The Work of the 2014 Fields Medalists

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L–R: Martin Hairer, Manjul Bhargava, South Korean president Park Geun-hye, Maryam Mirzakhani, IMU president Ingrid Daubechies, and Artur Avila.

The *Notices* solicited the following articles about the works of the four individuals to whom Fields Medals were awarded at the International Congress of Mathematicians in Seoul, South Korea, in August 2014. This was a historic occasion, as it marked the first time since the medal was established in 1936 that a woman was among the recipients. The International Mathematical Union also issued news releases describing the medalists' work, and these appeared in the October 2014 issue of the *Notices*.

—*Allyn Jackson*

Welington de Melo

The Work of Artur Avila

Artur Avila was awarded the Fields Medal in 2014 for his deep contributions to dynamical systems and to the spectral theory of one-frequency Schrödinger operators. Many of his profound achievements—in one-dimensional dynamics, both real and complex, as well as in flat billiards and in the spectral theory of Schrödinger operators—are characterized by an intense use of the powerful ideas of renormalization. He has also made deep advances in the theory of conservative dynamical systems in any dimension and in the stable ergodicity of partially hyperbolic systems.

Artur Avila was born in Rio de Janeiro, Brazil, where he lived until he finished his PhD studies at the Instituto de Matemática Pura e Aplicada (IMPA) in 2001. Before starting his postdoc position in the Collège de France in Paris, Artur, with collaborators, obtained a complete description of typical dynamics of unimodal interval maps, i.e., smooth interval maps with a unique nonflat critical point. It was already known that the space of unimodal maps contains two disjoint regions where the dynamics is well understood. For maps in the regular region, there exists a unique attracting fixed point, and the trajectories of almost all initial conditions, in the Lebesgue measure sense, are asymptotic to this periodic point. The typical dynamical behavior is therefore periodic. For maps in the stochastic region the dynamics is chaotic but with a good statistical description: there exists an absolutely continuous invariant measure that controls the dynamics of a typical orbit in the sense that the time average of any physical observable is equal to the space average of the observable with respect to this measure. In particular, the frequency of visits of a typical orbit to any interval is equal to the measure of this interval. In the complement of these two regions there are maps where the dynamics can be completely described but there are other maps whose dynamics is pathological. These different dynamical behaviors were already present in the famous quadratic family $x \in [0, 1] \mapsto q_\mu = \mu x(1 - x)$, where the parameter μ belongs to the interval $(0, 4]$. The set of parameter values corresponding to regular maps is open and dense by a difficult result proved in [18] and [20]. On the other hand, the set of parameter values corresponding to stochastic maps has positive Lebesgue measure, as proved in [20], and the complement has measure zero [22] but positive Hausdorff dimension. The main result in [5], [9]

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is that, for typical one-dimensional families of unimodal maps, one does not see the third region; i.e., the set of parameter values corresponding to maps in the third region has Lebesgue measure zero. For a typical one-parameter family of unimodal maps we have a decomposition of the parameter space with the same properties as the quadratic family.

Another important result in his early career is the unexpected rigidity in the space of unimodal maps obtained in [10]. In this article it is shown that there is a large set R of unimodal maps such that any typical one-parameter family intersects R in a set of parameters of full Lebesgue measure in the complement of the regular parameters and any two maps in R that are topologically conjugate are in fact smoothly conjugate. In particular, the authors provide a combinatorial formula to calculate the multipliers of all periodic points for maps in R.

More recently, in complex one-dimension dynamics, Avila and Lyubich described several fundamental properties on the geometry of the Julia set of Feigenbaum-type quadratic polynomials [6], [7], [2], [3], [4]. In particular, the existence of examples of such quadratic polynomials with Julia sets of positive Lebesgue measure is proven.

In most of these results, Avila uses as a fundamental tool the renormalization theory. This theory started with experimental discovery by the physicists Feigenbaum and Coullet-Tresser in the 1970s on the transition from simple to chaotic dynamics in families of unimodal maps. They formulated a conjecture that involves a nonlinear operator in the space of unimodal maps. The proof of the conjecture and some generalizations involve the work of several mathematicians such as Sullivan, McMullen, and Lyubich. Finally, Avila and Lyubich in [16] gave a very conceptual and much simpler proof of the general conjecture that holds also in the space of unimodal maps with higher criticality.

A second area of dynamical systems where Avila made fundamental contributions is the dynamics of interval exchange transformations, regular polygonal billiards, and ergodic properties of the geodesic Teichmüller flow in the moduli space of Abelian differentials on Riemann surfaces.

Given a partition $I_1, \ldots, I_n$ of the interval $[0, 1]$ in $d \geq 2$ intervals and a permutation σ of $\{1, \ldots, d\}$, we can define the mapping $T: [0, 1] \rightarrow [0, 1]$ by

$$T(x) = x - \sum_{j < i} \lambda_j + \sum_{\sigma(j) < \sigma(i)} \lambda_j$$

if $x \in I_i$, where λ_j is the length of the interval I_j. This is what is called an *interval exchange transformation*, and it is completely characterized by the permutation σ and by the vector $(\lambda_1, \ldots, \lambda_d)$ that belongs to a simplex in $\mathbb{R}^d_+$. The space of such maps is therefore finite dimensional. We say that a subset is typical if the complement has

zero Lebesgue measure. Veech proved that for any irreducible permutation the typical interval exchange transformation is uniquely ergodic. In 1980 Katok proved that no interval exchange transformation is mixing. Recall that a measure-preserving transformation $f: (X, \mu) \to (X, \mu)$ is mixing if

$$\lim_{n \to \infty} [\mu(f^{-n}(A) \cap B - \mu(A)\mu(B)] = 0$$

for any measurable sets A and B, and it is weak mixing if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} [\mu(f^{-n}(A) \cap B - \mu(A)\mu(B)] = 0.$$

The joint work of Avila and Forni [11] solved the main problem in the ergodic theory of interval exchange transformation: a typical interval exchange transformation is weakly mixing for any irreducible permutation that is not a rotation. This work is connected to a recent result of Avila and Delecroix proving that almost all regular polygonal billiards are weakly mixing. Again, one of the main tools for these results is a renormalization operator that consists of considering the first return map to a smaller interval and rescaling to the original size. It is called the Rauzy-Veech renormalization operator. Veech and Masur proved that it has an absolutely continuous invariant measure, and the ergodic properties of the operator give important information about the ergodic properties of a typical interval exchange transformation. The invariant measure is not finite, but in [25] Zorich defined an accelerated version of the renormalization operator, called the Rauzy-Veech-Zorich operator, that does have a finite absolutely continuous invariant measure. The new operator maps each interval exchange transformation into an iterate of the previous operator, the iterate depending on the interval exchange transformation. The dynamics of this new operator is closely related to the so-called Teichmüller geodesic flow that acts in the moduli space of Abelian differentials in compact Riemann surfaces of a given genus. Zorich experimentally discovered the existence of d exponents that for a typical interval exchange transformation with d intervals describes the deviation of the ergodic average from the mean and found that these exponents are given by the Lyapunov spectrum of the so-called Rauzy-Veech-Zorich cocycle over the renormalization operator. In [12] Avila and Viana proved the Kontsevich-Zorich conjecture stating that the Lyapunov spectrum of the Rauzy-Veech-Zorich cocycle is simple. Another important result in this area was obtained by Avila and collaborators establishing the exponential decay of correlation of the Teichmüller flow on strata of the moduli space of Abelian differentials that was conjectured by Veech.

