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Problems on Billiards, Flat Surfaces and Translation Surfaces

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Part of this list of problems grew out of a series of lectures given by the authors at the Luminy confrence in June 2003. These lectures appeared as survey articles in [**HuSt3**], [**Ma1**], see the related [**Es**] and [**Fo2**].

Additional general references for the background material for these problems are the survey articles [**MaTa**], [**Sm**], [**Gu1**], [**Gu2**], [**Zo4**] and the book [**St**].

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1. Flat surfaces and billiards in polygons

The first set of problems concerns general flat surfaces with nontrivial holonomy and billiards in polygons.

Using any of various variations of a standard unfolding construction one can glue flat surfaces from several copies of the billiard table. When the resulting surface is folded back to the polygon, the geodesics on the surface are projected to billiard trajectories, so billiards in polygons and flat surfaces are closely related.

A quintessential example of a flat surface is given by the surface of the standard cube in real three-space. With its induced metric it is a flat sphere with eight singularities. Indeed, at each corner, three squares meet so that each corner has a neighborhood that is isometric to a Euclidean cone, with cone angle $3\pi/2$. These are indeed the only singularities of the flat metric; we call them *conical singularities* of the flat surface. Since the cone angle is not a multiple of 2π, parallel transport of a non-zero tangent vector about a simple closed curve around a corner will result in a distinct tangent vector; thus the holonomy is nontrivial. In general, a "flat surface" here refers to a surface of zero Gaussian curvature with isolated conical singularities.

Having a Riemannian metric it is natural to study geodesics. Away from singularities geodesics on a flat surface are (locally isometric to) straight lines. The geodesic flow on the unit tangent bundle is then also presumably well behaved. For simplicity, let "ergodic" here mean that a typical geodesic visits any region of the surface, and furthermore (under unit speed parametrization) spends a time in the region that is asymptotically proportional to the area of the region.

PROBLEM 1 (Geodesics on general flat surfaces). Describe the behavior of geodesics on general flat surfaces. Prove (or disprove) the conjecture that the geodesic flow is ergodic on a typical (in any reasonable sense) flat surface. Does any (almost any) flat surface have at least one closed geodesic which does not pass through singular points?

If the answer is positive then one can ask for the asymptotics for the number of closed geodesics of bounded length as a function of the bound.

Note that typically a geodesic representative in a homotopy class of a simple closed curve is realized by a broken line containing many geodesic segments going from one conical singularity to the other. The counting problem for regular closed geodesics (ones which do not pass through singularities) is quite different from the counting problem for geodesics realized by broken lines.

The following questions treat billiards in arbitrary polygons in the plane.

PROBLEM 2 (Billiards in general polygons). Does every billiard table have at least one regular periodic trajectory? If the answer is affirmative, does this trajectory persist under deformations of the billiard table?

If a periodic trajectory exists, find the asymptotics for the number of periodic trajectories of bounded length as a function of the bound.

Describe the behavior of a generic regular billiard trajectory in a generic polygon; in particular, prove (or disprove) the assertion that the billiard flow is ergodic.[1]

We note that the case of triangles is already highly non-trivial. For recent work on billiards in obtuse triangles see [**Sc1**] and [**Sc2**].

In the case of triangles the notion of generic can be interpreted as follows. The space of triangles up to similarity can be parametrized as the set of triples $(\theta_1, \theta_2, \theta_3)$ with $\sum \theta_i = \pi$ and each $\theta_i > 0$. It is naturally an open simplex. Generic then refers to the natural Lebesgue measure.

To motivate the next problem we note that there is a close connection between the study of interval exchange transformations and billiards in rational polygons (defined below). An important technique in the study of interval exchange maps is that of renormalization. Given an interval exchange on the unit interval one can take the induced transformation on a subinterval. The resulting map is again an interval exchange map, and if one renormalizes so that the new interval has length one, then this gives a transformation on the space of unit interval exchange maps. This transformation is called the Rauzy-Veech induction ([**Ra**], [**Ve2**]), and it has proved to be of fundamental importance. There is a corresponding notion for the renormalization of translation surfaces given by the Teichmüller geodesic flow.

