\widehat{\mathcal{M}_{g,n}}$ with values in $\mathbb{R}^n_+$ induced by the Hamiltonian torus action defined above. Applying the hamiltonian reduction as in the previous section she proves that each *reduced space* $M_a = \left(H^{-1}(a_1,\dots,a_n)\right)/\mathbb{T}^n$ is symplectomorphic to $\overline{\mathcal{M}}_{g,n}(b_1,\dots,b_n)$ (where a_i and b_i are related by (8)). Applying formula (6) (where an abstract manifold M_a becomes in our context $\overline{\mathcal{M}}_{g,n}(b_1,\dots,b_n)$ with a_i and b_i related by (8)) Maryam expresses the volume $V_{g,n}(b_1,\dots,b_n)$ of $\overline{\mathcal{M}}_{g,n}(b_1,\dots,b_n)$ as a polynomial in squares of b_i. When

$$|\alpha| = m = \frac{1}{2}\dim_{\mathbb{R}}\mathcal{M}_{g,n}(b) = 3g-3+n$$

the coefficients $C(\alpha)$ of the polynomial are expressed by formula (7) as intersection numbers

$$C(\alpha) = c_{comb}\cdot\int_{M_0}\phi_1^{\alpha_1}\cdots\phi_n^{\alpha_n} \tag{9}$$

in terms of the characteristic classes ϕ_i of circle bundles.

Now Maryam makes the last two crucial remarks. Recall the definition of the tautological bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$: the fiber of $\mathcal{L}_i$ over $X \in \overline{\mathcal{M}}_{g,n}$ is the (co)tangent space to the Riemann surface X at the marked point x_i. Choosing an infinitesimal circle in the corresponding (co)tangent space we get a circle bundle over $\overline{\mathcal{M}}_{g,n}$. It is a standard fact that the first Chern class ψ_i of the line bundle $\mathcal{L}_i$ and the characteristic class of the associated circle bundle are the same. It remains to recognize in the newly constructed circle bundle, the one which Maryam defined above (by chopping the cusps of a hyperbolic surface in $\mathcal{M}_{g,n}(0,\dots,0)$ at the level of a canonically chosen horocycle around the cusp and by marking a point on every such horocycle). In other words, the characteristic classes ϕ_i in (9) coincide with the tautological classes ψ_i in (5). The tie between the volumes $V_{g,n}(b) = \operatorname{Vol}\mathcal{M}_{g,n}(b)$ as in (2) and the intersection numbers (correlators) $\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g$ is established. Now recall, that we have already seen certain recursive relations for the volumes $V_{g,n}(b)$ proved by Maryam (see the short description in the paragraph preceding the Theorem embodying formula (2)). Since the intersection numbers are now related to volumes, the recursion for volumes provides certain recursion for the intersection numbers. Maryam proves that the latter recursion implies the one encoded by Witten's conjecture.

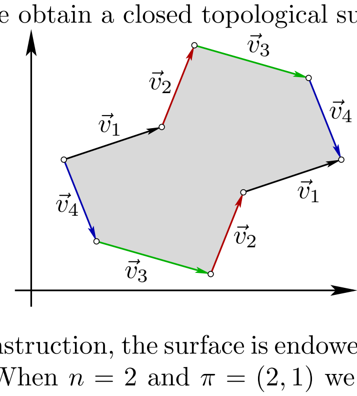
The recursive relation for the Weil–Peterson volumes of the moduli spaces $\mathcal{M}_{g,n}(b)$ and for the intersection numbers $\langle\tau_{d_1}\dots\tau_{d_n}\rangle$ obtained by M. Mirzakhani can be seen as one of the first manifestations of a new technique which got the name of *topological recursion* and which proves to be extremely fruitful in numerous other topics in mathematics and physics (see survey [Ey] of B. Eynard for more details).

3. Flat World

Very flat surfaces.

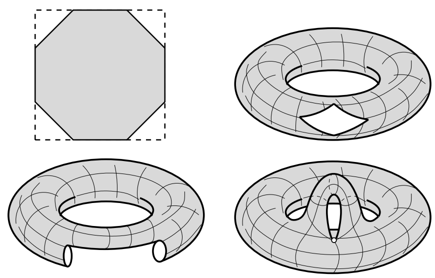
Now we return from the hyperbolic and symplectic worlds to the flat world discussed in the introduction. We have seen that to construct a hyperbolic metric on a torus, we have to allow to the hyperbolic metric to have a cusp. To construct a flat metric on a surface of genus different from one we have to allow to the flat metric to have several isolated conical singularities. Actually, we consider only those flat metrics which mimic a flat metric on a torus: namely, a parallel transport in such *very flat metric* of any tangent vector along any closed curve avoiding singularities brings the vector to itself.