The area of one-frequency Schrödinger operators was very active before the arrival of Avila. In particular, it was known that if one multiplies an analytic potential by a coupling constant generating a one-parameter family of operators, then for very small values of the coupling constant the spectrum of the operator is absolutely continuous, and for very large values the spectrum is pure point for typical values of the frequency. Not much was known for values of the coupling constant in between. However, it was known that to an operator corresponds a one-parameter family of $SL(2, \mathbb{R})$ cocycles, parametrized by the energy, and it was known that the spectrum coincides with the bifurcation set of the dynamical object. These are the values of the parameter such that the cocycle is not uniformly hyperbolic. Probably inspired by the results in one-dimensional dynamics, Avila described three regions in this space of cocycles: UR (uniform hyperbolic), SpC (supercritical), and SbC (subcritical) and called the complement of these C (critical). The cocycles in SpC are not uniformly hyperbolic but have positive Lyapunov exponents and were already well understood. The part of the spectrum of the operator that lies in this region corresponds typically to the pure point spectrum. Avila proved that the part of the spectrum that lies in SbC corresponds to the absolutely continuous part of the spectral measure. Finally, he proved the existence of a stratification of the space into submanifolds of positive codimension and that the set of critical cocycles is a small subset, in the measure-theoretic sense, of these submanifolds [23]. As a consequence, Avila proved that a typical one-parameter family of cocycles does not contain critical cocycles at all. Also as a consequence of Avila's construction, the spectrum of an operator corresponding to a typical potential, in a measure theoretical-sense (prevalence), decomposes into a finite number of disjoint open sets, and the spectral measure is absolutely continuous or pure point on a given open set if and only if the Lyapunov exponent is zero or positive in this region. This result justifies the intuition from physics that typically one is either in the conductor regime (absolutely continuous spectrum) or the insulator regime (pure point spectrum).

An intensely studied family of Schrödinger operators is the family of almost Mathieu operators, which is related to a celebrated physical phenomena, the quantum Hall effect. This family is not typical in the above sense because it has a critical cocycle where the coupling constant is equal to one. This is forced by a symmetry, the Aubry duality. Before the arrival of Avila in this area, much was known about this family. For example, for almost all frequencies and phases the spectrum is absolutely continuous in the subcritical region (coupling constant smaller than one) and pure point in the supercritical region. Also, the spectrum has positive Lebesgue measure in both regions. Still, some important questions remained open. Avila solved all the

remaining questions about the spectral measure of the almost Mathieu system: the spectrum is a Cantor set for critical values of the coupling constant for all irrational frequencies [14]; it has zero Lebesgue measure [24]; and for all irrational frequencies and almost all phases, the spectrum is continuous singular. To summarize, one can now say with Avila that the spectral measure is absolutely continuous precisely in the subcritical case and pure point precisely in the supercritical case.

In another area of dynamical systems, Avila proved that a C^1 volume-preserving diffeomorphism of a compact manifold can be approximated in the C^1 topology by a C^∞ volume-preserving diffeomorphism [1]. This was an open question for more than thirty years, and Avila's proof involves solving a partial differential equation with low regularity. The result is very relevant for the description of generic properties of conservative dynamical systems, because another fundamental tool, the closing lemma, is known only in the C^1 topology, and for good distortion estimates one needs higher regularity.

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Bjorn Poonen

The Work of Manjul Bhargava

Anyone solving equations over $\mathbb{Z}$ or $\mathbb{Q}$ soon recognizes the need to study the arithmetic of higher number fields and algebraic varieties. Manjul Bhargava has found new and improved ways to count many such objects by understanding the orbits of $G(\mathbb{Z})$ on $V(\mathbb{Z}) := \mathbb{Z}^n$ for various representations of algebraic groups $G \to \mathrm{GL}(V)$ over $\mathbb{Z}$. This has led to dramatic consequences in arithmetic geometry concerning the average behavior of elliptic curves and higher-genus curves over $\mathbb{Q}$.

Counting Number Fields

Consider the problem of estimating the number of isomorphism classes of degree n number fields k whose absolute discriminant D_k satisfies $|D_k| \le X$ as $X \to \infty$. The primitive element theorem guarantees that each such k is obtained by adjoining a root of a degree n irreducible polynomial over $\mathbb{Q}$. But there are many polynomials giving rise to each number field, and eliminating the redundancy is tricky.

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One can attempt to solve this and other arithmetic counting problems by following the outline below:

(1) Identify a representation $G \to \mathrm{GL}(V)$ and local conditions (inequalities and congruence conditions) such that the nondegenerate $G(\mathbb{Z})$-orbits in $V(\mathbb{Z})$ satisfying the local conditions parametrize the objects of interest.

(2) Count the orbits satisfying the inequalities by counting lattice points inside a *fundamental domain* (a region in $V(\mathbb{R})$ consisting of one representative of each $G(\mathbb{Z})$-orbit).

(3) Sieve out the orbits that violate the congruence conditions.

Davenport and Heilbronn [19] successfully carried out this three-step approach to count degree 3 number fields of bounded discriminant (and they were preceded by Gauss, who studied the average behavior of class numbers of quadratic fields, possibly by a similar method). As will be explained, Bhargava has contributed significant ideas improving all three steps, enabling him to carry out this approach in much more complicated situations.

Consider Step 1. Davenport and Heilbronn used a bijection taking certain $\mathrm{GL}_2(\mathbb{Z})$-orbits in the space $\mathrm{Sym}^3 \mathbb{Z}^2$ of degree 3 homogeneous forms over $\mathbb{Z}$ to degree 3 number fields. For $n \geq 4$, however, the representation $\mathrm{Sym}^n \mathbb{Z}^2$ of GL_2 does not yield a similar parametrization of degree n number fields. Instead, one studies *prehomogeneous vector spaces*, vector spaces V equipped with an algebraic group action for which geometrically there is a Zariski dense orbit. (The Zariski dense orbit condition is natural in hindsight, since any two degree n number fields become isomorphic when tensored with $\mathbb{C}$.) Prehomogeneous vector spaces were classified in [27]. Wright and Yukie carried out a detailed study of their arithmetic over fields in [32]. In particular, they identified prehomogeneous spaces whose orbits over $\mathbb{Q}$ gave rise to number fields of degrees 4 and 5, but they did not realize their goal of using these to *count* the fields.

Bhargava has gone through the list of prehomogeneous vector spaces to discover what their nondegenerate *integral* orbits correspond to, often with unexpected answers. For instance, whereas Wright and Yukie found a surjection taking nondegenerate $\mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_3(\mathbb{Q})$-orbits in $\mathbb{Q}^2 \otimes \mathrm{Sym}^3 \mathbb{Q}^2$ to degree 4 number fields, Bhargava discovered that nondegenerate $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_3(\mathbb{Z})$-orbits in $\mathbb{Z}^2 \otimes \mathrm{Sym}^3 \mathbb{Z}^2$ correspond to commutative rings of rank 4 over $\mathbb{Z}$ *equipped with a cubic resolvent ring* (a notion he invented). There seems to be no recipe for predicting what extra arithmetic structure is rendered visible by considering integral orbits, but Bhargava has heuristics that have guided him to an answer in each case.

The extra structure can be a nuisance or a boon. It is a nuisance when counting degree 4 fields, for instance, because each field contains many orders (subrings that are free over $\mathbb{Z}$ of the same rank as the field), and each order can have many cubic resolvent rings. But it is a boon if the extra structure itself is worth counting; for instance, one of the prehomogeneous spaces enabled Bhargava [2, Theorem 5] to prove the first nontrivial case of the Cohen-Martinet generalization [18] of the Cohen–Lenstra heuristics on the distribution of class groups.

In Step 2, heuristically the number of lattice points inside a fundamental domain $\mathcal{F}$ should be approximately the volume. If $\mathcal{F}$ has narrowing tentacles stretching to infinity, however, this heuristic can be hard to justify or even false! Bhargava's brilliant solution was to average the counts over several fundamental domains (in fact, to integrate over a continuum of fundamental domains) [2], [4]. This effectively fattens the tentacles so that the volume heuristic can be justified farther out along the tentacle, out to the point beyond which the number of relevant lattice points is provably negligible.

