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Deviation for interval exchange transformations

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Abstract. Consider a long piece of a trajectory $x, T(x), T(T(x)), \dots, T^{n-1}(x)$ of an interval exchange transformation T. A generic interval exchange transformation is uniquely ergodic. Hence, the ergodic theorem predicts that the number $\chi_i(x, n)$ of visits of our trajectory to the ith subinterval would be approximately $\lambda_i n$. Here λ_i is the length of the corresponding subinterval of our unit interval X. In this paper we give an estimate for the deviation of the actual number of visits to the ith subinterval X_i from one predicted by the ergodic theorem.

We prove that for almost all interval exchange transformations the following bound is valid:

$$\max_{\substack{x\in X\\ 1\le i\le m}} \limsup_{n\to+\infty} \frac{\log|\chi_i(x,n)-\lambda_i n|}{\log n} = \frac{\theta_2}{\theta_1} < 1.$$

Roughly speaking the error term is bounded by n^{θ_2/θ_1}. The numbers $0 \le \theta_2 < \theta_1$ depend only on the permutation π corresponding to the interval exchange transformation (actually, only on the Rauzy class of the permutation). In the case of interval exchange of two intervals we obviously have $\theta_2 = 0$. In the case of exchange of three and more intervals the numbers θ_1, θ_2 are the two top Lyapunov exponents related to the corresponding generalized Gauss map on the space of interval exchange transformations.

The limit above 'converges to the bound' *uniformly for all* $x \in X$ in the following sense. For any $\varepsilon > 0$ the ratio of logarithms would be less than $\theta_2(\pi)/\theta_1(\pi) + \varepsilon$ for all $n \ge N(\varepsilon)$, where $N(\varepsilon)$ does not depend on the starting point $x \in X$.

1. *Introduction*

1.1. *Interval exchange transformations. General requirements on induction procedure.*

Recall the notion of an interval exchange transformation. Consider an interval X, and cut it into m subintervals of lengths $\lambda_1, \dots, \lambda_m$. Now glue the subintervals together in another order, according to some permutation $\pi \in \mathfrak{S}_m$ and preserving the orientation. We again obtain an interval X of the same length, and hence we get a mapping $T : X \to X$, which is called an interval exchange transformation. Our mapping is piecewise linear, and it preserves the orientation and Lebesgue measure. It is singular at the points of cuts, unless two consecutive intervals separated by a point of cut are mapped to consecutive intervals in the image.

An interval exchange transformation T is completely determined by a pair (λ, π), $\lambda \in \mathbb{R}^m_+$, $\pi \in \mathfrak{S}_m$. Let $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \lambda_j$, and $X_i = [\beta_{i-1}, \beta_i[$ so that $X = X_1 \sqcup \cdots \sqcup X_m$. Define a skew-symmetric $m \times m$ matrix:

$$\Omega_{ij}(\pi) = \begin{cases} 1 & \text{if } i < j \text{ and } \pi(i) > \pi(j) \\ -1 & \text{if } i > j \text{ and } \pi(i) < \pi(j) \\ 0 & \text{otherwise.} \end{cases}$$

Consider the translation vector $\tau = \Omega(\pi)\lambda$. Our interval exchange transformation T is defined as follows:

$$T(x) = x + \tau_i, \quad \text{for } x \in X_i,\ 1 \le i \le m.$$

Note that if for some $k < m$ we have $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$, then our map T decomposes into two interval exchange transformations. We consider only the class $\mathfrak{S}^0_m$ of *irreducible* permutations—those which have no invariant subsets of the form $\{1, \ldots, k\}$, where $1 \le k < m$.

Suppose that we have some induction procedure which assigns to a given interval exchange transformation T corresponding to the pair (λ, π) some subinterval $X^{(1)} \subset X$. Consider the induced map $T^{(1)} = T|_{X^{(1)}}$ of T to this subinterval. It is easy to see, that $T^{(1)}$ is again an interval exchange transformation (see [**2**]). Suppose that we managed to choose the induction procedure so that:

Requirement 1. The new interval exchange transformation $T^{(1)}$ is again an exchange of the same number m of subintervals $X^{(1)}_1, \ldots, X^{(1)}_m$.