PROBLEM 3 (Renormalization of billiards in polygons). Is there a natural dynamical system acting on the space of billiards in polygons so as to allow a useful renormalization procedure?

2. Rational billiards, translation surfaces quadratic differentials and $\mathrm{SL}(2,\mathbb{R})$ actions

An important special case of billiards is given by the *rational billiards* — billiards in polygonal tables whose vertex angles are rational multiples of π. There

[1] On behalf of the Center of Dynamics of Pennsylvania State University A. Katok promised a prize for a solution of this problem.

is a well-known procedure (see the surveys [**MaTa**], [**Gu1**]) which associates to a rational billiard an object called a translation surface. One labels the sides of the polygon and successively reflects the polygon across sides. The rationality assumption guarantees that after a certain number of reflections a labelled side appears parallel to itself. In that case the pair of sides with the same label is glued by a parallel translation. The result is a closed surface with conical singularities. The billiard flow on the polygon which involves reflection in the sides is replaced by a straight line flow on the glued surface; under the natural projection of the surface to the billiard table the geodesics are projected to billiard trajectories. It turns out that many results in rational billiards are found by studying more general translation surfaces.

A translation surface is defined by the following data:

- a finite collection of disjoint polygons $\Delta_1, \dots, \Delta_n$ embedded into the oriented Euclidean plane;
- a pairing between the sides of the polygons: to each side s of any Δ_i is associated a unique side $s' \neq s$ of some Δ_j in such way that the two sides s, s' in each pair are parallel and have the same length $|s| = |s'|$. The pairing respects the induced orientation: gluing Δ_i to Δ_j by a parallel translation sending s to s' we get an oriented surface with boundary for any pair s, s';
- a choice of the positive vertical direction in the Euclidean plane.

A classical example is the square with opposite unit sides identified, giving the flat torus. This example arises from billiards in a square of side length $1/2$. Another example is a regular octagon with opposite sides identified. It arises by the unfolding process from billiards in a right triangle whose other angles are $\pi/8, 3\pi/8$. When translation surfaces arise from billiards the polygons in the gluings can be taken to be congruent, so translation surfaces arising from rational billiards always have extra symmetries not possessed by general translation surfaces. In this sense translation surfaces coming from rational billiards are always rather special.

Note that a translation surface is in particular a flat surface in the sense described before. It is locally Euclidean except possibly at the points corresponding to the vertices of the polygons. These points can be conical singularities, but the total angle around such a vertex — its cone angle — is always an integer multiple of 2π. For example, in the case of the regular octagon, the 8 vertices are identified to a single point with cone angle 6π.

Since the gluing maps are translations which are of course complex analytic, the underlying structure is that of a Riemann surface X. Moreover since translations preserve the form dz in each polygon, these forms dz fit together to give a holomorphic 1-form ω on X. Thus translation surfaces are often denoted by (X, ω). In this language a cone angle $2k\pi$ at a singularity corresponds to a zero of order $k-1$ of ω. The orders of the zeroes form a tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\sum \alpha_i = 2g - 2$ and g is the genus of the surface. In the case of the regular octagon, there is a single zero of order 2 so $\alpha = (2)$. The set of all (X, ω) whose zeroes determine a fixed tuple α form a moduli space $\mathcal{H}(\alpha)$, called a stratum. We may think of the points of this moduli space as glued polygons where the vectors corresponding to the sides are allowed to vary. Since each (X, ω) has the underlying structure of a Riemann surface, remembering just the complex structure gives a projection from each stratum to the Riemann moduli space. One can introduce "markings" in order to get a well-defined projection map from spaces of marked Abelian differentials to Teichmüller space.

