Изв. Акад. Наук СССР
Сер. Матем. Том 51(1987), № 6

Math. USSR Izvestiya
Vol. 31(1988), No. 3

THE QUASIPERIODIC STRUCTURE OF LEVEL SURFACES OF A MORSE 1-FORM CLOSE TO A RATIONAL ONE — A PROBLEM OF S. P. NOVIKOV

UDC 514.76+515.16

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ABSTRACT. The topological structure is described of the level surfaces of a Morse 1-form which is close to a rational one on a closed oriented manifold.

Applications are indicated to the investigation of the motion of an electron in a reciprocal lattice in a homogeneous magnetic field.

Figures: 3. Bibliography: 11 titles.

§1. Introduction. A problem of S. P. Novikov

In 1981, in the paper [1] and the survey article [2], S. P. Novikov initiated the construction of an analogue of Morse theory, in which instead of functions one considers closed differential 1-forms with nondegenerate singularities (*Morse-Novikov theory*). In these papers a number of problems were formulated, and the important concept of a *quasiperiodic manifold* was introduced. This, as Novikov pointed out to the author, needs to be made rather more precise in the following way.

We consider a finite set A (the set of indices) and a sequence, infinite in both directions, of indices $\alpha\colon \mathbf{Z} \to A$ ($\alpha_i \in A$, $i \in \mathbf{Z}$). To any mapping $\varphi\colon A \to \mathbf{R}$ we assign the number-valued function on the line $f_{\alpha,\varphi}\colon \mathbf{R} \to \mathbf{R}$, the value of which on the interval $[i; i+1[$ is equal to $\varphi(\alpha_i)$.

DEFINITION 1. The sequence α of indices is called *quasiperiodic* if, for every mapping φ, the function $f_{\alpha,\varphi}$ is almost periodic in the sense of Weyl.

Let us now suppose that to every index $\beta \in A$ there is associated a compact manifold M_β whose boundary consists of two components $\partial M_\beta = N_{\beta,0} \cup N_{\beta,1}$ which are in general not connected. We shall call the sequence α of indices *admissible* if $N_{\alpha_i,0} = N_{\alpha_{i+1},1}$ for every $i \in \mathbf{Z}$. Any admissible sequence of symbols determines a glued manifold $M(\alpha) = \bigcup_i M_{\alpha_i}$ with the natural identification.

DEFINITION 2. A manifold M is said to be *quasiperiodic* if it can be glued together from a finite collection of compact manifolds M_β, $\beta \in A$, by means of an admissible quasiperiodic sequence of symbols $\alpha\colon \mathbf{Z} \to A$.

NOVIKOV'S CONJECTURE. *A nonsingular level surface of a Morse* 1*-form having degree of irrationality* 2 *is a quasiperiodic manifold.*

In this paper we give proof of Novikov's conjecture in the case when the Morse form ω of irrationality degree 2 is close to a rational one. The first results about level surfaces of such forms were obtained in the author's paper [11].

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58E05; Secondary 58F05.

The author thanks S. P. Novikov for posing the problem and for his interest in the work.

We now explain some concepts which will be used later.

A point x of a manifold M is called a *singular* or *critical point* of a given 1-form ω on M if the form $\omega(x)$ is degenerate on T_xM: that is, if it is equal to zero on each vector of the tangent space T_xM. From now on we shall consider only closed 1-forms. A closed 1-form ω is locally the differential of some function: $\omega = df$. The singular points of the form ω and of the function f coincide. A singular point of the closed form is called *nondegenerate* if it is a nondegenerate one for f. We shall be considering closed differential 1-forms with nondegenerate critical points on a closed connected orientable manifold. Such forms we shall call *Morse forms.*

A Morse form determines a distribution of codimension 1 on the manifold. By the theorem of Frobenius, the integrability of the distribution follows from the property that the form is closed. We shall include in a single class, which henceforth will be called a *level of the form*, all the integral surfaces which can be joined by a path in the manifold M along which the integral of the form ω is equal to zero. We note that, in contrast to paths on the integral surfaces of the distribution, on which the restriction of the form ω is identically zero, the restriction of the form ω to the path here referred to need not by any means be identically zero. Hence we shall call each of the indicated classes a *level surface* of the form ω, or a *leaf* of the form ω. It will be proved later that in the case when the Morse form ω has no critical points of indices $0, 1, n-1$, or n, and in particular when there are no singular points at all, a leaf of the form ω turns out to be connected; that is, every leaf consists of a single integral surface of the distribution. In the presence of points with extreme indices this is generally speaking not the case. The aim of the present paper is to describe the topological structure of the level surfaces of a Morse form ω.

We define the *degree of irrationality* of a closed 1-form to be the number of rationally independent integrals of it over all possible cycles. More precisely: let $\gamma_1, \dots, \gamma_s$ be a system of generators of the fundamental group $\pi_1(M)$, and let $u_i = \int_{\gamma_i} \omega$, where $\omega \in \Omega_1(M)$ and $d\omega = 0$. Then the dimension of the vector space $\langle u_1, \dots, u_s\rangle_{\mathbf{Q}}$ over the field $\mathbf{Q}$ is called the *degree of irrationality* of the form ω and is denoted by $\operatorname{rk}\omega$. Thus $\operatorname{rk}\omega = \operatorname{rk}\langle u_1, \dots, u_s\rangle$. We remark that, in fact, the degree of irrationality is determined, not by the whole system of generators of the fundamental group $\pi_1(M)$, but simply by a family of cycles $\beta_1, \dots, \beta_l$ which realize generators of the group $H_1(M; \mathbf{Z})$ or, more precisely still, $\tilde{H}_1(M; \mathbf{Z})$. If we consider the family of integrals $v_i = \int_{\beta_i} \omega$ of the form ω over these cycles, then

$$\operatorname{rk}\langle u_1, \dots, u_s\rangle = \operatorname{rk}\omega = \operatorname{rk}\langle v_1, \dots, v_l\rangle.$$

The structure of a leaf of the form ω is strongly dependent on the degree of irrationality of ω.

Forms with degree of irrationality 1 will be called *rational.* To every rational Morse form ω we shall associate a map $f_\omega \colon M \to S^1$, $f_\omega(x) = \int_{x_0}^{x} \omega \pmod{\text{g.c.d.}(u_1, \dots, u_s)}$, where x_0 is some fixed point of the manifold M, and the u_i are the integrals over the cycles of a basis. Since $\omega = df_\omega$, the points which are critical points of the map f_ω will be exactly those which are critical for the form ω. The level surfaces of the form ω will be the level surfaces of the map f_ω. On the circle S^1 there will be a finite collection of critical values $b_1, \dots, b_m$, to which correspond singular level surfaces. Assuming that an orientation on the circle has been fixed, between every two critical values b_{i-1} and b_i we choose one regular value: $b_{i-1} < a_i < b_i$, $b_m < a_1 < b_1$. The nonsingular leaves (that is, the inverse images $f_\omega^{-1}(a_i)$ of noncritical values) will be closed orientable manifolds of codimension 1. Just as in classical Morse theory, the manifold M^n is glued together out of *elementary bordisms* (see [7])

$$M^n = f_\omega^{-1}[a_1a_2] \cup f_\omega^{-1}[a_2a_3] \cup \cdots \cup f_\omega^{-1}[a_ma_1],$$

only in our case the "upper" boundary $f_\omega^{-1}(a_1)$ of the last bordism $f_\omega^{-1}[a_ma_1]$ is glued to the "lower" boundary $f_\omega^{-1}(a_1)$ of the first bordism $f_\omega^{-1}[a_1a_2]$. (Having fixed once for all an orientation of S^1, we can speak of the "upper" and "lower" boundaries of a bordism.) By cutting the manifold M^n along a nonsingular leaf $L^{n-1} = f_\omega^{-1}(a_i)$, we obtain a bordism V^n, not connected in general, $\partial V = L_0 \cup L_1$, where $L_0 = L_1 = L^{n-1}$, with a "genuine" (i.e. with values in $\mathbf{R}$) Morse function f_ω on it, such that $f_\omega|_{L_0} = \text{const}$ and $f_\omega|_{L_1} = \text{const}$. Thus in the case when the Morse form ω has degree of irrationality 1, $\operatorname{rk}\omega = 1$, the structure of the leaves of ω is described by classical Morse theory.

If one raises the degree of irrationality of the form ω, the picture becomes considerably more complicated. In the simplest case, when ω has no critical points, a leaf of ω will be a covering of some connected closed orientable manifold which is a leaf of an approximating rational form ω_0. The monodromy group of this covering will be the group $\mathbf{Z}^{k-1}$, where $k = \operatorname{rk}\omega$ (see [2] and [3]). (The topological construction presented in §2, Remark 3, is the basis for further constructions, and underlies the proof of this assertion.) In particular, if the degree of irrationality of ω is equal to 2 and ω has no critical points, then a leaf of ω is a $\mathbf{Z}$-covering of a compact manifold L^{n-1}. To put it differently, a leaf of the form ω can be glued together from $\mathbf{Z}$ copies of a bordism W^{n-1}, where W^{n-1} is obtained by cutting the manifold L^{n-1}, the base of the covering, along a submanifold of codimension 1. The leaf of the form ω will have a periodic structure. However, if we permit the existence of critical points for the form ω, $\operatorname{rk}\omega = 2$, then a leaf of it will no longer be a covering of a compact manifold. But it turns out that, in the case when ω is close to some form with degree of irrationality 1, it is possible to endow a leaf of the form ω with a *quasiperiodic structure* (cf. [2] and [3]). The precise formulation and proof of this theorem for degree of irrationality 2 are presented in §§3 and 4. A generalization to the case of arbitrary degrees of irrationality is presented in §5. In the final §6 we consider one application of the theory of Morse forms to the study of the motion of an electron in an inverse lattice in a uniform magnetic field (see also [2] and [11]).

§2. General construction. The Morse form without critical points

Suppose that on the connected closed oriented manifold M^n we are given a Morse form ω, i.e. a closed differential 1-form with nondegenerate singular points. In this section we consider the simplest case, when ω has no singular points at all. Suppose that the degree of irrationality of ω is equal to k. Henceforth we shall suppose throughout that a Riemannian metric on the manifold M^n has been specified. In the presence of a metric, the form ω defines a gradient vector field and a phase current on M^n. If ω has no critical points, then the phase current has no singularities.