We ask the reader to believe that similar to the torus case all such very flat surfaces can be obtained by the following construction. Consider a collection of vectors $\vec{v}_1,\dots,\vec{v}_n$ in $\mathbb{R}^2$ and arrange these vectors into a broken line. Construct another broken line starting at the same point as the first one arranging the same vectors in the order $\vec{v}_{\pi(1)},\dots,\vec{v}_{\pi(n)}$, where π is some permutation of n elements. By construction the two broken lines share the same endpoints; suppose that they bound a polygon as in the picture. Identifying the pairs of sides corresponding to the same vectors $\vec{v}_j$, $j = 1,\dots,n$, by parallel translations we obtain a closed topological surface.



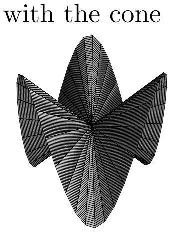
By construction, the surface is endowed with a flat metric. When $n = 2$ and $\pi = (2,1)$ we get a usual flat torus glued from a parallelogram. For larger number of vectors we might get a surface of higher genus, where the genus is determined by the permutation π. It is convenient to impose from now on some simple restrictions on the permutation π which guarantee, in particular, non degeneracy of the surface for an open set of parameters $\vec{v}_i$, see [Ma1] or [Ve].

For example, a regular octagon gives rise to a surface of genus two as illustrates the cartoon above. Indeed, identifying pairs of horizontal and vertical sides of a regular octagon we get a usual torus with a hole in the form of a square. We slightly cheat in the next frame, where we turn this hole by 45° and only then glue the next pair of sides. As a result we get a torus with two isolated holes as on the third frame. Identifying the remaining pair of sides (which represent the holes) we get a torus *with a handle*, or, in other words, a surface of genus two.



Similar to the torus case, the surface glued from the regular octagon also inherits a flat metric, but now the resulting flat metric has a singularity at the point obtained from identified vertices of the octagon.

Note that the flat metric thus constructed is, actually, *very flat*: since we identify the sides of the polygon only by translations, the parallel transport of any tangent vector along a closed cycle (avoiding conical singularities) on the resulting surface brings the vector back to itself. In other words, our flat metric has *trivial holonomy*. In particular, since a parallel transport along a small loop around any conical singularity brings the vector to itself, the cone angle at any singularity is an integer multiple of 2π. In the most general situation the flat surface of genus g would have several conical singularities with cone angles $2\pi(d_1+1), \dots 2\pi(d_m+1)$, where $d_1+\cdots+d_m = 2g-2$. The picture below shows a conical singularity with the cone angle 6π.



It is convenient to consider the vertical direction as part of the structure. A surface endowed with a flat metric with trivial holonomy and with a choice of a vertical direction is called a *translation surface*. Two polygons in the plane obtained one from another by a parallel translation give rise to the same translation surface, while polygons obtained one from another by a nontrivial rotation (usually) give rise to distinct translation surfaces.

We can assume that the polygon defining our translation surface is embedded into the complex plane $\mathbb{C} \simeq \mathbb{R}^2$ with coordinate z. The translation surface obtained by identifying the corresponding sides of the polygon inherits the complex structure. Moreover, since the gluing rule for the sides can be expressed in local coordinates as $z = \tilde{z} + const$, the closed 1-form dz is well-defined not only in the polygon, but on the surface. An exercise in complex analysis shows that the complex structure extends to the points coming from the vertices of the polygon, and that the 1-form $\omega = dz$ extends to the holomorphic 1-form on the resulting Riemann surface. This 1-form ω has zeroes of degrees $d_1, \dots, d_m$ exactly at the points where the flat metric has conical singularities of angles $2\pi(d_1+1), \dots, 2\pi(d_m+1)$.

Reciprocally, given a holomorphic 1-form ω on a Riemann surface one can always find a local coordinate z (in a simply-connected domain not containing zeroes of ω) such that $\omega = dz$. Such coordinate is defined up to an additive constant. It defines the translation structure on the surface. Cutting the surface along an appropriate collection of straight segments joining conical singularities we can unwrap the Riemann surface into a polygon as above.