Studying translation surfaces from the different viewpoints of geometry and complex analysis has proved useful.

If we loosen our restrictions on the gluings so as to allow reflections in the origin as well as translations, then there is still an underlying Riemann surface; the resulting form is a quadratic differential. The structure is sometimes also called a half-translation surface. Now the cone angles are integer multiples of π. Thus each quadratic differential determines a set of zeroes whose orders again give a tuple $\beta = (\beta_1, \ldots, \beta_n)$ with $\sum \beta_i = 4g - 4$. We similarly have strata of quadratic differentials $\mathcal{Q}(\beta)$. If we fix a genus g, and introduce markings, then the union of the strata of quadratic differentials (including those that are naturally seen as the squares of Abelian differentials) of genus g fit together to form the cotangent bundle over Teichmüller space.

Much of the modern treatment of the subject arises from the study of the action of the group $\mathrm{SL}(2,\mathbb{R})$ on each moduli space $\mathcal{H}(\alpha)$. Understanding the orbit of a translation surface allows one to understand much of the structure of the translation surface itself. For each (X,ω) realized as a union of glued polygons Δ_i, and $A \in \mathrm{SL}(2,\mathbb{R})$, let A act on each Δ_i by the linear action on $\mathbb{R}^2$. Since A preserves parallel lines, this gives a map of (X,ω) to some $A \cdot (X,\omega)$. We have a similar action for $A \in \mathrm{GL}^+(2,\mathbb{R})$. If we introduce markings then the projection of the orbit to Teichmüller space gives an isometric embedding of the hyperbolic plane into Teichmüller space equipped with the Teichmüller metric. The projection of the orbit of (X,ω) is called a Teichmüller disc. Similarly, we have Teichmüller discs for quadratic differentials. The image of the disc in the moduli space is typically dense. However there are (X,ω) whose orbit is closed in its moduli space $\mathcal{H}(\alpha)$. These are called Veech surfaces. We will discuss these in more detail in the next section.

2.1. Veech surfaces. This section discusses problems related to Veech surfaces and Veech groups. Given a translation surface (X,ω), or quadratic differential, one can discuss its affine diffeomorphism group; that is, the homomorphisms that are diffeomorphisms on the complement of the singularities, with constant Jacobian matrix (with respect to the flat metric). The group of Jacobians, $\mathrm{SL}(X,\omega) \subset \mathrm{SL}(2,\mathbb{R})$ can also be thought of as the stabilizer of (X,ω) in the moduli space under the action of $\mathrm{SL}(2,\mathbb{R})$ on the moduli space of all translation surfaces. (The $\mathrm{SL}(2,\mathbb{R})$ action is discussed in the next section). The Jacobians of orientation preserving affine diffeomorphisms form a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, also called a Fuchsian group. The image in $\mathrm{PSL}(2,\mathbb{R})$ is the Veech group of the surface. One can also think of this group as a subgroup of the mapping class group of the surface. Hyperbolic elements of this group correspond to pseudo-Anosovs in the mapping class group; parabolic elements to reducible maps and elliptics to elements of finite order.

The surface (X,ω) is called a Veech surface if this group is a lattice (that is of cofinite volume) in $\mathrm{PSL}(2,\mathbb{R})$. By a result of Smillie [**Ve4**] it is known that a surface is a Veech surface if and only if its $\mathrm{SL}(2,\mathbb{R})$-orbit is closed in the corresponding stratum.

In genus 2 it is known [**McM1**] that if $\mathrm{SL}(X,\omega)$ contains a hyperbolic element then in its action on the hyperbolic plane, it has as its limit set the entire circle at infinity. Consequently, it is either a lattice or infinitely generated. There are known examples of the latter, see [**HuSt2**] and [**McM1**].

PROBLEM 4 (Characterization of Veech surfaces). Characterize all Veech surfaces (for each stratum of each genus).