Step 3 is necessary to sieve out unwanted objects. For instance, each number field k has a unique maximal order $\mathcal{O}_k$, but the methods in Steps 1 and 2 lead more naturally to a count of *all* orders, so one must sieve out for each prime p the orders that fail to be maximal at p. Ekedahl [22] proved a general sieve result: for instance, given relatively prime $f, g \in \mathbb{Z}[x_1, \dots, x_n]$, he computed the density of what remains of $\mathbb{Z}^n$ after sieving out for each p the $\vec{a} \in \mathbb{Z}^n$ satisfying $f(\vec{a}) \equiv g(\vec{a}) \equiv 0 \pmod{p}$. But Bhargava's applications require sieving out also $\vec{a}$ satisfying congruences of the form $h(\vec{a}) \equiv 0 \pmod{p^2}$ for a certain polynomial h, and the existing results for the general problem of this kind rely on the *abc* conjecture [23], [24]. Bhargava circumvented this for the h arising in his application in a counterintuitive way: he mapped his problem to an Ekedahl-style counting problem in a higher-dimensional space. See [6] for the state of the art on this idea.

Ultimately, all these ideas enabled Bhargava to prove that for each $n \leq 5$, the number of isomorphism classes of degree n number fields k satisfying $|D_k| \leq X$ is asymptotic to $c_n X$ as $X \to \infty$ for an explicit constant c_n [2], [4]. He was also led to formulate a precise conjecture for higher n.

Elliptic Curves

If E is an elliptic curve $y^2 = x^3 + Ax + B$ over $\mathbb{Q}$, the set $E(\mathbb{Q})$ of rational points forms an abelian group that is finitely generated. The torsion subgroup of $E(\mathbb{Q})$ is well understood, but the rank as a function of (A, B) is so complicated that it is not even known whether it is computable in theory. Essentially the only known way to bound the rank is to observe that for any $n \geq 2$, the quotient

$E(\mathbb{Q})/nE(\mathbb{Q})$ injects into a finite computable group $\mathrm{Sel}_n E$ the *n-Selmer group*.

Birch and Swinnerton-Dyer showed that elliptic curves E equipped with a nonzero element of $\mathrm{Sel}_2 E$ are in bijection with nondegenerate $\mathbb{Q}^\times \times \mathrm{PGL}_2(\mathbb{Q})$-orbits in $\mathrm{Sym}^4 \mathbb{Q}^2$ and proved that each orbit has an (almost) integral representative [15]. This representation is not a prehomogeneous space, but at least it is *coregular*, meaning that its ring of polynomial invariants is isomorphic to a polynomial ring. Using knowledge of the invariants, Bhargava and Shankar [10] succeeded in adapting the three-step approach to prove that the average size of $\mathrm{Sel}_2 E$ as E varies is 3 and hence the average rank of $E(\mathbb{Q})$ is bounded! (See [25] for a summary of their proof.) Although de Jong [20] had proved analogous results over function fields by a similar approach of counting integral orbits and although Brumer [17] had proved that conjectures about elliptic curve L-functions would imply that the average rank over $\mathbb{Q}$ was bounded, Bhargava and Shankar were the first to prove *unconditionally* that the average rank over $\mathbb{Q}$ was bounded and also the first to obtain a *precise* value for the average of a Selmer group size over a global field.

In subsequent work, Bhargava and Shankar counted integral orbits in more complicated representations to show that for each $n \le 5$, the average of $\#\mathrm{Sel}_n E$ equals the sum of the divisors of n. They proved also that the average remains unchanged if finitely many congruence conditions are imposed upon A and B. This, combined with a theorem of Dokchitser and Dokchitser [21] relating Selmer groups to root numbers and Wong's method [31, §9] for constructing a positive-density family of elliptic curves in which the root number is equidistributed, implies that the average rank of $E(\mathbb{Q})$ is at most 0.885 [12, Theorem 3]. (Conjecturally the average is $1/2$.)

Recent advances in the arithmetic of elliptic curves allow results on Selmer groups of an elliptic curve over $\mathbb{Q}$ to be transferred to results about the rank and *analytic rank* (the order of vanishing at $s = 1$ of the L-function of E). Specifically, in addition to the Dokchitsers' work mentioned above, there is work on the Iwasawa main conjecture for GL_2 by Skinner and Urban [28] and by Wan [29]; work connecting p-adic L-functions to Heegner point heights by Bertolini, Darmon, and Prasanna [1] and by Brooks [16]; and work generalizing the Gross-Zagier formula by Yuan, Zhang, and Zhang [33]. Combining all of this with the work of Bhargava and Shankar, we now know that

- a positive fraction of elliptic curves over $\mathbb{Q}$ have rank 0 and analytic rank 0 [11] and
- a positive fraction of elliptic curves over $\mathbb{Q}$ have rank 1 and analytic rank 1 [13].

In particular, the *Birch and Swinnerton-Dyer conjecture* (that rank equals analytic rank) holds for a positive fraction of elliptic curves over $\mathbb{Q}$—in fact, more than 66 percent of them [14].

Higher-Genus Curves

Bhargava and Gross [7], using work of Wang [30], also found a coregular representation whose rational orbits parametrize 2-Selmer elements of Jacobians of hyperelliptic curves $y^2 = f(x)$, where f is a polynomial of degree $2g + 1$ over $\mathbb{Q}$ for some fixed g. They used this to prove that the average size of the 2-Selmer group of such Jacobians is 3, generalizing the result for elliptic curves. Combining this with Chabauty's p-adic method, they could prove that many odd degree hyperelliptic curves have few rational points. Later, Stoll and the present author [26] were able to combine the Bhargava-Gross results with new ideas on the 2-adic geometry of curves to show that as $g \to \infty$, the fraction of these curves having no rational points at all (other than the one at infinity) tends to 1.

Bhargava also found a representation providing information on *even* degree hyperelliptic curves, enabling him to show that for all $g \ge 0$, a positive fraction of curves $y^2 = f(x)$ with $\deg f = 2g + 2$ has no rational points (he showed that their 2-coverings fail to have local points) and the fraction tends to 1 as $g \to \infty$ [5]. Later, with Gross and Wang he showed that a positive fraction of these curves also has no points over any number field of odd degree despite having points over every completion of $\mathbb{Q}$ [8].

Combining the results for odd and even degrees, we now know that most equations of the form $y^2 = f(x)$ have no rational solutions.

Final Words

There are many other beautiful works of Bhargava we did not have space to discuss, such as a vast generalization [3] of Mahler's theorem on series expansions for p-adic continuous functions and a preprint with Hanke [9] on Conway's "290-conjecture" that a positive-definite quadratic form over $\mathbb{Z}$ represents all nonnegative integers if and only if it represents all nonnegative integers up to 290. But it is the counting techniques that have had the greatest impact. Bhargava's ideas will surely exert a strong influence on the future development of the subject.

Acknowledgments

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Jeremy Quastel

The Work of Martin Hairer

Martin Hairer of the University of Warwick was awarded a Fields Medal at the 2014 ICM in Seoul for his outstanding contributions to the theory of stochastic partial differential equations (SPDE). His most spectacular achievement has been his single-handed creation of the theory of regularity structures, which is a flexible analytic tool for giving sense to many ill-posed SPDEs. This breakthrough has opened up the entire field. Hairer is an Austrian who was raised in a mathematical family in Geneva, where he obtained a PhD in physics under Jean-Pierre Eckmann. But those who understand his work know that he is a mathematician in the best sense of the word: a quick study, patient, genial, and well rounded. His work ranges over many aspects of physically motivated problems in (stochastic) analysis. Here I will concentrate on explaining a few of his works in SPDE. First, some background.