This construction shows that the two structures are completely equivalent: the structure of a flat metric with trivial holonomy plus a choice of distinguished direction is equivalent to the structure of a Riemann surface endowed with a holomorphic 1-form.

Families of translation surfaces and dynamics in the moduli space. The polygon in our construction depends continuously on the vectors $\vec{v}_i$. This means that the topology of the resulting translation surface (its genus g, the number and the types of the resulting conical singularities) does not change under small deformations of the vectors $\vec{v}_i$. For every collection of cone angles $2\pi(d_1+1), \dots, 2\pi(d_m+1)$ satisfying $d_1+\cdots+d_m = 2g-2$ with integer d_i for $i = 1, \dots, n$, we get a *family* $\mathcal{H}(d_1, \dots, d_m)$ of translation surfaces. Vectors $\vec{v}_1, \dots, \vec{v}_n$ can be viewed as complex coordinates in this space, called *cohomological coordinates*. These coordinates define a structure of a *complex orbifold* on each space $\mathcal{H}(d_1, \dots, d_m)$. We have to confess that the geometry and topology of spaces of translation surfaces is not yet sufficiently explored.

Readers preferring algebro-geometric language may view a family of translation surfaces with fixed conical singularities $2\pi(d_1+1),\dots,2\pi(d_m+1)$ as the stratum $\mathcal{H}(d_1,\dots,d_m)$ in the moduli space $\mathcal{H}_g$ of all pairs (complex curve C; holomorphic 1-form ω on C), where the stratum is specified by the degrees $d_1,\dots,d_m$ of zeroes of ω, satisfying $d_1+\dots+d_m = 2g-2$. Note that while the moduli space $\mathcal{H}_g$ is a $\mathbb{C}^g$-bundle over the moduli space $\mathcal{M}_g$, individual strata do not carry a structure of a vector bundle over $\mathcal{M}_g$. For example, the minimal stratum $\mathcal{H}(2g-2)$ has complex dimension $2g$, while the moduli space $\mathcal{M}_g$ has complex dimension $3g-3$. The very existence of a holomorphic form with a single zero of degree $2g-2$ on a complex curve C is a strong condition on C.

To complete the description of the space of translation surfaces we need to present one more very important structure: the action of the group $\mathrm{GL}(2,\mathbb{R})$ on $\mathcal{H}_g$ preserving strata. The description of this action is particularly simple in terms of our polygonal model Π of a translation surface S. A linear transformation $g\in\mathrm{GL}(2,\mathbb{R})$ of the plane maps the polygon Π to a polygon $g\Pi$. The new polygon again has all sides arranged into pairs, where the two sides in each pair are parallel and have equal length. We can glue a new translation surface and call it $g\cdot S$. It is easy to see that unwrapping the initial surface into different polygons we get the same surface $g\cdot S$. Note also, that we explicitly use the choice of the vertical direction: any polygon is endowed with an embedding into $\mathbb{R}^2$ defined up to a parallel translation.

The subgroup $\mathrm{SL}(2,\mathbb{R})\subset\mathrm{GL}(2,\mathbb{R})$ preserves the flat area. This implies, that the action of $\mathrm{SL}(2,\mathbb{R})$ preserves the real hypersurface $\mathcal{H}_1(d_1,\dots,d_m)$ of translation surfaces of area one in any stratum $\mathcal{H}(d_1,\dots,d_m)$. The codimension-one subspace $\mathcal{H}_1(d_1,\dots,d_m)$ can be compared to the unit sphere (or rather to the unit hyperboloid) in the ambient stratum $\mathcal{H}(d_1,\dots,d_m)$.