This problem is trivial in genus one; in genus two K. Calta [**Ca**] and C. McMullen [**McM2**] have provided solutions. In the papers [**KnSm**] and Puchta [**Pu**] the acute rational billiard triangles that give rise to Veech surfaces were classified. In [**SmWe**] there is a criterion for a surface (X, ω) to be a Veech surface that is given in terms of the areas of triangles embedded in (X, ω).

PROBLEM 5 (Fuchsian groups). Which Fuchsian groups are realized as Veech groups? Which subgroups of the mapping class group appear as Veech groups? This is equivalent to asking which subgroups are the stabilizers of a Teichmüller disc.

PROBLEM 6 (Purely cyclic). Is there a Veech group that is cyclic and generated by a single hyperbolic element? Equivalently, is there a pseudo-Anosov map such that its associated Teichmüller disk, is invariant only under powers of the pseudo-Anosov?

PROBLEM 7 (Algorithm for Veech groups). Is there an algorithm for determining the Veech group of a general translation surface or quadratic differential?

An interesting class of Veech surfaces are the square-tiled surfaces. These surfaces can be represented as a union of glued squares all of the same size, see [**Zo3**], [**HuLe**].

PROBLEM 8 (Orbits of square-tiled surfaces). Classify the $\mathrm{SL}(2, \mathbb{R})$ orbits of square-tiled surfaces in any stratum. Describe their Teichmüller discs. A particular case of this problem is that of the stratum $\mathcal{H}(1, 1)$.

The above problem is solved only for the stratum $\mathcal{H}(2)$, see [**HuLe**], [**McM4**].

2.2. Minimal sets and analogue of Ratner's theorems. The next set of questions concern the $\mathrm{SL}(2, \mathbb{R})$ action. They are motivated by trying to find an analogue of the $\mathrm{SL}(2, \mathbb{R})$ action on the moduli spaces to Ratner's celebrated theorems on the actions of subgroups of a Lie group G on G/Γ where Γ is a lattice subgroup.

PROBLEM 9 (Orbit closures for moduli spaces). Determine the closures of the orbits for the $\mathrm{GL}^+(2, \mathbb{R})$-action on $\mathcal{H}(\alpha)$ and $\mathcal{Q}(\beta)$. Are these closures always complex-analytic (complex-algebraic?) orbifolds? Characterize the closures geometrically.

Note that by a theorem of Kontsevich any $\mathrm{GL}^+(2, \mathbb{R})$-invariant complex-analytic subvariety is represented by an affine subspace in period coordinates.

Consider the subset $\mathcal{H}_1(\alpha) \subset \mathcal{H}(\alpha)$ of translation surfaces of area one. It is a real codimension one subvariety in $\mathcal{H}(\alpha)$ invariant under the action of $\mathrm{SL}(2, \mathbb{R})$. In the period coordinates it is defined by a quadratic equation (the Riemann bilinear relation). It is often called a *unit hyperboloid.* It is worth noting that it is a manifold locally modelled on a paraboloid. The invariant measure on $\mathcal{H}(\alpha)$ gives a natural invariant measure on the unit hyperboloid $\mathcal{H}_1(\alpha)$. Similarly one can define the unit hyperboloid $\mathcal{Q}_1(\beta)$. It was proved by Masur and Veech that the total measure of any $\mathcal{H}_1(\alpha)$, $\mathcal{Q}_1(\beta)$ is finite.

PROBLEM 10 (Ergodic measures). Classify the ergodic measures for the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}_1(\alpha)$ and $\mathcal{Q}_1(\beta)$.

McMullen [**McM3**] has solved Problems 9 and 10 in the case of translation surfaces in genus 2.

A subset Ω is called minimal for the action of $\mathrm{SL}(2,\mathbb{R})$ if it is closed, invariant, and it has no proper closed invariant subsets. The $\mathrm{SL}(2,\mathbb{R})$ orbit of a Veech surface is an example of a minimal set.