The plan of our constructions is this. Using the form ω, we construct a form ω_0 approximating it, which has degree of irrationality 1 and is without critical points (Lemma 1). A leaf of this form is a connected closed oriented submanifold L^{n-1} of codimension 1. Since ω_0 has no critical points, a twisted product structure $L^{n-1} \to M^n \to S^1$ is defined on M^n. By using a certain special basis of the cycles in $\pi_1(M)$ (Lemma 2), we construct a $\mathbf{Z}$-covering $p\colon \hat{M}^n \xrightarrow{\mathbf{Z}} M^n$ (Lemma 3) under which the degree of irrationality of the forms ω_0 and ω is reduced by one (Lemma 4). We obtain an exact form $p^*\omega_0$ without critical points, which defines on $\hat{M}^n$ a direct product structure $\hat{M}^n = \hat{L}^{n-1} \times \mathbf{R}$, allowing us to project onto the leaf $\hat{L}^{n-1}$ of the form $p^*\omega_0$. The covering $p\colon \hat{M}^n \to M$ is constructed in such a way that the leaves of the forms $p^*\omega_0$ and $p^*\omega$ on $\hat{M}^n$ remain the same as those of ω_0 and ω on M^n (Lemma 4). The projection of a leaf of the form $p^*\omega$ onto the leaf $\hat{L}^{n-1}$ of the form $p^*\omega_0$ along the phase trajectories gives us a covering with monodromy group $\mathbf{Z}^{k-1}$, where $k-1 = \operatorname{rk}(p^*\omega) = \operatorname{rk}\omega - 1$ (Theorem). As an elementary illustrative example, one can consider an irrational spiral on a torus (a leaf of ω) covering a rational spiral.

We shall say that the form ω_0 *approximates* the form ω if the phase trajectories of ω_0 are everywhere transverse to the leaves of ω.

LEMMA 1 (construction of an approximating form). *For every Morse form ω of degree of irrationality k without critical points, there exists an approximating form ω_0 of degree of irrationality 1 which also has no critical points. The form ω_0 can be chosen in such a way that ω is represented as a sum of k rational forms, $\omega = \omega_0 + \alpha_1 + \cdots + \alpha_{k-1}$.*

◀ We express the cohomology class of ω in terms of an integral basis in $H^1(M;\mathbf{R})$, $[\omega] = \sum_1^l \lambda_i[\varepsilon^i]$, where $\lambda_i \in \mathbf{R}$, $\varepsilon^i \in \Omega_1(M)$, and all the integrals of the forms ε^i are whole numbers. Then $\operatorname{rk}\omega = \operatorname{rk}\langle\lambda_1,\dots,\lambda_l\rangle_{\mathbf{Q}} = k$. By reindexing the forms in the basis, if necessary, we arrange that the coefficients $\lambda_1,\dots,\lambda_k$ are rationally independent. In the sum $\sum_1^l \lambda_i[\varepsilon^i]$ we replace the coefficients $\lambda_{k+1}, \lambda_{k+2},\dots,\lambda_l$ by their rational expansions in terms of the first k coefficients. Grouping together the terms with the same λ_i, $1 \le i \le k$, we obtain

$$[\omega] = \sum_{i=1}^{l} \lambda_i[\varepsilon^i] = \sum_{j=1}^{k} \lambda_j[\varphi^j].$$

We note that all the integrals of the forms φ^j are rational. We represent λ_j as $p_j\lambda_1/q_j + \delta_j$, where $2 \le j \le k$, and $p_j/q_j \in \mathbf{Q}$ and $\delta_j \in \mathbf{R}$ are sufficiently small. For the forms $\alpha_1,\dots,\alpha_{k-1}$ we take the forms $\delta_2\varphi^2,\dots,\delta_k\varphi^k$ respectively, and we set $\omega_0 = \omega - \sum_2^k \delta_j\varphi^j$. The smallness of the numbers δ_j means first that the form ω_0 has the same critical points as ω (see the analogous theorem in [5]); that is, it has none at all; and second that ω_0 approximates ω. ▶

REMARK 1 (the minimal integral of a rational form). By definition, for a form ω_0 with degree of irrationality 1 we have $\operatorname{rk}\langle u_1,\dots,u_s\rangle = 1$, where $u_i = \int_{\gamma_i}\omega_0$ are the set of integrals over cycles of a basis. Thus there is a greatest common divisor $R(\omega_0) = \text{g.c.d.}(u_1,\dots,u_s)$ of the numbers $u_1,\dots,u_s$. Evidently $R(\omega_0)$ is a divisor of the integral of ω_0 along an arbitrary path γ, because γ is a composition of cycles in the basis, and the integral of ω_0 along γ is an integer linear combination of the numbers u_i. It is not difficult to construct a cycle along which the integral is equal to $R(\omega_0)$. In fact, since $R(\omega_0) = \text{g.c.d.}(u_1,\dots,u_s)$ there exists a collection of integers n_i such that $R(\omega_0) = n_1u_1 + \cdots + n_su_s$. Then the relation $\int_\gamma \omega_0 = R(\omega_0)$ will hold for the cycle $\gamma = \gamma_1^{n_1}\cdot\dots\cdot\gamma_s^{n_s}$, for instance. This also shows that for a form ω_0 with degree of irrationality 1 there exists a least (in absolute value) integral amongst all possible integrals over cycles: we denote this by $R(\omega_0)$. All the other integrals are divisible by $R(\omega_0)$.

LEMMA 2 (construction of a special basis of the cycles). *Suppose there exist a form ω with degree of irrationality k and a form ω_0 with degree of irrationality 1. If the degree of irrationality of the difference $(\omega - \omega_0)$ is equal to $k-1$, then one can choose a set of generators $\gamma_1,\dots,\gamma_t$ in the group $\pi_1(M)$ such that $\int_{\gamma_i}\omega_0 = 0$ for $i > 1$ and $\int_{\gamma_i}\omega = 0$ for $i > k$. (Nondegeneracy of the forms is not assumed.)*

◀ Starting from an arbitrary set of generators $\sigma_1,\dots,\sigma_s \in \pi_1(M)$ we construct a set of generators $\sigma_0', \sigma_1',\dots,\sigma_s' \in \pi_1(M)$, where the extra cycle σ_0' gives the minimal integral $\int_{\sigma_0'} \omega_0 = R(\omega_0)$ of the form ω_0, and the cycles σ_i' are obtained by "correcting" the corresponding cycles σ_i with the requisite power of σ_0': so $\sigma_i' := \sigma_i \cdot (\sigma_0)^{-n_i}$, where $n_i = (\int_{\sigma_i} \omega_0)/R(\omega_0)$. For the set of generators $\sigma_0', \sigma_1', \dots, \sigma_s'$ the relation $\int_{\sigma_i'} \omega_0 = 0$ holds for $1 \le i \le s$.

Using the decomposition of ω into the sum of rational forms $\omega = \omega_0 + \alpha_1 + \dots + \alpha_{k-1}$ obtained in Lemma 1, we repeat the constructions for the forms α_i in succession, and we obtain the required set of generators. ►

REMARK 2 (connection between rational forms and integral homology and cohomology). In the sequel, it will be very useful for us to know the precise connection between differential forms with degree of irrationality 1 and integral cohomology.

To a form ω_0 with degree of irrationality 1, $\omega_0 \in \Omega_1(M)$, $d\omega_0 = 0$, we associate the cohomology class of the form $[(1/R(\omega_0))\omega_0] \in H^1(M; \mathbf{R})$. The form $(1/R(\omega_0))\omega_0$ is integer-valued: that is, its integral along any path is equal to a whole number. Thus the cohomology class $[(1/R(\omega_0))\omega_0]$ can be regarded as a class from $H^1(M; \mathbf{Z})$, $[(1/R(\omega_0))\omega_0] \in H^1(M; \mathbf{Z})$.

We shall call a cycle $z \in H_i(M; \mathbf{Z})$ (or a cocycle $\varsigma \in H^i(M; \mathbf{Z})$) *indivisible* [10] if it is not an integer multiple of any other cycle (cocycle): $z \neq n \cdot z'$, where $n \in \mathbf{Z}$ and $|n| > 1$. For example, the cocycle $[(1/R(\omega_0))\omega_0] \in H^1(M; \mathbf{Z})$ is indivisible, since there exists a cycle upon which the cocycle $[(1/R(\omega_0))\omega_0]$ takes the value 1. The cycle $D[(1/R(\omega_0))\omega_0]$ which is Poincaré dual to it is, clearly, also indivisible. We make a convention that in future, when it will not cause confusion, we shall denote the cycle $D[(1/R(\omega_0))\omega_0]$ by $D[\omega_0]$.

If ω_0 is a Morse form, then any nonsingular leaf of it will be a realization by a manifold of the cycle $D[(1/R(\omega_0))\omega_0] \in H_{n-1}(M^n; \mathbf{Z})$, where we consider that $[(1/R(\omega_0))\omega_0] \in H^1(M; \mathbf{Z})$.

On the other hand, we can embed the group $H^1(M; \mathbf{Z})$ in $H^1(M; \mathbf{R})$, since $H^1(M; \mathbf{Z})$ has no torsion. This means that any cocycle z from $H^1(M; \mathbf{Z})$ can be regarded as a cocycle from $H^1(M; \mathbf{R})$. By the de Rham isomorphism this in turn can be realized by a closed 1-form on M, and the form which realizes it can be chosen to be a Morse form. A nonsingular leaf N of this form will be a realization by a manifold of the cycle $[N] = D((1/R(z))z) \in H_{n-1}(M^n; \mathbf{Z})$, which is the dual of the original cocycle $z \in H^1(M^n; \mathbf{Z})$, taken with the normalizing coefficient $1/R(z)$. The coefficient is chosen in such a way that the minimal value over all cycles of the cocycle $(1/R(z))z \in H^1(M^n; \mathbf{Z})$ is equal to one.

Suppose now that in the manifold M^n we are given a smooth orientable submanifold N^{n-1} of codimension 1 whose homology class $[N] \in H_{n-1}(M^n; \mathbf{Z})$ is not equal to zero. By cutting the manifold M^n along N^{n-1} we obtain a bordism W^n which is not connected in general, and has a boundary consisting of two identical components $\partial W^n = N' \cup N''$, where N' and N'' are diffeomorphic to N. On the bordism W^n we can construct a Morse function f which is constant on the upper and lower boundary and is such that the differential df is compatible with gluing W^n along the boundary (see [5]). On carrying out this gluing, we obtain a closed differential 1-form df on M^n with degree of irrationality 1 (df is not exact, because f is a function on the bordism W^n and not on the manifold M^n). The cohomology class of this form $[df] \in H^1(M^n; \mathbf{R})$ is given, apart from a real coefficient $\lambda \in \mathbf{R}$, by the cohomology class of the cocycle $D[N^{n-1}] \in H^1(M^n; \mathbf{Z})$, $[N^{n-1}] \in H_{n-1}(M^n; \mathbf{Z})$, where the cocycle $D[N^{n-1}]$ is regarded as a real-valued one: $[df] = \lambda \cdot D[N^{n-1}]$. We note that the original manifold N^{n-1} is not in general a leaf of the form df, even though $df|_N \equiv 0$. The point is that a leaf L of the form df realizes an indivisible cycle $[L] \in H_{n-1}(M; \mathbf{Z})$. The submanifold N realizes a cycle which is a multiple of the cycle $[L]$, $[N]=m[L]$. Therefore N is a disjoint union of m leaves of the form df (which also, in general, are not connected).

