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Flat Surfaces and Dynamics on Moduli Space

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Abstract. Dynamics of the Teichmüller geodesic flow on the moduli space of curves and asymptotic monodromy of the Hodge bundle along this flow have numerous applications to dynamics and geometry of measured foliations, to billiards in polygons, to interval exchange transformations, and to geometry of flat surfaces.

Mathematics Subject Classification (2010): 30F, 32G, 37D, 37E, 37H, 57M.

Introduction by the Organisers

The workshop *Flat surfaces and Dynamics on moduli space* brought together over 53 participants, many of them very young and active. (We did not take statistics, but we believe the average age of the participants to be below 35.) This area is currently extremely active. There were a number of major results presented at the conference that were obtained less than five months since the previous conference on a similar subject at ICERM (Providence). This included the example of a new $\mathrm{GL}_2(\mathbb{R})$-invariant suborbifold of an absolutely mysterious origin found by Mirzakhani and Wright, and a strategy for proving the existence (and, actually, genericity) of infinite cyclic Veech groups presented by Hamenstädt.

The background and topics of this workshop included dynamical systems, geometric topology, and algebraic geometry, as reflected in the following summary of some of the main lines of research.

The description of orbit closures of various flows on the moduli spaces is a guiding problem in this field. Major progress has been made in this direction recently. The talks of Aulicino, Bainbridge, Filip, Mohammadi, Nguyen and Wright presented progress on the $\mathrm{SL}_2(\mathbb{R})$-orbit closures. Smillie and Weiss reported on the corresponding problem for the horocycle flow. Counting problems on billards and the connection to intersection theory on moduli spaces were addressed by Athreya, Chen, Goujard, and Zograf.

The geometry of individual flat surfaces is also an active topic, both from the viewpoint of dynamics (as reported by Chaika, Lelièvre) and group theory (see the talks on Veech groups by Lehnert, Weitze-Schmitthüsen). The dynamics of translation surfaces of infinite type is a new branch in this field and has developed recently. Progress in this direction was reported on by Hooper, Treviño, and Valdez.

Ties with algebraic geometry varying from Shimura curves to p-adic origamis were discussed by Grushevsky, Herrlich, Kappes, Mondello, and Mukamel. The talks of Delecroix, Eskin, Fei Yu, Hubert were devoted to various aspects of the study of the Lyapunov exponents of the Hodge bundle and dynamical Hodge decomposition: from the relation to the Harder–Narasimhan filtration to applications to windtree billiards.

Talks in the conference unified the dynamical counterparts of such subjects as multidimensional diophantine approximations and random walks on groups as in the talks of Bufetov, Cheung, Gouëzel, to the geometric ones such as the counting of closed geodesics and study of nonclosed geodesics in the moduli space, and evaluating the lengths of the corresponding systols. These latter subjects were discussed in the talks of Boissy, Hamenstädt, and Lenzhen.

There was a broad variety of techniques in the presentations including a beautiful artistic movie of Davis using dancing to illustrate the cutting sequences of the flow in the double pentagon and multi-zooming software used by Athreya.

We have extraordinarily strong women colleagues in our area. Eight of them participated in the conference; the results of those who were unable to come, such as Maryam Mirzakhani, were presented by their collaborators.

The participants intensely discussed mathematics and worked between the talks and in the evenings. We expect to see many new results to soon emerge from these discussions.

Workshop: Flat Surfaces and Dynamics on Moduli Space

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Abstracts

Invariant measures for Linear Cocycles

Alex Eskin

(joint work with Christian Bonatti, Amie Wilkinson)

Let G denote the group $SL(2,\mathbb{R})$. (The results in this note apply to actions of any semisimple group with finite center, but since all of our applications involve $SL(2,\mathbb{R})$ we will restrict ourselves to that case). Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad U^+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \qquad U^- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix},$$

and let $A = \{g_t \ : \ t \in \mathbb{R}\}$, $P = AU^+$.

In this paper, we consider the following situation. Suppose G acts on a space X, preserving a measure ν. (In fact, we may relax the assumption that ν is G-invariant: some sort of product structure is enough). We will always assume that ν is ergodic with respect to the action of G. Let $\mathbf{H}$ be a vector space, and let $\alpha : G \times X \to GL(\mathbf{H})$ be a linear cocycle over the action of G, i.e α is a measurable map satisfying

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x)\alpha(g_2, x) \text{ for all } g_1, g_2 \in G \text{ and } \nu\text{-almost-all } x \in X.$$

Then G acts on $X \times \mathbf{H}$ via

$$g(x, \mathbf{v}) = (gx, \alpha(g, x)\mathbf{v}). \tag{1}$$

We think of $X \times \mathbf{H}$ as a bundle over X, and denote the fiber over $x \in X$ (which is isomorphic to $\mathbf{H}$) by $\mathbf{H}(x)$. We denote the action of $g \in G$ on $\mathbf{H}(x)$ by $(g)_*$, so for $\mathbf{v} \in \mathbf{H}(x)$, $(g)_*\mathbf{v} \in \mathbf{H}(gx)$.

The Osceledets multiplicative ergodic theorem states that for ν-almost every $x \in X$ there exists a g_t-equivariant splitting

$$\mathbf{H}(x) = \bigoplus_{j=1}^{m} \mathcal{V}_j(x), \tag{2}$$

and real numbers $\lambda_1 > \cdots > \lambda_m$ (called the Lyapunov exponents of α) such that for $v \in \mathcal{V}_j(x)$,

$$\lim_{|t| \to \infty} \frac{1}{t} \log \frac{\|(g_t)_* v\|}{\|v\|} = \lambda_j.$$

Definition 1 (ν-measurable almost invariant splitting)**.** *We say that the cocycle $\alpha : G \times X \to \mathbf{H}$ has a ν-measurable almost invariant splitting if there exists $n > 1$ and for ν-a.e $x \in$ there exist nontrivial subspaces $W_1(x), \dots, W_n(x) \subset \mathbf{H}(x)$ such that $W_i(x) \cap W_j(x) = \{0\}$ for $1 \le i < j \le n$ and also for a.e $g \in G$ and ν-a.e. $x \in X$,*

$$\alpha(g, x) W_i(x) = W_j(gx) \qquad \text{for some } 1 \le j \le n\ .$$

The map $x \to \{W_1(x), \dots, W_n(x)\}$ is required to be ν-measurable.

We note that the splitting (2) does not satisfy the conditions of Definition 1 since the $\mathcal{V}_j$ are (in general) equivariant only under g_t, and not under all of G.

Definition 2 (Strongly irreducible cocycle)**.** *A cocycle $\alpha : G \times X \to \mathbf{H}$ is strongly irreducible with respect to the measure ν if is does not admit a ν-measurable almost invariant splitting (as in Definition 1).*

Let $\mathbb{P}^1(H)$ be the projective space of $\mathbf{H}$ (i.e. the space of lines in $\mathbf{H}$). Then G also acts on $X \times \mathbb{P}^1(\mathbf{H})$ via the formula (1). The space $X \times \mathbb{P}^1(\mathbf{H})$ may not support a G-invariant measure, but since P is amenable and $\mathbb{P}^1(\mathbf{H})$ is compact, it will always support a P-invariant measure. In particular, for any P (or G) invariant measure ν on X, there will be a P-invariant measure $\hat{\nu}$ on $\mathbb{P}^1(\mathbf{H})$ which projects to ν under the natural map $X \times \mathbb{P}^1(\mathbf{H}) \to X$.

We can now state our main theorem.

Theorem 3. *Suppose G acts on X preserving an ergodic measure ν, and suppose $\alpha : G \times X \to GL(\mathbf{H})$ is a strongly irreducible cocycle. Let $\hat{\nu}$ be any P-invariant measure on $X \times \mathbb{P}^1(\mathbf{H})$ which projects to ν. Then, if we disintegrate*

$$d\hat{\nu}(x, \mathbf{v}) = d\nu(x)\, d\eta_x(\mathbf{v}),$$

then the measures η_x on $\mathbb{P}^1(\mathbf{H})(x)$ are in fact supported on $\mathbb{P}^1(\mathcal{V}_1)(x)$ where as in (2), $\mathcal{V}_1(x)$ is the Lyapunov subspace corresponding to the top Lyapunov exponent of α. In particular, if $\mathcal{V}_1$ is one-dimensional, $\hat{\nu}$ is unique.

Flat surfaces and strata. Suppose $g \geq 1$, and let $\beta = (\beta_1, \dots, \beta_m)$ be a partition of $2g-2$, and let $\mathcal{H}(\beta)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\beta_1 \dots \beta_m$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\beta)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo2].

Affine measures and manifolds. Let $\mathcal{H}_1(\beta) \subset \mathcal{H}(\beta)$ denote the subset of surfaces of (flat) area 1. There exists an natural $SL(2,\mathbb{R})$ action on $\mathcal{H}_1(\beta)$ which has been studied extensively.

An affine invariant manifold is a closed subset of $\mathcal{H}_1(\beta)$ which is invariant under the $SL(2,\mathbb{R})$ action and which in *period coordinates* (see [Zo2, Chapter 3]) looks like an affine subspace. Each affine invariant manifold $\mathcal{M}$ is the support of an ergodic $SL(2,\mathbb{R})$ invariant probability measure $\nu_{\mathcal{M}}$. Locally, in period coordinates, this measure is (up to normalization) the restriction of Lebesgue measure to the subspace $\mathcal{M}$, see [EMirz] for the precise definitions. It is proved in [EMM] that the closure of any $SL(2,\mathbb{R})$ orbit is an affine invariant manifold; this is analogous to one of Ratner's theorems in the theory of unipotent flows. (In genus 2, this result was previously proved by McMullen [McM]).

We will need the following:

Theorem 4 ([EMM, Theorem 2.3])**.** *Let $\mathcal{N}_n$ be a sequence of affine manifolds, and suppose $\nu_{\mathcal{N}_n} \to \nu$. Then ν is a probability measure. Furthermore, ν is the affine* *measure $\nu_{\mathcal{N}}$, where $\mathcal{N}$ is the smallest submanifold with the following property: there exists some $n_0 \in \mathbb{N}$ such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$.*

In particular, the space of ergodic P-invariant probability measures on $\mathcal{H}_1(\beta)$ is compact in the weak-star topology.

The Kontsevich-Zorich cocycle. We consider the *Hodge bundle* whose fiber above the point (M, ω) is the cohomology group $H^1(M, \mathbb{R})$ (viewed as a $2g$-dimensional real vector space). If we choose a fundamental domain for the action of the mapping class group Γ, then we have the cocycle $\tilde{\alpha} : SL(2,\mathbb{R}) \times \mathcal{H}_1(\beta) \to \Gamma$ where for x in the fundamental domain, $\tilde{\alpha}(g, x)$ is the element of Γ needed to return the point gx to the fundamental domain. Then, we define the Kontsevich-Zorich cocycle $\alpha(g, x)$ by

$$\alpha(g, x) = \rho(\tilde{\alpha}(g, x)),$$

where $\rho : \Gamma \to Sp(2g, \mathbb{Z})$ is the homomorphism given by the action on homology. The Kontsevich-Zorich cocyle can be interpreted as the monodromy of the Gauss-Manin connection restricted to the orbit of $SL(2,\mathbb{R})$, see e.g. [Zo2, page 64]. Recent papers in which the Lyapunov spectrum of the Kontsevich-Zorich cocyle over affine measures plays a major role include [Au1], [Au2], [Ba1], [Ba2], [BM], [CM1], [CM2], [DM], [EKZ], [EKZ2], [EMat], [Fo2], [F3], [FM], [FMZ1], [FMZ2], [FMZ3], [KZ1], [GH1], [GH2], [Ma], [MY], [MYZ], [MMY], [Mö], [T] and [W].

The following theorem answers a question asked in [MMY].

Theorem 5. *Let $\mathcal{N}_n$ be a sequence of affine manifolds, and suppose the affine measures $\nu_{\mathcal{N}_n}$ converge to the (affine) measure ν (as in Theorem 4). Then the Lyapunov exponents of $\nu_{\mathcal{N}_m}$ converge to the Lyapunov exponents of ν.*

The proof of Theorem 5 depends on Theorem 3 and the following theorem of S. Filip:

Theorem 6 ([Fi])**.** *Let $\alpha(\cdot,\cdot)$ denote (some exterior power of) the Kontsevich-Zorich cocycle restricted to an affine invariant submanifold $\mathcal{M}$. Let $\nu_{\mathcal{M}}$ be the affine measure whose support is $\mathcal{M}$, and suppose α has a $\nu_{\mathcal{M}}$-measurable almost-invariant splitting. Then, the subspaces $W_i(x)$ in Definition 1 can be taken to depend continuously on $x \in \mathcal{M}$.*

In fact it is proven in [Fi] that the dependence of the $W_i(x)$ on x is polynomial in the period coordinates.

Simplicity of Lyapunov Spectrum of Teichmüller curves. Recall that a *Teichmüller curve* is a closed $SL(2,\mathbb{R})$ orbit on $\mathcal{H}_1(\beta)$. Teichmüller curves (which are submanifolds of real dimension 3) are the smallest possible affine manifolds; any other type affine manifold has dimension greater than 3.

As a consequence of some recent results, we obtain the following:

Theorem 7. *All but finitely many Teichmüller curves in $\mathcal{H}(4)$ have simple Lyapunov spectrum.*

Theorem 7 was shown in [MMY] to follow from a conjecture of Delecroix and Lelièvre (stated in [MMY]). Our proof below is unconditional; however it is much less explicit and is completely ineffective. It also depends on the very recent results of Filip [Fi] and Nguyen-Wright [NW].

Proof of Theorem 7. Suppose there exist infinitely many Teichmüller curves $\mathcal{N}_n \subset \mathcal{H}_1(4)$ with multiplicities in the Lyapunov spectrum. By Theorem 4, the $\mathcal{N}_n$ have to converge to an affine manifold $\mathcal{N}$ (in the sense that the affine measures $\nu_{\mathcal{N}_n}$ will converge to the affine measure $\nu_{\mathcal{N}}$). Furthermore, $\mathcal{N}$ cannot be a Teichmüller curve (since by Theorem 4, $\mathcal{N}$ must contain all the $\mathcal{N}_n$ for n sufficiently large, and thus $\dim \mathcal{N} > 3$). By the main theorem of [NW], the only affine submanifolds of $\mathcal{H}_1(4)$ which are not Teichmüller curves are the connected components $\mathcal{H}(4)^{odd}$ and $\mathcal{H}(4)^{hyp}$ and the Prym locus. But, by by [AV], all of these have simple Lyapunov spectrum. This contradicts Theorem 5. □

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Is there a p-adic Wollmilchsau?

Frank Herrlich

The aim of this talk was to introduce a p-adic analogue of a special class of translation surfaces, namely *origamis* (or *square-tiled surfaces*). We want to know for which origamis such a p-adic analogue exists; specifically we ask this question for a prominent member of this class, the so called "Eierlegende Wollmilchsau" [2].

An *origami* is a (usually singular) translation surface which can be obtained by gluing finitely many squares. Equivalently it is a finite covering $\pi : X \to E$ of compact Riemann surfaces where E is a torus and π is ramified over at most one point (the image of the vertices of the squares). Recalling the equivalence between compact Riemann surfaces and complex projective nonsingular curves this definition carries over to arbitrary fields.

Let now K be a p-adic field, which in this talk means a field extension of $\mathbb{Q}_p$ which is contained in $\mathbb{C}_p$.

The theory of rigid analytic spaces which is based on the Tate algebra of convergent power series and the concept of affinoid domains endow every algebraic variety X with an analytic structure (called X^{an}); in particular it provides us with a p-adic analogue of a compact Riemann surface:

Definition/Theorem. A projective nonsingular curve X over K is called a *Mumford curve* if one of the following equivalent conditions holds:
(i) There is a subgroup $G \subset \mathrm{PGL}_2(K)$ acting discontinuously on some open domain $\Omega \subset \mathbb{P}^1(K)$ such that $X^{\text{an}} \cong \Omega/G$.
(ii) X^{an} can be covered by finitely many "disks with holes", i. e. subdomains of K of the form "large disk minus finitely many disks of smaller radius".
(iii) The reduction $\bmod p$ of X is totally degenerate.

Here the *reduction* of X is a projective curve $\bar{X}$ over the residue field of K; in a first approximation it is the variety defined by reducing $\bmod p$ the coefficients of the equation defining X. $\bar{X}$ is *totally degenerate* if all its irreducible components are rational.

Now we can define the key notion of this talk:

Definition. A *p-adic origami* over K is a covering $\pi : X \to E$ of Mumford curves over K such that E is of genus 1 and π is ramified over at most one point.

Any complex origami $O = (\pi : X \to E)$ defines a Teichmüller curve C_O in the moduli space M_g, where g is the genus of X. The curve C_O contains points defined over a number field, more precisely points corresponding to a covering which has a model $\pi_0 : X_0 \to E_0$ over a number field F. Since F is also contained in the p-adic field $\bar{\mathbb{Q}}_p$, we can tensor the model with $\bar{\mathbb{Q}}_p$ and obtain a covering $\pi_p : X_p \to E_p$ of curves over $\bar{\mathbb{Q}}_p$, and we can ask whether this is a p-adic origami.

More conceptually it is known that the curve C_O is defined over a number field F, i. e. is obtained from a curve $C_{O,0}$ in $M_g(F)$ by tensoring with $\mathbb{C}$. Then the question whether O defines a p-adic origami can be formulated as follows:

Question. Given an origami O, does $C_{O,p} = C_{O,0} \otimes \bar{\mathbb{Q}}_p$ intersect the locus of Mumford curves in $M_g(\bar{\mathbb{Q}}_p)$?

It should be noted that, due to a result of Lütkebohmert [4], a Teichmüller curve cannot completely be contained in the locus of Mumford curves. On the other hand, this locus is an analytically open subset, thus in particular of the full dimension $3g - 3$.

The main tool to answer the above question for a specific origami is the following result of K. Kremer which provides a complete classification of normal p-adic origamis:

Theorem. ([3], Thm. 5.1) Any normal p-adic origami is of the form $\Omega/\Gamma \to \Omega/G$ where G is a discontinuous subgroup of $\mathrm{PGL}_2(K)$ which is generated by a finite noncyclic group D and a hyperbolic element γ with a relation $\gamma\sigma\gamma^{-1} = \tilde{\sigma}$ for two (not necessarily distinct) elements $\sigma, \tilde{\sigma} \in D$; moreover Γ is a normal free subgroup of G of finite index.

For $p > 5$, the only candidates for D are the dihedral groups D_n (for $n \geq 3$), and the tetrahedral group A_4; for $p \in \{2, 3, 5\}$ there are a few more possible groups. Note that, since Γ is normal and free, the quotient homomorphism $\rho : G \to G/\Gamma$ must be injective on D. Using this observation we obtain the answer to the question in the title as a corollary to Kremer's theorem:

Corollary. The Teichmüller curve defined by the Wollmilchsau does not contain p-adic origamis (for any prime p).