PROBLEM 11 (Minimal sets). Describe the minimal sets for the $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{H}_1(\alpha)$ and $\mathcal{Q}_1(\beta)$. Since Veech surfaces give rise to minimal sets, this problem generalizes the problem of characterizing Veech surfaces.

The problem below is particularly important for numerous applications. One application is to counting problems.

PROBLEM 12 (Analog of Ratner theorem). Classify the ergodic measures for the action of the unipotent subgroup $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}_{t\in\mathbb{R}}$ on $\mathcal{H}_1(\alpha)$ and $\mathcal{Q}_1(\beta)$. We note that a solution of Problem 10 does not imply a solution of this problem. In particular this problem is open even in genus 2.

Similarly classify the orbit closures on these moduli spaces.

There are some results in special cases on this problem, see [**EsMaSl**] and [**EsMkWt**].

K. Calta [**Ca**] and C. McMullen [**McM2**] have found unexpected closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant sets in genus 2 which we now describe. One can form a family of translation surfaces from a given (X,ω) by varying the periods of the 1-form ω along cycles in the relative homology — those that join distinct zeroes — while keeping the "true" periods (that is, the absolute cohomology class of ω) fixed. One may also break up a zero of higher order into zeroes of lower order while keeping the absolute periods fixed. The resulting family of translation surfaces gives a leaf of the *kernel foliation* passing through (X,ω).

It follows from [**Ca**] and [**McM2**] that for *any* Veech surface $(X,\omega)\in\mathcal{H}(2)$, the union of the complex one-dimensional leaves of the kernel foliation passing through the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of (X,ω) is a closed complex orbifold $\mathcal{N}$ of complex dimension 3. By construction $\mathcal{N}$ is $\mathrm{GL}^+(2,\mathbb{R})$-invariant. Note that the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of the initial Veech surface (X,ω) is closed and has complex dimension 2, so what is surprising is that the union of the complex one-dimensional leaves passing through each point of the orbit $\mathrm{GL}^+(2,\mathbb{R})\cdot(X,\omega)$ is again closed.

We may ask a similar question in higher genus. Let $\mathcal{O}\subset\mathcal{H}(\alpha_1,\dots,\alpha_m)$ be a $\mathrm{GL}^+(2,\mathbb{R})$-invariant submanifold (suborbifold) on translation surfaces of genus g. Let $\mathcal{H}(\alpha'_1,\dots,\alpha'_n)$ be a stratum of surfaces of genus g that is adjacent, in that each α_i is the sum of corresponding α'_j. The complex dimension of the leaves of the kernel foliation in $\mathcal{H}(\alpha'_1,\dots,\alpha'_n)$ is $n-m$.

Consider the closure of the union of leaves of the kernel foliation in the stratum $\mathcal{H}(\alpha'_1,\dots,\alpha'_n)$ passing through $\mathcal{O}$; this is a closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant subset $\mathcal{N}\subset\mathcal{H}(\alpha'_1,\dots,\alpha'_n)$ of dimension at least $\dim_{\mathbb{C}}\mathcal{O}+n-m$.

PROBLEM 13 (Kernel foliation). Is $\mathcal{N}$ a complex-analytic (complex-algebraic) orbifold? When is $\dim_{\mathbb{C}} \mathcal{N} = \dim_{\mathbb{C}} \mathcal{O} + n - m$? On the other hand when does $\mathcal{N}$ coincide with the entire connected component of the enveloping stratum $\mathcal{H}(\alpha'_1, \dots, \alpha'_n)$?

One of the key properties used in [**McM3**] for the classification of the closures of orbits of $\mathrm{GL}(2, \mathbb{R})$ in each of $\mathcal{H}(1,1)$ and $\mathcal{H}(2)$ was the knowledge that on *any* translation surface of either stratum, one can find a pair of homologous saddle connections.