LEMMA 3. *Suppose that on the manifold K there is a closed 1-form φ_0 with degree of irrationality 1. There exists a covering map $p\colon \hat{K} \xrightarrow{\mathbf{Z}} K$ with monodromy group $\mathbf{Z}$ such that the form $p^*\varphi_0$ on the covering space $\hat{K}$ is exact,* $\operatorname{rk}(p^*\varphi_0)=0$. *(Nondegeneracy of φ_0 is not assumed.)*

◀ We shall find it useful to construct such a covering by several methods.

a) We can construct a formal covering from a normal subgroup (see [8]), namely the subgroup of cycles over which the integral of the form φ_0 is zero. When we quotient the group $\pi_1(K)$ by this subgroup, the cycles with identical integrals are identified together into a single class. Thus the quotient group is isomorphic to the group $\langle R(\varphi_0)\rangle_{\mathbf{Z}} \cong \mathbf{Z}$. The exactness of the form $p^*\varphi_0$ is evident.

b) It is known that an indivisible cycle of codimension 1 in a connected closed orientable manifold K^n can be realized by a connected submanifold. We consider the indivisible cycle $D[(1/R(\varphi_0))\varphi_0] \in H_{n-1}(K;\mathbf{Z})$ and we realize it by a connected submanifold N^{n-1}. By cutting the manifold K^n along the submanifold N^{n-1}, we obtain a bordism W with two identical boundary components $\partial W = N\cup N$. On W, the form φ_0 is exact. By gluing together $\mathbf{Z}$ copies of W, we obtain the required covering $\hat{K}=\cdots\cup W\cup W\cup W\cup\cdots$.

c) From the form φ_0 we construct the map $f_{\varphi_0}\colon K^n\to S^1$. The standard $\mathbf{Z}$-covering $\exp\colon \mathbf{R}\to S^1$ induces a $\mathbf{Z}$-covering $p\colon \hat{K}^n \xrightarrow{\mathbf{Z}} K^n$ and a map $F\colon \hat{K}\to\mathbf{R}$ which covers f_{φ_0}. We have $p^*(df_{\varphi_0})=dF$, and since $df_{\varphi_0}=\varphi_0$, the form $p^*\varphi_0$ is exact, $p^*\varphi_0=dF$ (see [10]), and

$$\begin{array}{ccc} \hat{K} & \xrightarrow{F} & \mathbf{R} \\ p\downarrow & & \downarrow \exp \\ K & \xrightarrow{f_{\varphi_0}} & S^1. \end{array}$$

It is not hard to see that $\hat{K}$ is exactly the fibered product $\hat{K}=K\times_{S^1}\mathbf{R}$. ▶

Suppose that on the connected closed orientable manifold M^n we are given a Morse form ω with degree of irrationality k without critical points. Let ω_0 be an approximating rational form as given by Lemma 1.

LEMMA 4. *On the covering space $\hat{M}$, where $p\colon \hat{M}\xrightarrow{\mathbf{Z}} M$ is constructed from the form ω_0 in accordance with one of the prescriptions in Lemma 3, the form $p^*\omega$ has a degree of irrationality which is one less than that of ω, $\operatorname{rk} p^*\omega = k-1 = \operatorname{rk}\omega - 1$. The restriction of the projection p to a connected component of a leaf of the form $p^*\omega$ is a diffeomorphism of connected components of leaves of the forms $p^*\omega$ and ω.*

◀ Using the decomposition of the form ω in Lemma 1, we have $\omega=\omega_0+\alpha_1+\cdots+\alpha_{k-1}$, where $\operatorname{rk}\alpha_i=1$. Since $[p^*\omega_0]=0$, we have $\operatorname{rk} p^*\omega\le k-1$. Since all the cycles except γ_1 in the set of generators $\gamma_1,\dots,\gamma_k\in\pi_1(M)$ constructed in Lemma 2 can be lifted to $\hat{M}$ (see Lemma 3a)), the relation $\operatorname{rk} p^*\omega = k-1$ holds.

It is clear that the restriction of the projection p to a connected component of a leaf of the form $p^*\omega$ is a covering over a connected component of a leaf of ω. The integral of ω along any closed path in a leaf of ω is equal to zero. From the rational independence of the forms occurring in the decomposition $\omega=\omega_0+\alpha_1+\cdots+\alpha_{k-1}$, it follows that any integral of ω_0 along a closed path in a leaf of ω is also equal to zero. Thus (see Lemma 3a)) our covering map is a diffeomorphism. ▶

THEOREM (see [3]). *Suppose that on the closed connected orientable manifold M^n we are given a closed 1-form ω of degree of irrationality k without critical points. Then a* *leaf of the form ω is connected, and is a covering space over a connected closed orientable submanifold, which is a leaf of an approximation form with degree of irrationality* 1. *The monodromy group of the covering is the group* $\mathbf{Z}^{k-1}$.

◄ We construct a covering $p\colon \hat{M} \xrightarrow{\mathbf{Z}} M$ over our manifold from the prescription in Lemma 3. Since the leaves of the forms ω_0 and ω are identical with those of $p^*\omega_0$ and $p^*\omega$ respectively, it is sufficient to show that a leaf of $p^*\omega$ is a $\mathbf{Z}$-covering of a leaf L^{n-1} of $p^*\omega_0$. We obtain this covering by projecting the leaf of $p^*\omega$ along phase trajectories onto a leaf L of $p^*\omega_0$. The projection is possible, since $\hat{M} = L \times \mathbf{R}$. The elements of the monodromy group are generated by the cycles $\gamma_2, \dots, \gamma_k$ from the set of generators $\gamma_1, \dots, \gamma_k$ of Lemma 2, when these have been lifted to $\hat{M}$ and realized in the leaf L of $p^*\omega_0$ ($\pi_1(\hat{M}) = \pi_1(L \times \mathbf{R}) = \pi_1(L)$). We note that any leaf of the covering we have constructed belongs entirely to a single phase trajectory. If we measure distance on this phase trajectory by the integral of ω along it, then an arbitrary cycle $\gamma \in \pi_1(L)$ acts on the leaf by a translation of $\int_\gamma \omega$ along the phase trajectory. ►

REMARK 3 (selection of the periodic structure). We now note a consequence, which we shall often use later on, of the theorem we have just proved. Suppose that the degree of irrationality of the form ω in the theorem is equal to 2. Then, on the covering manifold $\hat{M}$ of the $\mathbf{Z}$-covering $p\colon \hat{M}^n \to M$ which we constructed from the approximating form ω_0, the degree of irrationality of the form $p^*\omega$ is equal to 1 (Lemma 4), $\operatorname{rk} p^*\omega = 1$. Since the manifold $\hat{M}$ is the product of a leaf $\hat{L}$ of the form $p^*\omega_0$ with a line, $\hat{M}^n = \hat{L} \times \mathbf{R}$, it follows that $\hat{M}^n$ is contractible to $\hat{L}^{n-1}$, which means that the degree of irrationality of the restriction $p^*\omega|_{\hat{L}}$ of the form $p^*\omega$ to the leaf $\hat{L}^{n-1}$ is also equal to 1. It was proved in the theorem that a leaf of $p^*\omega$ is a $\mathbf{Z}$-covering over $\hat{L}^{n-1}$. When we use this covering π to lift the form $p^*\omega|_{\hat{L}^{n-1}}$ from the leaf $\hat{L}^{n-1}$ of the approximating form to the leaf of $p^*\omega$, it is not difficult to see that we obtain an exact form $\pi^*(p^*\omega|_{\hat{L}^{n-1}})$, $\operatorname{rk} \pi^*(p^*\omega|_{\hat{L}^{n-1}}) = 0$. Therefore π is the covering of the manifold $\hat{L}^{n-1}$ constructed from $p^*\omega|_{\hat{L}^{n-1}}$ by the prescription in Lemma 3, or, what is exactly the same, the covering of the manifold L^{n-1} constructed from the rational form $\omega|_{L^{n-1}}$. By using the second variant of the proof of Lemma 3, we are able to give a short description of the construction of the periodic structure of a leaf of ω. We recall that a generator of the monodromy group $\mathbf{Z}$ of our covering π acts by translation along phase trajectories by a distance $R(\omega|_{L^{n-1}})$ (see the theorem).

In order to select a periodic structure on a leaf of a form ω of degree of irrationality 2 without critical points, we have to consider the restriction of ω to a leaf L^{n-1} of an approximating rational form, and realize a cycle dual to the cocycle $[\omega|_{L^{n-1}}] \in H^1(L^{n-1}; \mathbf{Z})$ by a connected submanifold $N^{n-2} \subset L^{n-1}$. After that we have to transport N^{n-2} along phase trajectories onto the leaf of ω, and then transport it throughout the manifold M^n, moving it along phase trajectories by a distance which is a multiple of $R(\omega|_L)$. The leaf of ω turns out to be sliced into identical bordisms. We note that this bordism itself, and both components of its boundary, are connected.

§3. Absence of critical points of indices $0, 1, n-1, n$. Degree of irrationality 2

We now pass over to the fundamental case, where the Morse form ω has critical points. Now, in contrast with the case where there are no critical points, the form with degree of irrationality 1 playing the part of an approximating form will be fixed once for all. In the framework of our theory only those forms with degree of irrationality greater than 1 are considered which are sufficiently close to the chosen form with degree of irrationality 1. In attempts to carry over the method we used in the case without critical points—that is, the approximation of the given Morse form ω, $\operatorname{rk}\omega = k$, by a form with degree of irrationality 1—the constructions introduced below fail to work in general.

Therefore we suppose given a Morse form ω_0 with degree of irrationality 1 on the closed connected orienable manifold M^n, and we fix this form once and for all. We shall assume that to every critical value of the function $f_{\omega_0}\colon M^n \to S^1$ there corresponds only one critical point. We know that a nonsingular leaf of a Morse form with degree of irrationality 1 is closed and oriented. In this section we shall impose an additional condition on the form ω_0, requiring that it have no critical points of indices 0, 1, $n-1$, or n. In the first place, this condition ensures that we have a constant number of connected components in an arbitrary leaf of the form ω_0, since Morse surgery with indices different from 0, 1, $n-1$, and n has no effect on the number of connected components of a level surface. The other thing which the absence of critical points with extreme indices guarantees us is the following. As was shown in §2, specifying a Morse form without critical points on a manifold makes it possible (when a Riemannian metric has been chosen) to project along the phase trajectories of the form. Now, when the Morse form has acquired critical points, we can still project, but only locally: projection can be carried on only until we run into a critical point. If the Morse form ω_0 has no critical points with indices 0, 1, $n-1$, or n, then an arbitrary separatrix disk will intersect an arbitrary leaf of ω_0 in a sphere of codimension (in the leaf) not less than 2. This means that by slightly perturbing a general path in a leaf of ω_0, we can always move it off a given separatrix disk or family of disks. This gives us a way of transporting the displaced path onto another leaf along the phase trajectories, since none of the phase trajectories we need will run through the corresponding critical point or collection of points. In this section we shall make essential use of these two special properties of a Morse form ω_0 without critical points of extreme indices.