Proof. Recall that the Wollmilchsau is the normal origami $W = (\pi : X \to E)$ of degree 8 with Galois group Q_8, the quaternion group of order 8. This group consists of 6 elements of order 4 and one element of order 2, which is the square of every element of order 4. Thus Q_8 has no proper noncyclic subgroup, and it is itself not isomorphic to a subgroup of $\mathrm{PGL}_2(K)$ (not even for $p = 2$). On the other hand, if W would induce a p-adic origami, the finite noncyclic group D in the theorem would have to be isomorphic to a subgroup of G/Γ; we have seen that this is impossible.

More details and explicit examples of p-adic origamis can be found in the survey article [1] and in Kremer's paper [3].

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Critical Exponents and Schreier Graphs of Veech Groups

Ralf Lehnert

Veech groups are discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ assigned to translation surfaces. Hence they act by Moebius transformations on the hyperbolic plane $\mathbb{H}$ and are called Fuchsian groups. For a Fuchsian group Γ the Γ-orbit of any base point in $\mathbb{H}$ accumulates only at the boundary $\partial\mathbb{H}$. The set of these accumulation points is called the limit set $\Lambda(\Gamma)$ and Fuchsian groups can be characterized by its size. If $\Lambda(\Gamma)$ is finite, the group is called elementary, otherwise non-elementary. A non-elementary group is either of the first kind or of the second kind if the limit set is the whole boundary or not, respectively. A refinement of this classification is achieved by considering the critical exponent $\delta(\Gamma) := \inf\{a \in \mathbb{R} \mid \sum_{\gamma\in\Gamma} e^{-a\rho(i,\gamma i)} < \infty\}$, where ρ denotes the hyperbolic metric on $\mathbb{H}$.

While the critical exponent of elementary groups is at most $\frac{1}{2}$ the critical exponent of finitely generated Fuchsian groups of the first kind is exactly 1. It is well-known that finitely generated Fuchsian groups of the first kind have finite co-volume and thus are lattices. Translation surfaces having a lattice Veech group are called Veech surfaces. The main result of the talk is the following:

Theorem 1. *There exist translation surfaces, whose Veech group has critical exponent strictly between $\frac{1}{2}$ (maximal critical exponent of elementary groups) and 1 (critical exponent of lattices).*

Since there is no translation surface known having a Veech group of the second kind the natural candidates for the theorem are the infinitely generated Veech groups of the first kind. Translation surfaces with such Veech groups are constructed by McMullen ([1]) and independently by Hubert and Schmidt ([2]). We will concentrate on the second mentioned approach. Hubert and Schmidt looked at special points on the surface: a point P such that all geodesics emanating from a singularity and passing through P will end in a singularity is called a connection point. They proved that the Veech group of a translation cover of a Veech surface X ramified over the singularities and a non-periodic connection point P is infinitely generated of the first kind. We will denote the lattice Veech group of X by Γ.

A result of Gutkin and Judge ([3]) implies that this Veech group is commensurable to $\Pi := \mathrm{Stab}_\Gamma(P)$. Commensurable groups have the same critical exponent. This is why we want to estimate $\delta(\Pi)$.

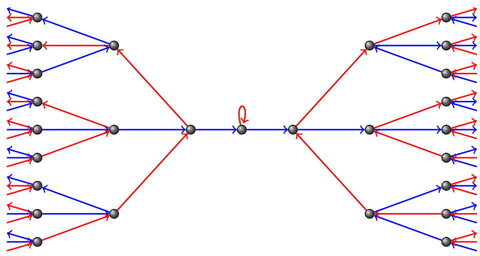

Figure 1. The root-looped 4-valent tree.

For the lower bound we observe that the property of the connection point guarantees that there are many parabolic elements in Π. By [4] the existence of a parabolic element and the fact that Π is non-elementary is sufficient for $\delta(\Pi) > \frac{1}{2}$.

Using graph-periodic manifolds Roblin and Tapie ([5]) find a criterion for the upper bound: if the Cheeger constant $c(G)$ of the Schreier graph G of $\Pi\backslash\Gamma$ is strictly positive, then $\delta(\Pi) < 1$.

The Cheeger constant of a graph G with vertex set V is defined as $c(G) := \inf_{\mathrm{finite} M\subset V}(\frac{||\partial M||}{||M||})$.

In our situation the vertices of the Schreier graph can be identified with the points of the Γ-orbit of P. We analyze translation coverings of translation surfaces L obtained from L-shaped polygons by identifying opposite sides ramified over the one singularity of L and a fixed connection point P. In [6] McMullen showed that for particular side lengths these surfaces are Veech surfaces and every Veech surface of genus 2 with one singularity up to the action of $\mathrm{GL}_2(\mathbb{R})$ is of this form. We prove 1 by proving that the Schreier graph of $\Pi\backslash\Gamma$ in this case indeed has strictly positive Cheeger constant. This is done in 4 steps:

(1) choose a finite generating set S of Γ containing large powers of the vertical and horizontal parabolic elements: $A^k, B^l \subset S$
(2) remove all edges but the edges corresponding to the elements A^k or B^l.
(3) Since this graph is no longer connected observe:
$$c(G) = \inf\{c(C) \mid C \text{ connected component of } G\}.$$
(4) define a complexity function on points of ΓP (= on the vertices) to prove that every connected component either is the infinite 4-valent tree or the root-looped 4-valent tree illustrated in Figure 1. Since both these graphs have Cheeger constant $\frac{2}{3}$ this finishes the proof.

More details to this talk can be found in my dissertation or in [7]

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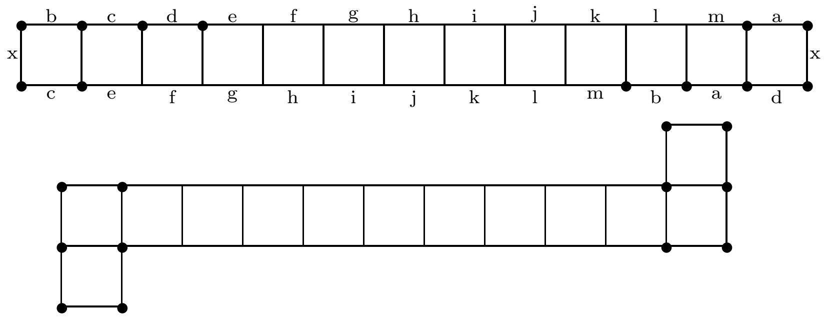
Totally non-congruence Veech groups in $H(4)$

Gabriela Weitze-Schmithüsen

We study the Veech groups of a special class of translation surfaces called *square-tiled surface* or *origamis*. They are always subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of finite index. How far are they from being a congruence group? Recall that a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ is a congruence group of level l if and only if the natural morphism $p_l : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/l\mathbb{Z})$ preserves the index, i.e. $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma] = [\mathrm{SL}(2,\mathbb{Z}/l\mathbb{Z}) : p_l(\Gamma)]$. For a general subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ we measure by the index of the image $p_l(\Gamma)$ in $\mathrm{SL}(2,\mathbb{Z}/l\mathbb{Z})$ the deficiency of Γ from being a congruence group of level l. If the index is small, then Γ is far away from being a congruence group. If the index equals 1 for all l, we call Γ a *totally non-congruence group*.

In [5] Klaus Wohlfahrt generalised the notion of the congruence level to arbitrary finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ which are not necessarily congruence groups. He assigned to such a group Γ the number L, today called *Wohlfahrt level*, which is the least common multiple of the cusp widths of Γ. If Γ is a congruence group, then its minimal congruence level and the Wohlfahrt level coincide. We have shown in [4] that in general the index $[\mathrm{SL}(2,\mathbb{Z}/l\mathbb{Z}) : p_l(\Gamma)]$ becomes maximal, if l equals the Wohlfahrt level L. In particular, Γ is a totally non-congruence group, if and only if $[\mathrm{SL}(2,\mathbb{Z}/L\mathbb{Z}) : p_L(\Gamma)] = 1$. We furthermore showed using the classification of origamis in $H(2)$ by Hubert/Lelièvre and McMullen (see [1] and [3]) that if X is an origami in $H(2)$, then $[\mathrm{SL}(2,\mathbb{Z}/L\mathbb{Z}) : p_L(\Gamma)]$ is 1 or 3. We generalise this statement using the classification of Lanneau/Nguyen (see [2]) to the Prym locus in $H(4)$. Recall that this locus consists of those translation surfaces X which allow an affine homeomorphism f_{-I} with derivative $-I$ which has four fixed points, i.e. the quotient X/f_{-I} is of genus 1.

The main tool in the proof is a criterion found in [4] which assures that a group Γ is a totally non-congruence group. Consider for this the cusps of Γ at ∞ and at 0. Let b_∞ and b_0 be the widths of these cusps, respectively, and let b be the least common multiple of b_∞ and b_0. Furthermore consider a conjugate $\Gamma' = A\Gamma A^{-1}$ ($A \in \mathrm{SL}(2,\mathbb{Z})$) of Γ and define b' for Γ' similarly to b. If b and b' are relatively prime, then Γ and its conjugate Γ' are totally non-congruence groups. This criterion is convenient for Veech groups of origamis, since two origamis in the same orbit have conjugated Veech groups. All we have to do is to find two representatives of the orbit, such that the horizontal and vertical cylinder decomposition assures that we have the desired property for the cusp widths. This works for about half of the cases. A modification of the criterion gives us for the remaining cases in the Prym locus of $H(4)$ that their Veech groups satisfy $[\mathrm{SL}(2,\mathbb{Z}/L\mathbb{Z}) : p_L(\Gamma)] = 3$. The figure above shows two representatives for one of the types of orbits that occur. Here the cusp widths b_∞ and b_0 are 13 and 13 in the first example, and 11 and 2 in the second example.



Two origamis in $H(4)$ in the same orbit: Edges labelled by the same letter and non-labelled opposite edges are glued

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Veech groups of infinite type surfaces

Ferrán Valdez

(joint work with Camilo Ramírez Maluendas)

Let S be a *translation surface*, that is, such that the transition functions of the corresponding structure are translations. We are interested in the so called *tame translations surfaces*. These are characterized by the following property: if $\hat{S}$ denotes the metric completion of S with respect to the natural translation invariant

flat metric inherited from the plane, then $\hat{S} \setminus S$ consists only of *cone angle* singularities (of finite or infinite total angle). As a consequence of Gauss-Bonnet's theorem, all compact translation surfaces are tame. On the other hand, a surface is said to be of *infinite type* if its fundamental group $\pi_1(S)$ is not finitely generated. Infinite type tame translation surfaces naturally appear when studying classical dynamical systems such as (irrational) polygonal billiards [5] or Wind-tree models [2]. *Wild* translation surfaces also exist and appear naturally when studying classical dynamical systems, such as Baker's maps [1]. Henceforth all translation surfaces taken into consideration will be tame and of infinite type, unless stated otherwise.
The main object of our interest regarding tame infinite type translation surfaces is the Veech group $\Gamma(S) < \mathrm{GL}_2^+(\mathbf{R})$ of S. This group of matrices is formed by all the differentials of affine diffeomorphisms of S that preserve orientation. More precisely, we are interested to describe, when the topological type of S is fixed, the different groups $\Gamma(S)$ that appear as we vary the tame translation surface structure. The following results describes what should be expected:

Theorem 1. [3] *The Veech group of a tame flat surface is either:*

(1) *countable and formed by matrices M satisfying $||Mv|| < ||v||$ for all $v \in \mathbf{R}^2$ (i.e. without contracting elements),*
(2) *conjugated to the group of matrices:*
$$P := \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} \mid t \in \mathbf{R},\ s \in \mathbf{R}^+ \right\},$$
(3) *conjugated to P', the group generated by P and $-\mathrm{Id}$ or,*
(4) *equal to $\mathrm{GL}_2^+(\mathbf{R})$.*

It is not difficult to see that if $\Gamma(S) = \mathrm{GL}_2^+(\mathbf{R})$, then S is either a plane or a ramified covering (over only one point) of a plane. The description we are aim to has already been carried out for the simplest infinite genus surface without planar ends:

Theorem 2. [3] *Any subgroup G of $\mathbf{GL}_2^+(\mathbf{R})$ satisfying assertion (1), (2) or (3) of Theorem 1 can be realized as Veech group of a tame flat surface S of infinite genus and only one end.*

Recall that infinite genus orientable surfaces without planar ends are characterized, up to homeomorphism, by the closed, compact, metrizable and totally disconnected topological space $\mathrm{Ends}(S)$, the *space of ends* of S [4]. Given its topological properties, we can always think of Ends(S) as a closed subspace of the Cantor set $\mathcal{C}$ that emerges middle-thirds removal procedure performed on an interval. Reciprocally, for every closed subspace X of the Cantor set $\mathcal{C}$ there exists a surface S such that $\mathrm{Ends}(S)$ is homeomorphic to X. Our first result states that all non countable Veech groups of tame translation surfaces appear in all infinite genus surfaces without planar ends.

Theorem 3. *Let X be any closed subspace of the Cantor set. Then there exists infinite genus tame translation surfaces S and S' such that their Veech groups*

$\Gamma(S_1)$ and $\Gamma(S_2)$ are conjugated to P and P' respectively. Moreover S and S' have no planar ends and $\mathrm{Ends}(S) = \mathrm{Ends}(S')$ *is homeomorphic to X.*

The proof of this theorem is based in a simple but delicate construction that starts with X and a connected subgraph $G(X)$ of the infinite binary tree whose space of ends is precisely X. Then, using $G(X)$ as guide we glue copies of an infinite genus and one end tame translation surface to obtain S_1 and S_2.

Question 4. *Let X be any closed subspace of the Cantor set. Can any countable subgroup G of $\mathrm{GL}_2^+(\mathbf{R})$ without contracting elements be realized as the Veech group of a tame translation surface of infinite genus, without planar ends and such that* $\mathrm{Ends}(S)$ *is homeomorphic to X?*

If we were to order infinite genus surfaces without planar ends by their topological complexity, we would obtain a spectrum with two extremities. At one extremity, say the left one, we would have the so called *Loch Ness monster*, which has only one end, and at the other extremity we would have the so called *blooming Cantor tree*, whose space of ends is homeomorphic to the Cantor set. Our main result implies that question 4 has a positive answer for both extremities of the aforementioned spectrum.

Theorem 5. *Let G be a countable subgroup of $\mathrm{GL}_2^+(\mathbf{R})$ without contracting elements. Then there exists a tame translation surface S, homeomorphic to the blooming Cantor tree, whose Veech group $\Gamma(S)$ is precisely G.*

The proof of this theorem is based in a simple but delicate construction used to prove theorem 2. The rough idea is to a produce Loch Ness monster whose Veech group is G, that will play the role of building block, and then glue countably many copies of this monster in a G equivariant affine way that mimics the structure of an infinite binary tree.

Final remarks. Surprisingly, the question we pose is still open for the so called *Jacob's ladder*, that is, S has two ends, each of which has genus. In particular, there is no known example of a tame Jacob's ladder whose Veech group is a lattice. On the other hand, the translation surfaces obtained in theorems 2, 3 and 5 have all infinite area. Almost nothing is know for finite area but infinite type translation surfaces. In particular:

Question 6 (Pascal Hubert)**.** *Does there exist an infinite genus but finite area translation surface S such that its Veech group is a lattice?*

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Orbit closures of translation surfaces

Alex Wright

We begin with a list of examples of orbit closures of translation surfaces.

(1) Spaces of covers of tori branched over 1 point.
(2) Spaces of covers of tori branched over more than 1 point.
(3) Closed orbits not arising from tori, for example the orbit of the regular octagon. Spaces of covers of surfaces on a given closed orbit.
(4) Eigenform loci in genus 2 (Calta [1], McMullen [4],). Prym eigenform loci in genus at most 5 (McMullen [5]).
(5) Strata. Spaces of double covers of surfaces in a given stratum of quadratic differentials. Branched covering constructions over these.

Recall

Theorem 1 (Eskin-Mirzakhani-Mohammadi [2])**.** *Every orbit closure is an affine invariant submanifold.*

For an orbit closure $\mathcal{M}$, we define

$$\mathrm{rank}(\mathcal{M}) = \frac{\dim(\mathcal{M}) - (\text{dimension of rel deformations in } \mathcal{M})}{2}.$$

The first four examples above have rank 1.

We also define the (affine) field of definition $\mathbf{k}(\mathcal{M})$ to be the smallest field such that $\mathcal{M}$ can be locally defined by linear equations with coefficients in this field. We say that $\mathcal{M}$ is arithmetic if $\mathbf{k}(\mathcal{M}) = \mathbb{Q}$. Above, examples 1, 2, and 5 are arithmetic.

Conjecture 2 (Mirzakhani's covering conjecture)**.** *If $\mathcal{M}$ is arithmetic, it arises from a covering construction.*

Conjecture 3 (Mirzakhani's arithmeticity conjecture)**.** *If $\mathcal{M}$ is higher rank (i.e., $\mathrm{rank}(\mathcal{M}) > 1$), then $\mathcal{M}$ is arithmetic.*

Some evidence for the covering conjecture will be given in David Aulicino's talk. This talk is about the arithmeticity conjecture.

Theorem 4 (Wright [8])**.** $\mathrm{rank}(\mathcal{M}) \cdot \deg \mathbf{k}(\mathcal{M}) \leq g$.

Here g is the genus. This result shows that larger orbit closures are indeed more arithmetic. An immediate corollary is that the arithmeticity conjecture is true in genus 3. (It was previously known to be true in genus 2 as a particular consequence of work of McMullen [6].)

Theorem 5 (Wright [9]). *Say $(X,\omega) \in \mathcal{M}$, and the cylinders in some direction on (X,ω) have circumferences $c_1,\dots,c_k$. Then*

$$\mathbf{k}(\mathcal{M}) \subset \mathbb{Q}[c_2/c_1,\dots,c_k/c_1].$$

An immediate corollary is that in a nonarithmetic orbit closure, every cylinder on every surface is parallel to another, with irrational ratio of circumferences.

Theorem 6 (Möller [7], Filip [3]). *Any nonarithmetic orbit closure parameterizes surfaces with "some real multiplication."*

The above three results indicate just a hint of why a nonarithmetic higher rank orbit closure would be strange and interesting.

Previously we had believed that intricate inductive arguments might establish the arithmeticity conjecture. However, we have found a counterexample.

Joint work in progress with Mirzakhani. *There is a nonarithmetic rank 2 orbit closure in the stratum of genus 4 translation surfaces with a single singularity.*

We had previously believed that, if such an object existed, it would have to have a rank 1 orbit closure in its boundary which displayed rank 2 behaviour. This is indeed what occurs in the new orbit closure: We find in the boundary a new rank 1 orbit closure of genus 2 translation surfaces with one marked point.