Markov Processes

Markov processes are used widely in the sciences to model dynamical processes whose evolution is in some way random. They are defined through transition probabilities $p(t, x, A)$ representing the probability for the state X_{t+s} at future time $t + s$ to be in a set A, given that at the present time s, $X_s = x$. If we had explicit transition probabilities and an initial measure μ_0 in hand, we could in principle compute everything. However, they are usually presented to us only implicitly, the state evolving according to some sort of stochastic equation. Even in the case of stochastic *ordinary* differential equations, $X_t \in \mathbb{R}^d$, $t > 0$,

$$(1) \qquad \dot{X}_t = b(X_t) + \sigma(X_t)\xi_t,$$

where in the natural simplest cases ξ_t is *Gaussian white noise*, it is not obvious how to make sense of the equation. White noise means that averages of ξ_t over nice disjoint sets are uncorrelated, or, in other words, $B_t = \int_0^t \xi_s ds$ is Brownian motion. One of the first things one learns is that with probability 1 it is nowhere differentiable; in fact, it just fails to be Hölder 1/2. From the equation, we expect the same of X_t. A classical fact is that the Riemann-Stieltjes integral $\int F dG$ is ill defined if the Hölder exponents of F and G sum to less than 1. Hence one cannot give unambiguous meaning for typical realizations of the noise to the *stochastic integral* which is the last term in the integrated version of (1),

$$(2) \qquad X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s.$$

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However, the difference between various approximations turns out to have a probabilistically well-defined limit, and therefore the ambiguity can be understood as a modelling issue. This was worked out by Itô and others, the key concept being martingales. A downside is lack of continuity: If ξ_t^ε are smooth approximations to the white noise, $\int_0^t \sigma(X_s^\varepsilon)\xi_s^\varepsilon ds$ may not even converge to $\int_0^t \sigma(X_s)\xi_s ds$. A modern point of view [8] due to Terry Lyons clarifies the approximation issue by noting that once one has made a choice of the simplest ambiguous quantity $\mathbb{B}_t := \int_0^t B_s dB_s$, the rest is ordained. In particular, the solution map $B_t \mapsto X_t$ factors into a probabilistic step in which one enhances the Brownian motion B_t with an admissible choice of $\mathbb{B}_t$, together with an explicit, deterministic, continuous map from the resulting *rough path* $(B_t, \mathbb{B}_t)$ to the solution X_t.

Another important question about Markov processes concerns long time behavior. A nice situation is when the process is ergodic—there exists a unique invariant measure—and after a long time basically finds itself in the temporally stationary process started with the invariant measure.

In the finite-dimensional setting of (1), these problems can be studied through the operators

$$(3) \qquad L = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}$$

where $a = \sigma\sigma^*$. The transition probabilities are the kernels of the semigroup e^{tL} and invariant measure solutions of $L^*\mu = 0$. If the variance matrix a is degenerate, the uniqueness follows from the Hörmander hypoellipticity theorem, probabilistic proofs of which can be obtained using the stochastic calculus of variations a.k.a. *Malliavin calculus*.

SPDE

However, many physically motivated problems are presented to us as stochastic *partial* differential equations, in particular, several equations for processes which arise as canonical representatives of huge *fluctuation universality classes*. Despite several decades of work, methods which could deal with these important examples proved elusive, and a general theory was not expected. Now d in (3) is replaced by a continuum $\mathbb{R}^d$, and there exists no useful infinite-dimensional PDE theory to help us. Many of these equations are *far* more ill-posed than (1), where we were just marginally below the regularity at which things would have made classical sense. These singular SPDEs are relations between nonlinear functions of various derivatives of nowhere differentiable functions which are not even close to making classical sense. They seemed to contain some magical hidden meaning which could not be made precise.

Ergodicity of Navier-Stokes

An SPDE problem where the long time behavior is of obvious interest (e.g. in turbulence) is that of a randomly forced fluid. We have the incompressible Navier-Stokes equations which ask for a divergence-free vector field evolving according to

$$(4) \qquad \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla P = \nu \Delta u + F.$$

P is the pressure, and F is the external force. We assume F is acting in just a few long wavelength modes (i.e. someone is stirring it). Physically, the injected energy cascades down to smaller scales in the *inertial range* until it gets to a smallest scale, at which it is dissipated by the viscosity ν, which should be thought of as small. A natural question then is whether there is a unique invariant measure supported in this inertial range. One works in two dimensions (in $d = 3$ simply making sense of the unforced equation for more than a short time is already a millennium problem). Since the forcing is in only a few modes, the problem is highly hypoelliptic and becomes that of obtaining results that compare favorably to the Hörmander condition, but now in infinite dimensions. A key difficulty is lack of natural reference measures—it is just too easy to be singular in infinite dimensions—and a subsequent lack of good norms for the dynamics. The problem attracted the interest of several groups in the early 2000s [7], [17], [3]; with the finite-dimensional forcing it was finally solved by Hairer and Mattingly [10] with a condition allowing a forcing on only two Fourier modes for any $\nu > 0$! An asymptotic Feller property is introduced which is sensitive to the regularization of the transition densities due to both probabilistic and dynamic mechanisms, and this in turn is verified through the nondegeneracy of the Malliavin covariance matrix. Naturally there are extremely interesting and important questions, such as the distribution of energy in these invariant measures, for which there are physical predictions which await mathematically rigorous work.

The KPZ Revolutions

In the mid-1980s, Kardar, Parisi and Zhang introduced the equation

$$(5) \qquad \partial_t h = \lambda(\partial_x h)^2 + \partial_x^2 h + \xi$$

to describe a randomly growing interface in $d = 2$, such as the boundary of a bacterial colony or a slowly burning front, idealized to be a height function $h(t, x)$, $t > 0$, $x \in \mathbb{R}$. The right-hand side identifies the three key mechanisms of *relaxation* $(\partial_x^2 h)$, *uncorrelated random forcing* (space-time white noise ξ), and, most important, the nonlinear slope dependent drift or *lateral growth* $(\partial_x h)^2$.

If the asymmetry λ is set to 0, the equation becomes linear (in the noise) and readily integrated, yielding a continuous function-valued Markov process which has Brownian motion as invariant measure. A smooth initial function evolves for $t > 0$ to a locally Brownian version and in the long time limit after centering, to Brownian motion.

Amazingly, the nonlinear dynamics *also* preserves Brownian motion (modulo an absolute height shift), but the fluctuations are otherwise non-Gaussian and of nonstandard superdiffusive size $t^{1/3}$ (compared to $t^{1/4}$ when $\lambda = 0$). At the physical level, this goes back to highly nonrigorous renormalization group computations on the stochastic Burgers equation satisfied by $u = \partial_x h$.

In a breakthrough that surprised both the mathematics and physics communities, Johannson [15] (and [1] for a related model) succeeded in computing the free energy fluctuations of geometric last passage percolation and showing that normalized by $t^{1/3}$ they converge to the Tracy-Widom law, which had arisen ten years earlier in the seemingly unrelated field of random matrices. These free energies satisfy discrete versions of the KPZ equation, and at the physical level one expects all such one-dimensional systems displaying the three basic mechanisms to have the same large time asymptotics, refined into asymptotic fluctuation classes depending only on initial data. This is the *strong KPZ universality conjecture*, and the poorly understood universal large time limit is known as the *KPZ fixed point*. There has been significant progress over the last fifteen years through the work of Spohn, Sasamoto, Tracy, Widom, le Doussal, Dotsenko, Borodin, Corwin, O'Connell, Seppäläinen, Ferrari, and others on a small number of exactly solvable models, including to some extent the KPZ equation itself, where a few exact one-point distributions have been computed. However, at this level KPZ is just one model within the huge universality class. The connection with random matrices and the origin of the exact solvability and its link to representation theory have been considerably clarified by the work of Borodin and coauthors, especially under the umbrella of Macdonald Processes [2].

However, the KPZ equation is also itself universal in the sense that it is expected to be the unique heteroclinic orbit connecting (by varying time t or equivalently λ) the Gaussian fixed point to the KPZ fixed point and consequently the universal fixed point of models with weakly tuned asymmetry or noise. This is the *weak KPZ universality conjecture*. In contrast to the strong conjecture, where at the present time we can rely only on fortuitous exact computations coupled with asymptotics, the weak conjecture is open to a general attack by mathematical analysis.