Suppose now that on the manifold there is another form ω_1 with degree of irrationality 1, which in general will not be a Morse form. The only case of interest to us is that where the cohomology classes $[\omega_0]$, $[\omega_1] \in H^1(M^n;\mathbf{R})$ of the forms ω_0 and ω_1 are independent. We consider the form $\omega_0 + \varepsilon\omega_1$, where $\varepsilon \in \mathbf{R}$. For sufficiently small ε, $\omega_0 + \varepsilon\omega_1$ is a Morse form, and has exactly as many critical points (with the same indices) as has ω_0. (The proof is completely analogous with the proof of the corresponding fact for functions; see [5].) We shall not be considering values of the parameter for which the degree of irrationality of $\omega_0 + \varepsilon\omega_1$ is equal to 1, because the structure of the leaves of such a form is completely described by classical Morse theory. (The set of such values of the parameter is countable.) Thus the degree of irrationality of $\omega_0 + \varepsilon\omega_1$ is equal to 2, $\operatorname{rk}(\omega_0 + \varepsilon\omega_1) = 2$. Suppose that the form ω_0 has m critical points, to which correspond the critical values $b_1, \dots, b_m$ of the function $f_{\omega_0}\colon M^n \to S^1$. We select a set of regular values $a_1, \dots, a_m$ such that $a_i < b_i < a_{i+1}$ for $1 \le i < m$ and $a_m < b_m < a_1$, and we consider the set of nonsingular leaves $L_i = f_{\omega_0}^{-1}(a_i)$ of ω_0. (We recall that, having fixed an orientation for the circle S^1 in the mapping $f_{\omega_0}\colon M^n \to S^1$, we have given a meaning to the double inequalities $x < y < z$.)

We consider the restriction $(\omega_0 + \varepsilon\omega_1)|_{L_i}$ of the form $(\omega_0 + \varepsilon\omega_1)$ to the level surface $L_i = f_{\omega_0}^{-1}(a_i)$ of ω_0. Then

$$\operatorname{rk}(\omega_0 + \varepsilon\omega_1)|_{L_i} = \operatorname{rk}(\varepsilon\omega_1|_{L_i}) = \operatorname{rk}\omega_1|_{L_i}.$$

It is asserted that for any leaf L_i the degree of irrationality of the restriction of ω_1 to L_i is equal to 1, $\operatorname{rk}\omega_1|_{L_i} = 1$. This is proved in exactly the same way as in Remark 3 of §2, with only this difference: when one wants to project a path in the covering space $\hat{M}^n$ along the phase trajectories of the form $p^*\omega_0$, one may have to perturb the path a little in order to move it off the required separatrix disks. With the same restriction, we can apply the theorem of §2 and prove that a leaf of the form $\omega_0 + \varepsilon\omega_1$ is connected.

We now explain the plan for the construction of a quasiperiodic structure in the case when the Morse form $\omega_0 + \varepsilon\omega_1$ with degree of irrationality 2 has no critical points with extreme indices.

As long as we are between two critical values b_{i-1} and b_i in the manifold M^n, the situation is no different from the case where there are no critical points. In the same way as there, we can project a leaf of the form $\omega_0 + \varepsilon\omega_1$ onto the leaf L_i of the form ω_0, and the projection will be a local diffeomorphism. The projection admits path lifting, provided only that we do not traverse the limits b_{i-1} and b_i in the course of the lifting. In the same way as in §2, in the leaf L_i^{n-1} of ω_0 we construct a submanifold which realizes a cycle dual to the cocycle $[\omega_1|_{L_i}]$, and we denote this submanifold by $N_{i,1}^{n-2}$: so $[N_{i,1}^{n-2}] = D[\omega_1|_{L_i}]$ (see Remark 3 in §2). We note that we can choose the submanifold $N_{i,1}$ to be connected. As before, we shall transport the submanifold $N_{i,1}$ into the leaf of $\omega_0 + \varepsilon\omega_1$, thus slicing out identical bordisms, which we denote by W_{2i}, in that leaf. The bordism W_{2i} is diffeomorphic to the leaf L_i^{n-1} cut along the submanifold $N_{i,1}^{n-2}$. Generally speaking, however, we can only transport the submanifold $N_{i,1}$ around the leaf of ω whilst we remain within the bounds of the critical values b_{i-1} and b_i. In the part of the leaf of $\omega_0 + \varepsilon\omega_1$ which lies between the bounds b_i and b_{i+1} we shall slice out another bordism $W_{2(i+1)}$. The problem is how to pass through the critical value. This will be done in the following fashion. The submanifold $N_{i,1}$ will be chosen in such a way that it does not intersect the left separatrix disk of the ith critical point of ω_0, that is, the critical point immediately above the leaf L_i.

We explain why such a choice is possible. As the level surface L_i of the form ω_0 approaches the ith critical point, the linear dimension of the sphere $S = L_i \cap D_{i,L}$ tends to zero, and with it the quantity

$$d := \max_{x_1, x_2 \in S} \left(\int_{x_1}^{x_2} \omega_1|_S \right)$$

also tends to zero. Since $R(\omega_1|_{L_i})$ does not change in this process, we can wait until d becomes less than $R(\omega_1|_{L_i})$, replace the form $\omega_1|_{L_i}$ by a cohomologous Morse form which is close to it, and choose a nonsingular leaf N^{n-2} of the latter which does not intersect the sphere S. It is clear that the relation $[N^{n-2}] = D[\omega_1|_{L_i^{n-1}}]$ holds in L_i, but in general the submanifold N^{n-2} is not as yet connected.

We now use the fact that the index of the critical point is different from 0, 1, $n-1$ and n, and that therefore the codimension in M^n of the left separatrix disk $D_{i,L}$ is greater than one. This indicates that the codimension of the sphere S in L_i is greater than one, and hence that we can always pull off S any path lying in L_i and joining two connected components of the submanifold N (see, for example, Lemma 4.5 in [5]). By considering a narrow tubular neighborhood (in L_i) of the new path, we can glue together the two connected components of N by means of the handle H_1^{n-1} which so arises. The surface $N' \subset L_i$ so constructed will comprise fewer connected components than N, but $[N'] = [N]$. By thoroughly applying the gluing technique, and verifying that the original submanifold N^{n-2} is a realization of an indivisible cycle, we finally obtain a connected submanifold which we denote by $N_{i,1}^{n-2}$. By construction $[N_{i,1}^{n-2}] = D[\omega_1|_{L_i}]$; and $N_{i,1}^{n-2} \cap S = \varnothing$. Thus the promised submanifold $N_{i,1} \subset L_i$ has been constructed.

We transport the submanifold $N_{i,1}$ along the phase trajectories of the form $\omega_0 + \varepsilon\omega_1$ into the leaf L_{i+1} of the form ω_0: we denote the result of the transport by $N_{i+1,0} \subset L_{i+1}$. The transport is well-defined, since the forms ω_0 and $\omega_0 + \varepsilon\omega_1$, being near one another, have nearby separatrix disks, and since the submanifold $N_{i,1}$ does not intersect the corresponding separatrix disk of $\omega_0 + \varepsilon\omega_1$. It is not hard to show, using the feasibility of transporting (perturbed) closed paths from the surface L_i to L_{i+1} and vice versa, that the submanifold $N_{i+1,0}^{n-2} \subset L_{i+1}^{n-1}$ we have constructed realizes a cycle which is dual to the cocycle $[\omega_1|_{L_{i+1}}]$.

In every pair L_i, L_{i+1} of leaves of ω_0, therefore, we have distinguished a pair of submanifolds $N_{i,1}^{n-2} \subset L_i^{n-1}$ and $N_{i+1,0}^{n-2} \subset L_{i+1}^{n-1}$. These submanifolds can be projected to one another along the phase trajectories of the form $\omega_0+\varepsilon\omega_1$, and each of them realizes in its own leaf a cycle dual to ω_1. On cutting the leaf L_i along the submanifold $N_{i,1}$, we obtain a bordism W_{2i}, which we shall call *regular*. We glue together sufficiently many of the bordisms W_{2i} to get the submanifold $N_{i,0}$ wholly inside the glued manifold, and we cut the assembly along the submanifold $N_{i,0}$. We obtain a bordism W_{2i-1} with "lower" boundary $N_{i,0}$ and "upper" boundary $N_{i,1}$. We shall call this a *transitional bordism.*

It is now not hard to describe how to pass through the critical point. On projecting the submanifold $N_{i,1}$ from the leaf L_i of the form ω_0 into a leaf of the form $\omega_0 + \varepsilon\omega_1$, we shall mark off regular bordisms W_{2i} in the leaf of $\omega_0 + \varepsilon\omega_1$, exactly as in Remark 3 of §2. The separatrix disk $D_{i,L}$ of the ith critical point of $\omega_0 + \varepsilon\omega_1$ intersects the chosen leaf of $\omega_0 + \varepsilon\omega_1$ in a finite collection of spheres. (By "leaf" here one understands, in actual fact, a specific connected part of a leaf cut out by the bordism $f_{\omega_0}^{-1}[b_{i-1}b_i]$.) Each sphere is wholly contained in its own regular bordism W_{2i}^{n-1}. Moving "upwards from below", and marking off bordisms W_{2i}, we at some instant cover the sphere near the critical point. Thereupon we mark off a transitional bordism W_{2i+1} on the leaf of the form $\omega_0 + \varepsilon\omega_1$, and then begin to mark off successive regular bordisms W_{2i+2}. The passage through the critical point has been accomplished.

The construction we have carried out demonstrates the truth of the

THEOREM (degree of irrationality 2; no critical points with extreme indices). *For sufficiently small $\varepsilon \in \mathbf{R}$, a nonsingular leaf of the form $\omega_0 + \varepsilon\omega_1$ is glued together from bordisms W_j, $1 \le j \le 2m$, according to the following scheme*:

$$\underline{\cdots \cup W_{2i-2}} \cup W_{2i-1} \cup \underline{W_{2i} \cup W_{2i} \cup \cdots \cup W_{2i}} \cup W_{2i+1} \cup \underline{W_{2i+2} \cup \cdots}$$

Two series of identical bordisms W_{2i} and W_{2i+2} are glued together by a single transitional bordism W_{2i+1}. After the last series of bordisms, which is a series of W_{2m}'s, a transitional bordism W_1 is glued on, and then a new cycle begins.

We note that the series consisting of consecutive regular bordisms W_{2i} (the leaf L_i cut along a submanifold) is a "sufficiently large piece" of a $\mathbf{Z}$-covering over the leaf L_i of the form ω_0.

We now demonstrate the properties of continuity and quasiperiodicity possessed by the structure which we have assembled.

We take a small piece of a phase trajectory of the form $\omega_0 + \varepsilon\omega_1$, located sufficiently far from the critical points. On it we shall measure distance by the integral of the form $\omega_0 + \varepsilon\omega_1$. Then if on the chosen piece of phase trajectory we space out a number of points at equal intervals of length $\varepsilon \cdot R(\omega_1|_{L_i})$, all the points will turn out to be on one leaf of the form $\omega_0 + \varepsilon\omega_1$ (see the proof of the theorem in §2). The quantity $R(\omega_1|_{L_i})$ does not depend upon the choice of the leaf L_i of the form ω_0. We shall assume that a quasiperiodic structure has been assigned to the above-mentioned leaf of $\omega_0 + \varepsilon\omega_1$. Then to each of the selected points on the chosen phase trajectory there corresponds a regular bordism W_{2i} containing that point; to an ordered set of the points there corresponds a certain portion of the quasiperiodic structure.