For a point $x \in X^{(1)}_j$ in the 'new' subinterval $X^{(1)}_j$ define B_{ij} to be the number of intersections of the trajectory $x, Tx, T(T(x)), \ldots, T^{l-1}(x)$ of x with the 'old' subinterval X_i before the first return $T^l(x) \in X^{(1)}$ to the 'new' subinterval $X^{(1)}$. We assume that:

Requirement 2. For any pair $1 \le i, j \le m$ the number B_{ij} is the same for all $x \in X^{(1)}_j$.

'Induction procedures' as specified above really exist, for example *Rauzy induction* [**7**]. Note that any induction procedure leads to a mapping of the space of interval exchange transformations to itself (see [**7**], [**8**]): given a pair (λ, π) which determines an interval exchange transformation T we assign to it a pair $(\lambda^{(1)}, \pi^{(1)})$, where $\lambda^{(1)}$ is the vector of the lengths of subintervals $X^{(1)}_1, \ldots, X^{(1)}_m$, and $\pi^{(1)}$ is the new permutation. Define the norm of a vector $v \in \mathbb{R}^m$ to be $\|v\| = |v_1| + \cdots + |v_m|$. Having an interval exchange transformation $T(\lambda, \pi)$, we can renormalize the domain of $T(\lambda, \pi)$ to have the unit length. Thus fixing the permutation π we identify the space of all interval exchange transformations $T(\lambda, \pi)$ with the standard $(m-1)$-dimensional simplex $\Delta^{m-1} = \{\lambda \in \mathbb{R}^m_+ \mid \|\lambda\| = 1\}$. Any induction procedure as described above leads to a mapping $\Delta^{m-1} \times \mathfrak{S}_m \to \Delta^{m-1} \times \mathfrak{S}_m$, under additional renormalization

$$(\lambda, \pi) \mapsto \left(\frac{\lambda^{(1)}}{\|\lambda^{(1)}\|}, \pi^{(1)} \right).$$

In Rauzy induction the induced permutation $\pi^{(1)}$ is always irreducible, provided that the initial permutation π is irreducible, $\pi \in \mathfrak{S}_m^0$. Denote by $\mathfrak{R}(\pi)$, $\pi \in \mathfrak{S}_m^0$, all permutations accessible from the given one by iterations of Rauzy induction. The finite set $\mathfrak{R}(\pi)$ is called the *Rauzy class* of permutation π. Subsets $\Delta^{m-1} \times \mathfrak{R}(\pi)$ are invariant under the map $\mathcal{T} : \Delta^{m-1} \times \mathfrak{S}_m^0 \to \Delta^{m-1} \times \mathfrak{S}_m^0$ corresponding to Rauzy induction. W. Veech proved in [**8**] that for any irreducible permutation π the map $\mathcal{T}$ is ergodic with respect to the absolutely continuous invariant measure on $\Delta^{m-1} \times \mathfrak{R}(\pi)$.

Modifying Rauzy induction in [**11**] we have constructed the induction procedure which leads to another map $\mathcal{G} : \Delta^{m-1} \times \mathfrak{R}(\pi) \to \Delta^{m-1} \times \mathfrak{R}(\pi)$. Closely following the original proof in [**8**] we proved ergodicity of this map with respect to absolutely continuous invariant *probability* measure μ on $\Delta^{m-1} \times \mathfrak{R}(\pi)$ (the invariant measure corresponding to the map $\mathcal{T}$ is infinite). The relation between maps $\mathcal{T}$ and $\mathcal{G}$ is similar to the relation between *additive* and *multiplicative* continued fraction algorithms in [**1**].

Let us discuss what we can gain from the generalized Gauss map $\mathcal{G}$ in our problem.

1.2. *Recursive bound for the deviation.* The induction procedure $\mathcal{G}$ satisfies both of the conditions formulated above. Consider the matrix (actually, a matrix-valued function) as described above: $B = B(\lambda, \pi)$ corresponding to this induction procedure. We will show that both matrices $B^{-1}(\lambda, \pi)$ and ${}^{t}B(\lambda, \pi)$ define measurable cocycles on $\Delta^{m-1} \times \mathfrak{R}$ with respect to μ, i.e. $\int \log^+ \|B^{-1}\| \, d\mu$ and $\int \log^+ \|{}^{t}B\| \, d\mu$ are both finite. Here and below we denote by ${}^{t}A$ the matrix transposed to matrix A.