Recall that under appropriate assumptions on the permutation π, the n vectors

$$\vec{v}_1=\begin{pmatrix}v_{1,x}\\ v_{1,y}\end{pmatrix}\dots,\vec{v}_n=\begin{pmatrix}v_{n,x}\\ v_{n,y}\end{pmatrix}$$

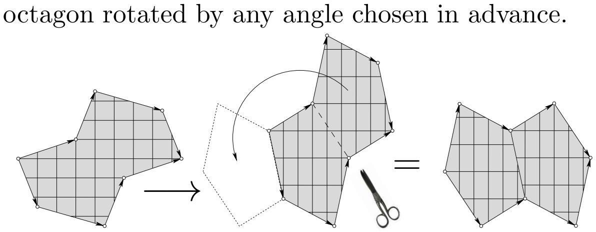
as in the picture with a polygon define local coordinates in the embodying family $\mathcal{H}(d_1,\dots,d_m)$ of translation surfaces. Let $d\nu := dv_{1x}dv_{1y}\dots dv_{nx}dv_{ny}$ be the associated volume element in the corresponding coordinate chart $U\subset\mathbb{R}^{2n}$. It is easy to verify, that $d\nu$ does not depend on the choice of "coordinates" $\vec{v}_1,\dots,\vec{v}_n$, so it is well-defined on $\mathcal{H}(d_1,\dots,d_m)$. Similarly to the case of the Euclidian volume element, we get a natural induced volume element $d\nu_1$ on the unit hyperboloid $\mathcal{H}_1(d_1,\dots,d_m)$. A simple exercise shows that the action of the group $\mathrm{SL}(2,\mathbb{R})$ preserves the volume element $d\nu_1$.

The following Theorem proved independently and simultaneously by H. Masur [Ma1] and W. Veech [Ve] is the keystone of the theory of flat surfaces.

Theorem (H. Masur; W. A. Veech)**.** *The total volume of every stratum $\mathcal{H}_1(d_1,\dots,d_m)$ is finite.*

The group $\mathrm{SL}(2,\mathbb{R})$ and its diagonal subgroup act ergodically on every connected component of every stratum $\mathcal{H}_1(d_1,\dots,d_m)$.

Here "ergodically" means that any measurable subset invariant under the action of the group has necessarily measure zero or full measure. Ergodic theorem claims that in such situation the orbit of almost every point homogeneously fills the ambient connected component. In plain terms, the ergodicity of the action of the diagonal subgroup implies, in particular, the following. Having almost any polygon as above, we can choose appropriate sequence of times t_i such that contracting the polygon horizontally with a factor e^{t_i} and expanding it vertically with the same factor e^{t_i} and modifying the resulting polygonal pattern of the resulting translation surface by an appropriate sequence of cut-and-paste transformations we can get arbitrary close to, say, regular octagon rotated by any angle chosen in advance.



Philosophy of "almost all" versus "all".

Even for those classes of dynamical systems which are sufficiently well understood, the only kind of predictions of "what would happen after sufficiently long time" always contain some version of the word "typically" usually meaning "for a full measure set of initial data". The trouble (which, depending on the taste, might be considered as an advantage: "do not get distracted by details") is that even for those dynamical systems which are very well studied, and where one knows, basically, everything about "typical behavior" of trajectories, one can say almost nothing about behavior of any concrete particular trajectory: there is no way to tell, whether your particular starting data are "typical" or not. If you repeat thousands of experiments with random starting data and you want to establish some statistics, you do not care about rare nontypical fluctuations. But if you are interested in the future of some very special asteroid B612, and only by this, most of the methods of dynamical systems become completely useless for you.

The difficulty is conceptual; it is neither related to lack of knowledge at the current state of development of mathematics, nor to the presence of noise, or friction, etc in realistic dynamical systems. Even for absolutely deterministic systems, and even assuming all necessary mathematical abstractions like absence of any noise or friction, the trouble persists. The reason is that for the vast majority of dynamical systems (including very smooth and nice ones) certain individual trajectories might be extremely sophisticated: they can cover extremely fractal sets on a large scale of time.

For example, the map $f : x \mapsto \{2x\}$ homogeneously twisting the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ twice around itself has orbits filling Cantor sets of, basically, arbitrary Hausdorff dimension between zero and one; has nonclosed orbits avoiding certain arcs of the circle, etc. In other words, this extremely nice map, clearly, has trajectories with very peculiar properties. All these properties become much more visible using the binary representation of a real number $x \in [0, 1[$ instead of the usual decimal one. If

$$x = \frac{n_1}{2} + \cdots + \frac{n_k}{2^k} + \cdots ,$$

where all binary digits n_k are zeroes or ones, then the map f acts on the sequence $(n_1, n_2, \ldots, n_k, \ldots)$ by erasing the first digit. (This operation on the space of semi-infinite sequences of zeroes and ones is called the *Bernoulli shift*).