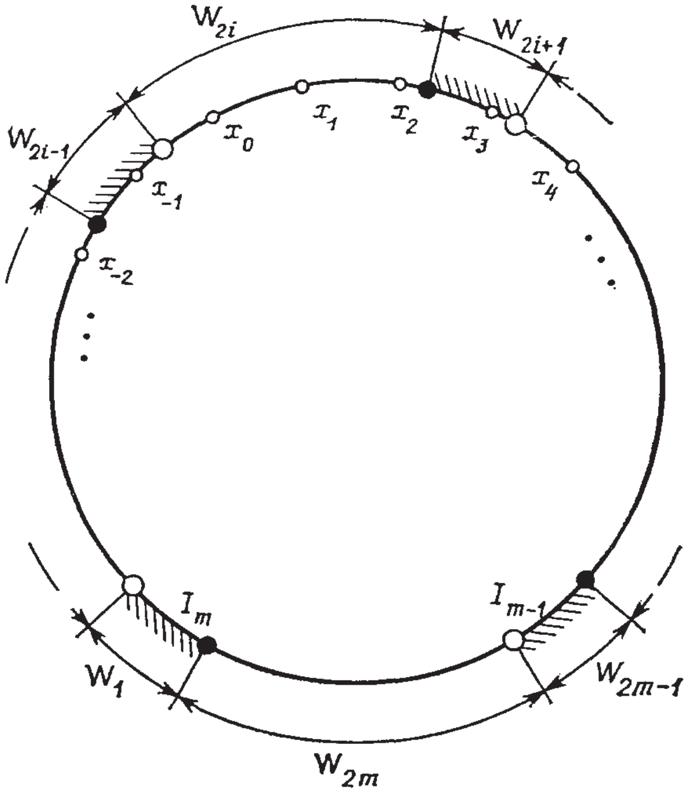

FIGURE 1. To the sequence of points

$$\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, x_4, \dots\}$$

there corresponds the portion

$$\cdots \cup W_{2i-2} \cup W_{2i-1} \cup W_{2i} \cup W_{2i} \cup W_{2i} \cup W_{2i+1} \cup W_{2i+2} \cup \cdots$$

of the quasiperiodic structure

Using the remark we have made, we can give the following description of the *quasiperiodic structure* of the leaves of the form $\omega_0 + \varepsilon\omega_1$.

We construct a circle S^1 on which m disjoint intervals I_i of unit length are marked out (see Figure 1). Here m is the number of critical points of ω_0. We consider some point x_0 on the circle, and we construct a sequence of points $\{x_j\}$, $-\infty < j < \infty$, where each point x_j is obtained from x_0 by turning through an angle j/l, l being the length of the circle. To each point x_j we associate one of the bordisms W_i, $1 \leq i \leq 2m$, according to the following rule: if x_j lies in one of the intervals I_i, we assign to it the transitional bordism W_{2i+1}; if x_j is enclosed between I_i and I_{i+1}, then we assign to it the regular bordism W_{2i+2}. To the boundary point of the interval I_i denoted by a heavy dot in Figure 1 there corresponds a bordism W_{2i} with glued-in singularity. By choosing other points of the circle in place of the initial point x_0, we obtain different sequences.

ASSERTION. *With every such sequence, considered as a set, one can associate in a unique way a leaf of the form $\omega_0+\varepsilon\omega_1$. The correspondence between points of the sequence $\{x_j\}$ and bordisms W_i describes the quasiperiodic structure of the leaf associated with the set $\{x_j\}$. In this sense the correspondence between sequences $\{x_j\}$ and leaves of $\omega_0+\varepsilon\omega_1$ is continuous.*

Finally, we note that as $\varepsilon \to 0$ the ratio of the lengths of the segments enclosed between the intervals I_{i-1} and I_i (that is, the relative numbers of identical bordisms in the series of W_{2i}'s) tends to the ratio of the lengths of the corresponding segments $[b_{i-1}, b_i]$ on the circle in the map $f_{\omega_0}: M^n \to S^1$.

§4. Critical points with indices 1, $n-1$ and $0, n$. Degree of irrationality 2

In this section we remove the restrictions on the indices of the critical points of the Morse form ω_0. The critical points with indices 0 and n, however, give essentially nothing new, so we shall assume for the present that only critical points with indices 1 and $n-1$ have been introduced. We are interested in knowing what happens to a leaf of the form $\omega_0 + \varepsilon\omega_1$ as one passes through critical points with these indices.

Now a leaf of the form ω_0, like one of the form $\omega_0 + \varepsilon\omega_1$, is nonconnected in general. Moreover, the restriction of the form $\omega_1|_{L_i}$ to certain leaves of ω_0, or to their connected components, may happen to be an exact form, and in this case compact connected components may occur in a leaf of $\omega_0 + \varepsilon\omega_1$. Let us consider an illustrative example. We consider the height function h on the bordism W (Figure 2).

On gluing together the boundary components $h^{-1}(0)$ and $h^{-1}(1)$ of W, we obtain a pretzel with the Morse form $\omega_0 = dh$ on it, $\operatorname{rk}\omega_0 = 1$. In Figure 2 the level lines of a certain form $\omega_0 + \varepsilon\omega_1$ are drawn at constant "distance" (in the sense of the new form $\omega_0 + \varepsilon\omega_1$) from one another. Although the level lines themselves remain exactly the same as those of ω_0, the "heights" (in the sense of the form $\omega_0 + \varepsilon\omega_1$) of the bordisms V' and V'' (shaded in Figure 2) have become unequal. No path γ along which the integral of ω_1 is nonzero can be pushed down into any leaf of ω_0: it "hangs up" on the critical points with indices 1 and $n-1$. Therefore, in spite of the fact that $\operatorname{rk}(\omega_0 + \varepsilon\omega_1) = 2$, the restriction of ω_1 to an arbitrary leaf of ω_0 is exact, and every leaf of $\omega_0 + \varepsilon\omega_1$ consists of a countable collection of compact connected components.

Our problem is to assign a quasiperiodic structure to any noncompact connected component of $\omega_0 + \varepsilon\omega_1$.

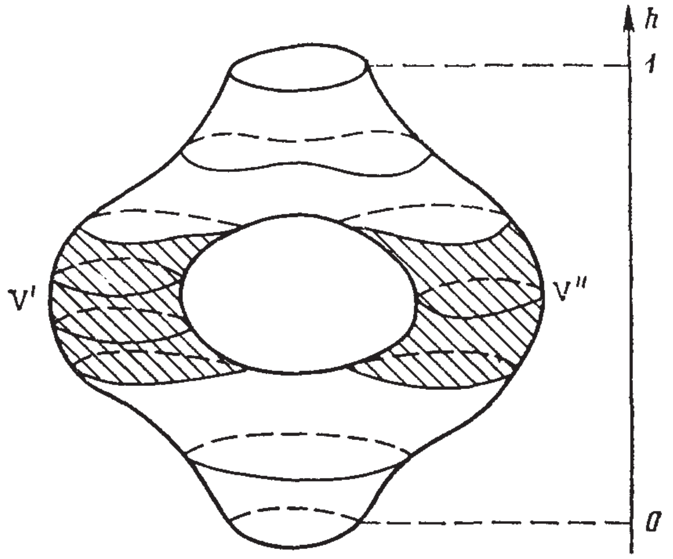

FIGURE 2

Let us consider a connected component V of an elementary bordism $f_{\omega_0}^{-1}[a_i a_{i+1}]$, $a_i < b_i < a_{i+1}$, containing a critical point of index 1. (A point of index $n-1$ becomes a point of index 1 if one interchanges the "top" and "bottom", i.e. if one changes the orientation of the circle S^1 in the mapping $f_{\omega_0}: M^n \to S^1$.) If the index of the critical point is equal to one, then the "upper" boundary $L_{i+1}^{n-1} = (f_{\omega_0}|_V)^{-1}(a_{i+1})$ of the bordism V intersects the separatrix disk $D^{n-1}_{i,R}$ in a sphere S^{n-2}, and the "lower" boundary $L^{n-1}_i = (f_{\omega_0}|_V)^{-1}(a_i)$ intersects the separatrix disk $D^1_{i,L}$ in a pair of points $S^0 = x_0 \cup x_1$. We are only interested in the case where $\omega_1|_{L_{i+1}}$ is not exact. (Exactness of $\omega_1|_{L_{i+1}}$ entails exactness of $\omega_1|_{L_i}$; but this means that the leaves of $\omega_0 + \varepsilon\omega_1$ in the bordism V will be compact.)

Let us consider the sphere S^{n-2} formed by the intersection of the separatrix disk $D^{n-1}_{i,R}$ with the leaf L_{i+1} of ω_0. It is important that we distinguish four types of critical points of index 1 $(n-1)$:

a) The cycle realized by the sphere $[S^{n-2}]$ is not homologous to zero in the group $H_{n-2}(L^{n-1}_{i+1}; \mathbf{Z})$ and is not homologous to the cycle $D[\omega_1|_{L_{i+1}}]$.

b) The cycle realized by the sphere $[S^{n-2}]$ is not homologous to zero, but is homologous to $D[\omega_1|_{L_{i+1}}]$.

If the cycle $[S^{n-2}] \in H_{n-2}(L_{i+1}; \mathbf{Z})$ is homologous to zero, then the leaf L_i of the form ω_0 consists of two connected components, and we consider the following two possibilities:

c) The form $\omega_1|_{L_i}$ is exact on one of the components.

d) $\omega_1|_{L_i}$ is nonexact on both components.

Before we move on to a description of each of these cases, we remark upon the special nature of critical points of index 1 $(n-1)$ in relation to our constructions. The leaves L_i and L_{i+1} of ω_0 do not now enjoy the same properties, because an arbitrary path can be transported from L_i to L_{i+1} along the phase trajectories (after possibly being moved off the points $(x_0 \cup x_1) = S^0 = \partial D_{i,L}$), while it is not always possible to transport a path from L_{i+1} to L_i: the path may be linked with the sphere $S^{n-2} = \partial D_{i,R}$. We shall select the submanifolds $N_{i,1} = D[\omega_1|_{L_i}]$ in the leaves L_i, in accordance with general principle, and then lift them to L_{i+1}. The submanifolds $N_{i+1,0}$ so obtained may prove to be nonconnected, and the relation $[N_{i+1,0}] = D[\omega_1|_{L_{i+1}}]$ may fail to hold.

a) $D[\omega_1|_{L_{i+1}}] \neq [S^{n-2}] \neq 0$. In this case, the component L_i of the leaf of ω_0 is connected. We show that the restriction $\omega_1|_{L_i}$ is not exact. In fact, since the cycles $[S^{n-2}]$ and $D[\omega_1|_{L_{i+1}}]$ are not homologous, their independence follows from their indivisibility. But then we can construct a cycle γ which does not intersect the sphere S^{n-2} and is nontrivially linked with $D[\omega_1|_{L_{i+1}}]$. On lowering γ from L_{i+1} to L_i, we obtain a cycle along which the integral of $\omega_1|_{L_i}$ is not equal to zero.