Let $X^{(k)} = X_1^{(k)} \sqcup \cdots \sqcup X_m^{(k)}$ be the interval and corresponding subintervals under the exchange obtained after k steps of the induction procedure $\mathcal{G}$. We assume that the initial interval exchange transformation is normalized, i.e. $\|\lambda\| = 1$, and we do not normalize the vector $\lambda^{(k)}$ whose components are represented by the lengths of subintervals $X_j^{(k)}$, $1 \le j \le m$. Denote by $B^{(k)}(\lambda, \pi)$ the product

$$B^{(k)}(\lambda, \pi) := B(\lambda, \pi) \cdot B(\mathcal{G}(\lambda, \pi)) \cdot B(\mathcal{G}(\mathcal{G}(\lambda, \pi))) \cdots B(\mathcal{G}^{(k-1)}(\lambda, \pi)).$$

By definition of our induction procedure we have

$$\lambda = B^{(k)}(\lambda, \pi) \cdot \lambda^{(k)}. \tag{1.1}$$

Now consider a *very long* piece of trajectory $x, T(x), T(T(x)), \ldots, T^{n-1}(x)$ of a point $x \in X$. We want to get an upper bound for the absolute value of the error term

$$E_i(X, x, T, n) := \chi_i(X, x, T, n) - \lambda_i n, \tag{1.2}$$

where

$$\chi_i(X, x, T, n) := \sum_{l=0}^{n-1} \chi_{X_i}(T^l(x)) \tag{1.3}$$

and $\chi_Y(x)$ is the characteristic function of the subset $Y \subset X$.

Let $x_{(1)}$ be the first visit of our piece of trajectory to the subinterval $X_{(1)} := X^{(k)}$. Let

$$n_{(1)} := \sum_{l=0}^{n-1} \chi_{X_{(1)}}(T^l(x)) \tag{1.4}$$

be the number of visits of our piece of trajectory to the subinterval $X_{(1)}$.

Consider the interval exchange transformation

$$T_{(1)} := T|_{X^{(k)}} \tag{1.5}$$

induced by T on the subinterval $X^{(k)}$. Note that the successive visits to $X^{(k)}$ correspond to a piece of trajectory $x_{(1)}, T_{(1)}(x_{(1)}), \ldots, T_{(1)}^{n_{(1)}-1}(x_{(1)})$ of the induced transformation $T_{(1)}$ on $X_{(1)} = X^{(k)}$.

Consider the error term for the number of visits of this induced piece of trajectory of $T_{(1)}$ to the subinterval $X_j^{(k)} \subset X^{(k)} = X_{(1)}$:

$$E_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)}) = \sum_{l=0}^{n_{(1)}-1} \chi_{X_j^{(k)}}(T_{(1)}^l(x_{(1)})) - \frac{\lambda_j^{(k)}}{\lambda^{(k)}} n_{(1)}.$$

Denote by e_i the covector $(0, \ldots, 0, 1, 0, \ldots, 0)$ having the only nontrivial entry at the ith place, $1 \le i \le m$. Denote $e_0 = (1, 1, \ldots, 1)$.

PROPOSITION 1. *Assume that the interval exchange transformation T is uniquely ergodic. Then the error term for the number of visits of a finite piece of trajectory to the subinterval $X_i \subset X$, $1 \le i \le m$, satisfies the following recursive relation:*

$$\begin{aligned} |E_i(X, x, T, n)| \quad \le \quad & 2 \cdot \max_{1 \le j \le m} \|B^{(k)}(\lambda, \pi) \cdot e_j\| \\ & + m \cdot \|{}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0)\| \cdot \max_{1 \le j \le m} |E_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)})|. \end{aligned}$$

We prove this proposition in §2.

1.3. *Formulation of results.* Let $\theta_1 \ge \theta_2 \ge \cdots \ge \theta_m$ be the collection of Lyapunov exponents of the cocycle $B^{-1}(\lambda, \pi)$.

PROPOSITION 2. *For almost all $\lambda \in \Delta_{m-1} \times \mathfrak{R}(\pi)$ the following limit exists and is equal to $\theta_1(\pi)$:*

$$\lim_{k \to +\infty} \frac{1}{k} \log \|B^{(k)}(\lambda, \pi) e_j\| = \theta_1$$

for all $1 \le j \le m$.

PROPOSITION 3. *For almost all $\lambda \in \Delta_{m-1} \times \mathfrak{R}(\pi)$ the limits below exist and satisfy the following relation:*

$$\max_{1 \le i \le m} \lim_{k \to +\infty} \frac{1}{k} \log \|{}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0)\| = \theta_2,$$

where $e_0 = (1, 1, \ldots, 1)$.