For example, cutting a surface (X, ω) in $\mathcal{H}(1,1)$ along two homologous saddle connections joining distinct zeroes decomposes the surface into two tori, allowing one to apply the machinery of Ratner's Theorem.

PROBLEM 14 (Decomposition of surfaces). Given a connected component of the stratum $\mathcal{H}(\alpha)$ of Abelian differentials (or of quadratic differentials $\mathcal{Q}(\beta)$ find those configurations of homologous saddle connections (or homologous closed geodesics), which are present on every surface in the stratum.

For quadratic differentials the notion of homologous saddle connections (homologous closed geodesics) should be understood in terms of homology with local coefficients, see [**MaZo**].

The last two problems in this section concern the Teichmüller geodesic flow on the moduli spaces. This is the flow defined by the 1-parameter subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}_{t \in \mathbb{R}}$ on $\mathcal{H}(\alpha_1, \dots, \alpha_m)$.

In any smooth dynamical system the Lyapunov exponents (see [**BaPe**], [**Fo2**]) are important. Recently, A. Avila and M. Viana [**AvVi**] have shown the simplicity of the spectrum for the cocycle related to the Teichmüller geodesic flow (strengthening the earlier result of Forni on positivity of the smallest Lyapunov exponent).

PROBLEM 15 (Lyapunov exponents). Study *individual* Lyapunov exponents of the Teichmüller geodesic flow:
– for all known $\mathrm{SL}(2; \mathbb{R})$-invariant subvarieties;
– for strata;
– for strata of large genera as the genus tends to infinity.

Are they related to characteristic numbers of any natural bundles over appropriate compactifications of the strata?

The motivation for this problem is a beautiful formula of Kontsevich [**Ko**] representing the sum of the first g Lyapunov exponents in differential-geometric terms.

It follows from the Calabi Theorem [**Cb**] that given a *real* closed 1-form ω_0 with isolated zeroes Σ (satisfying some natural conditions) on a smooth surface S of real dimension two, one can find a complex structure on S and a holomorphic 1-form ω such that the ω_0 is the real part of ω. Consider the resulting point (X, ω) in the corresponding stratum. For generic (X, ω) the cocycle related to the Teichmüller geodesic flow acting on $H^1(X, \mathbb{R})$ defines a pair of transverse Lagrangian subspaces $H^1(X, \mathbb{R}) = L_0 \oplus L_1$ by means of the Oseledets Theorem ([**BaPe**]). These subspaces correspond to contracting and to expanding directions.

Though the pair (X, ω) is not uniquely determined by ω_0, the subspace $L_0 \subset H^1(X, \mathbb{R})$ does not depend on (X, ω) for a given ω_0. Moreover, L_0 does not change under small deformations of ω_0 that preserve the cohomology class $[\omega] \in H^1(S, \Sigma; \mathbb{R})$. We get a topological object L_0 defined in implicit dynamical terms.

PROBLEM 16 (Dynamical Hodge decomposition). Study properties of distributions of the Lagrangian subspaces in $H^1(S;\mathbb{R})$ defined by the Teichmüller geodesic flow, in particular, their continuity. Is there any topological or geometric way to define them?

The Lagrangian subspaces are an interesting structure relating topology and geometry.

2.3. Geometry of individual flat surfaces. Let (X,ω) be a translation surface (resp. quadratic differential). Fix a direction $0 \leq \theta < 2\pi$ and consider a vector field (resp. line field) on each polygon of unit vectors in direction θ. Since the gluings are by translations (or by rotations by π about the origin followed by translations) which preserve this vector field, there is a well-defined vector field on (X,ω) (resp. line field) defined except at the zeroes. There is a corresponding flow ϕ_t^θ (resp. foliation). A basic question is to understand the dynamics of this flow or foliation. In the case of a flat torus this is classical. For any direction either every orbit in that direction is closed or every orbit is dense and uniformly distributed on the surface. This property is called unique ergodicity.