In L_i, as always, we select a submanifold $N_{i,1}$, $[N_{i,1}] = D[\omega_1|_{L_i}]$, which does not contain the points x_0 and x_1 of the left separatrix disk, $x_0 \cup x_1 = D_{i,L} \cap L_i$. On projecting the submanifold $N_{i,1}$ onto L_{i+1}, we obtain the submanifold $N_{i+1,0}$. Now, however, we obtain only the following relation:

$$D[\omega_1|_{L_{i+1}}] = n_1[N_{i+1,0}] + n_2[S^{n-2}],$$

where $\text{g.c.d.}(n_1, n_2) = 1$.

We now show how we can choose a transitional bordism W_{2i+1} in the given case. As before, we glue together sufficiently many of the regular bordisms W_{2i+2} (the leaf L_{i+1} cut along the connected submanifold $N_{i+1,1}$). Then we make n_1 cuts along a submanifold $N_{i+1,0}$ and n_2 cuts along spheres S^{n-2} in such a way that we obtain a bordism with "upper" boundary $N_{i+1,1}$ and "lower" boundary consisting of n_1 copies of $N_{i+1,0}$ and n_2 copies of S^{n-2}. After gluing over all the spheres S^{n-2} with disks, we obtain the transitional bordisms W_{2i+1}.

In the course of the transition through a critical point of index 1 of this type, the following takes place in a leaf of the form $\omega_0 + \varepsilon\omega_1$ (see Figure 3). A sequence of regular bordisms W_{2i+2} (the leaf L_{i+1}, cut along $N_{i+1,1}$) is glued to a transitional bordism W_{2i+1} along its "upper" boundary $N_{i+1,1}$; and n_1 sequences of regular bordisms W_{2i} (the leaf

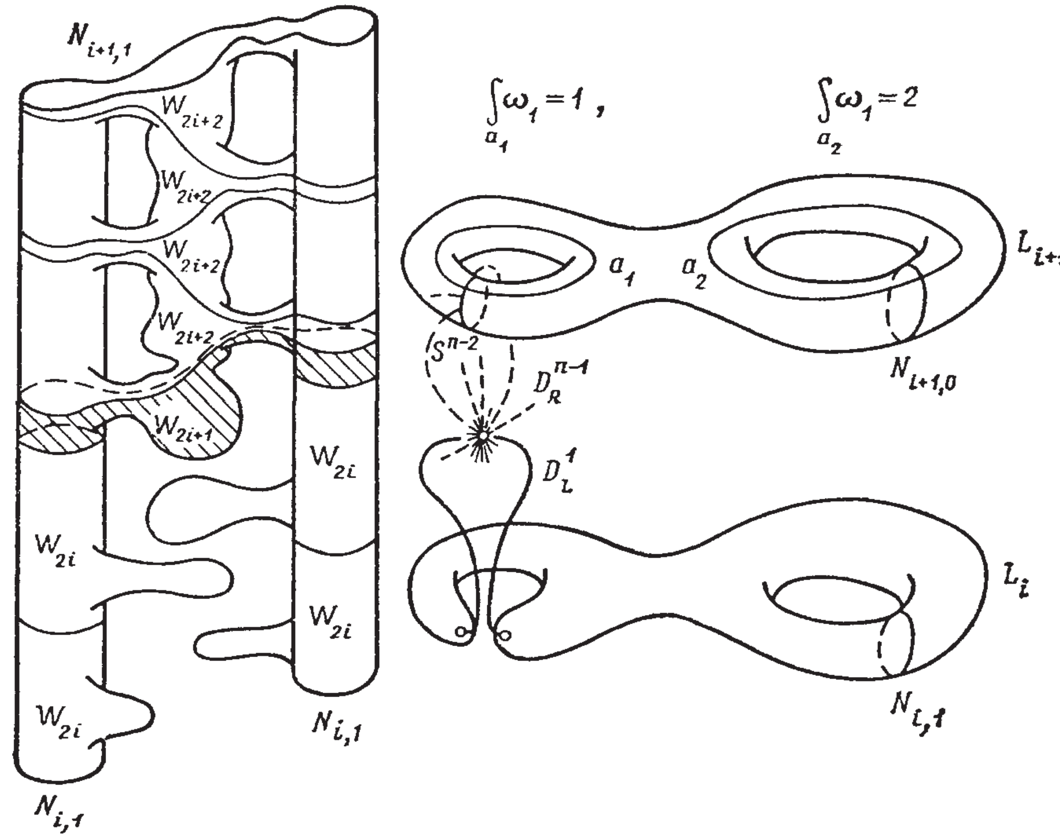

FIGURE 3

L_i, cut along $N_{i,1}$) are glued to its lower boundary, each sequence being glued to its own component $N_{i+1,0}$ of the "lower" boundary of the bordism W_{2i+1},

$$\partial W_{2i+1} = N_{i+1,1} \cup \left(\bigcup_{j=1}^{n_1} N_{i+1,0} \right).$$

Thus, a leaf of the form $\omega_0 + \varepsilon\omega_1$ can "split" into several series of the same type at transitional bordisms of this kind.

b) $D[\omega_1|_{L_{i+1}}] = [S^{n-2}]$. In the case when the cycle realized by S^{n-2} is not homologous to zero but is homologous to $D[\omega_1|_{L_{i+1}}]$, the component L_i of the leaf of the form ω_0 will be connected, but the form $\omega_1|_{L_i}$ will be exact. The boundary of the transitional bordism W_{2i+1} will consist of the submanifold $N_{i+1,1}$ alone. Moving "downwards" in the leaf of $\omega_0 + \varepsilon\omega_1$, we glue the last ("lowest") regular bordism W_{2i+2} to the film W_{2i+1}. The leaf of $\omega_0 + \varepsilon\omega_1$ "terminates" at a critical point of this form. To a component L_i of a leaf of ω_0 there will correspond components diffeomorphic to L_i of leaves of $\omega_0 + \varepsilon\omega_1$.

c) $[S^{n-2}] = 0$. In this case the leaf L_i consists of two connected components L_i' and L_i'', and the form ω_1 is exact on one of them. Compact connected components occur in the leaf of $\omega_0 + \varepsilon\omega_1$: they correspond to the component L_i'' of the leaf of ω_0 and are diffeomorphic to this. In the component L_i' we select, as always, a submanifold $N_{i,1}$, $[N_{i,1}] = D[\omega_1|_{L_i'}]$. Then we transport it to L_{i+1}, and obtain a submanifold $N_{i+1,0}$ there, $[N_{i+1,0}] = D[\omega_1|_{L_{i+1}}]$, after which we carry out the usual construction. Thus the transition through a critical point of this type, in the sense of determining a quasiperiodic structure, is completely analogous to the transition through a critical point of index different from 0, 1, $n-1$, and n.

d) $[S^{n-2}] \neq 0$. If the restriction of the form ω_1 is not exact on both the components L_i' and L_i'' of the leaf of ω_0, then we choose one submanifold in each component: $N_{i,1}' \subset L_i'$,

$[N'_{i,1}] = D[\omega_1|_{L'_i}]$ and $N''_{i,1} \subset L''_i$, $[N''_{i,1}] = D[\omega_1|_{L''_i}]$. When projected into the leaf L_{i+1}, these submanifolds give a pair of submanifolds $(N'_{i+1,0} \cup N''_{i+1,0}) \subset L_{i+1}$. It is not hard to prove the validity of a relation

$$D[\omega_1|_{L_{i+1}}] = n_1[N'_{i+1,0}] \cup n_2[N''_{i+1,0}].$$

The transitional bordism W_{2i+1} is constructed in complete analogy with case a), only now none of the components of the boundary will be glued over with disks any more. To each of the n_1 components $N'_{i+1,0}$ of the lower boundary of the bordism W_{2i+1} we glue its own series of bordisms W'_{2i} (the leaf L'_i, cut along $N'_{i,1}$); and to each of the n_2 components $N''_{i+1,0}$ we glue a series of bordisms W''_{2i} (the leaf L''_i, cut along $N''_{i,1}$). To the upper boundary $N_{i+1,1}$ of the transitional bordism W_{2i+1} we glue a series of bordisms W_{2i+2} (the leaf L_{i+1}, cut along $N_{i+1,1}$). In passing through a critical point of this type, the leaf again "splits".

Now that we have shown how the transition through a critical point of index 1 $(n-1)$ is carried out locally, we are in a position to assign a quasiperiodic structure to a level surface of a form ω.

We set

$$\mathrm{Ann}(\omega_0) := \left\{ c \in \tilde{H}_1(M;\mathbf{Z}) \,\middle|\, \int_c \omega_0 = 0 \right\}.$$

We also set

$$h := \underset{c\in \mathrm{Ann}(\omega_0)}{\mathrm{G.C.D.}} \int_c \omega_1; \qquad H := \underset{i,j}{\mathrm{L.C.M.}}\, R(\omega_1|_{L_{ij}}),$$

where L_{ij} is the jth connected component of the ith leaf of the form ω_0. We shall construct the bordisms from which we shall glue together the level surface of ω. We begin with the regular ones. The number of these will, as before, be equal to the number of critical points. Now, however, they will generally speaking be nonconnected.

We consider a leaf L_i of the form ω_0, $L_i = \bigcup_j L_{ij}$. We suppose that the restriction $\omega_1|_{L_{ij}}$ of the form ω_1 to the jth connected component L_{ij} of the leaf L_i is not exact. As before, we cut L_{ij} along the submanifold $N_{ij,1}$. We obtain a bordism W_{ij}. After this, we glue together $H/R(\omega_1|_{L_{ij}})$ copies of W_{ij}, and we take the disjoint union of $R(\omega_1|_{L_{ij}})/h$ copies of the bordism so obtained. Such a bordism W_{ij} can be cut out of the leaf of ω by lifting the submanifold $N_{ij,1}$ from L_{ij} to the leaf of ω, displacing the cut so obtained along the phase trajectories through a height εH (in the sense of the integral of the form ω) and replicating this pair $\tilde{N}_{ij,1}$, $\tilde{N}'_{ij,1}$ of cuts $R(\omega_1|_{L_{ij}})/h$ times by successive displacements through the height εh.

If $\omega_1|_{L_{ij}}$ is exact, then L_{ij} is covered by a compact component $\tilde{L}_{ij}$ of the leaf of ω. For $\tilde{W}_{ij}$ we take H/h copies of the component $\tilde{L}_{ij}$ of ω, suspended one above another at a height of εh.

Taking the disjoint union $\bigcup_j \tilde{W}_{ij}$ over all the connected components j of L_i, we obtain the ith regular bordism. The transitional bordisms are constructed analogously, by use of the "local stockpiles" made above. To each critical point, however, there will now correspond H/h different transitional bordisms.