To attempt it, one needs an analytic theory of KPZ. But if $h(t, x)$ in (5) is locally Brownian in x, what could the nonlinear term $(\partial_x h)^2$ possibly mean? Of course, the problem of finding ways to multiply genuine Schwartz distributions is an old

one, and various idiosyncratic techniques existed, but here one has to find the definition imposed by the physics. It was far from clear how to accomplish this, and earlier attempts turned out *not* to be physical [4].

Actually, for the KPZ equation, there is a cheap way out. At a completely formal level,

$$(6) \qquad Z(t, x) = \exp h(t, x)$$

satisfies

$$(7) \qquad \partial_t Z = \partial_x^2 Z + \xi Z.$$

Due to the roughness of ξ, there is really an infinite absolute height shift. But we are just looking at fluctuations, so a huge change of reference frame doesn't matter. The *multiplicative stochastic heat equation* (7) was one of the few truly infinite-dimensional nonlinear SPDEs which could be handled by the Itô techniques. The resulting $h := \log Z$ is known as the *Hopf-Cole solution* of KPZ, and it is now known to have the expected physical behavior arise in the weakly asymmetric/weak noise limits and is what the exact computations refer to, the remarkable fact being that the correlations, i.e. the noise averages $E[Z(t, x_1) \cdots Z(t, x_n)]$, satisfy closed Bethe ansatz solvable *delta-Bose gas* equations.

This suggests that one way to prove the weak universality conjecture would be to show that the appropriate partition functions converge to (7). This was done first by Bertini and Giacomin for a model called asymmetric simple exclusion and then later for a few others. But only very special models can be handled in this manner; one is faced again with the problem that the Itô theory is not well set up for approximations, and it just begs the question of finding a more robust theory for nonlinear SPDE.

Regularity Structures

A natural approach to make sense of a singular SPDE is through regularization. For example, we could take smooth approximations ξ_ε to our noise ξ. If we can somehow find a limit of the resulting solutions as the regularization is removed that doesn't depend on the way we regularize, we can say it is the solution. But typically the solution map lacks continuity in the necessary topology, and convergence is false. However, it may converge after renormalization by some probabilistically well-defined but now *divergent* quantity. For KPZ (5), Hairer proved (so far only on the torus $\mathbb{T}$) that *there is a diverging C_ε such that the solution $h_\varepsilon(t, x)$ of*

$$(8) \qquad \partial_t h_\varepsilon = \lambda[(\partial_x h_\varepsilon)^2 - C_\varepsilon] + \partial_x^2 h_\varepsilon + \xi_\varepsilon$$

converges to the Hopf-Cole solution, in probability, locally uniformly as continuous functions. If the regularization of the noise were just in space, this could be done using the Hopf-Cole + Itô

approach, but the key point is that the new method is *general and robust*. To give just one example, if the regularization of the noise is on a scale ε in space and ε^2 in time, the weak KPZ universality suggests that the classical solution to

$$(9) \qquad \partial_t \hat{h}_\varepsilon = \varepsilon (\partial_x \hat{h}_\varepsilon)^4 - C'_\varepsilon + \partial_x^2 \hat{h}_\varepsilon + \xi_\varepsilon$$

should converge to the Hopf-Cole solution of (5) with a highly nontrivial λ, the exponent 4 miraculously becoming a 2 in the limit. Hairer's method can be adapted to prove such things (see [13] for a review) as well as providing a good approximation theory for equations like (7) [14]. The interested reader is invited to write the equation for $\hat{Z} = \log \hat{h}$ and try to understand why on Earth it would converge to (7).

Motivated by rough paths, the idea is to factor the solution map into a deterministic part and a probabilistic part. The deterministic part is provided by an abstract solution map which acts on an abstract version of the equation together with a lift of the noise. The probabilistic part, which has some degrees of freedom, is how you lift the noise to the abstract framework.

The main new tool to construct the abstract solution map is the notion of *regularity structures*. This is a vector space tailor-made for the equation, with sufficient information to provide a local description of the solution at each point at various levels of regularity. It contains polynomials in the time and space variables, like a classical Taylor expansion. But our solution is going to be built out of various functionals of the noise which are too rough to represent using only these. So it should also contain symbols representing the input noise and enough rules for multiplication and versions of the heat operator and differentiation so that we can represent our equation there. One also needs a group of transformations from the description at one point to another point, and it is here that a highly nontrivial algebraic structure arises. Next one builds a space of functions $\mathcal{D}$ taking values in the regularity structure that generalizes the classical Hölder spaces and on which there are analogues of the Schauder estimates for the heat operator. These allow one to obtain the solution there as a fixed point.

Finally, a *reconstruction operator* pastes together the expansions at different points so that the abstract solution is realized as a genuine Schwartz distribution, which (one hopes!) is the solution to the original equation.

It is indeed the solution for the naïve lift of the regularized noise. But as we remove the regularization, things do not converge. Fortunately, a renormalization group acts on the space of lifts of the noise/multiplication rules, and by a careful choice one can find a sequence which does converge as one removes the regularization. These renormalizations are realized as concrete

renormalizations of the equations satisfied by the reconstruction; in the case of KPZ we get (8), and the Hopf-Cole limit can be thought of as at best satisfying

$$(10) \qquad \partial_t h = \lambda[(\partial_x h)^2 - \infty] + \partial_x^2 h + \xi.$$

In this way, the problem is reduced to finding the correct renormalization, usually by educated guesswork, and then proving convergence of the corresponding finite collection of multilinear transformations of the noise.

If the smooth approximating noises are Gaussian, we are in a situation of Wiener chaoses, and there are well-known criteria for convergence based on L^2 norms. While these are completely explicit, they are generally complicated convolutions of singular kernels, and convergence depends on a careful counting of singularities in integrals over a resulting graph. This is highly reminiscent of earlier calculations in quantum field theory, and Feynman-like diagrams are used to keep track of the cumbersome computations.

Dynamic Φ_3^4

The Φ_d^4 model is supposed to be the Gibbs probability measure on Schwartz distributions Φ on $\mathbb{R}^d$, or the torus $\mathbb{T}^d$ or some subset, where the probability of Φ is proportional to $e^{-\beta H(\Phi)}$ where $H(\Phi) = \int \left[|\nabla \Phi|^2 + \frac{C}{2}|\Phi|^2 - \frac{\lambda}{4}\Phi^4 \right]$. It is a universal model for phase coexistence in near-critical phenomena, closely related to SPDE and similarly ill-posed in $d \geq 2$, since the regularity of the Gaussian field obtained by setting $\lambda = 0$ is insufficient to define $\int \Phi^4$. So we take some regularization and try to find a nontrivial (i.e. non-Gaussian) limit as it is removed, perhaps after some renormalization. The problem was well studied, the main result being that it is only nontrivial in $d \leq 3$. An outstanding issue was whether the limits resulting from different regularizations in the difficult $d = 3$ case were really the same. Similar to the SPDE problems, what was missing was an *intrinsic* characterization of the limit.

It was suggested that by analogy with the Ising model, where information about the static model had been obtained from the Glauber dynamics, one might approach this through a study of the stochastic dynamics ("stochastic quantization") with Φ_3^4 as invariant measure:

$$(11) \qquad \partial_t \Phi = \Delta \Phi + \lambda \Phi^3 - C\Phi + \xi.$$

It is expected to be a universal model for such dynamical models near criticality in a limit similar to the weak limits for KPZ.

But (11) just opened a new can of worms; for twenty years nobody could even make sense of it in $d = 3$ until Hairer constructed the appropriate regularity structure, in which the solution exists as an abstract model to which lifts of appropriate renormalized solutions of regularized versions of (11) converge.