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Higher Rank Orbit Closures in $\mathcal{H}^{\mathrm{odd}}(4)$

David Aulicino
(joint work with Duc-Manh Nguyen, Alex Wright)

1. Introduction

It is a long standing problem to understand the $\mathrm{SL}(2,\mathbb{R})$ orbit closures of translation surfaces in moduli space. Most recently, a significant breakthrough was made in the work of Eskin and Mirzakhani, and Eskin, Mirzakhani, and Mohammadi, [4, 5] where they showed that every orbit closure is an affine invariant manifold. This made the work stated in this abstract possible.

Before the theorems of [4, 5], there was already progress in genus two. Work of McMullen [6] and Calta [3] showed that in $\mathcal{H}(2)$ either the orbit is closed, i.e. it is a Teichmüller curve, or it is dense in the stratum. In $\mathcal{H}(1,1)$, they showed that there are intermediate dimensional orbit closures given by Prym eigenforms. Otherwise, the orbit is again either a Teichmüller curve or dense in the stratum.

In $\mathcal{H}(4)$ there are two connected components: $\mathcal{H}^{\mathrm{hyp}}(4)$ and $\mathcal{H}^{\mathrm{odd}}(4)$. It was shown by Nguyen and Wright [7], that all orbits in $\mathcal{H}^{\mathrm{hyp}}(4)$ are either Teichmüller curves or dense in $\mathcal{H}^{\mathrm{hyp}}(4)$.

We prove that the Prym locus is the unique intermediate dimensional orbit closure in $\mathcal{H}^{\mathrm{odd}}(4)$. Recall that the Prym locus $\tilde{\mathcal{Q}}(3,-1^3)$ is the canonical double covering of the stratum of quadratic differentials on the torus with a triple zero and three simple poles.

Theorem 2 ([1] Thm. 1.1)**.** *The only proper higher rank affine invariant submanifold of $\mathcal{H}^{odd}(4)$ is the Prym locus $\tilde{\mathcal{Q}}(3,-1^3)$.*

This theorem and its complete proof can be found in [1].

3. Background

Let (X,ω) be a translation surface written as a pairing of a Riemann surface carrying an Abelian differential. Let Σ denote the finite set of singularities of ω on X. Let $\{\gamma_1,\ldots,\gamma_n\} \subset H_1(X,\Sigma,\mathbb{Z})$ be a basis for relative homology. Period coordinates are defined by the map

$$\Phi : (X,\omega) \mapsto \left(\int_{\gamma_i} \omega\right) \in \mathbb{C}^n.$$

By [4, 5], period coordinates provide local coordinate charts for the orbit closure $\mathcal{M}$ of (X,ω).

We consider the tangent space to an affine manifold in a subspace of relative cohomology, i.e. $T_{\mathbb{C}}(\mathcal{M}) \subset H^1(X,\Sigma,\mathbb{C})$. We can restrict to the real tangent space corresponding to $T_{\mathbb{R}}(\mathcal{M}) \subset H^1(X,\Sigma,\mathbb{R})$. Then there is a canonical projection

$$p : H^1(X,\Sigma,\mathbb{R}) \to H^1(X,\mathbb{R}),$$

which can be restricted to the tangent space to yield $p(T_{\mathbb{R}}(\mathcal{M})) \subset H^1(X, \mathbb{R})$. It was proven in [2], that $p(T_{\mathbb{R}}(\mathcal{M}))$ is symplectic so, in particular, it has even dimension. We define the rank of an orbit closure following [8], to be half the dimension of $p(T_{\mathbb{R}}(\mathcal{M}))$.

In the stratum $\mathcal{H}(4)$, p is an isomorphism. Hence, the dimension of $T_{\mathbb{R}}(\mathcal{M})$ is either 2, 4, or 6. If $p(T_{\mathbb{R}}(\mathcal{M}))$ has rank one, then the orbit is a Teichmüller curve, and if it has rank three, then the orbit is dense in the connected component. Therefore, it suffices to study orbit closures with rank two.

4. Sketch of the Proof

The main tool used in the proof of Theorem 2 is the cylinder deformation theorem of [8]. In order to take advantage of the extra dimensions in the orbit closure, [8] is used to guarantee the existence of a translation surface on which we can stretch and twist some, but not all, of the cylinders. Then we consider all possible cylinder diagrams and prove that every rank two orbit closure contains a translation surface that decomposes into three cylinders, the maximum possible in $\mathcal{H}(4)$.

By analyzing all of the three cylinder diagrams, we are able to show that some cannot lie in a rank two orbit closure, and those that do, must exhibit symmetries that allow the translation surface to admit a double cover to a torus carrying a quadratic differential in the stratum $\mathcal{Q}(3, -1^3)$.

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$\mathbf{GL}_2^+(\mathbb{R})$ -orbit closures in Prym eigenform loci

Duc-Manh Nguyen

(joint work with Erwan Lanneau)

Given a positive integer $g \geq 2$, and let $\kappa = (k_1, \dots, k_n)$ be an integral vector such that $k_i > 0$ and $k_1 + \cdots + k_n = 2g - 2$, we denote by $\mathcal{H}(\kappa)$ the moduli space of pairs (X, ω), where X is a Riemann surface of genus g, and ω is a holomorphic one-form having exactly n zeros with orders $k_1, \dots, k_n$. An element of $\mathcal{H}(\kappa)$ is called a *translation surface.* There exists a "natural" action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}(\kappa)$. For numerous applications, especially the computation of various invariants related to the dynamics of this action, the knowledge of orbits closures is crucial. It has been observed that $\mathcal{H}(\kappa)$ has several similarities with locally homogeneous spaces. It was conjectured that any orbit closure is a "nice" submanifold of $\mathcal{H}(\kappa)$. This conjecture was verified in genus two by the work of McMullen [7], its resolution in full generality has been announced recently by Eskin, Mirzak hani, and Mohammadi [1, 2], and Filip [3, 4].

It is well known that there exist in any stratum $\mathcal{H}(\kappa)$ two types of orbits, those are dense (the closure is the whole stratum) and those are closed (the closure is the orbit itself). For a long time, little is known about the existence of $\mathrm{GL}_2^+(\mathbb{R})$-orbits that are neither closed nor dense. In his classification of orbit closure in genus two, McMullen pointed out the existence of such intermediate orbits, they belong to the set or Riemann surfaces whose Jacobi variety admits a real multiplication by $\mathbb{Q}$ or a quadratic field. Following McMullen, we call such subsets *Prym eigenform loci.* Latter McMullen showed that similar loci also exist in genus $3, 4, 5$, and there are infinitely many *primitive* closed orbits contained in those loci (see [6]).

The techniques developed by McMullen to classify the orbit closures in genus two do not generalize easily to higher genus. The purpose of this talk is to introduce an alternative proof of McMullen's classification in genus two (for the case of Prym eigenforms) not involving the result by Eskin-Mirzakhani-Mohammadi, which can be used to obtain the same classification in the Prym eigenform loci in genus $3, 4, 5$. Our proof relies on the following properties of the Prym eigenforms

(1) Prym eigenforms are completely periodic in the sense of Calta, which means that whenever one has a regular closed geodesic in a Prym eigenform, the direction of this geodesic is actually periodic, *e.g.* any geodesic ray in this direction is either a closed geodesic or terminates at a singularity.
(2) Those loci are invariant by moving in the leaves of the kernel foliation, *e.g.* up to the action of $\mathrm{GL}_2^+(\mathbb{R})$, a neighborhood of any Prym eigenform consists of surfaces having the same absolute coordinates and the relative coordinates are slightly changed.

The key idea of the proof is to use the action of the horocycle flows in two distinct directions, and a result on the topological stability of cylinder decompositions.

It is worth noticing that the result is not new, since it can be easily obtained by using Eskin-Mirzakhani-Mohammadi's result. Nevertheless, we would like to emphasize on the fact that our proof is "almost" elementary, and can be used to show that there only exist finitely many closed orbits in each component of the Prym eigenform loci in genus three.

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Eigenform certification in three ways

Ronen E. Mukamel
(joint work with Abhinav Kumar)

A particularly rich class of flat surfaces arises from the study of eigenforms for Jacobian endomorphisms. For a compact Riemann surface X of genus g and a totally real field K of degree g over $\mathbb{Q}$, we will say that the Jacobian $\mathrm{Jac}(X)$ admits *real multiplication by* K if there is a holomorphic homomorphism $T : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$, self-adjoint with respect to the Rosati involution, which generates a ring isomorphic to an order in K. We will call a holomorphic one-form $\omega \in \Omega(X)$ an *eigenform for real multiplication* if $\mathrm{Jac}(X)$ admits real multiplication by a field K generated by T with T stabilizing ω up to scale.

McMullen described genus two eigenforms as polygon cut-and-paste and showed that they exhibit remarkable dynamical properties.

Theorem 1 (McMullen)**.** *If* (X,ω) *is a genus two eigenform with double zero, then the Veech group* $\mathrm{SL}(X,\omega)$ *is a lattice.*

Since then, there have been many results demonstrating that eigenforms are central examples in the study of translation surfaces and dynamics on moduli spaces of quadratic differentials. For instance, Möller proved a strong converse to McMullen's theorem for higher genus surfaces with lattice Veech group.

Eigenforms for real multiplication can also be described as algebraic one-forms on algebraic curves. In this talk, we will demonstrate several methods for verifying that an algebraic one-form on an algebraic curve is an eigenform and prove the following theorem.

Theorem 2. *The algebraic one-form* $\omega = dw/z$ *on the genus two algebraic curve* X *defined by the Weierstrass equation*

$$z^2 = 24 + 52w - 8w^2 - 12w^3 - 2w^4 + w^5$$

is an eigenform for real multiplication by $\mathbb{Q}(\sqrt{3})$.

Our first method is numerical. We compute the periods of dw/z to high precision and verify that they satisfy an appropriate $\mathbb{Q}(\sqrt{3})$-linear relationship. This method is relies the tools in Magma related to analytic Jacobians and provides strong evidence that our Main Theorem is true even though it does not give a proof.

In our second method, we numerically sample an explicit algebraic correspondence on X using the tools related to the Abel-Jacobi map in Magma. Interpolation then gives an exact equation with integer coefficients which can be used to prove our Main Theorem using only rigorous arithmetic in number fields.

In our final method, we use Teichmüller theory and the equations for the Hilbert modular surfaces given by Elkies and Kumar. This method can also be used to prove our Main Theorem as well as to give an explicit algebraic model for the the image of $\mathrm{SL}_2(\mathbb{R})$-orbit of (X, ω) in the moduli space of genus two Riemann surfaces.

Shimura curves in the locus of Jacobians of curves of low genus

Samuel Grushevsky
(joint work with Martin Möller)

In this talk we presented our joint work, partly in progress, with Martin Möller, on constructing infinitely many examples of Shimura curves contained in the locus of Jacobians. Some results have appeared in [1], with further work forthcoming.

Working over the complex numbers, we recall that a Shimura subvariety of the moduli space $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash \mathrm{Sp}_{2g}(\mathbb{R})/U(g)$ of principally polarized abelian varieties is a subvariety induced by a homomorphism $G \to \mathrm{Sp}_{2g}(\mathbb{R})$ for some algebraic group G. A well-known conjecture states that for genus g sufficiently large there do not exist (non-zero dimensional) Shimura subvarieties contained generically in the locus of Jacobians of smooth curves. This conjecture has seen a lot of work, with progress towards proving it by Viehweg, Zuo, Möller, and most recently by Lu and Zuo who proved that such one-dimensional subvarieties do not exist under some additional hypothesis (being of Mumford type).

On the other hand, a number of Shimura curves contained in the loci of Jacobians of low genus curves were described as loci of curves admitting certain automorphisms by many authors including Mumford, Oort, de Jong, Moonen, Noot, and others — but the list of such examples is finite.

Working in genus 3, where the locus of Jacobians is dense in $\mathcal{A}_3$, we construct *infinitely many* Shimura curves contained in the locus of hyperelliptic Jacobians, which is codimension one in $\mathcal{A}_3$. We prove

Theorem 1 ([1]). *For any fixed $u \in \mathbb{Q}[i]$, the family of abelian threefolds given by period matrices of the form* $\begin{pmatrix} t+iu^2 & u^2/2 & iu \\ u^2/2 & t & u \\ iu & u & i \end{pmatrix}$, *where t is the parameter lying in a suitable translate of the upper half-plane, defines a Shimura curve contained in the locus of hyperelliptic Jacobians of genus 3.*

We further conjecture that these are the only Shimura curves contained in the locus of hyperelliptic Jacobians of genus 3. To discover these curves, and to have a basis for the conjecture, we develop new techniques and criteria allowing us to obtain restrictions on possible Shimura curves contained in the zero loci of modular forms. Further applying these techniques and expanding modular forms in Fourier series near various strata of the toroidal compactification of $\mathcal{A}_3$ in particular shows that any such Shimura curve must have degeneration of the type above.

Working in genus 4, which is the first case when the locus of Jacobians is not dense in $\mathcal{A}_g$, we construct, generalizing a geometric construction of Pirola using Prym varieties of triple covers, the following examples:

Theorem 2. *There exist* infinitely many *Shimura curves contained in the locus of Jacobians of genus 4 curves.*

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Cutting out arithmetic Teichmüller curves in genus two with theta functions

André Kappes
(joint work with Martin Möller)

A square-tiled surface is a flat surface (X, ω) that admits a map $p : X \to E$ to an elliptic curve E (usually taken to be the square torus) such that p is branched over at most one point and $p^*dz = \omega$. Affinely deforming the square-tiled surface yields an algebraic curve in $\mathcal{M}_g$, an arithmetic Teichmüller curve. It is a hard combinatorial problem to decide, which square-tiled surfaces generated the same Teichmüller curve.

So far, a classification has only been obtained for square-tiled surfaces in the stratum $\Omega\mathcal{M}_2(2)$, i.e. whose map p is doubly ramified at one point [4], [2]; the invariants in this case are the degree d of a minimal map to a torus and a second invariant ε, the spin. The Euler characteristics of the associated Teichmüller curves $W_{d^2,\varepsilon}$ been computed in [1], [3].

In our work in progress, we give an approach to describe the Teichmüller curves in $\Omega\mathcal{M}_2(1,1)$, i.e. generated by square-tiled surfaces, where p is simply ramified at two points. There is an additional invariant: M, the torsion order in $\mathrm{Jac}(E)$ of the difference of the images of the ramification points. Conjecturally, d, M and ε classify all arithmetic Teichmüller curves in $\Omega\mathcal{M}_2(1,1)$. A refined version of this conjecture, due to Zmiaikou [6], also gives precise values of the Euler characteristics in the case $M = 1$.

If (X, ω) is a square-tiled surface of genus 2, then its Jacobian $\mathrm{Jac}(X)$ is isogenous to a product of elliptic curves. A reformulation of this fact says that $\mathrm{Jac}(X)$

admits multiplication by the pseudo-quadratic order

$$\mathfrak{o}_{d^2} = \{(a,b) \in \mathbb{Z}^2 \mid a \equiv b \bmod d\},$$

where d is the degree of a minimal map $p : X \to E$. Thus, $\mathrm{Jac}(X)$ corresponds to a point in a pseudo-Hilbert modular surface $X_{d^2} = \mathbb{H}^2/\mathrm{SL}(\mathfrak{o}_{d^2} \oplus \mathfrak{o}_{d^2}^\vee)$ parametrizing abelian surfaces whose endomorphism ring contains $\mathfrak{o}_{d^2}$.

We formulate algebraic conditions that cut out the Teichmüller curve generated by (X, ω) inside X_{d^2}. In the case of $\Omega\mathcal{M}_2(2)$, this has been done by Möller and Zagier [5] using derivatives of theta functions evaluated at Weierstraß points. The case $\Omega\mathcal{M}_2(1,1)$ is more involved as we need to pass to the universal family of abelian surfaces $A_{d^2} \to X_{d^2}$. We cut out the branch locus of a square-tiled surface with torsion order M in A_{d^2} by the following three conditions: that such points be on the curve, embedded via the Abel-Jacobi map into its Jacobian, that they be a zero of the first eigendifferential with respect to multiplication by $\mathfrak{o}_{d^2}$, and that they map to M-torsion points under the projection to the elliptic curve. Note that the push-forward of the branch locus to X_{d^2} gives the class of the union of Teichmüller curves (possibly with multiplicities) – at least if $M > 2$. For $M \in \{1, 2\}$, we also cut out the reducible locus and the curves $W_{d^2,\varepsilon}$.

The three conditions can be recast in terms of the classical Riemann theta function

$$\vartheta : \mathbb{H}_2 \times \mathbb{C}^2 \to \mathbb{C}, (z,u) \mapsto \vartheta(z,u), (Z,u) \mapsto \sum_{x \in \mathbb{Z}^2} e^{\pi i x^T Z x + 2\pi i x^T u}$$

pulled back to A_{d^2}: A point $x \in A_{d^2}$ is on the curve if and only if it is in the vanishing locus of ϑ, and x is a zero of the first eigendifferential if and only if $\frac{\partial \vartheta}{\partial u_2}(x) = 0$.

Having defined a suitable compactification of A_{d^2} and X_{d^2}, we can compute the class of the branch locus by writing it as the triple intersection of three divisor classes in the Chow ring of A_{d^2}. These classes can be explicitly calculated, and the push-forward is a linear combination of the Hodge classes coming from the two projections of A_{d^2} to the universal family of elliptic curves. As one result of our methods, we are able to compute the Euler characteristic of the union of Teichmüller curves with invariant d (for odd d), ε and $M = 1$, and thereby confirm the counting part of Zmiaikou's conjecture; the problem of irreducibility of this locus remains however to be settled.

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On cohomological dimension of the moduli space of Riemann surfaces

Gabriele Mondello

The moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g \geq 2$ is can be looked at from several perspectives.

As an orbifold, it make sense to consider its de Rham cohomology. Because $\mathcal{M}_g$ is not compact, its top-degree de Rham cohomology group vanishes and so one could wonder what is the highest degree for which it does not. In order to make this quantity invariant under finite unramified covers, one needs to consider de Rham cohomology with coefficients in a (real or complex) flat vector bundle.

This invariant $\mathrm{cohdim}_{dR}(\mathcal{M}_g)$, called *de Rham cohomological dimension* of $\mathcal{M}_g$, was computed by Harer in [4]. It turns out that $\mathrm{cohdim}_{dR}(\mathcal{M}_g) = 4g-5$, and so quite smaller than $\dim_{\mathbb{R}}(\mathcal{M}_g) = 6g-6$.

But $\mathcal{M}_g$ has also the structure of complex analytic orbifold, and so it makes sense to speak of its Dolbeault cohomology with coefficients in a holomorphic vector bundle and of its *Dolbeault cohomological dimension.* As before, being $\mathcal{M}_g$ non-compact and connected, we know that the top-dimensional group always vanishes but quite a stronger vanishing is suspected to hold.

Expectation (Looijenga): $\mathrm{cohdim}_{Dol}(\mathcal{M}_g) = g-2$.