In every connected component of every cylinder $f_{\omega_0}^{-1}]b_i, b_{i+1}[$ we now distinguish, as before, a segment of a phase trajectory. Using the transitional bordisms, we glue together the segments along the leaves of the form ω, and we obtain a graph of the connected components of the level surfaces of the map f_{ω_0}. For any closed path c in the graph which is such that $[c] \in \mathrm{Ann}(\omega_0)$, a relation $\int_c \omega_1 = k \cdot \varepsilon h$ is satisfied, where $k \in \mathbf{Z}$. It is not hard to show that, by modifying the transitional bordisms (which results in changes to the gluing of the segments in the graph of connected components), we can arrange that $\int_c \omega_1 = 0$ for every closed path c in the graph such that $[c] \in \mathrm{Ann}(\omega_0)$. But this permits us, when we are selecting the quasiperiodic structure, to replace the graph of the connected components of the level surfaces of f_{ω_0} by a single circle.

Therefore a quasiperiodic structure on the level surfaces of the form ω can be described in the following way. On a circle, m segments of length H are selected, and each of them is subdivided into H/h equal parts. The length of the circle is much greater than H, and is incommensurable with H. The totality of segments of the circle plays the role of the indexing set (see Definition 1 in the Introduction). To each segment on the circle, a bordism is assigned: to the longer segments there correspond regular bordisms, and to each of the short segments of length h we assign its own transitional bordism. The correspondence looks like that in Figure 1, but now every short segment is divided into H/h equal parts, and to each of these parts there corresponds its own transitional bordism. We fix a point x_0 on the circle, and start moving in both directions along the circle from the point x_0 in steps of length H, marking out points $x_1, x_2, \dots$ in one direction and points $x_{-1}, x_{-2}, \dots$ in the other. By assigning to every point the selected interval in which it lies, we obtain an admissible quasiperiodic sequence, from which we glue up a quasiperiodic manifold which is a leaf of the form ω. Every such sequence of points $\{x_i\}$ uniquely defines a leaf of ω. The sequences of points $\{x_i\}$ and $\{x_i'\}$ which thus correspond to the same leaf are those, any only these, for which the point x_0' is obtained from x_0 by a displacement through kh, where $k \in \mathbf{Z}$. Thus on each leaf we can define H/h nonequivalent quasiperiodic structures.

We have therefore proved the

THEOREM. *For any rational Morse form ω_0 on a compact manifold, and any rational form ω_1, a level surface of the form $\omega_0+\varepsilon\omega_1$ for sufficiently small $\varepsilon \in \mathbf{R}$ is a quasiperiodic manifold.*

§5. Morse forms with an arbitrary degree of irrationality

In this section we describe the structure of a leaf of a form ω which is close to a rational one, when ω has an arbitrary degree of irrationality k. As the constructions below will make clear, the special case $\operatorname{rk}\omega = 2$ considered above turns out to be a key one, since almost all the constructions of the general case $\operatorname{rk}\omega = k$ reduce to it.

Form without critical points. Description of the analogues of bordisms. We begin the study of the structure of a leaf in the case of degree of irrationality k, as in the foregoing, with an analysis of the simplest case, where the Morse form ω, $\operatorname{rk}\omega = k$, has no critical points at all. According to the theorem in §2, a level surface of such a form is a $\mathbf{Z}^{k-1}$-covering over a compact leaf of an approximating rational form ω_0, and thus has a "$\mathbf{Z}^{k-1}$-periodic structure". We shall make this structure explicit (see the analogous construction for $k = 2$ in Remark 3 of §2).

By applying Lemma 1 from §2, we express the form ω as a sum of k independent rational forms $\omega = \omega_0 + \alpha_1 + \dots + \alpha_{k-1}$. The restriction of ω to a leaf L of the approximating rational form ω_0 will be represented as a sum of $k-1$ independent rational forms, $\omega|_L = \alpha_1|_L + \dots + \alpha_{k-1}|_L$.

In the cohomology classes of the forms $\alpha_i|_L$ we consider Morse forms α_i', so that $H^1(L;\mathbf{R}) \ni [\omega|_L] = [\alpha_1'] + \dots + [\alpha_{k-1}']$. For each rational form α_i' we construct a map $f_{\alpha_i'}\colon L \to S^1$. We consider the map

$$F_\omega\colon L \to \underbrace{S^1 \times \dots \times S^1}_{k-1} = T^{k-1},$$

in which $F_\omega\colon x \mapsto f_{\alpha_1'}(x) \times \dots \times f_{\alpha_{k-1}'}(x)$.

Now in the torus T^{k-1}, which is the range of the map F_ω, we shall define a "singular set" X. We denote by $X_1 \subset T^{k-1}$ the union of all the sets

$$\underbrace{S^1 \times \cdots \times S^1}_{j-1} \times b \times \underbrace{S^1 \times \cdots \times S^1}_{k-1-j},$$

where b is any critical value of the map $f_{\alpha'_j}$, and where $1 \le j \le k-1$. We denote by $X_2 \subset T^{k-1}$ the union of all the subsets of the form

$$\underbrace{\underbrace{S^1 \times \cdots \times S^1}_{j-1} \times b' \times S^1 \times \cdots \times S^1}_{i-1} \times b'' \times S^1 \times \cdots \times S^1,$$

where $b' \times b''$ is any critical value of the map $f_{\alpha'_j} \times f_{\alpha'_i} \colon L \to S^1 \times S^1$, and $1 \le j < i \le k-1$. In the same manner we define the sets $X_3, X_4, \ldots, X_{k-1}$. The set X_{k-1} is the set of critical values of the map $F_\omega = f_{\alpha'_1} \times \cdots \times f_{\alpha'_{k-1}}$. It follows from Sard's lemma that each of the sets X_i, and therefore also their union $X = \bigcup_1^{k-1} X_i$, has measure zero in T^{k-1}. Thus the set $T^{k-1} \setminus X$ has measure 1. The points of $T^{k-1} \setminus X$ will for us play the role of the regular values of a Morse function.

We choose a point $x \in T^{k-1} \setminus X$. We denote by x_i the projection of x on the circle S^1 corresponding to the map $f_{\alpha'_i}$, so that $x = x_1 \times \cdots \times x_{k-1} \in S^1 \times \cdots \times S^1 = T^{k-1}$. The inverse image $F_\omega^{-1}(T^{k-2})$ of the torus $T^{k-2} = S^1 \times \cdots \times S^1 \times x_i \times S^1 \times \cdots \times S^1$ will be called the ith boundary or ith cut, and denoted by ${}_iN^{n-2}$. The cut with index i realizes a cycle which is dual (in L) to the cocycle $[(1/R(\alpha'_i))\alpha'_i]$. Every cut is a nonsingular level surface of the corresponding form. We note that, by construction, an arbitrary subset of the cuts we have chosen will either intersect in a submanifold, or not intersect at all. We consider the universal covering $\mathrm{Exp}\colon \mathbf{R}^{k-1} \to T^{k-1}$ over the torus T^{k-1} which is the range of the function F_ω, and we consider the lattice $\mathrm{Exp}^{-1}(x) \subset \mathbf{R}^{k-1}$ which is the inverse image of the selected point x. Given the covering $\mathrm{Exp}\colon \mathbf{R}^{k-1} \xrightarrow{\mathbf{Z}^{k-1}} T^{k-1}$, the map $F_\omega \colon L \to T^{k-1}$ induces a covering $\pi\colon \hat{L} \xrightarrow{\mathbf{Z}^{k-1}} L$ and a map $\hat{F}_\omega \colon \hat{L} \to \mathbf{R}^{k-1}$. The following diagram is commutative:

$$\begin{array}{ccc} \hat{L} & \xrightarrow{\hat{F}_\omega} & \mathbf{R}^{k-1} \\ \pi \downarrow \mathbf{Z}^{k-1} & & \mathrm{Exp} \downarrow \mathbf{Z}^{k-1} \\ L & \xrightarrow{F_\omega} & T^{k-1}. \end{array}$$

In other words, $\hat{L}$ is the fibered product $L \times_{T^{k-1}} \mathbf{R}^{k-1}$ with respect to the morphisms $F_\omega \colon L \to T^{k-1}$ and $\mathrm{Exp}\colon \mathbf{R}^{k-1} \to T^{k-1}$. All the forms $\pi^*\alpha'_i$, and therefore also the form $\pi^*(\omega|_L)$, are exact on the total space $\hat{L}$ of the covering π. Thus the covering space $\hat{L}$ is diffeomorphic to a leaf of the form ω in the manifold M. The inverse image $F_\omega^{-1}(I^{k-1})$ of an elementary cell I^{k-1} of the lattice $\mathrm{Exp}^{-1}(x)$ in the space $\mathbf{R}^{k-1}$ will be called an "elementary piece". The leaf $\hat{L}$ of ω is glued together from elementary pieces in a way which is induced by the gluing together of $\mathbf{R}^{k-1}$ from elementary cells. We note that the "elementary piece" (the analogue of a bordism) is the manifold L (the leaf of the form ω_0) sliced open along all the $k-1$ cuts.

We have therefore proved that a leaf of a Morse form ω with degree of irrationality k, $\operatorname{rk}\omega = k$, which has no critical points, possesses a $(k-1)$-periodic structure. An "elementary piece" is obtained by slicing a leaf L of an approximating rational form ω_0 along $k-1$ cuts. The cuts are chosen in such a way that any subset of them either does not intersect, or intersects in a submanifold. In other words, all the "edges" of the elementary piece are submanifolds.

Transition from one periodic structure to another. By choosing the "regular" point x in the torus T^{k-1}, we obtained the lattice $\mathrm{Exp}^{-1}(x)$ in the space $\mathbf{R}^{k-1}$. By taking another "regular" point y, we obtain a new lattice $\mathrm{Exp}^{-1}(y)$. An elementary cell I_0 of the original lattice turns out to be cut into 2^{k-1} parallelepipeds. We shall assume that the point y was chosen in such a way that all the vertices of the parallelepipeds so obtained correspond to "regular" points of the torus. We shall denote by $\{{}_jN_0\}_{1\le j\le k-1}$ the family of cuts corresponding to the point x and the cell I_0, and by $\{{}_jN_1\}_{1\le j\le k-1}$ the family corresponding to the point y and the cell I_1. On the manifold $\hat{L}^{n-1}$ two periodic structures appear, generated by the lattices $\mathrm{Exp}^{-1}(x)$ and $\mathrm{Exp}^{-1}(y)$. By a transition from one periodic structure to the other we shall mean the following construction. In the space $\mathbf{R}^{k-1}$ we take a hyperplane dividing $\mathbf{R}^{k-1}$ into two half-spaces $\mathbf{R}_0^{k-1}$ and $\mathbf{R}_1^{k-1}$. We fill the half-spaces incompletely with cells I_0 and I_1 respectively. We fill the remaining "strip" with parallelepipeds obtained by intersecting I_0 and I_1. The structure we have indicated on $\mathbf{R}^{k-1}$ will induce a transition from one periodic structure on the manifold $\hat{L}$ to the other. The "elementary pieces" corresponding to I_0 and I_1 will be called *regular*, and the "pieces" corresponding to the intersections $I_0 \cap I_1$ will be called *transitional.* The "regularity" of the vertices of the parallelepipeds $I_0 \cap I_1$ implies that all the "edges" are submanifolds for the transitional "pieces" also. (In other words, every subset of every mixed family $\{{}_jN_{i,q}\}_{1\le j\le k-1}$, $q = 0, 1$, either intersects in a submanifold or does not intersect at all; that is, the mixed family $\{{}_jN_{i,q}\}_{1\le j\le k-1}$ also gives a "family of cuts".)