For a general translation surface a saddle connection is defined to be a leaf joining a pair of conical singularities. There are only countably many saddle connections in all possible directions. For any direction which does not have a saddle connection, the flow or foliation is minimal, which means that for any point, if the orbit in either the forward or backward direction does not hit a conical singularity then it is dense. Veech [**Ve3**] showed that as in the case of a flat torus, every Veech surface satisfies the dichotomy that for any θ, the flow or foliation has the property that either:

- Every leaf which does not pass through a singularity is closed. This implies that the surface decomposes into a union of cylinders of parallel closed leaves. The boundary of each cylinder is made up of saddle connections. The directional flow is said to be *completely periodic* if it has this property.
- The foliation is minimal and uniquely ergodic.

PROBLEM 17 (Converse to dichotomy). Characterize translation surfaces for which

(1) the set of minimal directions coincides with the set of uniquely ergodic directions;

(2) the set of completely periodic directions coincides with the set of non-uniquely ergodic directions.

Note that Property (2) implies Property (1).

In genus $g = 2$ it is known that for every translation surface which is not a Veech surface there is a direction θ which is minimal and not uniquely ergodic [**ChMa**]. On the other hand, using work of Hubert–Schmidt [**HuSt2**], J. Smillie and B. Weiss [**SmWe**] have given an example of a surface which is not a Veech surface and yet for which Property (2) holds. (This surface is obtained as a ramified covering over a Veech surface with a single ramification point.)

As mentioned above, a closed orbit avoiding the conical singularities of the flat surface determines a cylinder of parallel lines, all of the same length. It is of interest to find the asymptotics for the number of cylinders (in all possible directions) of lengths less than a given number. In the case of the standard flat torus the number of cylinders of length at most L is asymptotic to

$$\frac{1}{\zeta(2)}\pi L^2.$$

Each Veech surface also has quadratic asymptotics [**Ve3**] and the same is true for generic surfaces in each stratum [**EsMa**].

PROBLEM 18 (Quadratic asymptotics for any surface). Is it true that *every* translation surface or quadratic differential has exact quadratic asymptotics for the number of saddle connections and for the number of regular closed geodesics?

PROBLEM 19 (Error term for counting functions). What can be said about the error term in the quadratic asymptotics for counting functions

$$N((X,\omega),L) \sim c\cdot L^2$$

on a generic translation surface (X,ω)? In particular, is it true that

$$\limsup_{L\to\infty}\frac{\log|N(S,L)-c\cdot L^2|}{\log L} < 2?$$

Is the lim sup the same for almost all flat surfaces in a given connected component of a stratum?

The classical Circle Problem gives an estimate for the error term in the case of the torus.

Veech proved that for Veech surfaces the limsup in the error term is actually a limit, see [**Ve3**]. However, nothing is known about the value of this limit. One may ask whether there is a uniform bound for this limit for Veech surfaces in a given stratum or even for the square-tiled surfaces in a given stratum.

2.4. Topological and geometric properties of strata.

PROBLEM 20 (Topology of strata). Is it true that the connected components of the strata $\mathcal{H}(\alpha)$ and of the strata $Q(\beta)$ are $K(\pi,1)$-spaces (i.e. their universal covers are contractible)?

It is known [**KoZo**], [**La**] that the strata $\mathcal{H}(\alpha)$ and $\mathcal{Q}(\beta)$ need not be connected. With the exception of the four strata listed below, there are intrinsic invariants that allow one to tell which component a given translation surface or quadratic differential belongs to.

PROBLEM 21 (Exceptional strata). Find a geometric invariant which distinguishes different connected components of the four exceptional strata $\mathcal{Q}(-1,9)$, $\mathcal{Q}(-1,3,6)$, $\mathcal{Q}(-1,3,3,3)$ and $\mathcal{Q}(12)$.

At the moment the known invariant (called the extended Rauzy class) distinguishing connected components is given in combinatorial and not geometric terms. ([**La**])

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