The geodesic flow on any compact Riemann surface of constant negative curvature has similar behavior. It was observed long ago by H. Furstenberg and B. Weiss that the closures of individual trajectories might have arbitrary (or almost arbitrary) Hausdorff dimension in the range from 1 (closed trajectories) to 3 (typical trajectories).

A straight-line flow on a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is an example of a (very rare) dynamical system, where the closure of *any* orbit is a nice submanifold. Let $\vec{V} = (V_1, \ldots, V_n)$ denote the direction of the flow. The closure of *any* trajectory in direction $\vec{V}$ is a subtorus $\mathbb{T}^k$, where $1 \le k \le n$ is the degree of irrationality $k = \dim_{\mathbb{Q}}\{V_1, \ldots, V_n\}$ of the direction. Say, in the particular case of a two-dimensional torus, when $n = 2$, all trajectories of the flow in a rational direction are already closed — they are circles $\mathbb{S}^1 = \mathbb{T}^1$, and the closure of any trajectory of the flow in an irrational direction is the entire torus $\mathbb{T}^2$. Actually, this is not surprising at all: torus is a homogeneous space, and the group of automorphisms of the torus preserving the flow acts transitively on the torus.

In a sense, up to know, there was only one known class of dynamical systems, for which one could find the closure of any single trajectory, and for which all possible closures were described by a short list of possible simple cases like in the example above. This class is restricted to certain very special dynamical systems in homogeneous spaces. One of the key statements in this theory was proved by Marina Ratner; extremely important contributions to this theory as well as fantastic applications to the number theory, were developed by S. G. Dani, G. Margulis, and by other great mathematicians, including A. Eskin, S. Mozes, and N. Shah. The scale of applications of this theory to different areas of mathematics continues to extend. Indeed, homogeneous spaces naturally appear in various domains of mathematics. (Both the theory and the list of major contributors merit a separate paper rather than a short paragraph.)

Magic Wand Theorem.

Now everything is ready to present the result of Alex Eskin and Maryam Mirzakhani [EMi] (incorporating the joint results of these authors and of Amir Mohammadi [EMiMh]).

It is known that the moduli space *is not* a homogeneous space. Nevertheless, the theorem stated below proves that the orbit closures of $\mathrm{GL}(2,\mathbb{R})$ in the space of translation surfaces are as nice as one can only hope: they are complex manifolds possibly with very moderate singularities (i.e. they are complex orbifolds). In this sense the action of the $\mathrm{GL}(2,\mathbb{R})$ and of $\mathrm{SL}(2,\mathbb{R})$ on the space of translation surfaces mimics certain properties of the dynamical systems in homogeneous spaces mentioned at the end of the previous section.

Theorem ([EMi], [EMiMh])**.** *The closure of any* $\mathrm{GL}(2,\mathbb{R})$-*orbit is a complex suborbifold (possibly with self-intersections); in cohomological coordinates* $\vec{v}_1, \ldots, \vec{v}_n$ *in the ambient space* $\mathcal{H}(d_1, \ldots, d_m)$ *of translation surfaces it is locally represented by an affine subspace.*

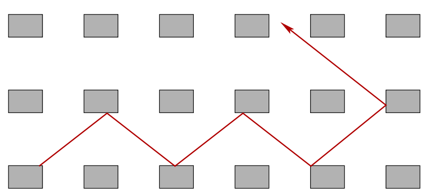
Any ergodic $\mathrm{SL}(2,\mathbb{R})$-*invariant measure is supported on a suborbifold. In coordinates* $\vec{v}_1, \ldots, \vec{v}_n$ *this suborbifold is represented by the intersection of* $\mathcal{H}(d_1, \ldots, d_m)$ *with an affine subspace, and the invariant measure is induced from the usual affine measure to this intersection.*

As a vague conjecture (or, better say, as a very optimistic dream) these properties were discussed since long ago, and since long ago, there was not a slightest hint for a general proof. The only exception is the case of surfaces of genus two, for which ten years ago C. McMullen proved a very precise statement [McM1] classifying all possible orbit closures. He used, in particular, the hard artillery of Ratner's results which are applicable here. However, the theorem of McMullen is based on very special properties of surfaces of genus two, which do not generalize to higher genera.