Since Hairer's first well-posedness results, several proofs have appeared for KPZ [9] and dynamic Φ_3^4 [16], [5]. However these appear to be far more closely tied to the particular regularization as well as the specifics of the equation. One of the key points of the regularity structures is that they do give an *intrinsic* characterization of solutions for a large class of equations like (5) or (11) as the solution in the appropriate $\mathcal{D}$ with the lift of the rough noise. The condition is that the equation should be *subcritical*, meaning that as you zoom in, the nonlinearity vanishes. In addition, regularity structures provide a precise description of the solution, level by level of (ir)regularity.

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Anton Zorich

The Work of Maryam Mirzakhani

On August 13, 2014 (the opening day of ICM at Seoul), Maryam Mirzakhani received the Fields Medal "for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces," becoming the first woman to win the Fields Medal. We outline several directions of Mirzakhani's research, which ranges from geometry of hyperbolic surfaces to dynamics in moduli spaces passing through applications of symplectic topology to algebraic geometry of moduli spaces. More scientific details can be found in the presentation of C. McMullen for the ICM [McM] and in a very accessible article [W] of A. Wright. For a more personal biographical note we recommend the paper of E. Klarreich [Kl].

Moduli Spaces

We are used to the fact that geometric objects might form continuous families with a rich topology. For example, the family of all straight lines in the plane passing through the origin forms a circle; the family of all complex lines passing through the origin of the vector space $\mathbb{C}^{n+1}$ forms the complex projective space $\mathbb{C}\mathrm{P}^n$.

One can consider continuous families of certain geometric structures on a fixed manifold. For example, the family of all possible complex structures on a smooth compact surface S of genus g forms the *moduli space* $\mathcal{M}_g$ of complex dimension $3g - 3$. By the uniformization theorem complex structures on a smooth surface are in natural bijection with hyperbolic metrics of constant curvature, so the moduli space $\mathcal{M}_g$ can also be seen as the family of nonsingular metrics of a fixed negative curvature on the surface S.

During the last several decades various moduli spaces became very common in mathematics and theoretical physics. When working on extremely naive objects, such as graphs or interval exchange transformations, one might run into questions related to moduli spaces.

Theoretical physics continually develops its opinion on the nature of the relevant space which we inhabit. Strings give way to d-branes, and moduli spaces of Riemann surfaces give way to moduli spaces of Calabi–Yau manifolds.

While the geometry of the world most of us inhabit is not yet clear, what is clear is that Maryam Mirzakhani has spent many years in the worlds of hyperbolic and flat surfaces.

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The Hyperbolic World
Weil-Petersson Volumes

Consider a smooth surface S of genus g with n holes (where $n = 0$ is not excluded). A closed curve α on S is called *simple* if it does not have self-intersections. Speaking about *simple closed curves* on a surface S we always tacitly assume that they are not contractible either to a point or to one of the boundary components (if there are any).

Suppose now that the surface S is endowed with a metric of constant curvature -1. By convention, we always assume that the boundary components of the resulting *hyperbolic surface X* are realized by geodesics β_i in the hyperbolic metric, where $i = 1, \ldots, n$. The hyperbolic lengths of the geodesic boundary components β_i are denoted by $b_i(X)$ or by $L_i(X) = |\beta_i|_X$. By convention, the zero value $L_i = b_i = 0$ corresponds to a cusp of the hyperbolic metric.

Fixing the hyperbolic lengths b_i of all boundary components and varying the hyperbolic metric, we get a continuous family of hyperbolic metrics on S. The real $(6g - 6 + 2n)$-dimensional space of all such metrics is called the *moduli space* $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ *of bordered hyperbolic surfaces*. By the work of W. Goldman and S. Wolpert it carries a natural closed 2-form ω_{WP} called the *Weil-Petersson symplectic form*.

The wedge power ω^n of a symplectic form on a manifold M^{2n} defines a volume form. The volume of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ with respect to the volume form ω_{WP}^{3g-3+n} is called the *Weil-Petersson* volume of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$. To give an account of Mirzakhani's work on Weil-Petersson volumes, we start with the mysterious identity of G. McShane.

Theorem (McShane). *Let $f(x) = (1 + e^x)^{-1}$ and let X be a hyperbolic torus with a cusp. Then*

$$\sum_{\gamma} f(\ell_{\gamma}(X)) = \frac{1}{2},$$

where the sum is taken over all simple closed geodesics γ on X, and $\ell_{\gamma}(X)$ is the length of the geodesic γ.

This identity is in some sense a miracle: though the length spectrum of simple closed geodesics is different for different hyperbolic tori with a cusp, the sum above is identically $1/2$ for any $X \in \mathcal{M}_{1,1}(0)$. Ten years after the work of McShane, Mirzakhani discovered a remarkable generalization of McShane's identity to hyperbolic surfaces of any genus with any number of boundary components.

Let us discuss why such identities are relevant to the Weil-Petersson volumes of the moduli spaces. Integrating the right-hand side of McShane's identity over the moduli space $\mathcal{M}_{1,1}(0)$ with respect to the Weil-Petersson form, one obviously gets

$\frac{1}{2}\operatorname{Vol}\mathcal{M}_{1,1}(0)$. What is not tautology is that the integral of the sum on the left-hand side admits a geometric interpretation as the integral of f over the natural cover $\mathcal{M}_{1,1}^*(0)$ of the initial moduli space $\mathcal{M}_{1,1}(0)$. This cover is already much simpler than the original moduli space: it admits global coordinates in which the integral of f can be easily computed.

Mirzakhani's more general identity does not immediately yield the volume. However, cutting the initial surface by simple closed geodesics involved in her identity and developing the idea of averaging over all possible hyperbolic surfaces, Mirzakhani gets a recursive relation for the volume $V_{g,n}(L_1, \ldots, L_n) := \operatorname{Vol}\mathcal{M}_{g,n}(L)$ in terms of analogous volumes of simpler moduli spaces. These relations allow Mirzakhani to prove the following statement and to compute the volumes explicitly.

Theorem ([M2]). *The volume $V_{g,n}(L_1, \ldots, L_n)$ is a polynomial in $L_1^2, \ldots, L_n^2$; namely, we have*

$$(12) \qquad V_{g,n}(L) = \sum_{|\alpha| \leq 3g-3+n} C_\alpha \cdot L^{2\alpha},$$

where $C_\alpha > 0$ lies in $\pi^{6g-6+2n-2|\alpha|} \cdot \mathbb{Q}$.

Simple recursive formulae for volumes in genera $0, 1, 2$ were found earlier by P. Zograf. Very precise asymptotics of volumes for large genera were recently proved by M. Mirzakhani and P. Zograf [MZg] (up to a multiplicative constant conjecturally equal to $\frac{1}{\sqrt{\pi}}$ which still resists a rigorous evaluation).

Counting Simple Closed Geodesics

Consider a hyperbolic surface X of finite area. In this section there are no boundary components, but we still allow the hyperbolic metric of constant curvature -1 to have cusps. We denote the corresponding moduli space by $\mathcal{M}_{g,n}$. It is known that the growth rate of the number of closed geodesics of length at most L on any such surface X has the rate e^L/L when the bound L grows.

It is not surprising that most long closed geodesics have self-intersections. A quantitative estimate of "most" is much more subtle: M. Rees and I. Rivin showed that the number $s_X(L)$ of *simple* closed geodesics of length at most L grows polynomially in L.

In counting geodesics, one can count separately those simple closed geodesics which separate X into two connected components and those which do not. More generally, two simple closed curves on a topological surface S have the same *topological type* if one curve can be transformed to another by a homeomorphism of S. For any surface of a fixed genus g with a fixed number n of punctures the number of different topological types of simple closed curves is finite. M. Mirzakhani counted simple closed hyperbolic geodesics type by type.