While $\mathrm{cohdim}_{Dol}(\mathcal{M}_g) < 3g-3$ is the only known bound for general g, still some evidence supports this conjecture:

(a) it is verified for $g = 2, 3, 4, 5$ (see [7], [3]);
(b) it implies Diaz's result [2] that no compact holomorphic subvariety of $\mathcal{M}_g$ can have (complex) dimension bigger than $g-2$;
(c) it implies Looijenga's vanishing [5] of tautological cohomology classes above degree $g-2$;
(d) it implies the above mentioned Harer's vanishing result for de Rham cohomology.

In this talk we sketch the proof of the following result (which is clearly non-optimal, at least for $g = 2, 3, 4, 5$).

Theorem A. For all $g \geq 2$ we have

$$\mathrm{cohdim}_{Dol}(\mathcal{M}_g) \leq 2g-2\,.$$

As the Dolbeault cohomological dimension behaves additively for projectivized vector bundles, the above statement is equivalent to the following.

Theorem A'. For all $g \geq 2$ we have

$$\mathrm{cohdim}_{Dol}(\mathbb{P}\mathcal{H}_g) \leq 3g-3$$

where $\mathcal{H}_g = \{(C,\varphi) \,|\, C \in \mathcal{M}_g,\ \varphi \in H^{1,0}(C)\}$ is the holomorphic Hodge bundle.

Now, $\mathbb{P}\mathcal{H}_g$ has a well-known holomorphic stratification by multiplicities of the zeroes of φ: the typical locally closed stratum of (complex) codimension $2g-2-k$ (up to finite unramified cover) looks like $\mathbb{P}\mathcal{H}(m_1, m_2, \ldots, m_k)$, that is the space of triples $(C, P, [\varphi])$, where $C \in \mathcal{M}_g$, $P \subset C$ is a subset of $k \geq 1$ distinct points $P = \{p_1, \ldots, p_k\}$ and φ vanishes at p_i of order $m_i \geq 1$, so that $m_1 + \cdots + m_k = 2g-2$.

The first step is to show the following result, which might be interesting on its own (though it is non-optimal already in genus 2, and for $\mathbb{P}\mathcal{H}(4)$ and $\mathbb{P}\mathcal{H}(3,1)$ in genus 3, when the strata are affine and so have $\mathrm{cohdim}_{Dol} = 0$, see [6]).

Theorem B. For all $g \geq 2$ and all k and $m_\bullet$ we have

$$\mathrm{cohdim}_{Dol}(\mathbb{P}\mathcal{H}(m_1, \ldots, m_k)) \leq g\,.$$

The second step is to find an open cover of $\mathbb{P}\mathcal{H}_g$ adapted to the above stratification (i.e. with the same combinatorics) and such that the statement of Theorem B extends to each open subset. Because the stratification goes down to codimension $2g-3$ at most, one can conclude that the projectivized Hodge bundle has cohomological dimension at most $(2g-3)+g = 3g-3$.

Due to [1], the proof of Theorem B can be reduced to producing a real-valued exhaustion (i.e. bounded from below and proper) function ψ on $\mathbb{P}\mathcal{H}(m_1, \ldots, m_k)$ such that a semi-definite negative subspace $W_- \subset T_{(C,P,[\varphi])}\mathbb{P}\mathcal{H}(m_1, \ldots, m_k)$ of its complex Hessian $i\partial\overline{\partial}\psi$ is at most g-dimensional at every point $(C, P, [\varphi])$.

Such a ψ is constructed using the area function of the flat metric $|\varphi|^2$ on C with conical singularities at P and the length functions (still relative to the metric $|\varphi|^2$) associated to short saddle connections on C (i.e. smooth $|\varphi|^2$-geodesics on C joining couples of conical points).

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Orbit closures in Teichmüller dynamics and extremal cycles in algebraic geometry

Dawei Chen
(joint work with Izzet Coskun)

Let X be a normal projective variety. Let $D = \sum_{i=1}^n a_i Z_i$ be a divisor, i.e. a linear combination of codimension-one subvarieties $Z_i \subset X$. Define an equivalence relation "$\equiv$" for divisors, called *numerical equivalence*: $D \equiv D'$ if $D \cdot C = D' \cdot C$ for every curve $C \subset X$, where $D \cdot C$ is the intersection number of D and C. Define the *effective cone* of X by

$$\mathrm{Eff}(X) = \{D = \sum_{i=1}^n a_i Z_i \mid a_i \geq 0\}/\equiv .$$

By definition, $\mathrm{Eff}(X)$ has a convex structure. If the class of a divisor D spans a one-dimensional face of $\mathrm{Eff}(X)$, we call D an *extremal effective divisor*. Understanding $\mathrm{Eff}(X)$ amounts to finding out all the extremal effective divisors on X.

The effective cone governs the birational geometry of X. For instance, the canonical divisor class of X is contained in the interior of $\mathrm{Eff}(X)$ if and only if X is of *general type*, which is a higher dimensional analogue of curves of genus ≥ 2. On the other hand, $\mathrm{Eff}(X)$ may fail to be closed or finite polyhedral, see [K, II 4.16] for an example of the latter.

Here we focus on the case when X is the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus g curves with n ordered marked points. Since the 1980s, motivated by the problem of determining the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$, many authors have constructed families of effective divisors on $\overline{\mathcal{M}}_{g,n}$. For example, Harris, Mumford and Eisenbud [HM, H, EH], using Brill-Noether and Gieseker-Petri divisors showed that $\overline{\mathcal{M}}_g$ is of general type for $g > 23$.

Although we know many examples of effective divisors on $\overline{\mathcal{M}}_{g,n}$, the structure of $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$ remains mysterious in general. In particular, for a long time it was not known whether there exist g and n such that $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$ is not finitely generated.

In [CC], we study the case of genus one. By exhibiting infinitely many extremal effective divisors on $\overline{\mathcal{M}}_{1,n}$ for every $n \geq 3$, we are able to show that $\mathrm{Eff}(\overline{\mathcal{M}}_{1,n})$ is not finitely generated. The construction of those extremal divisors is motivated by the strata of quadratic differentials in genus one.

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a collection of n integers satisfying $\sum_{i=1}^n a_i = 0$, not all equal to zero. Consider the stratum of quadratic differentials $\mathcal{Q}(\mathbf{a})$ parameterizing quadratic differentials q on smooth genus one curves E such that

$$(q)_0 - (q)_\infty = \sum_{i=1}^{n} a_i p_i,$$

where $p_1, \ldots, p_n \in E$ are distinct. We do not require q to have simple poles only. Denote by $D_{\mathbf{a}}$ the closure of the projection of $\mathcal{Q}(\mathbf{a})$ in $\overline{\mathcal{M}}_{1,n}$. Then $D_{\mathbf{a}}$ is an effective divisor on $\overline{\mathcal{M}}_{1,n}$.

Assume that $n \geq 3$ and that $\gcd(a_1, \ldots, a_n) = 1$. The main result of [CC] says that $D_{\mathbf{a}}$ is an extremal effective divisor on $\overline{\mathcal{M}}_{1,n}$. Moreover, these $D_{\mathbf{a}}$ provide infinitely many extremal divisors, and hence $\mathrm{Eff}(\overline{\mathcal{M}}_{1,n})$ is not finitely generated.

Here we present the outline of a proof that works for the nonvarying strata $\mathcal{Q}(k, -1^k)$ and $\mathcal{Q}(k, 1, -1^{k+1})$ in the sense of [CM] (see [CC] for a different proof that works in general). Take a Teichmüller curve C in such a nonvarying stratum $\mathcal{Q}(\mathbf{a})$. Since we know the sum of Lyapunov exponents of C by [CM, Section 8], one can check that $C \cdot D_{\mathbf{a}} < 0$ in this case. Since the union of Teichmüller curves forms a (Zariski) dense subset in $D_{\mathbf{a}}$, it follows that $D_{\mathbf{a}}$ is extremal.

As a concluding remark, $\mathrm{SL}(2, \mathbb{R})$-orbit closures in the strata of abelian and quadratic differentials provide a number of algebraic cycles in $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$, based on the recent breakthrough [EM] and [F]. These cycles are insufficiently studied from the viewpoint of algebraic geometry. It would be interesting to figure out their intersection-theoretic properties and extremality in the cone of effective cycles.

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Counting special trajectories for right-angled billiards

Jayadev S. Athreya

(joint work with Alex Eskin and Anton Zorich)

In our work [1, 2] we study counting problems for special trajectories on families of right-angled billiards. Namely, we assume that the billiard table is a topological disk endowed with a flat metric, and that the boundary of the disk is piecewise geodesic such that the angle at every corner of the boundary is an integer multiple of $\frac{\pi}{2}$. We consider families of polygons sharing the same interior corner angles $\left(\frac{\pi}{2}k_1, \frac{\pi}{2}k_2, \ldots, \frac{\pi}{2}k_n\right)$. Actually, it will be convenient to consider a slightly larger space $\mathcal{B}(k_1, \ldots, k_n)$ of "directional billiards" distinguishing a billiard table Π and the same table turned by angle ϕ. The measure in the space $\mathcal{B}(k_1, \ldots, k_n)$ is the product measure of Lebesgue measure arising from the side lengths and the angular measure $d\phi$.

We count the number of generalized diagonals of bounded length in such billiards (that is, the number of trajectories of bounded length which start in some fixed corner P_i and arrive to some fixed corner P_j, and the number of closed billiard trajectories of bounded length. Note, that closed regular trajectories are never isolated in rational billiards: they always form bands of "parallel" closed trajectories of the same length. Thus, when counting closed trajectories one actually counts the number of such bands. By convention we always count *non-oriented* generalized diagonals and *non-oriented* closed billiard trajectories.

In this report, we mention two representative results.

Theorem 1. *For any right-angled billiard Π outside of a zero measure set in any family $\mathcal{B}(k_1, \ldots, k_n)$ the number $N_{ij}(\Pi, L)$ of generalized diagonals of length at most L joining a pair of fixed corners P_i, P_j with angles $\frac{\pi}{2}$ has the following quadratic asymptotics as $L \to \infty$:*

$$N_{ij}(\Pi, L) \sim \frac{1}{2\pi} \cdot \frac{L^2}{\textit{Area of the billiard table}}\,. \tag{1}$$

The fact that this asymptotics does not depend at all on the billiard table is at the first glance counterintuitive. What is even more surprising is that it is universal: it is the same not only for almost all billiard tables inside each family, but it does not vary even from one family to another! In particular, though the shape of two polygons of the same area can bbe quite different, the number of trajectories of length at most L joining the right-angle corner P_i to the right-angle corner P_j is approximately the same for both polygons, and in fact is approximately the same as the number of trajectories of length at most L joining two corners of the usual rectangular billiard of the same area when $L \gg 1$.

The situation becomes more complicated when we consider other types of corners of the billiard. Consider, for example, an L-shaped billiard table. Let $P_1, \ldots, P_5$ be the right-angle corners of the L-shaped billiard, and let P_0 be the corner with the interior angle $\frac{3\pi}{2}$.

Theorem 2. *For almost any* L*-shaped billiard Π the number $N_{i0}(\Pi, L)$ of generalized diagonals of length at most L joining a fixed corner P_i with angle $\frac{\pi}{2}$ and the corner P_0 with angle $\frac{3\pi}{2}$ has the following quadratic asymptotics as $L \to \infty$:*

$$N_{i0}(\Pi, L) \sim \frac{2}{\pi} \cdot \frac{L^2}{\textit{Area of the billiard table}}\,. \tag{2}$$

The naive intuition does not help: the angle $\frac{3\pi}{2}$ at the corner P_0 is *three* times larger than in the previous case, while the constant in the asymptotics for the number of generalized diagonals is *four* times larger than in the previous statement.

Currently we have no idea how to obtain this factor 4 without using techniques of the Teichmüller geodesic flow, Lyapunov exponents of the Hodge bundle, and the computation of volumes of the moduli spaces of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$.

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Cries and whispers in windtree models

Vincent Delecroix

(joint work with Anton Zorich)

The windtree model is a billiard in the plane introduced by P. and T. Ehrenfest [3] and J. Hardy and J. Weber [5]. Identical rectangular scatters of size $a \times b$ are disposed regularly along the lattice $\mathbb{Z}^2$. A particle moves in the complement of these rectangles and bounces elastcally off on the rectangular scatterers (see Figure 1). We denote by $\phi_t^{a,b}(x,\theta)$ the *flow* of the billiard.

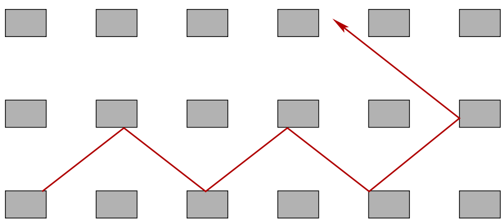

Figure 1. Original windtree model.

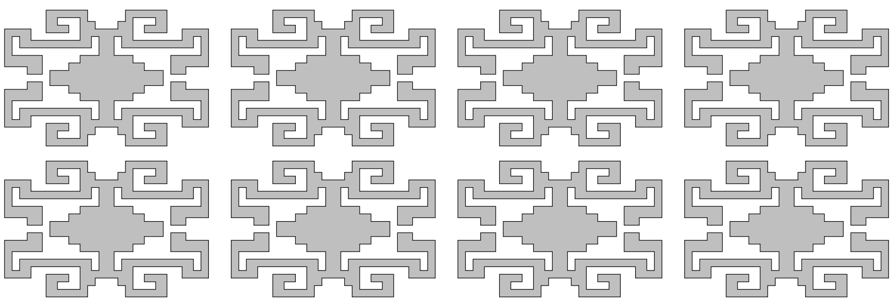

Figure 2. An exotic windtree model.

It was shown by V. Delecroix, P. Hubert and S. Lelièvre[2] that the escape rate of particles is of the order of $t^{2/3}$, much faster than for a random walk for which it is of the order of $t^{1/2}$. And more precisely:

$$\limsup_{t\to\infty} \frac{\mathrm{d}((x,\theta), \phi_t^{a,b}(x,\theta))}{\log t} = 2/3.$$

where $\mathrm{d}(.,.)$ is any reasonable distance on $\mathbb{R}^2 \times S^1$. This result is true with respect to any parameters a, b of the rectangular scatterers, for almost every angle θ and any departure position x.

The exponent $2/3$ is an exponent of the Kontsevich-Zorich cocycle. The fact that we have an explicit number in that case comes from the work of A. Eskin, M. Kontsevich and A. Zorich [4].

In 2012, J.-C. Yoccoz asked to the author what diffusion coefficients arise when one changes the shape of obstacles. Using volume computations in [1] we were able to express this exponent for obstacles with only right angles and a horizontal and vertical symmetry (see an example in Figure 2). Because of the 4-fold symmetry, we have $4m$ corners with angle $\pi/2$ and $4(m-1)$ corners with angle $3\pi/2$ for some positive integer m. We proved together with A. Zorich that the diffusion gets smaller as the obstacle gets more complicated and more precisely

$$\limsup_{t\to\infty} \frac{\mathrm{d}((x,\theta), \phi_t((x,\theta))}{\log t} = \frac{(2m)!!}{(2m+1)!!}.$$

Where $k!! = k \cdot (k-2) \cdot (k-4) \cdots$. The above result is valid for almost all lengths of the obstacle once the shape is fixed. Let us remark that we recover the exponent $2/3$ of the windtree model when $m = 1$.

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Homologous cylinders in flat surfaces: Geometry of configurations and Siegel–Veech constants

Elise Goujard

(joint work with Max Bauer)

We study the geometry of configurations of homologous cylinders in a flat surface using ratios of Siegel–Veech constants. These ratios can be interpreted geometrically as the mean area of a cylinder in a configuration $\mathcal{C}$, the proportion of large cylinders (that fill a large part of the area of the surface) of type $\mathcal{C}$ and the probability for a cylinder to be large. For a fixed configuration of homologous (resp. $\hat{\text{h}}$omologous) saddle connections in a stratum of Abelian (resp. quadratic) differentials, we obtain formulas for these ratios, that depend only on the dimension of the stratum and on the number of cylinders in the configuration.

For the family $\mathcal{Q}(1^k, -1^l)$ in genus g, we obtain more precise results: we are able to explicit the relation between Siegel–Veech constants and volumes of the boundary strata. This extends the results of Athreya, Eskin and Zorich in [AEZ] to genera $g \geq 1$.

Statement of results:

Let S be a flat (translation or half-translation) surface. Let $\mathcal{C}$ be an admissible configuration of saddle connections for the stratum containing S (see [EMZ] and [MZ] for the definition and the classification of configurations of saddle connections for Abelian and quadratic differentials).

We consider the following numbers:

- $N_{cyl}(S, R, \mathcal{C}) = \text{Card}\{\gamma \text{ of type } \mathcal{C}, \gamma \text{ fills a cylinder}, |\gamma| \leq R\}$
- $N_{area}(S, R, \mathcal{C}) = \frac{1}{\text{Area of } S} \sum\limits_{\substack{\gamma \text{ of type } \mathcal{C}, \\ |\gamma| \leq R}} \text{area of the cylinder(s) filled by } \gamma$
- $N_{cyl}(S, R, \mathcal{C}, p) = \text{Card}\{\gamma \text{ of type } \mathcal{C},\ |\gamma| \leq R$, area of the cylinders filled by γ is at least part p of the area of $S\}$
- $N^1_{cyl}(S, R, \mathcal{C}, p) = \text{Card}\{\gamma \text{ of type } \mathcal{C},\ |\gamma| \leq R$, area of one distinguished cylinder filled by γ is part p of the area of $S\}$

Then the associated Siegel–Veech constants

$$c_*(\mathcal{C}) = \lim_{R \to \infty} \frac{N_*(S, R, \mathcal{C})}{\pi R^2}$$

exist for almost every flat surface S, and depend only on the connected component of the stratum containing S (and $\mathcal{C}$, p), following [EM].

The ratios of Siegel–Veech constants can be interpreted geometrically as follows:

- $\dfrac{c_{area}(\mathcal{C})}{c_{cyl}(\mathcal{C})}$: Mean area of a cylinder in configuration $\mathcal{C}$
- $\dfrac{c_{cyl}(\mathcal{C}, p)}{c_{cyl}(\mathcal{C})}$: Proportion of large cylinders in configuration $\mathcal{C}$

- $\dfrac{c^1_{cyl}(\mathcal{C},p)}{c_{cyl}(\mathcal{C})}$: Probability for a fixed cylinder in configuration $\mathcal{C}$ for being large.

Theorem 1. *Consider a stratum of Abelian or quadratic differentials, of complex dimension d. Let $\mathcal{C}$ be an admissible configuration with q cylinders for this stratum. Let n be the complex dimension of the boundary strata for this configuration ($n = d - q - 1$). Then:*

$$\frac{c_{area}(\mathcal{C})}{c_{cyl}(\mathcal{C})} = \frac{1}{d-1} \tag{1}$$

$$\frac{c_{cyl}(\mathcal{C},p)}{c_{cyl}(\mathcal{C})} = \frac{B(1-p;n,q)}{B(n,q)} = (1-p)^n \sum_{k=0}^{q-1} \binom{n-1+k}{k} p^k \tag{2}$$

$$\frac{c^1_{cyl}(\mathcal{C},p)}{c_{cyl}(\mathcal{C})} = (1-p)^{d-2} \tag{3}$$

Note that since ratios (1) and (3) depend only on the dimension of the stratum, summing on all configurations in the stratum we obtain the formulas already found by Vorobets in [Vo].