General results in [**10**] immediately imply the following.

PROPOSITION 4. *For any Rauzy class $\mathfrak{R}$, the largest Lyapunov exponent $\theta_1(\mathfrak{R})$ of the Gauss map is strictly greater than the next one:*

$$\theta_1(\mathfrak{R}) > \theta_2(\mathfrak{R}).$$

By combining the above propositions with some additional tricks and using the unique ergodicity of almost all interval exchange transformations [**5**, **8**], we will prove the main theorem of this paper.

THEOREM 1. *For any irreducible permutation π of more than two elements and for any λ from a set of full Lebesgue measure in Δ_{m-1} the following property is valid. Consider interval exchange transformation $T(\lambda, \pi)$ defined on the unit interval X. For any $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that for any $n > N(\varepsilon)$ and for any $1 \le i \le m$*

$$\frac{\log|\chi_i(x, n) - \lambda_i n|}{\log n} \le \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))} + \varepsilon,$$

where

$$\chi_i(x, n) = \sum_{l=0}^{n-1} \chi_{X_i}(T^l(x))$$

and χ_{X_i} is the characteristic function of the subset $X_i \subset X$. The number $N(\varepsilon)$ depends on ε and on the pair (λ, π) but does not depend on the point $x \in X$.

Hence the following limits exist and satisfy the following bound:

$$\limsup_{n\to+\infty} \frac{\log|\chi_i(x, n) - \lambda_i n|}{\log n} \le \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))},$$

where

$$0 \le \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))} < 1.$$

For the dense set of points $x_l \in X$ of the form $x_l = T^l(0)$, $l \in \mathbb{Z}$, the equality is valid:

$$\max_{1\le i\le m} \limsup_{n\to+\infty} \frac{\log|\chi_i(x_l, n) - \lambda_i n|}{\log n} = \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))}.$$

Remark 1. The case $m = 2$ is exceptional for us since this is the only case when $\theta_2(\{2, 1\}) = -\pi^2/(12\log 2) < 0$. But an interval exchange transformation of two intervals is equivalent to a rotation of a circle. The equality

$$\limsup_{n\to+\infty} \frac{\log|\chi_i(x, n) - \lambda_i n|}{\log n} = 0 \quad \text{for } i = 1, 2$$

is well known in this case.

We prove Proposition 1 in the next section. In §3 we give some general information concerning dual cocycles. In §4 we discuss some basic properties of the cocycles B^{-1}, ${}^tB$, and ${}^tB|_{\mathrm{Ann}(\lambda)}$, and prove Proposition 3. In §5 we show that the columns of $B^{(k)}$ are asymptotically well-distributed and prove Proposition 2. The main theorem is proved in §6.

2. *Recursive bound for the deviation: proof of Proposition 1*

In this section we prove Proposition 1. Let X be a unit interval and let $x \in X$ be a point on it. Let $T : X \to X$ be a uniquely ergodic interval exchange transformation; let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and π be the corresponding vectors of lengths of subintervals and permutations. Let $n, k \in \mathbb{N}$ be positive integers and let $X^{(k)} \subset X$ be the subinterval obtained after k iterations of the induction corresponding to the map $\mathcal{G}$. Define

$$\begin{aligned} n^+(X, x, T, n, k) &:= \begin{cases} n & \text{if } T^l(x) \notin X^{(k)} \text{ for all } 1 \le l < n \\ \min_{0 \le l < n} l \mid T^l(x) \in X^{(k)} & \text{otherwise} \end{cases} \\ n^-(X, x, T, n, k) &:= \begin{cases} 0 & \text{if } T^l(x) \notin X^{(k)} \text{ for all } 1 \le l \le n \\ \min_{0 < l} l \mid T^{n+l}(x) \in X^{(k)} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1}$$

We denote

$$x_{(1)} := T^{n^+}(x), \quad \tilde{n} := n - n^+ + n^-.$$

We will extend our piece of trajectory up to the time $n + n^- - 1$, and then we will consider three parts of it. The first part $x, T(x), \ldots, T^{n^+-1}(x)$ is the part before the first visit to the interval $X^{(k)}$; we let it be empty if $n^+ = 0$. The third part is the extension part $T^n(x), \ldots, T^{n+n^--1}$; we let it be empty if $n^- = 0$. The second part is $T^{n^+}(x), \ldots, T^{n+n^--1}(x)$; we let it be empty if $n^+ = n$. We let $E_i(X, y, T, 0) := 0$ for any X, y, T by convention; $E_i(X, y, T, n)$ is defined for $n > 0$ by (1.2). Note that $E_i(X, x, T, n)$ is an additive cocycle, i.e.