The general case. We now pass on to the general case, in which the rational Morse form ω_0 has singularities. We investigate the structure of a level surface of a form ω with degree of irrationality k under the hypothesis that ω is close to ω_0. As we did earlier, we stipulate that to each critical value of the function $f_{\omega_0}\colon M \to S^1$ constructed from ω_0 there corresponds a single critical point. To avoid unwieldy constructions, we restrict ourselves to the case when ω_0 has no critical points of extreme indices 0, 1, $n-1$, and n. We shall also assume that ω is represented in the form $\omega = \omega_0 + \varepsilon\omega_{k-1}$, where $\operatorname{rk}\omega = k$, $\operatorname{rk}\omega_0 = 1$, and $\operatorname{rk}\omega_{k-1} = k-1$.

We represent the form ω_{k-1} as a sum of $k-1$ independent rational forms, $\omega_{k-1} = \alpha_1 + \cdots + \alpha_{k-1}$ (see Lemma 1 in §2). Then $\omega = \omega_0 + \varepsilon(\alpha_1 + \cdots + \alpha_{k-1})$, and the restriction of ω to an arbitrary leaf L_i of ω_0 is a sum $\omega|_{L_i} = \varepsilon(\alpha_1|_{L_i} + \cdots + \alpha_{k-1}|_{L_i})$, where all the forms $\alpha_j|_{L_i}$ remain independent on every leaf L_i. For each form α_j we construct pairs of submanifolds ${}_jN_{i,0}$ and ${}_jN_{i,1}$ in the nonsingular leaves L_i of ω_0 (exactly as we constructed the submanifolds $N_{i,0}$ and $N_{i,1}$ from the rational form ω_1 in §3),

$$[{}_jN_{i,0}] = [{}_jN_{i,1}] = D[\alpha_j|_{L_i}] \in H_{n-2}(L_i^{n-1}; \mathbf{Z}).$$

We shall suppose, for simplicity, that no two submanifolds ${}_jN_{i,0}$ and ${}_jN_{i,1}$ intersect. Then, in the cohomology class of each form $[\alpha_j|_{L_i}] \in H^1(L_i; \mathbf{R})$ we can choose a Morse from α_{ji} for which the submanifolds ${}_jN_{i,0}$ and ${}_jN_{i,1}$ are leaves. On the leaves L_i of ω_0 we define maps $F_i\colon L_i \to T^{k-1}$, where

$$F_i = f_{\alpha_{1,i}} \times \cdots \times f_{\alpha_{k-1,i}}\colon L_i \to S^1 \times \cdots \times S^1.$$

Since the "sets of regular values" of the functions F_i have measure 1, we can arrange by means of a small perturbation of the submanifolds ${}_jN_{i,0}$ and ${}_jN_{i,1}$ (simultaneously in all the leaves L_i of ω_0) that every family $\{{}_jN_{i,q}\}_{1\le j\le k-1}$, $q = 0, 1$, shall define a family of cuts of the corresponding manifold L_i, for $1 \le i \le m$, m being the number of critical points of ω_0.

By combining the methods of §3 with constructions from the first part of the present section, we obtain the following description of the structure of a leaf of ω. In $\mathbf{R}^{k-1}$ we select a family of parallel hyperplanes generated by periodic repetition of a collection of m planes. (Here m is the number of critical points of ω_0.) Between two neighboring hyperplanes a lattice in $\mathbf{R}^{k-1}$ is selected. The transition from lattice to lattice through a hyerplane is described above. The cells of lattices enclosed between similar planes are similar (that is, the cells also repeat after every m planes).

Every such lattice structure describes the structure on the corresponding leaf of ω. This correspondence is continuous. To the regular cells of a lattice there correspond the regular "elementary pieces" K_{2i}, and to the transitional ones there correspond the transitional "pieces". In contrast with the case $\operatorname{rk} \omega = 2$, there will be $2^{k-1} - 1$ "transitional pieces" corresponding to each critical point. We recall that the cuts were chosen in such a way that any "edge" of either a regular or a transitional piece is a submanifold.

§6. Application. The motion of an electron in a reciprocal lattice in a homogeneous magnetic field

In this section, some physical applications of the theory of Morse forms will be demonstrated. It was shown in [2] that the problem of investigating the semiclassical motion of an electron in a reciprocal lattice in a homogeneous magnetic field (a problem of S. P. Novikov) can be reformulated in terms of a Morse 1-form on a two-dimensional submanifold M^2 of the torus T^3, and this reformulation allows one to answer a number of questions. In the present section we shall describe the motion of an electron in the case when the homogeneous magnetic field is close to a *rational* one (the field is called rational if it is aligned with some vector of the reciprocal lattice).

We elucidate the physical nature of the problem (see [2]). In the absence of a magnetic field, an electron in the lattice Γ moves with constant quasimomenta p_1, p_2, p_3, defined modulo the reciprocal lattice, and has a dispersion law $\varepsilon(p_1, p_2, p_3)$. The collection of quasimomenta can thus be regarded as a point of the torus $T^3 = \mathbf{R}^3/\Gamma$. In a weak magnetic field, one can consider the semiclassical motion of the electron in the space of quasimomenta, given by the Hamiltonian $\varepsilon = \varepsilon(p + \frac{e}{c}A)$, where A is the vector potential. Let $p' = p + \frac{e}{c}A$. The motion of the electron in a uniform magnetic field $\vec{H}$ is determined by the intersection of a surface $\varepsilon(p') = \text{const}$ with a plane orthogonal to the magnetic field.

ASSERTION. *Suppose that the weak magnetic field in the lattice is rational* (*that is, the vector $\vec{H}$ is aligned with some vector of the dual lattice*) *and is orthogonal to the nonsingular Fermi surface $\varepsilon(p') = \text{const}$ only at isolated points. Suppose further that for any two such points lying in a single plane orthogonal to the vector $\vec{H}$, the vector joining them belongs to the reciprocal lattice. Then in a nearby uniform magnetic field the motion of a particle in p'-space occurs in a flat strip of finite width.*

We shall denote a nonsingular level surface $\varepsilon(p') = \text{const}$ by $\hat{M}^2 \subset \mathbf{R}^3$, and call it a *Fermi surface.* The plane, orthogonal to the magnetic field $\vec{H}$, in which the motion of the electron occurs can be considered as a level surface of the form $\hat{\Omega} = H_1 dp'_1 + H_2 dp'_2 + H_3 dp'_3$ with constant coefficients. We denote the restriction of the form $\hat{\Omega}$ to the Fermi surface by $\hat{\omega} = \hat{\Omega}|_{\hat{M}^2}$. We assume that the form $\hat{\omega}$ has only nondegenerate critical points (the field $\vec{H}$ is orthogonal to the Fermi surface only at isolated points).

So our problem is to investigate the behavior of the level surfaces of the restriction $\hat{\omega} = \hat{\Omega}|_{\hat{M}^2}$ of the form $\hat{\Omega}$ with constant coefficients on a triply periodic surface $\hat{M}^2$ in $\mathbf{R}^3$, under the hypothesis that $\hat{\omega}$ is a Morse form. We note that the objects we are considering admit quotienting by the lattice Γ. As a result of the quotienting we obtain a compact manifold M^2 (the quotient of the Fermi surface $\hat{M}^2$) in the torus $T^3 = \mathbf{R}^3/\Gamma$, $M^2 \subset T^3$. To the exact form $\hat{\Omega}$ there will correspond a closed form $\Omega = H_1 dp'_1 + H_2 dp'_2 + H_3 dp'_3$ on the torus T^3, and to the exact form $\hat{\omega}$ on $\hat{M}_2$ there will correspond a closed form ω on M^2. If the field $\vec{H}$ is rational, then the degree of irrationality of the form Ω is equal to 1, and then $\operatorname{rk}\omega \leq \operatorname{rk}\Omega = 1$. As was shown in §1, from this follows the compactness of the level surfaces of the form ω on the Fermi surface M^2 in the torus T^3. Thus a leaf of the form ω is a collection of circles, and in the universal covering space $\mathbf{R}^3 \to T^3$ the level surfaces of the form ω will be either closed or periodic. So the trajectories of an electron in p'-space in the original rational field are either closed or periodic (see [2]).

We can now proceed to the proof of the assertion formulated above. We shall denote by $\hat{\Phi}$ the form corresponding to a uniform magnetic field which is close to the rational field $\vec{H}$, and by $\hat{\varphi} = \hat{\Phi}|_{\hat{M}^2}$ its restriction to a Fermi surface. Just as in the general situation (see §1), we construct from the rational from ω a map $f_\omega \colon M^2 \to S^1$ which determines a dissection of the Fermi surface into a composite of elementary bordisms. The condition in the assertion guarantees that to every critical value of f_ω there corresponds only one critical point. We consider a connected component W of the elementary bordism determined by the map f_ω, with a saddle-shaped critical point. The boundary of W has the form of a union of three components, each of which realizes some cycle in T^3. There is the following

LEMMA. *At least one of the three cycles is homologous to zero in the torus T^3.*

Suppose, for example, that the cycles α and β lie on one level surface. Suppose neither of them is homologous to zero in T^3. Since α and β do not intersect, and both are embedded in T^2 (the leaf $\Omega = 0$), we have $\alpha = \pm\beta$. So γ is either 0 or 2α. But since γ is embedded without self-intersections in T^2 (the leaf $\Omega = 0$) in which we can also realize the cycle α, the possibility $\gamma = 2\alpha$ is excluded, and this proves the lemma.

It follows from the proof of the lemma that the remaining two cycles are homologous. Further, a connected component of the elementary bordism can be represented as a cylinder with a hole cut out (the boundary of this hole being our null-homologous cycle).

In every elementary bordism we now take one connected component of a level surface of the form ω, realizing a cycle which is homologous to zero in T^3. Then for every form φ close to ω there will exist closed level surfaces which are close to the one chosen for ω and which also realize cycles homologous to zero in the torus. We cut the Fermi surface along such leaves of the form φ. We assert that for any connected component N, the image of the map $i_* \colon H_1(N;\mathbf{Z}) \to H_1(T^3;\mathbf{Z})$ induced by the embedding $i \colon N \to T^3$ has at most two generators. In fact, N can be represented as the result of gluing together a collection of cylinders and cylinders with holes (the top of the last one being glued to the bottom of the first), i.e. it is a torus with a number of holes. This means that when we transfer to the covering space $\mathbf{R}^3$ over T^3, for any connected component $\hat{N}$ of the surface covering N, there exist two parallel planes between which $\hat{N}$ is enclosed. Since any leaf $\varphi = 0$ is contained in some connected component N, this leaf in $\mathbf{R}^3$ will be enclosed in the strip formed by the intersection of the plane $\hat{\Phi} = 0$ with the pair of planes chosen for $\hat{N}$.

Received 18/JUNE/85

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