Theorem ([M3]). *For any hyperbolic surface $X \in \mathcal{M}_{g,n}$ the number $s_X(L, type)$ of simple closed geodesics on X of length at most L and of a fixed topological type has exact polynomial asymptotics:*

$$\lim_{L \to +\infty} \frac{s_X(L, type)}{L^d} = c(type) \cdot \operatorname{Vol} B_1^{Th}(X),$$

where $d = 6g - 6 + 2n = \dim_\mathbb{R}\mathcal{M}_{g,n}$.

For geometers we add that the quantity $\operatorname{Vol} B_1^{Th}(X)$ is the normalized volume of the "unit ball" in the space of *measured laminations* $\mathcal{ML}_{g,n}$ where the "unit ball" is defined in terms of the hyperbolic metric on X and its volume is computed in terms of the Thurston measure on $\mathcal{ML}_{g,n}$.

Note that the asymptotic proportion of simple closed geodesics of fixed type is the same for all hyperbolic surfaces X in $\mathcal{M}_{g,n}$:

$$\lim_{L \to +\infty} \frac{s_X(L, type_1)}{s_X(L, type_2)} = \frac{c(type_1)}{c(type_2)}.$$

For example, Mirzakhani shows that nonseparating simple closed geodesics on *any* hyperbolic surface of genus two are six times more frequent than the separating ones.

The proof of the theorem combines methods from two domains. On the one hand, technology elaborated by Mirzakhani in [M2] allows one to compute *averages* over $\mathcal{M}_{g,n}$ of all kinds of counting functions of simple closed geodesics (and not only Weil-Petersson volumes). To prove asymptotic formulae for *individual* hyperbolic surfaces, Mirzakhani elegantly relates the counting problems to the Thurston measure on the space of *measured laminations* $\mathcal{ML}_{g,n}$ and deduces the desired results from the ergodicity of the action of the mapping class group on $\mathcal{ML}_{g,n}$ with respect to the Thurston measure, a result proved by H. Masur.

The Symplectic World
Witten's Conjecture

Complex projective space $\mathbb{C}P^n$ mentioned in the introduction carries the natural *tautological line bundle*: its fiber over a "point" $[\mathcal{L}] \in \mathbb{C}P^n$ is the line $\mathcal{L}$ considered as a vector space. Any complex line bundle ξ over a compact manifold M can be induced from the tautological bundle by an appropriate map $f_\xi : M \to \mathbb{C}P^n$ (for a sufficiently large n depending on M). The second cohomology of the complex projective space $H^2(\mathbb{C}P^n; \mathbb{Z}) \simeq \mathbb{Z}$ has a distinguished generator c_1. The induced element $f_\xi^* c_1 \in H^2(M; \mathbb{Z})$ is called the *first Chern class* of the line bundle ξ.

There is a natural bijective correspondence between hyperbolic metrics of constant negative curvature with n cusps and complex structures endowed with n distinct marked points $x_1, \ldots, x_n$ on a closed smooth surface of genus g. In this section we use this latter interpretation of the moduli space $\mathcal{M}_{g,n}$.

Consider the (co)tangent space $\mathcal{L}(C, x_i)$ to the Riemann surface C at the marked point x_i. Varying $(C, x_1, \ldots, x_n)$ in $\mathcal{M}_{g,n}$ we get a family of complex lines $\mathcal{L}(C, x_i)$ parameterized by the points of $\mathcal{M}_{g,n}$. This family forms a line bundle $\mathcal{L}_i$ over the moduli space $\mathcal{M}_{g,n}$. This *tautological line bundle* $\mathcal{L}_i$ extends to the natural *Deligne-Mumford compactification* $\overline{\mathcal{M}}_{g,n}$ of the initial moduli space. The space $\overline{\mathcal{M}}_{g,n}$ is a nice compact complex orbifold, so for any $i = 1, \ldots, n$ one can define the first Chern class $\psi_i := c_1(\mathcal{L}_i)$. Recall that cohomology has a ring structure, so taking a product of k cohomology classes of dimension 2 (as the first Chern class) we can integrate the resulting cohomology class over a compact complex manifold of complex dimension k. In particular, for any partition $d_1 + \cdots + d_n = 3g - 3 + n$ of $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$ into the sum of nonnegative integers, one can integrate the product $\psi_1^{d_1} \ldots \psi_n^{d_n}$ over the orbifold $\overline{\mathcal{M}}_{g,n}$. By convention, the *intersection number* (or the "correlator" in a physical context) is defined as

$$(13) \qquad \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}.$$

As always, when there are plenty of rational numbers indexed by partitions or such, it is useful to wrap them into a single generating function. The resulting generating function is really famous. For physicists it is a *partition function in two-dimensional quantum gravity*. In mathematical terms, E. Witten conjectured in 1991 a certain recursive formula for the numbers (13) and interpreted this recursion in the form of KdV differential equations satisfied by the generating function. The conjecture caused an explosion of interest in the mathematical community: a single formula interlaced quantum gravity, algebraic geometry, enumerative geometry, combinatorics, topology, and integrable systems.

The first proof of Witten's conjecture is due to M. Kontsevich, who used metric ribbon graphs as a "combinatorial model" of the moduli space to express the intersection numbers (13) as a sum over 3-valent ribbon graphs. Maryam Mirzakhani suggested in [M1] an alternative proof. She ingeniously applied techniques of symplectic geometry to moduli spaces of bordered Riemann surfaces $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ discussed in the previous section. Namely, Mirzakhani recognized the intersection numbers (13) in the coefficients C_α from formula (12) for the Weil-Petersson volumes $V_{g,n}(L)$ (up to a routine normalization factor). This allowed Mirzakhani to reduce the recurrence relations for the intersection numbers contained in Witten's formula to recurrence relations for the volumes $V_{g,n}(L)$ discussed above and thus prove Witten's conjecture. (Yet other proofs are due to A. Okounkov and R. Pandharipande, who used the *Gromov-Witten theory* of $\mathbb{P}^1$ and to M. Kazarian and S. Lando, who used the *ELSV-formula*.)
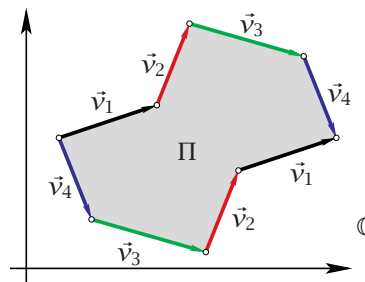

Figure 1.

The Flat World
Families of Translation Surfaces

To construct a flat metric on a surface of genus different from one, we have to allow isolated conical singularities. We consider only those flat metrics which mimic a flat metric on a torus: namely, parallel transport of any tangent vector along any closed curve avoiding singularities is required to bring the vector to itself. A surface endowed with this kind of flat metric is called a *translation* surface. Since parallel transport along a small loop around any conical singularity brings the vector to itself, the cone angle at any singularity on a translation surface is an integer multiple of 2π.

Similar to what one finds in the case of the torus, all translation surfaces can be obtained by the following construction. Consider a collection of vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^2$ and arrange these vectors into a broken line. Construct another broken line starting at the same point as the first one, arranging the same vectors in the order $\vec{v}_{\pi(1)}, \ldots, \vec{v}_{\pi(n)}$, where π is some permutation of n elements. By construction the two broken lines share the same endpoints; suppose that they bound a polygon as in Figure 1. Identifying the pairs of sides corresponding to the same vectors $\vec{v}_j$, $j = 1, \ldots, n$, by parallel translations, we obtain a closed topological surface.

It is convenient to consider the vertical direction as part of the structure. Under this convention, the structure of a translation surface is equivalent to the structure of a pair (Riemann surface C, holomorphic 1-form ω on it). As a complex coordinate on C one can use the coordinate z in the complex plane $\mathbb{C}$ where the polygon Π lives, the holomorphic 1-form ω is the form dz in these coordinates, ω has zeroes exactly at the conical singularities, and the order of zero at a singularity with the cone angle $2\pi(d_i + 1)$ is d_i.