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Enumeration of 1-cylinder configurations and distribution of genomic distance

Peter Zograf

A 1-cylinder configuration is a gluing of the boundary circles of the cylinder by means of an interval exchange map. Such a gluing is described by a pair of permutations $\sigma, \tau \in S_n$, where σ is a cycle of length n and τ is arbitrary. The cycle type of the commutator $\sigma\tau\sigma^{-1}\tau^{-1}$ is called the type of the configuration and is denoted by $\mu = [1^{m_1}2^{m_2}\ldots]$.

Two different approaches to the enumeration of 1-cylinder configurations of a given type μ are proposed. The first one is based on the Frobenius formula [4]:

Proposition 1. *The number $N(\mu)$ of 1-cylinder configurations of type μ is given by*

$$N(\mu) = \frac{1}{|\mathrm{Aut}(\mu)|} \sum_{r=1}^{n} r!(n-1-r)!\, \chi_r(\mu) \;,$$

where $|\mathrm{Aut}(\mu)| = 1^{m_1} m_1! 2^{m_2} m_2! \ldots$, and $\chi_r(\mu)$, $r = 0, \ldots, n-1$, are the coefficients of the polynomial

$$\sum_{r=1}^{n} (-1)^r \chi_r(\mu) x^r = \frac{(1-x)^{m_1}(1-x^2)^{m_2} \ldots}{1-x} \;.$$

Another approach is in the spirit of [1] and utilizes a combinatorial recursion for the numbers $N(\mu)$. In terms of the generating function

$$F(s, t_1, t_2, \ldots) = \sum_{n=1}^{\infty} \sum_{\mu \vdash n} N(\mu) s^{n-1} t_1^{m_1} t_2^{m_2} \ldots$$

one has

Proposition 2. *Put*

$$M_1 = \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \left((i-1) t_j t_{i-j} \frac{\partial}{\partial t_{i-1}} + j(i-j) t_{i+1} \frac{\partial^2}{\partial t_j \partial t_{i-j}} \right) \;.$$

Then F satisfies the evolution equation

$$\frac{\partial F}{\partial s} = M_1 F$$

that, together with the initial condition $F|_{s=0} = t_1$, determines F uniquely. (Equivalently, F is explicitly given by the formula $F = e^{sM_1} t_1$.)

Put $t_i = t$ and consider the following specialization of F:

$$F(s, t, t, \ldots) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} h_{n,k} s^{n-1} t^k \;.$$

The numbers $h_{n,k}$ first appeared in genomics in the Bafna-Pevzner approach to genome comparison and genome rearrangements [3] and were later called the Hultman numbers (cf. http://oeis.org/A164652). Namely, $h_{n,k}$ is the number of genomes build from the same set of n genes at the genomic (2-break) distance $n-k$ from the original one. The limiting distribution of the Hultman numbers is a relevant question in bioinformatics [2]:

Proposition 3. *As $n \to \infty$ the numbers $h_{n,k}$ become normally distributed in k with the mean and variance both equal to $\log n$.*

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The set of uniquely ergodic interval exchanges with a fixed permutation on 4 (or more) letters is path connected

Jon Chaika

(joint work with Sebastian Hensel)

Theorem 1. *Let π be an irreducible, non-degenerate permutation on* 4 *or more letters. The set of uniquely ergodic interval exchange transformations with permutation π is path connected as a subset of Δ.*

Remarks (1) This is not a statement about foliations. 4 interval exchange transformations (IETs) correspond to foliations in $\mathcal{H}(2)$. The set of arational foliations or uniquely ergodic foliations is not path connected in $\mathcal{H}(2)$.
(2) The theorem does not hold for permutations on 3 or fewer letters.
(3) We prove the theorem for the permutation (4321). Then we work to extend to more general permutations. Indeed the paths we build in permutations on more letters spend most of their time in 3-dimensional slices that are secretly 4-IETs.

2. Strategy

2.1. **Basic idea.** We have two uniquely ergodic IETs with permutation (4321). We start with the straight line path $\mathcal{P}_1$ connecting them. Inductively given a path connecting $\mathcal{P}_n$ that connects the two IETs we improve it to a path $\mathcal{P}_{n+1}$ that "looks better". All of these paths probably have plenty of points that are not even minimal. However, in the limit:
(a) We get a set that only contains uniquely ergodic points.
(b) This limiting set is a path.
These two issues are overcome by the same mechanism.

2.2. **Paths that look better.** It is easier to tackle approximate minimality then unique ergodicity. Indeed for an interval exchange to be non-minimal it has to have a periodic orbit or split into two disjoint components. These can be seen in a finite number of steps. This requires that the IET has two distinct discontinuities in the same orbit. When this occurs we can check whether this will cause non-minimality. So if $T \in \mathcal{P}_n$, δ, δ' are discontinuities of T so that $T^i(\delta) = \delta'$ for $i < m_{T,n}$ then this is not a cause of non-minimality. To do this we use Rauzy induction.

2.3. **Properties of Rauzy induction.** We refer to the excellent introduction [2] for a comprehensive treatment of Rauzy induction. We restrict out attention to stating some properties:

Rauzy induction is an almost everywhere defined map on the space of IETs. It is like a Gauss map for IETs; the gauss map can provide the terms in the continued fraction expansion of a number. These numbers tell us finer and finer information about the rotation: the behavior of its orbits, the region the rotation angle could be chosen in, nearby periodic rotations that are close to the rotations, etc. The same is true for Rauzy induction which provides matrices $M(T, n)$ that capture much of this data. For instance:

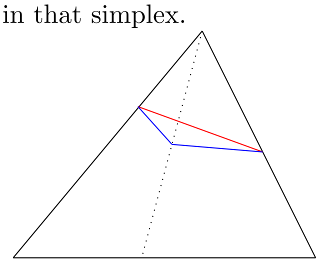
$$\begin{aligned} M(T,n)\Delta &= \{M(T,n)\mathbb{R}^4_+ \cap \Delta\} \\ &= \{S : R^i(S) \text{ is coarsely the same as } R^i(T) \text{ for } i \leq n\}. \end{aligned}$$

Moreover non-minimal IETs are guaranteed to be points where Rauzy induction is not defined, though unlike the Gauss map the converse is not true. This is necessary for our construction.

So the planes where some power of Rauzy induction is not defined identify candidates for non-minimality. Moreover they are stratified by the power of Rauzy induction that is not defined on them.

Our procedure is to improve our paths by making them cross the fail planes of higher powers of Rauzy induction at good points. In fact we have an algorithm. The input are two "adjacent" fail planes of Rauzy induction. The output is the next fail plane of Rauzy induction the path will cross and the point at which it crosses it.

FIGURE 1. A schematic: The red line is $\mathcal{P}_i$. The blue is $\mathcal{P}_{i+1}$. The solid lines denote a simplex of Rauzy induction. The dotted line is a fail plane in that simplex.



2.4. **Features.** We choose the point of crossing carefully. So a crossing of a fail plane we have chosen will be on all future paths. The crossings cause columns to be added together frequently and many columns to interact. The consecutive failure points get all the columns to interact.

To obtain unique ergodicity we use a criterion of Veech:

Theorem 3 (Veech [1] p.225)**.** *Suppose that T is an IET so that $R^n(T)$ is defined for all $n \geq 1$, and such that $\bigcap_{n\geq 1} M(T, n)\Delta = \{T\}$. Then T is uniquely ergodic.*

This criterion is also connected to how we get paths in the limit: we define to greater and greater depth the sub-simplices that the path travels through. If our simplices are getting smaller than the paths can wiggle less.

3.1. **Some issues.** Some of our simplices which border early crossed fail planes will essentially be very close to being lines. In order to have our limit object be a path at the point on such a early fail plane we use the fact that consecutive failure points get all the columns to interact. Indeed, as we follow the point on this early fail plane, 3 columns are pointing in roughly the same direction. When we cross to the other side of some future fail plane the other column interacts with them and then points in roughly the same direction.

3.2. **Generalizing to higher permutations.** There is a key definition for generalizing:

Definition 4. *An IET S is called a* secret 4-IET at level k *if there exists $M(T, r)$ a matrix of Rauzy induction with $r \leq k$, v a nonnegative vector with at most 4 non-zero entries so that $L(S) = M(T, r)v$.*

We prove that the set of Secret 4-IETs at level k are path connected. Moreover the paths we build are unions of paths of secret IETs at level k_i except possibly at the endpoints. Also the paths extend to paths at the endpoints.

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Unique ergodicity without moduli spaces

Rodrigo Treviño

Let (S, α) be a flat surface. By that I mean that S is a Riemann surface and α a 1-form on S which is holomorphic. That this defines a flat surface is a standard fact; one can consult Zorich's excellent introduction to the area to see how this is done [7]. The form α defines two dynamical systems on S, called the *horizontal and vertical* flows. One obtains them by integrating the unit vector fields in $\ker \Im(\alpha)$ and $\ker \Re(\alpha)$, respectively, away from the zeros of α. This is well-defined by a theorem of Frobenius since α is holomorphic and therefore closed. Both of these flows preserve the Lebesgue measure on S since, away from the zeros of α, they are locally defined as isometries. Since we can rotate the vector fields and obtain one from the other, it does not matter much which one we study. However, to be consistent, whenever I mention a *translation flow* I will refer to the horizontal flow. I will make the following two assumptions about the surfaces (S, α) which I am willing to consider:

Finite area: Specificaly, we have that $\frac{2}{i}\int_S \alpha \wedge \bar{\alpha} < \infty$.

Non-facetiousness: For almost every point $p \in S$ we have that the translation flow starting at p is defined for all time.

The so-called *Barak Weiss surface*, $(S = \{z \in \mathbb{C} : |z| < 1\}, dz)$, is facetious since the translation flow for *any* point is defined only for some time $T < 2$. Note that compact flat surfaces satisfy the above assumptions. As for non-compact surfaces of finite area, by a theorem of Strebel, *parabolicity* is sufficient to satisfy the non-facetious condition. See [6, Remark 1].

Deformations of the flat metric turn out to be very profitable tools to study translation flows. In particular, for any (S, α) satisfying the above assumptions, we can define a one-parameter family of flat surfaces obtained by shortening the direction of the horizontal flow while stretching the direction of the vertical flow, thereby deforming the flat metric of (S, α). More specifically,

$$g_t : (S, \alpha) \mapsto (S, \alpha_t) = g_t(S, \alpha) \qquad \text{where} \qquad \alpha_t = e^{-t}\Re(\alpha) + ie^t\Im(\alpha),$$

is called the *Teichmüller deformation*.

Recently I have been trying to understand when the translation flows on surfaces satisfying the above two conditions are uniquely ergodic, and this is what my talk is about. It highlights the use of the evolving geometry given the Teichmüller deformation. Moreover, it is motivated by the recent trend of studying the geometry and dynamics of flat surfaces of infinite genus.

Suppose $(S, \alpha) \in \mathcal{H}_g$ is a compact flat surface of genus g belong to the moduli space $\mathcal{H}_g$ of flat surfaces of genus g. In this case, the one-parameter family of surfaces (S, α_t) is called the *Teichmüller orbit* of (S, α). In this case, the answer to the unique ergodicity question is answered by a theorem of Masur [5].

Masur's criterion. *Suppose the translation flow on (S, α) is not uniquely ergodic. Then the Teichmüller orbit $g_t(S, \alpha)$ leaves every compact subset of $\mathcal{H}_g$.*

The geometric spirit of Masur's criterion is the following: *if the geometry of our one-parameter family of surfaces does not degenerate as the deformation continues, then the translation flow is uniquely ergodic.* Indeed, recurrence to a compact subset of $\mathcal{H}_g$ implies that the geometry converges along a subsequence (S, α_{t_k}). It is this geometric spirit which I aim to generalize for surfaces satisfying the finite and non-facetious assumptions above.

Theorem 1 ([6])**.** *Let (S, α) be a flat surface of finite area. Suppose that for any $\eta > 0$ there exist a function $t \mapsto \varepsilon(t) > 0$, a one-parameter family of subsets*

$$S_{\varepsilon(t),t} = \bigsqcup_{i=1}^{C_t} S_t^i$$

of S made up of $C_t < \infty$ path-connected components, each homeomorphic to a closed orientable surface with boundary, and functions $t \mapsto \mathcal{D}_t^i > 0$, for $1 \leq i \leq C_t$, such that for

$$\Gamma_t^{i,j} = \{\textit{paths connecting } \partial S_t^i \textit{ to } \partial S_t^j\}$$

and

$$\delta_t = \min_{i\neq j} \sup_{\gamma\in\Gamma_t^{i,j}} dist_t(\gamma,\Sigma)$$

the following hold:

(1) $\mathrm{Area}(S\backslash S_{\varepsilon(t),t}) < \eta$ *for all* $t>0$,
(2) $dist_t(\partial S_{\varepsilon(t),t},\Sigma) > \varepsilon(t)$ *for all* $t>0$,
(3) *the diameter of each* S_t^i, *measured with respect to the flat metric on* (S,α_t), *is bounded by* $\mathcal{D}_t^i$ *and*

$$\int_0^\infty \left(\varepsilon(t)^{-2}\sum_{i=1}^{C_t}\mathcal{D}_t^i + \frac{C_t-1}{\delta_t}\right)^{-2} dt = +\infty.$$

Moreover, suppose the set of points whose translation trajectories leave every compact subset of S *has zero measure. Then the translation flow is ergodic.*

Theorem 1, although it may seem quite complicated, it is not, and it is expressed purely in terms of evolving geometric quantities. As such, it is independent of moduli spaces and of topological type of the surface being considered, as long as it has finite area. It should be mentioned that the proof of the theorem uses Forni's set-up and, moreover, is inspired by his proof of the spectral gap for the Kontsevich-Zorich cocycle for g_t-invariant measures [2, §2]. It should also be mentioned that, usually, one can expect to be able to upgrade the result to unique ergodicity. This is done in all examples of applications mentioned below.

In the case that (S,α) is a compact surface, Theorem 1 can be expressed as follows. Denote by δ_t the *systole* of (S,α_t), i.e., the length of the shortest closed geodesic on (S,α_t).

Theorem 2 ([6])**.** *Let* S *be a compact flat surface of finite area. If*

$$\int_0^\infty \delta_t^2\, dt = \infty,$$

then the translation flow is uniquely ergodic.

Theorem 2 generalizes Masur's criterion and a theorem of Cheung-Eskin [1].

It is in contexts where non-compact surfaces appear that Theorem 1 is most useful and showcases its strength, since there is no moduli space for such surfaces. In any case, having something playing the role of a moduli space to help keep track of the evolving geometry is always useful. I will mention two applications of Theorem 1 which have been implemented.

Flat surface models of ergodic systems [4]**.** In joint work with K. Lindsey, we create a dictionary that brings together Bratteli diagrams, cutting and stacking transformations and flat surfaces. Although a very special version of this has been considered by Bufetov before, our general point of view is a new way of constructing flat surfaces which satisfy both conditions above. In fact, one could argue that *any* flat surface satisfying the two conditions above can be obtained through this construction using a Bratteli diagram, and that a generic flat surface from this construction is of infinite genus. It is possible to control the deforming geometry by the shift operation on the Bratteli diagram and thus Theorem 1 has been translated to this setting and new unique ergodicity results have been obtained. One can ask whether the space of all Bratteli diagrams serves a good role as a moduli space for all flat surfaces with the above properties and whether a version of Masur's criterion holds in this setting. The answer is a careful and self-restrained yes, since there are a few surprises. This is work in progress.

Covers of surfaces of infinite type [3]**.** In joint work with P. Hooper, we study flows on surfaces of infinite genus and finite area and bundles of covers thereof. The usurper moduli space in this case is $(SL(2,\mathbb{R}) \times \mathcal{C})/SL(S,\alpha)$, where $SL(S,\alpha)$ is the Veech group and $\mathcal{C}$ is a Cantor set which depends on the types of covers one is considering. One of the main results is that, if the translation flow on the base surface has a non-divergent Teichmüller orbit on $SL(2,\mathbb{R})/SL(S,\alpha)$, then almost every cover in $\mathcal{C}$ (with respect to some $SL(S,\alpha)$-invariant probability measure) has a uniquely ergodic translation flow. One of the reasons this may be significant is that pseudo-Anosov maps in general will not lift to a cover. However, we can still retrieve unique ergodicity for almost every cover.

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Generalized Gauss map for simultaneous approximation

Yitwah Cheung

(joint work with Nicolas Chevallier)

Associated to any $\theta \in \mathbb{R}^d$ is an increasing sequence of positive integers defined by

$$q_0 = 1, \quad q_{k+1} = \min\{n > q_k : \operatorname{dist}(n\theta, \mathbb{Z}^d) < \operatorname{dist}(q_k\theta, \mathbb{Z}^d)\}$$

where the distance is taken with respect to the Euclidean norm on $\mathbb{R}^d$.

We also have an associated sequence

$$r_k = \operatorname{dist}(q_k\theta, \mathbb{Z}^d), \quad k = 0, 1, \ldots$$

which strictly decreases to zero.

Define

$$C(\theta) = \lim_{k\to\infty} \frac{\ln q_k}{k} \quad \text{and} \quad C^*(\theta) = \lim_{k\to\infty} \frac{-\ln r_k}{k}$$

whenever the limits exist. In the case $d = 1$, the existence of the limit $C(\theta)$ for a.e. $\theta \in \mathbb{R}$ (with respect to Lebesgue measure) was shown by Khintchine; the precise value of the constant was computed by Levy. The Khintchine-Levy result is nowadays a standard exercise in textbooks on ergodic theory. Indeed, we have

$$C(\theta) = C^*(\theta) = \frac{\pi^2}{12\ln 2}$$

for Lebesgue almost every $\theta \in \mathbb{R}$.

Our work-in-progress extends Khintchine-Levy theorem to higher dimensions.

Theorem 1. *There are constants C and $C*$ such that for almost every $\theta \in \mathbb{R}^d$, $C(\theta) = C$ and $C^*(\theta) = C^*$.*

Let X denote the space of unimodular lattices in $\mathbb{R}^{d+1}$. There is a natural identification of X with $\mathrm{SL}_{d+1}\mathbb{R}/\mathrm{SL}_{d+1}\mathbb{Z}$. Let μ_H denote the pullback of Haar measure under this identification. There is a well-known normalization that yields

$$\mu_H(X) = \frac{\zeta(2)\cdots\zeta(d+1)}{d+1}.$$

Let S be the subset of X consisting of lattices $\Lambda \subset \mathbb{R}^{d+1}$ that have a pair of vectors $e_1 = (u_1, h_1)$ and $e_2 = (u_2, h_1) \in \mathbb{R}^d \times \mathbb{R}$ such that

$$\|u_1\| = |h_2|, \quad |h_1| < \|u_1\|, \quad \text{and} \quad \|u_2\| < |h_2|$$

and such that $\pm e_1$ and $\pm e_2$ are the only nonzero elements of Λ in $\bar{B}(0, r) \times [-r, r]$ where $r = \|u_1\| = |h_2|$. Note that S is an (open) submanifold of codimension one. Moreover, it is transverse the flow induced by the left action of $g_t = \mathrm{diag}(e^t, \ldots, e^t, e^{-dt})$, allowing us to realize X as a suspension over S.