$$E_i(X, x, T, n + l) = E_i(X, x, T, n) + E_i(X, T^n(x), T, l) \quad 1 \le i \le m.$$

Hence we can use the following representation:

$$\begin{aligned} &|E_i(X, x, T, n)| \\ &\quad = \ |E_i(X, x, T, n^+) + E_i(X, x_{(1)}, T, \tilde{n}) - E_i(X, T^n(x), T, n^-)| \\ &\quad \le \ n^+(X, x, T, n, k) + n^-(X, x, T, n, k) + |E_i(X, x_{(1)}, T, \tilde{n})|. \end{aligned} \tag{2.2}$$

Now we will treat the last term of the above expression.

Note that by Requirement 2 on the induction procedure the first return time $n_j^{(k)} = n_j^{(k)}(\lambda, \pi)$ of the trajectory $y, T(y), T(T(y)), \ldots$ to the subinterval $X^{(k)}$ is the same for all $y \in X_j^{(k)}$, and is equal to

$$n_j^{(k)}(\lambda, \pi) := B_{1j}^{(k)} + \cdots + B_{mj}^{(k)} = \|B^{(k)}(\lambda, \pi) e_j\|, \tag{2.3}$$

where the only nonzero component of the vector e_j is 1 at the jth place. Moreover, the number of visits of this trajectory to the subinterval X_i is

$$\chi_i(X, y, T, n_j^{(k)}) = B_{ij}^{(k)}(\lambda, \pi). \tag{2.4}$$

Hence, for any $y \in X_j^{(k)}$

$$E_i(X, y, T, n_j^{(k)}) = \chi_i(X, y, T, n_j^{(k)}) - \lambda_i n_j^{(k)} = B_{ij}^{(k)}(\lambda, \pi) - \lambda_i \cdot n_j^{(k)}(\lambda, \pi). \tag{2.5}$$

Let us keep track of successive intersections of the piece of a trajectory $x_{(1)}, T(x_{(1)}), \ldots, T^{\tilde{n}-1}$ with the subinterval $X^{(k)}$. We consider only the nontrivial case, when this intersection is nonempty. Let $j_0, j_1, \ldots$, where $1 \le j_q \le m$, enumerate corresponding subintervals $X_j^{(k)} \subset X^{(k)}$, so that the first visit to the interval $X^{(k)}$ occurs at the subinterval $X_{j_0}^{(k)}$, the next at $X_{j_1}^{(k)}$, etc.

Since E_i is an additive cocycle we can rewrite $E_i(X, x_{(1)}, T, \tilde{n})$ as the sum of $n_{(1)}$ terms (see (1.4) and (1.5) for notation):

$$E_i(X, x_{(1)}, T, \tilde{n}) = E_i(X, x_{(1)}, T, n^k_{j_0}) + E_i(X, T_{(1)}(x_{(1)}), T, n^k_{j_1}) + \cdots. \tag{2.6}$$

Note that due to (2.5) there are at most m different patterns corresponding to $j_q \in \{1, \dots, m\}$ in the above sum. The patterns are represented by (2.5). In our notation (see (1.3)–(1.5)) the number of visits to subinterval $X^{(k)}_j$ is equal to $\chi_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)})$. Hence, using (2.5) we can rewrite (2.6) as

$$E_i(X, x_{(1)}, T, \tilde{n}) = \sum_{1\le j\le m} \chi_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)}) \cdot (B^{(k)}_{ij} - \lambda_i n^{(k)}_j). \tag{2.7}$$

Using the notation of (1.2) we rewrite (2.7) as

$$\begin{aligned} E_i(X, x_{(1)}, T, \tilde{n}) &= \sum_{1\le j\le m} \left(E_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)}) + \frac{\lambda^{(k)}_j}{\|\lambda^{(k)}\|} n_{(1)} \right) \cdot (B^{(k)}_{ij} - \lambda_i n^{(k)}_j) \\ &= \sum_{1\le j\le m} E_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)}) \cdot (B^{(k)}_{ij} - \lambda_i n^{(k)}_j) \\ &\quad + \frac{n_{(1)}}{\|\lambda^{(k)}\|} \sum_{1\le j\le m} \lambda^{(k)}_j (B^{(k)}_{ij} - \lambda_i n^{(k)}_j), \end{aligned} \tag{2.8}$$

where $\lambda^{(k)}$ is the vector of lengths of subintervals $X^{(k)}_1, \dots, X^{(k)}_m$, i.e. $\|\lambda^{(k)}\| = \lambda^{(k)}_1 + \cdots + \lambda^{(k)}_m$.