The proof of Alex Eskin and of Maryam Mirzakhani is a titanic work which took many years. It absorbed numerous fundamental developments in dynamical systems which do not have any direct relation to moduli spaces. To mention only a few, it incorporates certain ideas of low entropy method of M. Einsiedler, A.Katok, E. Lindenstrauss; results of G. Forni and of M. Kontsevich on Lyapunov exponents of the Teichmüller geodesic flow; the ideas from the works of Y. Benoit and J.-F. Quint on stationary measures; iterative improvement of the properties of the invariant measure inspired by the approach of G. Margulis and G. Tomanov to the actions of unipotent flows on homogeneous spaces; some fine ergodic results due to Y. Guivarch and A. Raugi.

Applications.

What can we do now, when this theorem is proved? For example, in certain situations this theorem works really like a Magic Wand, which allows to touch *any* given billiard in certain class of billiards and in theory (and more and more often in practice) find the corresponding orbit closure in the moduli space of translation surfaces. The geometry of this orbit closure tells you, basically, everything you want to know about the initial billiard.



Consider, for example, the following *windtree model* introduced by physicists P. and T. Ehrenfest more than a century ago [Eh]. We study the billiard in the plane filled periodically with the identical rectangular obstacles as in the picture. A trajectory might go far away, then return relatively close back to the starting point, then make other long trips. The *diffusion rate* ν describes the average rate T^ν with which the trajectory expands in the plane on a long range of time $T \gg 1$. More formally,

$$\nu := \lim_{T\to\infty} \frac{\log(\text{diameter of trajectory of length } T)}{\log T}\,.$$

For the usual random walk in the plane, or for a billiard with periodic circular obstacles the diffusion rate is known to be $1/2$: the most distant point of a piece of trajectory corresponding to segment of time $[0, T]$ would be located roughly at a distance $\sqrt{T}$. V. Delecroix, P. Hubert, and S. Lelièvre have recently showed that for the windtree model as in the picture the diffusion rate is $2/3$. They used combination of methods of dynamics in the moduli space and several extremely lucky coincidences.

Now, due to the work of Eskin, Mirzakhani, and Mohammadi, one gets a feeling that a Magic Wand about which you dreamed reading “Cinderella”, became an every day tool accessible at a department store. Moreover, it gets enhanced every month by new results like those of A. Avila, M. Möller, A. Eskin [AEMö] J. Chaika and A. Eskin [CkE], S. Filip [Fi1], A. Write [Wr1].

You want to find diffusion rate for generalized wind-tree model with periodic scatterers of the shape of a more complicated rational polygon? To perform this task you proceed as follows. Replace the periodic billiard with an associated compact flat surface. Touch it with the Magic Wand of Eskin–Mirzakhani–Mohammadi and find its $\mathrm{SL}(2,\mathbb{R})$-orbit closure in the space of flat surfaces. Run the geodesic flow to compute the mean monodromy (Lyapunov exponents) of the appropriate block of the complex Hodge bundle, and you are done.

To be honest, in full generality, this strategy is a new dream (though in some situations it already works perfectly well [DZ]). We do not have yet a classification of $\mathrm{SL}(2,\mathbb{R})$-invariant orbifolds except for genus two. This is a next challenging problem which might be full of mysteries and marvels as indicate most recent results of M. Mirzakhani and of A. Write, who have recently found an invariant suborbifold of completely enigmatic origin in the stratum $\mathcal{H}(6)$ (see [Wr2]).

One should not have an impression that the theory developed by A. Eskin, M. Mirzakhani, A. Mohammadi and other researchers in this area is designed to serve billiards. A billiard in a polygon is just a cute way to describe certain class of dynamical systems; the same kind of dynamical systems appear in solid state physics, in conductivity theory, in the theory of surface foliations. (For more ample presentation of the context and of the applications of the Magic Wand Theorem, see [Zor])

The result of A. Eskin and M. Mirzakhani also opens a new way to study moduli spaces. We do not know yet all possible applications of the Magic Wand Theorem which might be obtained in this direction.

The integral calculus was partly developed by Kepler (a century before Newton and Leibniz) in order to measure the volume of wine barrels. Who could imagine at that time that volume of a solid of revolution would be discussed in any textbook of mathematics for beginners and that the integral calculus would become an essential part of all contemporary engineering. The theorem proved by Alex Eskin and Maryam Mirzakhani is so beautiful and powerfull that, personally, I have no doubt that it would find numerous applications far beyond our current imagination.

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References

[AEMö] A. Avila, A. Eskin, M. Möller, *Symplectic and isometric* $\mathrm{SL}(2,\mathbb{R})$*-invariant subbundles of the Hodge bundle*, `arXiv:1209.2854`, to appear in J. Reine Angew. Math.