Let μ be the measure on S induced by μ_H. Then it can further be shown that

$$C + C^* = \frac{\zeta(2)\cdots\zeta(d+1)}{\mu(S)}.$$

The measure $d\mu$ admits a surprisingly simple explicit representation with suitably chosen coordinates.

Examples of horocyle invariant measures on the moduli space of translation surfaces, I and II

John Smillie, Barak Weiss

We give examples showing that for the horocycle flow acting on the moduli space of translation surfaces, invariant measures and orbit-closures need not be given by linear equations. We describe orbit closures which are manifolds with non-empty boundary and with infinitely generated fundamental group, and also examples which at almost every point are described by non-linear equations. This is in contrast with the situation for the action of $G = \mathrm{SL}_2(\mathbb{R})$ on these spaces, for which work of McMullen (in genus two), and Eskin-Mirzakhani-Mohammadi (for arbitrary stratum) showed that such examples do not arise.

Classification results for the G-action on the two genus two strata $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ were obtained in celebrated work of McMullen. In two recent breakthrough papers, Eskin-Mirzakhani (for measures) and Eskin-Mirzakhani-Mohammadi (for orbit-closures) obtained a local picture valid in every stratum. They showed that orbit-closures are cut out of an immersed linear submanifold (which may have self-intersections) by one quadratic equation corresponding to the area-one condition, and the measures are obtained from the Lebesgue measure on a linear submanifold by a coning off construction. In fact the results of Eskin-Mirzakhani-Mohammadi yield new information in the genus-two case as well, as they also deal with all stationary measures and all B-orbit-closures, where $B \subset G$ is the upper triangular group. We show that the picture for the horocycle action is significantly more complicated.

Rigidity of the Kontsevich-Zorich cocycle and Hodge theory

Simion Filip

In this talk, I will discuss some applications of Hodge-theoretic methods to Teichmüller dynamics. It is based on the papers [1, 2].

The main tool is the natural variation of Hodge structure over a Teichmüller disk. The local differential geometry is the same as in the classical case, going back to Griffiths. However, to obtain consequences at the global level one needs the finite measure invariant under $\mathrm{SL}_2 \mathbb{R}$.

Several results follow. On the dynamical side, one proves a version of Deligne semisimplicity adapted to the $\mathrm{SL}_2 \mathbb{R}$ setting, as well as the usual one. Using further dynamical tools, one finds that measurable invariant bundles are necessarily rigid - in period coordinates they must be polynomial. This implies that measurable and real-analytic algebraic hulls must coincide.

On the algebraic side, we first find that affine manifolds parametrize Riemann surfaces whose Jacobians have extra endomorphisms (typically, real multiplication). For Teichmüller curves, this was first proved by Möller in [4].

One then considers the mixed Hodge structure that comes from the zeros of the 1-form. This is an extra structure on the relative cohomology, coming from the relative periods of the holomorphic 1-forms. Using again subharmonicity techniques, we find that the mixed Hodge structure splits. Geometrically, this means that certain naturally defined points on (factors of) the Jacobians must be torsion. In the case of Teichmüller, this result was proved by Möller in [3].

The torsion interpretation of the last result implies that affine manifolds are given by Hodge-theoretic conditions. In particular, they must be algebraic varieties.

To discuss the methods, recall first that we have a family of Riemann surfaces over some stratum. Their first cohomology groups form a local system called the Kontsevich-Zorich cocycle. It also has a holomorphic subbundle, given by the holomorphic 1-forms on the Riemann surfaces. This endows the cocycle with a natural positive-definite metric, called the Hodge metric.

The curvature of the bundles considered, as well as their tensor powers, are very special. Specific parts of the bundle have negative curvature. This, in turn, implies that norms of sections are locally subharmonic functions.

With a version of the maximum principle, one could conclude that global sections of the bundle must have constant norm. This is not directly available because Teichmüller disks are not compact. The method thus uses subharmonicity for random walks on the group $\mathrm{SL}_2\,\mathbb{R}$. The finite invariant measure replaces compactness.

The above method proves the semisimplicity properties of the Kontsevich-Zorich cocycle. Further rigidity follows from considering the stable and unstable foliations on the affine manifold. Namely, one finds that *measurable* $\mathrm{SL}_2\,\mathbb{R}$-invariant bundles must be polynomial along each of the foliations. Assembling these facts, it follows that the bundles are given by polynomials in period coordinates, in particular are real-analytic.

The paper [1] deals with the mixed Hodge structure on the cohomology of flat surfaces. Namely, recall that period coordinates are given by the first relative cohomology group of the surfaces, denoted H^1_{rel}. The holomorphic 1-forms also have relative periods, thus providing a canonical subspace inside the relative cohomology group, denoted F^1.

The space H^1_{rel} has a rational structure coming from the integer cohomology, while the space F^1 is of transcendental origin, given by integrals of 1-forms. However, over an affine manifold these spaces are forced to be in a rather good position. Namely, for 1-forms which are in the tangent space of the affine manifold, the relative and absolute periods are constrained by linear equations.

The constrains are obtained as follows. Given a holomorphic 1-form α living in the *pure* cohomology H^1, there are two ways to lift it to H^1_{rel}. One is using the tangent space of the affine manifold, the other is using the space F^1 naturally living in H^1_{rel}. Using subharmonicity techniques, one finds that these two lifts must agree. This yields the claimed linear relations.

Finally, one can put all the above equations into a global geometric form. Namely, ones has a bundle of (factors of) the Jacobians, and canonically defined points on them. The linear relations can be interpreted as those points being torsion.

This geometric interpretation implies that affine manifolds are completely determined by algebro-geometric conditions. Therefore, they must be algebraic varieties, in fact defined over $\overline{\mathbb{Q}}$.

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Loci in strata of meromorphic quadratic differentials with fully degenerate Lyapunov spectrum

Pascal Hubert
(joint work with Julien Grivaux)

Lyapunov exponents of the Teichmüller flow have been studied a lot since the work of Zorich [8] and Forni [5]. Their understanding is important for applications to the dynamics of interval exchange transformations and polygonal billiards. A big breakthrough is the Eskin-Kontsevich-Zorich formula for the sum of positive Lyapunov exponents [4]. Given a $\mathrm{SL}(2;\mathbb{R})$ invariant suborbifold of a stratum of quadratic differentials, they relate the sum $\lambda_1^+ + \cdots + \lambda_g^+$ to the Siegel-Veech constant of the invariant locus. *We will only be interested in the bundle with fiber $H^1(X,\mathbb{R})$ over a Riemann surface X.*

By a theorem of Kontsevich and Forni, the sum $\lambda_1^+ + \cdots + \lambda_g^+$ is also the integral over the invariant locus of the curvature of the Hodge bundle along Teichmüller discs ([5], [4]). Using this interpretation, every Lyapunov exponent is computed for cyclic covers of the sphere branched over 4 points ([3], [6], [2]). For some cyclic covers, Forni-Matheus-Zorich remarked that the sum $\lambda_1^+ + \cdots + \lambda_g^+$ is equal to zero. This surprizing fact means that the complex structure of the underlying Riemann surface is constant along the Teichmüller disc. Forni-Matheus-Zorich ask whether it is possible to construct other invariant loci with this property. We give a simple explanation of this phenomenon discovered by Forni-Matheus-Zorich.

Theorem 1. *There exist closed* $\mathrm{GL}(2;\mathbb{R})$ *invariant loci of quadratic differentials of arbitrarily large dimension with zero Lyapunov exponents.*

This result can be interpreted in the following way: the projection of such a locus to the moduli space of *compact* Riemann surfaces is a point. The situation for strata of *abelian* differentials is completely different: there are finitely many invariant suborbifolds with fully degenerate Lyapunov spectrum, and they are arithmetic Teichmüller curves (see [7], and [1]).

Proof. To prove Theorem 1, one has to construct $\mathrm{SL}(2;\mathbb{R})$ a suborbifold of a stratum with a constant complex structure. To provide examples, we start with a locus in genus zero (for instance a stratum), and make a cover ramified over three points. The branched locus moves in an equivariant way when the quadratic differentials varies on the sphere. For instance, one can choose poles of the differentials. Since, there is only one complex structure on $\mathbb{CP}^1$ minus three points, the complex structure is constant on the cover. This implies that the curvature of the Hodge bundle is equal to zero, therefore the Lyapunov exponents satisfy

$$\lambda_1^+ = \cdots = \lambda_g^+ = 0.$$

□

A more precise statement can be given for pillow-tiled surfaces (covers of the sphere ramified over 4 points).

Corollary 2. *Let (X, q, π) be a pillow-tiled surface such that the covering map π is Galois. Then the Lyapunov exponents of the Teichmüller disc of (X, q) are equal to zero if and only the branching locus of π contains at most three points.*

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Lyapunov exponents and Harder-Narasimhan filtrations on Teichmüller curves

Fei Yu

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the bundle of pairs (X, ω), where $\omega \neq 0$ is a holomorphic 1-form on $X \in \mathcal{M}_g$. Denote $\Omega\mathcal{M}_g(m_1, ..., m_k) \hookrightarrow \Omega\mathcal{M}_g$ the stratum of pairs (X, ω), where $\omega(\neq 0)$ have k distinct zeros of order $m_1, ..., m_k$ respectively([KZ03]).

There is a nature action of $GL_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g(m_1, ..., m_k)$, whose orbits project to complex geodesics (Teichmüller geodesic flows) in $\mathcal{M}_g$. The projection of an orbit is almost always dense. Fix an $SL_2(\mathbb{R})$-invariant, ergodic measure μ on $\Omega\mathcal{M}_g$. The Lyapunov exponents for the Teichmüller geodesic flow on $\Omega\mathcal{M}_g$ measure the logarithm of the growth rate of the Hodge norm of cohomology classes under the parallel transport along the geodesic flow ([KZ97],[Zo06])):

$$1 = \lambda_1 \geq \lambda_2 \geq ...\lambda_g \geq 0$$

If the stabilizer $SL(X, \omega) \subset SL_2(\mathbb{R})$ of a given form is a lattice, then the projection of its orbit gives a closed, algebraic Teichmüller curve C. After suitable base change and compacfication, we can get a universal family $f : S \to C$, which is a relative minimal semistable model with disjoint sections $D_1, ..., D_k$; here $D_i|_X$ is a zero of ω when restrict to each fiber X ([CM11]).

We can use the algebraic geometry technique to study the question from dynamical system since the following formula links the sum of Lyapunov exponents and degree of Hodge bundle on Teichmüller curves (originally by Kontesvich for Teichmüller geodesic flows. cf.[KZ97],[Fo02],[BM10]).

$$\sum_{i=1}^{g}\lambda_i = \frac{2\deg f_*\omega_{S/C}}{2g(C)-2+|\Delta|}$$

It is surprising that the relative canonical bundle formula of the Teichmüller curve is very simple and elegant([CM11], [EKZ11]):

$$\omega_{S/C} \simeq f^*\mathcal{L}\otimes\mathcal{O}(\sum_i m_iD_i)$$

Here $\mathcal{L}\subset f_*\omega_{S/C}$ be the line bundle whose fiber over the point corresponding to X is $\mathbb{C}\omega$, the generating differential of Teichmüller curves.

There are many nature vector subbundles of the Hodge bundle $f_*\omega_{S/C}$:

$$\mathcal{L}\otimes f_*\mathcal{O}(\sum d_iD_i)\subset \mathcal{L}\otimes f_*\mathcal{O}(\sum m_iD_i) = f_*\omega_{S/C}$$

One can construct many filtrations of Hodge bundle by using these subbundles ([YZ12a]).

By using those filtrations, we can get an upper bound of the slope of each graded quotient for the Harder-Narasimhan filtration of $f_*(\omega_{S/C})$ of Teichmüller curves in each stratum. For a vector bundle V, define $\mu_i(V)=\mu(gr_j^{HN})$ if $rk(HN_{j-1}(V)) < i \le rk(HN_j(V))$. Write w_i for $\mu_i(f_*(\omega_{S/C}))/deg(\mathcal{L})$.

$$1 = w_1 \ge w_2 \ge ...w_g$$

For a Teichmüller curve which lies in $\Omega\mathcal{M}_g(m_1,...,m_k)$, we have inequalities ([YZ12b]):

$$w_i \le 1 + a_{H_i(P)}$$

Here a_i is the i-th largest number in $\{-\frac{j}{m_i+1}|1\le j\le m_i, 1\le i\le k\}$, P is the special permutation and $H_i(P)\ge 2i-2$.

TABLE 1. Genus 3 case: λ_i for whole stratum(cf.[KZ97]) and w_i for all Teichmüller geodesic curves in this stratum(cf.[YZ12a], [YZ12b]).

zeros	component	λ_2	λ_2	w_2	w_3	$\sum w_i$
(4)	hyp	0.6156	0.1844	3/5	1/5	9/5
(4)	odd	0.4179	0.1821	2/5	1/5	8/5
(3,1)		0.5202	0.2298	2/4	1/4	7/4
(2,2)	hyp	0.6883	0.3117	2/3	1/3	2
(2,2)	odd	0.4218	0.2449	1/3	1/3	5/3
(2,1,1)		0.5397	0.2936	1/2	1/3	11/6
(1,1,1,1)		0.5517	0.3411	$\le 1/2$	$\le 1/2$	≤ 2

Now we have λ_i measuring the stability of dynamical system and w_i measuring the stability of algebraic geometry. The numerical λ_i for whole stratum and w_i

for all Teichmüller curves in this stratum are listed in Table 1. Are there some relations between them? In fact we make the following conjecture:

Conjecture: The polygon of Lyapunov exponents lies below the Harder-Narasimhan polygon on Teichmüller curves.

That is for any $1 \leq k \leq g$

$$\sum_{i=k}^{g} \lambda_i \leq \sum_{i=k}^{g} w_i$$

(or equivalence to say $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} w_i$). When the equality is reached, we also conjecture if $\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} w_i$ (and $w_k \neq w_{k+1}$), then $f_* \omega_{S/C}$ split to SL_2-invariant of rank k and rank $g-k$ vector bundle. It can be considered as the inverse of Kontsevich's formula.

We can prove two simple corollaries 1)$w_g \geq 0$; 2)$w_i = 0 \implies \lambda_i = 0$ by using the Simpson corresponding and Kontsevich's formula.

We also hope that w_i can be defined for any Teichmüller geodesic flows, and it is compatible with the definition on Teichmüller curves (some continuity properties), then we can discuss the same conjecture for all Teichmüller geodesic flows.

The conjecture is inspired by the Katz-Mazur theorem(cf.[Ma72],[Ma73]) which tells us that the Hodge polygon lies below the Newton polygon on the crystalline cohomology.

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Entropy of random walks on hyperbolic groups

Sébastien Gouëzel

(joint work with Frédéric Mathéus and François Maucourant)

Let Γ be a nonamenable finitely generated group. The main examples we have in mind are the free group $\mathbb{F}_2$; the surface group $\pi_1(M_g)$ with $g \geq 2$; the modular group Mod_g; higher rank groups such as $\mathrm{SL}(n, \mathbb{Z})$. We are interested in the following question: how can one construct random elements of Γ? Moreover, if there are different answers to this question, how are they related?

The first answer is geometric. Assume that Γ acts by isometries on a space (X, d_X). Taking a basepoint $x_* \in X$, this gives a left-invariant distance $d(\gamma, \gamma') = d_X(\gamma x_*, \gamma' x_*)$ on Γ. A random point in Γ can be constructed by taking a large n, and then choosing a point uniformly in the ball $B(e, n)$, i.e., according to the measure $\nu_n = \frac{1}{|B(e,n)|} \sum_{\gamma \in B(e,n)} \delta_\gamma$. Usually (for instance in all the above examples), the space X comes with a natural compactification, and the sequence of measures ν_n converges to a measure ν_∞ supported on $\partial\Gamma$.

The second answer is probabilistic. Consider a probability measure μ, supported on the generators of the group. Let n be large, and let $g_1, \dots, g_n$ be i.i.d. elements of Γ, chosen according to μ. The product $Z_n = g_1 \cdots g_n$ is a random element of the group, distributed according to μ^{*n}. When n tends to infinity, this sequence of measures converges to a measure on the boundary ∂X, denoted by μ_∞, called the hitting measure of the random walk. The *drift* ℓ of the random walk is defined as the almost sure limit of $\lim d(e, Z_n)/n$ (it exists by Kingman's theorem).

To compare these two notions, one can try to compare at finite time, i.e., see how μ^{*n} and $\nu_{\ell n}$ are related. Or one can compare the resulting measures μ_∞ and ν_∞ on ∂X.

In $\mathbb{F}_2$, with its standard set of generators and the corresponding word distance, let μ be the uniform measure on the generators. It is then easy to check that the measures μ_∞ and ν_∞ coincide (they are the uniform measure on the boundary). It is widely expected that this is essentially the only case where this happens, and that in general μ_∞ and ν_∞ should be singular. This is easy to check in the case of $\mathrm{SL}(n, \mathbb{Z})$: the measure ν_∞ is supported on measures with maximally degenerate Lyapunov spectrum, while the Lyapunov exponents of a random walk are all different (if the support of the random walk generates the group, which we always assume). However, this is generally hard to prove, since the measures μ^{*n} share many qualitative and quantitative properties.

Our main result is that, in hyperbolic groups away from the free group, the measures are mutually singular, as expected:

Theorem 1. *Let Γ be a surface group, or more generally any hyperbolic group which is not virtually free, endowed with a word distance. Let μ be a finitely supported symmetric probability measure on Γ. Then the measures μ_∞ and ν_∞ are mutually singular.*

This result is obtained as a consequence of a stronger quantitative theorem, involving the notion of *entropy*. The entropy h of a random walk is the exponential growth rate of its typical support, i.e.,

$$h = \lim_{n\to\infty} \frac{\log \min\{|K| \,:\, \mu^{*n}(K) \geq 1/2\}}{n}.$$

Let v denote the growth rate of balls, i.e., $v = \lim n^{-1} \log |B(e,n)|$. Since the walk at time n is essentially supported by the ball $B(e, \ell n)$, one has $h \leq \ell v$ (this is the fundamental inequality of Guivarc'h). If the inequality is strict, this means that the walk is supported by an exponentially small subset of $B(e, \ell n)$, i.e., it is concentrated in a very narrow subset of the group (and, therefore, μ^{*n} and $\nu_{\ell n}$ are very different). This is the content of the following theorem (which implies the previous one):

Theorem 2. *Let Γ be a surface group, or more generally any hyperbolic group which is not virtually free, endowed with a word distance. Let μ be a finitely supported symmetric probability measure on Γ. Then it satisfies $h < \ell v$.*

The proofs rely on rigidity properties of cocycles on the boundary of hyperbolic groups and, in a crucial way, on the fact that the word distance is integer-valued.