The polygon in our construction depends continuously on the vectors $\vec{v}_i$. This means that the topology of the resulting translation surface (its genus g, the number and the types of the resulting conical singularities) do not change under small deformations of the vectors $\vec{v}_i$. Fixing a collection of cone angles $2\pi(d_1 + 1), \ldots, 2\pi(d_m + 1)$

with integer d_i for $i = 1, \ldots, n$, we get a *family* $\mathcal{H}(d_1, \ldots, d_m)$ of translation surfaces. The vectors $\vec{v}_1, \ldots, \vec{v}_n$ can be viewed as complex coordinates in this family, called *period coordinates*. These coordinates define an orbifold structure and a natural volume element on every $\mathcal{H}(d_1, \ldots, d_m)$.

Readers preferring algebro-geometric language may view any such family as a stratum in the moduli space $\mathcal{H}_g$ of pairs (Riemann surface C, holomorphic 1-form ω on C), where the stratum is specified by the degrees $d_1, \ldots, d_m$ of the zeroes of ω.

Each stratum admits a natural action of the group $\mathrm{GL}(2, \mathbb{R})$. A linear transformation $g \in \mathrm{GL}(2, \mathbb{R})$ of the ambient $\mathbb{R}^2 \simeq \mathbb{C}$ maps the polygon Π to a polygon $g\Pi$. Identifying pairs of parallel sides of $g\Pi$ by translations, we get a translation surface $g \cdot S$.

The subgroup $\mathrm{SL}(2, \mathbb{R}) \subset \mathrm{GL}(2, \mathbb{R})$ preserves the flat area. This implies that the action of $\mathrm{SL}(2, \mathbb{R})$ preserves the real hypersurface $\mathcal{H}_1(d_1, \ldots, d_m)$ of translation surfaces of unit area. The latter codimension one subspace can be compared to the unit sphere (or rather to the unit hyperboloid) in the ambient stratum $\mathcal{H}(d_1, \ldots, d_m)$. The action of the group $\mathrm{SL}(2, \mathbb{R})$ preserves the naturally induced volume element on our "unit hyperboloid." The flow induced by the action of the diagonal subgroup is called the *Teichmüller geodesic flow*.

Theorem (H. Masur, W. A. Veech). *For any $(d_1, \ldots, d_m)$, the stratum $\mathcal{H}_1(d_1, \ldots, d_m)$ has finite volume. The Teichmüller geodesic flow is ergodic on every connected component of every stratum.*

Here "ergodic" means that any measurable subset invariant under the action of the group has necessarily measure zero or full measure. The ergodic theorem says that in such situations the orbit of almost every point homogeneously fills the ambient connected component. Namely, for almost all starting data the "time" average of an integrable function f over the corresponding trajectory coincides with the "space" average, that is, with the integral of f over the ambient connected component of $\mathcal{H}_1(d_1, \ldots, d_m)$.

We state the ergodic theorem for *almost all* starting data, because even for extremely nice and smooth maps (like a map homogeneously winding
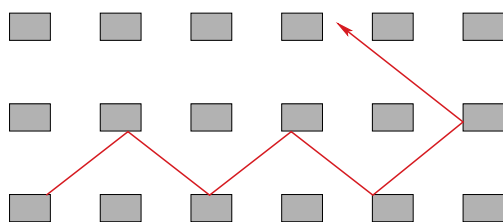

Figure 2.

a circle twice around itself) certain rare trajectories behave highly nonhomogeneously, accumulating on some peculiar (often fractal) sets. Though these atypical starting data have measure zero, the multitude of types of the corresponding atypical trajectories is enormous. For example, for the map of the circle mentioned above and for almost any real number d between 0 and 1, one can find a trajectory whose closure has Hausdorff dimension d.

The Magic Wand

Dynamical systems for which all possible orbit closures and all possible ergodic measures are described by a short list of possible simple cases are very rare. An extremely important class of examples comes from very special dynamical systems on homogeneous spaces. The fundamental theorem of M. Ratner describing possible measures and orbit closures has had fantastic applications to number theory developed by S. G. Dani, G. Margulis, and others. The theorem below shows that though the moduli space *is not* a homogeneous space, it is almost one.

Theorem ([EM], [EMM]). *The closure $\mathcal{L}$ of any $\mathrm{GL}(2, \mathbb{R})$-orbit in any stratum $\mathcal{H}(d_1, \ldots, d_m)$ is a complex orbifold given locally by linear equations in period coordinates.*

Every $\mathrm{SL}(2, \mathbb{R})$-orbit closure supports a unique ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant probability measure, and any ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant measure is supported on a suborbifold.

As vague conjectures (or better, very optimistic dreams), these properties were discussed for a long time without the slightest hint of a general proof. The only exception is the case of surfaces of genus two, for which ten years ago C. McMullen completely classified all possible orbit closures. Applying clever geometric construction, he managed to reduce the problem to genus one, where Ratner's theorem becomes applicable.

The proof of A. Eskin, M. Mirzakhani, and A. Mohammadi is a titanic work which took many years. It absorbed numerous fundamental developments in dynamical systems which mostly do not have any direct relation to moduli spaces. To mention only a few, it incorporates certain ideas of the low entropy method of M. Einsiedler, A. Katok, E. Lindenstrauss; results of G. Forni and of M. Kontsevich on the Lyapunov exponents of the Teichmüller geodesic flow; the ideas from the work of Y. Benoit and J.-F. Quint on stationary measures; iterative improvement of the properties of the invariant measure inspired by the approach of G. Margulis and G. Tomanov to the actions of unipotent flows on homogeneous spaces; some fine ergodic results due to Y. Guivarch and A. Raugi.

In some situations the theorem really serves as a Magic Wand. Consider, for example, the *windtree*

model introduced by physicists P. and T. Ehrenfest more than a century ago. We study the billiard in the plane filled periodically with the identical rectangular obstacles as in Figure 2. A trajectory might go far away, then return relatively close back to the starting point, then make other long trips. The *diffusion rate* ν describes the average rate T^ν with which the trajectory expands in the plane on a long range of time $T \gg 1$. More formally,

$$\nu := \lim_{T \to \infty} \frac{\log(\text{diameter of trajectory of length } T)}{\log T} \, .$$

For the random walk in the plane or for a billiard with periodic circular obstacles, the diffusion rate is known to be $1/2$: the most distant point of a piece of trajectory corresponding to segment of time $[0, T]$ would be located roughly at a distance $\sqrt{T}$. It was recently discovered in [DHL] that for the windtree model as in Figure 2 the diffusion rate is $2/3$.

Suppose now that we want to find the diffusion rate for a generalized windtree model with periodic scatterers of the shape of a more complicated rational polygon. Replace the periodic billiard with an associated compact flat surface. Touch it with the Magic Wand of Eskin-Mirzakhani-Mohammadi and find its $\mathrm{SL}(2, \mathbb{R})$-orbit closure in the space of flat surfaces. Run the geodesic flow to compute the mean monodromy (Lyapunov exponents) of the appropriate block of the complex Hodge bundle, and we get the answer.

To be honest, in full generality, this strategy is a new dream (though in some situations it already works; see [DZ]). We do not yet have a classification of $\mathrm{SL}(2, \mathbb{R})$-invariant orbifolds except in genus two. This presents a new challenge, which might be full of mysteries and marvels, as indicated by recent results of M. Mirzakhani and A. Wright, who have found a $\mathrm{GL}(2, \mathbb{R})$-invariant suborbifold of completely enigmatic origin in the family $\mathcal{H}(6)$.

A billiard in a polygon is just an elegant way to describe a certain class of dynamical systems; the same kind of dynamical systems appear in solid-state physics, in conductivity theory, in the theory of surface foliations, and the Magic Wand is extremely useful for the related problems in these areas (see [Zor] for details on applications of the Magic Wand). It also opens a new way to study moduli spaces.

Acknowledgments

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