[CkE] J. Chaika and A. Eskin, *Every flat surface is Birkhoff and Osceledets generic in almost every direction*, `arXiv:1305.1104` (2013), to appear in JMD.

[DZ] V. Delecroix and A. Zorich, *Cries and Whispers in Windtree Forests*, `arXiv` (2015).

[Eh] P. and T. Ehrenfest, Begriffliche Grundlagen der statistischen Auffassung in der Mechanik Encykl. d. Math. Wissensch. IV 2 II, Heft 6, 90 S (1912) (in German, translated in:) The conceptual foundations of the statistical approach in mechanics, (trans. Moravicsik, M. J.), 10-13 Cornell University Press, Itacha NY, (1959).

[ELSV] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), no. 2, 297–327.

[EMi] A. Eskin, M. Mirzakhani, *Invariant and stationary measures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on Moduli space*, `arXiv:1302.3320` (2013), 1–172.

[EMiMh] A. Eskin, M. Mirzakhani, A. Mohammadi, *Isolation, equidistribution, and orbit closures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on Moduli space*, `arXiv:1305.3015` (2013), 1–46.

[Ey] B. Eynard, *A short overview of the "Topological recursion"* `arXiv:1412.3286`, to appear in Proceedings of ICM (2015).

[Fi1] S. Filip, *Semisimplicity and rigidity of the Kontsevich–Zorich cocycle*, `arXiv:1307.7314` (2013).

[G] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225.

[KaL] M. Kazarian, S. Lando, *An algebro-geometric proof of Witten's conjecture*, J. Amer. Math. Soc. **20** (2007), no. 4, 1079–1089.

[Kl] E. Klarreich, *A Tenacious Explorer of Abstract Surfaces*, Quanta Magazine, August 12, 2014.

[Ko] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1-23

[Ma1] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. **115** (1982), 169–200.

[Ma2] H. Masur, *Ergodic actions of the mapping class group*, Proc. Amer. Math. Soc. **94** (1985), 455–459.

[McM1] C. McMullen, *Dynamics of* $\mathrm{SL}(2,\mathbb{R})$ *over moduli space in genus two*, Ann. of Math. (2) **165** (2007), no. 2, 397–456.

[McM2] C. McMullen, *The mathematical work of Maryam Mirzakhani*, Proceedings of the International Congress of Mathematicians, (Seoul, 2014), to appear. `http://www.math.harvard.edu/~ctm`

[McS] G. McShane, *Simple geodesics and a series constant over Teichmüller space*, Invent. Math. **132**, 607–632 (1998).

[Mi1] M. Mirzakhani, *Weil–Petersson volumes and intersection theory on the moduli space of curves*, Journal of Amer. Math. Soc. **20** (2007), no. 1, 1–23.

[Mi2] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222.

[Mi3] M. Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Annals of Math. (2) **168** (2008), no. 1, 97125.

[MiZ] M. Mirzakhani, P. Zograf, *Towards large genus asymtotics of intersection numbers on moduli spaces of curves*, `arXiv:1112.1151` (2011).

[OP] A. Okounkov and R. Pandharipande, *Gromov–Witten theory, Hurwitz theory, and matrix models*, Algebraic geometry — Seattle 2005. Part 1, 325–414, Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.

[R] I. Rivin, *Simple curves on surfaces*, Geom. Dedicata **87**, 345–360 (2001).

[Ve] W. A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals of Math., **115** (1982), 201–242.

[Wi] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, in Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991

[Wo1] S. Wolpert, *An elementary formula for the Fenchel–Nielsen twist*, Comment. Math. Helv. **56** (1981), 132–135.

[Wr1] A. Wright, *Cylinder deformations in orbit closures of translation surfaces*, Geometry and Topology, to appear.

[Wr2] A. Wright, *Orbit closures of translation surfaces*, Oberwolfach Report **15** (2014), 884–885.

[Zog] P. Zograf, *The Weil–Petersson volume of the moduli space of punctured spheres*, In: "Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 367–372, Contemp. Math., **150**, AMS, Providence, RI, 1993.

[Zor] A. Zorich, *Le théorème de la baguette magique de A. Eskin et M. Mirzakhani*, Gazette des mathématiciens, **142** (2014) 39–54; English translation: `arXiv:`.

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