On the modulus of continuity for spectral measures in substitution dynamics

Alexander Bufetov
(joint work with Boris Solomyak)

The talk gives first quantitative estimates on the modulus of continuity of the spectral measure for weak mixing suspension flows over substitution automorphisms, which yield information about the "fractal" structure of these measures. The main results are, first, a Hoelder estimate for the spectral measure of almost all suspension flows with a piecewise constant roof function; second, a log-Hoelder estimate for self-similar suspension flows; and, third, a Hoelder asymptotic expansion of the spectral measure at zero for such flows. Our second result implies log-Hoelder estimates for the spectral measures of translation flows along stable foliations of pseudo-Anosov automorphisms. A key technical tool in the proof of the second result is an "arithmetic-Diophantine" proposition, which has other applications. In the appendix this proposition is used to derive new decay estimates for the Fourier transforms of Bernoulli convolutions.

Small dilatations of pseudo-Anosov homeomorphisms on hyperelliptic strata of Abelian differentials

Corentin Boissy
(joint work with Erwan Lanneau)

1. Introduction

1.1. Pseudo-Anosov homeomorphisms and translation surfaces. By definition, a pseudo-Anosov homeomorphism ϕ on a surface S defines a pair of measured foliations μ_S, μ_U on S: the stable one, contracted by a factor $\theta > 0$ by ϕ, and the unstable one, dilated by the same factor. Such pair of measured foliation defines a pair (X, q), where X is a Riemann surface homeomorphic to S, and q is a quadratic differential, that we usually call *flat surface*. In particular, the set of conjugacy classes of pseudo-Anosov homeomorphism is in one-to-one bijection with closed orbits of the Teichmüller geodesic flow.

When the foliations μ_S, μ_U are orientable, the corresponding quadratic differential is the square of an Abelian differential ω, and (X, ω) is a *translation surface.*

1.2. Minimization problem. The set of dilatatations for fixed genus is a discrete subset of $\mathbb{R}$, and hence, admits a minimum δ_g. The value δ_g is unknown except for $g = 1$ and $g = 2$. The value δ_g^+, which corresponds to pseudo-Anosov homeomorphism with orientable foliations, is also unknwown, except for some small values of g (see [8, 6]). From [10], we know that δ_g tends to one when g tends to infinity. Note that a recent result of McMullen [9] gives the minimal value of the spectral radius of matrices in $M_{2g}(\mathbb{Z})$, with a reciprocal characteristic polynomial, although it is not known if this minimum corresponds to a pseudo-Anosov homeomorphism or not.

In [5], Farb proposes a natural refinement of the minimization problem: compute the minimal dilatation $\delta(\mathcal{H}(k_1, \ldots, k_r))$ associated to a stratum $\mathcal{H}(k_1, \ldots, k_r)$ of the moduli space of translation surface. Note that these strata are not connected in general (see [7]), so it is natural to ask for the minimal dilatation $\delta(\mathcal{C})$, for $\mathcal{C} \subset \mathcal{H}(k_1, \ldots, k_r)$ a connected component.

1.3. Constructing pseudo-Anosov homeomorphisms in a stratum. For a fixed connected component of a stratum, there is a well known construction of pseudo-Anosov homeomorphisms due to Veech (see [11]). We give a sketch of this construction. Consider a Rauzy diagram $\mathcal{D}$ associated to the component, and let γ be a closed path in $\mathcal{D}$, with endpoints a permutation π. By the usual Rauzy–Veech operations, we associate to γ a matrix M. Choose γ such that the matrix M is primitive. Let λ be a positive eigenvector for the Perron-Frobenius eigenvalue θ of M and let τ be an eigenvector for the eigenvalue θ^{-1} of M. One can show that the pair $(\pi, \lambda + i\tau)$ is a *suspension data* which defines a translation surface S. By construction and the usual properties of the Rauzy–Veech induction, the pair $(\pi, \frac{1}{\theta}\lambda + i\theta\tau)$ defines the same surface S, and therefore, one gets a pseudo-Anosov homeomorphism of dilatation θ.

This construction provides an easy way to build pseudo-Anosov homeomorphisms in a given connected component of stratum. However, any pseudo-Anosov homeomorphism constructed in this way fixes a horizontal separatrix. Hence, not all pseudo-Anosov homeomorphisms of a stratum can be obtained in this way.

2. Minimization of the dilatation in hyperelliptic strata

2.1. Statement of the result.

Theorem 3 (B-Lanneau)**.**

- *Let $g \geq 2$. The smallest dilatation of a pseudo-Anosov homeomorphism in $\mathcal{H}^{hyp}(2g-2)$ is the largest root of the polynomial:*
$$X^{2g+1} - 2X^{2g-1} - 2X^2 + 1$$
- *Let $2 \leq g \leq 15$. The smallest dilatation of a pseudo-Anosov homeomorphism in $\mathcal{H}^{hyp}(g-1, g-1)$ is the largest root of the polynomial:*
$$X^{2g+2} - 2X^{2g} - 2X^{g+1} - 2X^2 + 1 \quad \textit{for } g \textit{ even}$$
$$X^{2g+2} - 2X^{2g} - 4X^{g+2} + 4X^g + 2X^2 + 1 \quad \textit{for } g \textit{ odd}$$

3.1. **A variation of the Veech construction.** A variation of the construction given in Paragraph 1.3 is the following: for a permutation $\pi = \left(\begin{smallmatrix} a_1,\dots,a_d \\ b_1,\dots,b_d \end{smallmatrix}\right)$, we define $s(\pi) = \left(\begin{smallmatrix} b_d,\dots,b_1 \\ a_d,\dots,a_1 \end{smallmatrix}\right)$. Instead of considering closed paths in the Rauzy diagram, we consider nonclosed paths from a permutation π to $s(\pi)$. We obtain, similarly as previously, the datum (π, ζ) that defines a translation surface, with $\begin{pmatrix} -\frac{1}{\theta} & 0 \\ 0 & -\theta \end{pmatrix}$ in the Veech group.

We have the following result.

Proposition 4 (B-Lanneau)**.** *Any pseudo-Anosov homeomorphism on a hyperelliptic connected component can be obtained by the previous construction provided that its dilatation is smaller than 2.*

See also [3, 4] for another construction of pseudo-Anosov homeomorphisms on a hyperelliptic connected component.

4.1. **Sketch of the proof.** Now we give a short sketch of the proof of Theorem 3. From the previous proposition, it follows that we need to study all paths γ in the Rauzy diagram, from a permutation π to $s(\pi)$. There are two cases when the associated dilatation can be easily controlled.

(1) When the path γ passes through the "center" of the Rauzy diagram, *i.e.* the permutation $\pi_n = \left(\begin{smallmatrix} 1 & \dots & n \\ n & \dots & 1 \end{smallmatrix}\right)$. In this case, we show that the pseudo-Anosov can also be constructed by the usual Veech construction. In this case, it was shown in [1] that the dilatation is bounded from below by 2.

(2) When the path γ starts from one of the two "main loops" of the Rauzy diagram, *i.e.* permutations obtained from π_n by the Rauzy moves $\mathcal{R}_t^k$ or $\mathcal{R}_b^k$. In this case, we can bound from below the dilatation by the one corresponding to a finite number of paths, for which the matrix can be explicitely computed. Then, in the case when n is even (corresponding to

the stratum $\mathcal{H}(2g-2)$), one can actually compute the minimum (when n is odd, this corresponds to $\mathcal{H}(g-1,g-1)$, and the computations are more complicated. In this case, a computer assisted proof gives the result for small n).

Of course, there are many paths that correspond to neither of the two previous situations. In this case, there is a natural way to change the path γ (in fact, its starting point). This is done by a kind of "left-right" Rauzy–Veech induction on (π, ζ), that preserves the corresponding translation surface, and therefore the pseudo-Anosov homeomorphism. We have the following result, which concludes the proof of the Theorem 3.

Proposition 5. *If the dilatation is smaller than 2, the above left-right Rauzy–Veech induction eventually leads to a starting point in the big loop of the Rauzy diagram.*

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Typical properties of periodic orbits for the Teichmüller flow

Ursula Hamenstädt

Consider the moduli space $\mathcal{H}$ of area one abelian differentials for a closed oriented surface S of genus $g \geq 2$. The *Teicmüller flow* Φ^t acts on $\mathcal{H}$ preserving a Borel probabiliy measure λ in the Lebesgue measure class.

Periodic orbits γ for the Teichmüller flow on $\mathcal{H}$ are countable and can be ordered by their length. Indeed, if $h = 4g-4$ is th entropy of the Lebesgue measure λ

with respect to the flow Φ^t then as $R \to \infty$, the number of periodic orbits γ of period at most R is asymptotic to $\frac{e^{hR}}{hR}$.

Let $\mathcal{P}$ be the set of all periodic orbits on $\mathcal{H}$.

Definition 1. *A subset $\mathcal{Q} \subset \mathcal{P}$ is called* typical *if*

$$\sharp\{\gamma \in \mathcal{Q} \mid \ell(\gamma) \leq R\} \sim \frac{e^{hR}}{hR}.$$

In the talk, we introduce three different properties for periodic orbits.

First, the *Lyapunov exponents* of the Kontsvich Zorich cocycle on $\mathcal{H}$. are defined. It was shown by Avila and Viana that the Lyapunov spectrum is simple. This means that there are precisely g different positive exponents $1 = \lambda_1 > \cdots > \lambda_g > 0$.

A periodic orbit γ determines up to conjugation a matrix $A(\gamma) \in Sp(2g, \mathbf{Z})$. Let $e^{\alpha_1(\gamma)} \geq \cdots \geq e^{\alpha_g(\gamma)}$ be the absolute values of the g largest eigenvalues, in decreasing order and define

$$\mathcal{A}_\epsilon = \{\gamma \mid \ell(\gamma) = R, \frac{1}{R}|\alpha_i(\gamma) - \lambda_i| < \epsilon.$$

A symplectic matrix $A \in Sp(2g, \mathbf{Z})$ determines a trace field $K(A)$ which is a number field of degree at most g over $\mathbf{Q}$. The Galois group of a number field of degree $k \leq g$ over $\mathbf{Q}$ field is a subgroup of the symmetric group in g elements. Call the field *full* if its degree equals g and if its Galois group equals the symmetric group. Define

$$\mathcal{B} = \{\gamma \mid K(A(\gamma)) \text{ is full}\}.$$

If the trace field $A \in Sp(2g, \mathbf{Z})$ if full then A defines a Hilbert modular variety which is contained in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of rank g. The composition $\mathcal{J}$ of the canonical projection of $\mathcal{H}$ onto the moduli space of Riemann surfaces with the Torelli map maps the *Teichmüller disc $T(\gamma)$* generated by a periodic orbit γ into $\mathcal{A}_g$. Define

$$\mathcal{C} = \{\gamma \mid \mathcal{J}T(\gamma) \text{ is not contained in a Hilbert modular variety}\}.$$

We discuss the following.

Theorem 2. *The following are typical properties for periodic orbits.*

(1) *For $\epsilon > 0$ the set $\mathcal{A}_\epsilon$.*
(2) *The set $\mathcal{B}$.*
(3) *For $g \geq 3$ the set $\mathcal{C}$.*

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Limit sets of Teichmüller geodesic rays in the Thurston boundary

Anna Lenzhen
(joint work with Chris Leininger, Kasra Rafi)

We describe a method for constructing measured laminations which are minimal but not uniquely ergodic. The laminations are constructed as limits of explicit sequences of simple closed curves on the surface that form a quasi-geodesic in the curve complex and hence, by a theorem of Klarreich, are minimal, filling and measurable. Analogous to continued fraction coefficients associated to an irrational real number, any lamination ν in our family has an associated infinite sequence of positive integers $\{r_i\}$ that can be chosen essentially arbitrarily and encode the *arithmetic properties* of ν. One important consequence of the explicit nature of our construction is that we can adjust the values of $\{r_i\}$ to produce different interesting examples. In particular, we show that if the sequence r_i grows fast enough then the lamination ν is not uniquely ergodic.

Although our construction is general in spirit, we carry out detailed computations in the case of the five-times punctured sphere. This case is already rich enough for us to observe some interesting phenomena. Our lamination ν is a limit of a sequence of simple closed curves γ_i defined as follows. Let ρ be a finite order homeomorphism (rotation counter-clockwise by $4\pi/5$) of a five-times punctured sphere S and D a Dehn twist. We then set $\phi_r = D^r \circ \rho$ and define

$$\Phi_i = \phi_{r_1} \circ \ldots \circ \phi_{r_i} \qquad \text{and} \qquad \gamma_i = \Phi_i(\gamma_0)$$

Theorem 1. *There exists $R > 0$ so that if the powers r_i are larger than R, then the path $\{\gamma_i\}$ is a quasi-geodesic in the curve complex and hence the limiting lamination exists, is minimal and filling. Furthermore, if r_i grow fast enough, namely if $r_{i+1} \geq 4^{i+2} r_i$, then the limiting lamination ν is not uniquely ergodic.*

We are interested in understanding a Teichmüller geodesic where ν is topologically equivalent to the vertical foliation of the associated quadratic differential. (Recall that there is a one-to-one correspondence between measured laminations and singular measured foliations). Let $\Delta(\nu)$ be the simplex of all possible projectivized measures on ν. In the case of the five-times punctured sphere, if ν is minimal and not uniquely ergodic then $\Delta(\nu)$ is one dimensional, that is, it is homeomorphic to an interval. The endpoints of this interval are projective classes associated to ergodic measures on ν, denoted ν_α and ν_β. Every other measure is in the form $\bar{\nu} = c_\alpha \nu_\alpha + c_\beta \nu_\beta$ for positive real numbers c_α and c_β. Note that the projective class of $\bar{\nu}$ depends only on the ratio of c_α and c_β.

Now fix any point X in Teichmüller space. There is a unique Teichmüller geodesic

$$g = g(X, \bar{\nu}) : [0, \infty) \to T(S)$$

starting from X so that the vertical foliation associated to g is in the projective class of $\bar{\nu}$. We examine the limit set $\Lambda(g)$ of this geodesic in the Thurston boundary of Teichmüller space.

By appealing to the results of Rafi [2, 3], we can determine how the coefficients $\{r_i\}$ effect the behavior of this Teichmüller geodesic. In particular, at any time t, we can describe the geometry of the surface $g(t)$ using the numbers r_i. As a result, we can control the limit set of g.

Theorem 2. *Let $\bar{\nu} = c_\alpha \nu_\alpha + c_\beta \nu_\beta$ and $g = g(X, \bar{\nu})$ be as above. If r_i satisfy certain growth conditions, then*

$$\Lambda(g) = \Delta(\nu),$$

for any value of c_α and c_β.

Note that, in particular, even when $\bar{\nu} = \nu_\alpha$ is an ergodic measure, the limit set still includes the other ergodic measure ν_β.

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Torsion and the Veech dichotomy

Matt Bainbridge

(joint work with Philipp Habegger, Martin Möller)

In this talk, we discuss some of the ideas which go into proving the following finiteness statement for Teichmüller curves in the generic stratum of genus-three holomorphic one-forms.

Theorem 1. *There are at most finitely many Teichmüller curves in the generic stratum $\Omega\mathcal{M}_3(1, 1, 1, 1)$ having cubic trace field.*

There first ingredient in the proof is a global torsion bound for Teicumüller curves in this stratum. Given (X, ω) generating an algebraically primitive (meaning the degree of the trace field is equal to the genus of X) Teichmüller curve, Möller's torsion condition says that for any two zeros p and q of ω, we have $n(p - q) = 0$ in $\mathrm{Jac}(X)$ for some $n \in \mathbb{N}$. We call a n which works for any two zeros a *torsion bound* for (X, ω).

Theorem 2. *There is a uniform torsion bound for all algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_3(1, 1, 1, 1)$.*

The proof is heavily computer aided and roughly amounts to finding all subtori of a certain subvariety of an algebraic torus.

The finiteness theorem then follows from:

Theorem 3. *For any g, a uniform torsion bound for all algebraically primitive Teichmüller curves in $\Omega\mathcal{M}_g(1^{2g-2})$ implies that this stratum only contains finitely many such curves.*

The main idea is to use conformal geometry together with the torsion bound to constrain the geometry of any one-form (X, ω) generating a Teichmüller curve in this stratum. Any cylinder on (X, ω) is parallel to a finite collection of cylinders which fill X. Veech showed that these cylinders have rationally commensurable moduli. We use the above torsion bound to give strong constraints on these moduli. More precisely, fix a periodic direction on (X, ω). Let Γ be the dual graph, whose edges correspond to cylinders of (X, ω). We call a maximal set of edges which is not disconnected by removing any vertex a *block* of cylinders.

Theorem 4. *Suppose (X, ω) satisfies the torsion condition with torsion bound N. Then for each block of cylinders, there are at most finitely many possibilities for the vector of moduli (up to scaling).*

The idea is to consider a different one-form η on X defined by a torsion divisor D supported on the zeros of ω. Specifically, η is the pullback of the form dz/z by the meromorphic function f with $(f) = ND$, This form η has cylinders which satisfy obvious linear equations with integral coefficients bounded by N. Near the boundary of moduli space, these cylinders have moduli close to those of corresponding cylinders of ω, yielding many linear equations which these moduli satisfy.

Orbit closures for the $\mathrm{SL}(2, \mathbb{R})$ action on moduli space

Amir Mohammadi

(joint work with A. Eskin, M. Mirzakhani)

We prove results about orbit closures and equidistribution for the $\mathrm{SL}(2, \mathbb{R})$ action on the moduli space of compact Riemann surfaces, which are analogous to the theory of unipotent flows. The proofs of the main theorems rely on the measure classification theorem of [6] and a certain isolation property of closed $\mathrm{SL}(2, \mathbb{R})$ invariant manifolds which are developed in our joint work.

Suppose $g \geq 1$, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a partition of $2g - 2$, and let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, i.e. the space of pairs (M, ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\alpha_1, \ldots, \alpha_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\alpha)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [13].

The space $\mathcal{H}(\alpha)$ admits an action of the group $\mathrm{SL}(2, \mathbb{R})$ which generalizes the action of $\mathrm{SL}(2, \mathbb{R})$ on the space $\mathrm{GL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$ of flat tori.

Affine measures and manifolds. The area of a translation surface is given by

$$a(M, \omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(\alpha)$ is defined as a subset of translation surfaces in $\mathcal{H}(\alpha)$ of area one. For a subset $\mathcal{N}_1 \subset \mathcal{H}_1(\alpha)$ we write

$$\mathbb{R}\mathcal{N}_1 = \{(M, t\omega) \mid (M,\omega) \in \mathcal{N}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(\alpha).$$

Definition 1. *An ergodic* $\mathrm{SL}(2,\mathbb{R})$*-invariant probability measure* ν_1 *on* $\mathcal{H}_1(\alpha)$ *is called* affine *if the following hold:*

(i) *The support* $\mathcal{M}_1$ *of* ν_1 *is an* immersed submanifold *of* $\mathcal{H}_1(\alpha)$*, i.e. there exists a manifold* $\mathcal{N}$ *and a proper continuous map* $f : \mathcal{N} \to \mathcal{H}_1(\alpha)$ *so that* $\mathcal{M}_1 = f(\mathcal{N})$*. The self-intersection set of* $\mathcal{M}_1$*, i.e. the set of points of* $\mathcal{M}_1$ *which do not have a unique preimage under* f*, is a closed subset of* $\mathcal{M}_1$ *of* ν_1*-measure* 0*. Furthermore, each point in* $\mathcal{N}$ *has a neigborhood* U *such that locally* $\mathbb{R}f(U)$ *is given by a complex linear subspace defined over* $\mathbb{R}$ *in the period coordinates.*

(ii) *Let* ν *be the measure supported on* $\mathcal{M} = \mathbb{R}\mathcal{M}_1$ *so that* $d\nu = d\nu_1 da$*. Then each point in* $\mathcal{N}$ *has a neighborhood* U *such that the restriction of* ν *to* $\mathbb{R}f(U)$ *is an affine linear measure in the period coordinates on* $\mathbb{R}f(U)$*, i.e. it is (up to normalization) the restriction of the Lebesgue measure* λ *to the subspace* $\mathbb{R}f(U)$*.*

Definition 2. *We say that any suborbitfold* $\mathcal{M}_1$ *for which there exists a measure* ν_1 *such that the pair* $(\mathcal{M}_1, \nu_1)$ *satisfies (i) and (ii) an* affine invariant submanifold.

Note that in particular, any affine invariant submanifold is a closed subset of $\mathcal{H}_1(\alpha)$ which is invariant under the $\mathrm{SL}(2,\mathbb{R})$ action, and which in period coordinates looks like an affine subspace. We also consider the entire stratum $\mathcal{H}(\alpha)$ to be an (improper) affine invariant submanifold. It follows from [7, Thm. 2.2] that the self-intersection set of an affine invariant manifold is itself a finite union of affine invariant manifolds of lower dimension.

Notational Conventions. In case there is no confusion, we will often drop the subscript 1, and denote an affine manifold by $\mathcal{N}$. Also we will always denote the affine probability measure supported on $\mathcal{N}$ by $\nu_{\mathcal{N}}$.

Let $P \subset \mathrm{SL}(2,\mathbb{R})$ denote the subgroup $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. The following theorem is the main result of [6]:

Theorem 3 ([6])**.** *Let* ν *be any* P*-invariant probability measure on* $\mathcal{H}_1(\alpha)$*. Then* ν *is* $SL(2,\mathbb{R})$*-invariant and affine.*

Theorem 3 is a partial analogue of Ratner's celebrated measure classification theorem in the theory of unipotent flows, see [10].

We now state the following orbit closure theorem from [7].

Theorem 4 ([7])**.** *Suppose* $x \in \mathcal{H}_1(\alpha)$*. Then, the orbit closure* $\overline{Px} = \overline{\mathrm{SL}(2,\mathbb{R})x}$ *is an affine invariant submanifold of* $\mathcal{H}_1(\alpha)$*.*

The proof of this theorem utilizes Theorem 3 together with an "avoidance" principle which is established in [7]. In the theory of unipotent flows such avoidance principles are obtained using polynomial like behavior of unipotent flows, [11] and [2]. Here, however, the polynomial like behavior of unipotent flows is poorly understood the following serves as a replacement.

The Main Proposition and Countability For a function $f : \mathcal{H}_1(\alpha) \to \mathbb{R}$, let

$$(A_t f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\theta x).$$

Following the general idea of Margulis, the strategy of the proof is to define a function which will satisfy a certain inequality involving A_t, [4] and [3]. In fact, the main technical result of [7] is the following:

Proposition 5. *Let $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ be an affine submanifold. (In this proposition $\mathcal{M} = \emptyset$ is allowed). Then there exists an* $\mathrm{SO}(2)$*-invariant function $f_{\mathcal{M}} : \mathcal{H}_1(\alpha) \to [1, \infty]$ with the following properties:*

(a) *$f_{\mathcal{M}}(x) = \infty$ if and only if $x \in \mathcal{M}$, and $f_{\mathcal{M}}$ is bounded on compact subsets of $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$. For any $\ell > 0$, the set $\overline{\{x : f(x) \leq \ell\}}$ is a compact subset of $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$.*

(b) *There exists $b > 0$ (depending on $\mathcal{M}$) and for every $0 < c < 1$ there exists $t_0 > 0$ (depending on $\mathcal{M}$ and c) such that for all $x \in \mathcal{H}_1(\alpha)$ and all $t > t_0$,*
$$(A_t f_{\mathcal{M}})(x) \leq c f_{\mathcal{M}}(x) + b.$$

(c) *There exists $\sigma > 1$ such that for all $g \in \mathrm{SL}(2, \mathbb{R})$ with $\|g\| \leq 1$ and all $x \in \mathcal{H}_1(\alpha)$,*
$$\sigma^{-1} f_{\mathcal{M}}(x) \leq f_{\mathcal{M}}(gx) \leq \sigma f_{\mathcal{M}}(x).$$

The proof uses the recurrence properties of the $\mathrm{SL}(2, \mathbb{R})$ action proved by in [1], and also the fundamental result of Forni on the uniform hyperbolicity in compact sets of the Teichmüller geodesic flow [8, Cor. 2.1]; combining these two facts we get "expansion on average", see [9].

In the case $\mathcal{M}$ is empty, a function satisfying the conditions of Proposition 5 has been constructed in [5] and used in [1].

In fact, we show that the constant b in Proposition 5 (b) depends only on the "complexity" of $\mathcal{M}$, see [7, §8]. This fact is used in for the proof of the following. This proposition is needed for the proof of Theorem 4, Another proof of Proposition 6 is given in [12].

Proposition 6. *There are at most countably many affine manifolds in each stratum.*

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Cutting sequences on Bouw-Möller surfaces

Diana Davis

Given a translation surface with edges labeled, consider a geodesic flow in some direction, and record the bi-infinite sequence of edges that the trajectory cuts through. This is called a *cutting sequence*.

For a given translation surface, our goal is to characterize all possible cutting sequences. For the square torus, the cutting sequences are called *Sturmian sequences*, and the characterization is well known [4]. For the regular octagon surface, John Smillie and Corinna Ulcigrai have given a characterization of the closure of the space of all cutting sequences [5], [6].

The strategy in each of these cases and in the present work, to work towards our eventual goal of characterizing all possible cutting sequences, is to use symmetries of the surface to obtain new trajectories from a given trajectory, and thus obtain new cutting sequences from a known cutting sequence. We study *Veech surfaces* [7], whose group of symmetries has three types of elements: rotations and reflections, which induce a permutation on the edge labels, and the parabolic element, which acts as a "twist," or "shear." We can do this because the cylinders of Veech surfaces have commensurable moduli.

Our specific goal is to determine the effect of the parabolic element of the Veech group on trajectories and their associated cutting sequences on surfaces whose cylinders all have the same modulus.

Definition 1. *Let τ be a trajectory on a Veech surface whose cylinders all have modulus M, and let $c(\tau)$ be its associated cutting sequence. Let τ' be the image of τ under the parabolic whose derivative is $\left(\begin{smallmatrix}1 & M\\ 0 & 1\end{smallmatrix}\right)$, and let $c(\tau')$ be its associated cutting sequence. A* sandwiched *edge label is one that has the same label on either side, as does A and only A in the sequence $\ldots CACDDBA\ldots$.*

Theorem 2 ([2]). *Let τ be a trajectory on a double regular n-gon surface, n odd, with $\theta \in [0, \pi/5]$. To obtain $c(\tau')$ from $c(\tau)$, keep the sandwiched edge labels. (In this talk, we show a video that demonstrates this result using human dancers.)*

The double regular polygon surfaces are an n-indexed family of Veech surfaces. One might ask if there is any larger family of such surfaces. The so-called *Bouw-Möller surfaces* are such a family, an (m, n)-indexed family of Veech surfaces wherein every cylinder has the same modulus. These surfaces were discovered algebraically by Irene Bouw and Martin Möller [1], and then Pat Hooper gave a polygon decomposition for the surfaces [3], and then Alex Wright showed that the surfaces are the same in all cases [8].

Definition 3. *A* semi-regular *polygon is an equiangular $2n$-gon whose edge lengths alternate between two different values, possibly 0. The polygon decomposition of the (m, n) Bouw-Möller surface is a collection of m semi-regular $2n$-gons, whose edge lengths are chosen so that each cylinder has the same modulus. The even-numbered edges of each polygon are glued to one other polygon, and the odd-numbered edges to another polygon; see* [3].

The Bouw-Möller surfaces are a large family of Veech surfaces, and the double regular polygon surfaces are a subfamily, the case $m = 2$. Our goal is to characterize all cutting sequences on these surfaces, using their symmetries, as above.

These surfaces exhibit a surprising and beautiful symmetry, which is that the (m, n) Bouw-Möller surface is affinely equivalent to the (m, n) Bouw-Möller surface. In this talk, we explicitly demonstrate how to cut up the $(3, 5)$ surface, which consists of two regular pentagon and a regular decagon, and reassemble it into a sheared version of the $(5, 3)$ surface, which consists of two equilateral triangles, two semi-regular hexagons, and a regular hexagon. We also show how to cut up the $(5, 3)$ surface and reassemble it into a sheared version of the $(3, 5)$ surface. A dissection of this kind does not appear anywhere in the literature, though a recipe is given in [3].

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Immersions and the space of all translation surfaces

W. Patrick Hooper

For this talk, a *translation surface* is a topological surface equipped with an atlas of charts to the plane so that the transition function are translations. Note that this definition means that our surfaces have no singularities, and all translation surfaces other than quotients of the plane are incomplete. Our translation surfaces will be pointed, i.e., they come with a choice of a basepoint. Our goal is to place a topology on the space $\mathcal{M}$ of all translation surfaces, which includes incomplete surfaces of infinite topological type, and to draw connections to associated dynamical systems.[1]

Immersions

Let D be a simply connected subset of a translation surface containing the basepoint. An *immersion* of D into a (pointed) translation surface S is a continuous map $D \rightsquigarrow S$ which sends the basepoint of D to the basepoint of S, and which acts as a translation in local coordinates.

We remark that the restriction of the notion of immersion to the space $\tilde{\mathcal{M}}$ of all pointed simply connected translation surfaces yields a partial ordering on $\tilde{\mathcal{M}}$. The following result discusses the structure of this ordering.

Theorem 1. *Let* 0 *denote the degenerate translation surface consisting of a single point; it immerses in everything. The set* $\tilde{\mathcal{M}} \cup \{0\}$ *equipped with the partial order* $\rightsquigarrow$ *forms a complete lattice, i.e., each subset of* $\tilde{\mathcal{M}} \cup \{0\}$ *has a supremum and an infimum in* $\tilde{\mathcal{M}} \cup \{0\}$.

Topologies on moduli spaces

We use immersions to define the *immersive topology* on $\tilde{\mathcal{M}}$. A sequence of simply connected translation surfaces $\tilde{S}_n \in \tilde{\mathcal{M}}$ converges to $\tilde{S} \in \tilde{\mathcal{M}}$ if the following two statements hold:

- For every $K \subset \tilde{S}$ homeomorphic to a closed disk which contains the basepoint in its interior, there is an N so that $K \rightsquigarrow \tilde{S}_n$ for $n > N$.
- For every $\tilde{U} \in \tilde{\mathcal{M}}$, if $\tilde{U} \rightsquigarrow \tilde{S}_n$ for infinitely many n, then $\tilde{U} \rightsquigarrow \tilde{S}$.

(We just state the notion of convergence of sequences here because sequences are more relevant to this discussion, but each statement above corresponds to collection of open sets. See [Hoo13b] for a formal definition of the topology.)

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There is a canonical disk bundle over $\tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$, where the fiber over a surface $\tilde{S} \in \tilde{\mathcal{M}}$ is a copy of $\tilde{S}$. As a set, we have

$$\tilde{\mathcal{E}} = \{(\tilde{S}, p) \ : \ \tilde{S} \in \tilde{\mathcal{M}} \text{ and } p \in \tilde{S}\}.$$

In the *immersive topology* on $\tilde{\mathcal{E}}$, a sequence $(\tilde{S}_n, p_n) \in \tilde{\mathcal{E}}$ converges to $(\tilde{S}, p) \in \tilde{\mathcal{E}}$ if both of the following hold:

- The sequence $\tilde{S}_n$ converges to $\tilde{S}$ in the immersive topology on $\tilde{\mathcal{M}}$.
- For one (or equivalently all) closed disk $K \subset S$ containing p and the basepoint, the immersions $\iota_n : K \rightsquigarrow \tilde{S}_n$ (which exist for n sufficiently large) satisfy $d_n\big(p_n, \iota_n(p)\big) \to 0$ as $n \to \infty$, where d_n is the metric on $\tilde{S}_n$.

(Again, see [Hoo13b] for a formal definition.)

Given topologies on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$, there is a canonical way to topologize the space $\mathcal{M}$ of all (pointed) translation surfaces. Namely given a translation surface $S \in \mathcal{M}$, we consider its universal cover $\tilde{S} \in \tilde{\mathcal{M}}$ and consider all lifts of the basepoint of S to $\tilde{S}$. So, a sequence of translation surfaces $S_n \in \mathcal{M}$ converges to $S \in \mathcal{M}$ if both of the following hold:

- The sequence of universal covers $\tilde{S}_n$ converges to $\tilde{S}$ in $\tilde{\mathcal{M}}$.
- A point $\tilde{p} \in \tilde{S}$ is a lift of the basepoint of S if and only if there is a sequence $\tilde{p}_n \in \tilde{S}_n$, with each $\tilde{p}_n$ a lift of the basepoint of S_n, so that $(\tilde{S}_n, \tilde{p}_n)$ converges to $(\tilde{S}, \tilde{p})$ in $\tilde{\mathcal{E}}$.

We also topologize the translation surface bundle $\mathcal{E}$ over $\mathcal{M}$, but we will not define it here. (See [Hoo13a] for formal definitions.)

We will now highlight some of the main results from [Hoo13a] involving these topologies.

Theorem 2. *The immersive topologies on $\tilde{\mathcal{M}}$, $\tilde{\mathcal{E}}$, $\mathcal{M}$ and $\mathcal{E}$ are second countable and Hausdorff.*

In particular, note that convergent sequences have unique limits. Furthermore, there is only one obstruction to finding a convergent subsequence of a sequence in $\mathcal{M}$:

Theorem 3. *For any $\epsilon > 0$, the set of surfaces in $\mathcal{M}$ for which the basepoint's open ϵ-neighborhood is isometric to the open ϵ-ball in the plane is compact.*

Dynamics

We would also like to utilize this topology to understand related dynamical systems. The straight line flow on $\mathcal{M}$ in the direction of θ moves the basepoint in direction θ at unit speed. Similar straight-line flows can be defined on the spaces $\tilde{\mathcal{M}}$, $\tilde{\mathcal{E}}$, and $\mathcal{E}$, and it can be shown that these flows are continuous wherever they are defined. The general linear group $\mathrm{GL}(2, \mathbb{R})$ also acts on these spaces, and it can be shown that the actions are jointly continuous. As a consequence to this, we can prove a result about how affine automorphisms behave under limits.

Theorem 4. *If S_n converges to S in $\mathcal{M}$, and there is a sequence $\phi_n : S_n \to S_n$ of affine automorphisms whose derivatives converge in $GL(2,\mathbb{R})$ and the the images under ϕ_n of the basepoints converge to some point $(S,p) \in \mathcal{E}$, then S has an affine automorphism with the limiting derivative which sends the basepoint to p.*

We have the following analog of Masur's criterion for unique ergodicity [Mas92].

Theorem 5. *Suppose $S \in \mathcal{M}$ is a translation surface of area one. If there is a sequence of times $t_n \to \infty$ and a sequence of basepoints s_n of S so that under the Teichmüller flow $g^{t_n}(S,s_n)$ converges to a unit area surface in $\mathcal{M}$, then the vertical straight line flow in the vertical direction is uniquely ergodic.*

This result can be deduced from work of Treviño [Tre14], but has not yet appeared.

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Slope gaps for golden L diagonals

Samuel Lelièvre

(joint work with Jayadev Ateyra, Jon Chaika)

There is a limiting distribution for (appropriately renormalized) gaps between successive slopes of diagonals on the golden L in the following sense:

Theorem 1. *Let Λ_R be the set of diagonals with slope $0 \le s \le 1$ and x coordinate in $[0,R]$, S_R the set of their slopes and G_R the set of gaps between successive slopes in S_R. There exists a function $f : [0,+\infty) \to [0,+\infty)$ such that $\forall a,b \in \mathbb{R}$ with $a < b$,*

$$\lim_{R\to+\infty} \frac{\sharp\,(G_R \cap (a,b))}{\sharp G_R} = \int_a^b f.$$

This probability density function for the limiting distribution of slope gaps is computed explicitely. It is piecewise analytic with 8 pieces (7 jumps in the derivative), and is zero on $[0,1]$.

This is proved by constructing a Poincaré section for the horocyclic flow $h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$ acting on the $\mathrm{SL}(2,\mathbb{R})$ orbit of L, which is $G_H \cdot L \sim G/\Gamma$, where $G = \mathrm{SL}(2,\mathbb{R})$ and Γ is the Veech group of the golden L, which is the Hecke group H_5, a $(2,5,\infty)$ triangle group. In coordinates (a,b), where $g_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ represents

$g_{a,b} \cdot L$, the section, made of surfaces $M \in G \cdot L$ with a horizontal long diagonal of length < 1, corresponds to the triangle Ω with vertices $(0,1)$, $(1,1)$, $(1,-\bar{\varphi})$, where $\bar{\varphi}$ denotes $\frac{1}{\varphi}$ $(= \varphi - 1)$.

The return map is piecewise affine in these coordinates and the return time is piecewise defined by

$$R(a,b) = \begin{cases} \frac{1}{a(a+b)} & \text{if } (a,b) \in \Omega_1 \\ \frac{1}{a(a\bar{\varphi}+b)} & \text{if } (a,b) \in \Omega_\varphi \\ \frac{1}{ab} & \text{if } (a,b) \in \Omega_\infty \end{cases}.$$

The function f is then given by

$$f(t) = \int_{\{(a,b),\ R(a,b)<t\}} R(a,b)\ dadb,$$

which gives explicit piecewise defined expressions.

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