Recall that by e_i we denote the covector $(0, \dots, 0, 1, 0, \dots, 0)$ having the only nontrivial entry at the ith place, $1 \le i \le m$. By e_0 we denote $e_0 = (1, 1, \dots, 1)$.

According to (2.5),

$$((B^{(k)}_{i1} - \lambda_i n^{(k)}_1), \dots, (B^{(k)}_{im} - \lambda_i n^{(k)}_m)) = {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0) \in \mathrm{Ann}(\lambda^{(k)}). \tag{2.9}$$

The fact that the covector above belongs to the annihilator $\mathrm{Ann}(\lambda^{(k)})$ of vector $\lambda^{(k)}$ follows from (1.1):

$$\begin{aligned} \langle \lambda^{(k)}, {}^{\mathrm{t}}B^{(k)}(e_i - \lambda_i e_0)\rangle &= \langle (B^{(k)})^{-1} \cdot \lambda, {}^{\mathrm{t}}B^{(k)} \cdot (e_i - \lambda_i e_0)\rangle \\ &= \langle \lambda, e_i - \lambda_i e_0\rangle = \langle e_i, \lambda\rangle - \lambda_i \langle e_0, \lambda\rangle \\ &= \lambda_i - \lambda_i(\lambda_1 + \cdots + \lambda_m) = \lambda_i - \lambda_i \cdot 1 = 0. \end{aligned} \tag{2.10}$$

Hence the second sum in (2.8) vanishes and we finally get

$$\begin{aligned} &|E_i(X, x_{(1)}, T, \tilde{n})| \\ &\quad = \left| \sum_{1\le j\le m} (B^{(k)}_{ij} - \lambda_i n^{(k)}_j) \cdot E_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)}) \right| \\ &\quad \le m \cdot \|{}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0)\| \cdot \max_{1\le j\le m} |E_j(X_{(1)}, x_{(1)}, T_{(1)}, n_{(1)})|. \end{aligned} \tag{2.11}$$

Now let us estimate the values of n^+ and n^-.

LEMMA 2.1. *Assuming that the interval exchange transformation T is uniquely ergodic the following bound is valid:*

$$n^{\pm}(X, x, T, n, k) \le \max_{1\le j\le m} n^{(k)}_j(\lambda, \pi) = \max_{1\le j\le m} \|B^{(k)}(\lambda, \pi) \cdot e_j\|.$$

Proof. Note that according to results in [**2**] unique ergodicity of T implies minimality of T and T^{-1}. Hence any point $x \in X$ belongs to a piece of trajectory of some point $\tilde{x} \in X^{(k)}$ before the first return to $X^{(k)}$. Hence $n^{\pm}$ is less than or equal to the maximal first return time to the interval $X^{(k)}$. But the possible values of the first return time are $n_j^{(k)}$, where $j = 1, \dots, m$. □

Combining (2.2) with (2.11) and with Lemma 2.1 we complete the proof of Proposition 1.

3. *Dual cocycle*

A pair (g, A) consisting of a map $g : Y \to Y$ preserving a probability measure on the space Y, and of a measurable cocycle $A(y)$, $y \in Y$, with the values in the group $GL(m)$ defines a fiberwise linear mapping $A(y) : \mathbb{R}^m_y \to \mathbb{R}^m_{g(y)}$ on the total space of the trivialized vector bundle over the base Y with the fiber $\mathbb{R}^m$. This mapping is a fiberwise isomorphism. Hence it induces the dual fiberwise-linear mapping in the total space of the adjoint trivialized vector bundle with the fiber $(\mathbb{R}^m)^*$. This mapping corresponds to the cocycle ${}^{t}A^{-1}$, which we will call the *dual cocycle*. By ${}^{t}A$ we denote the matrix transposed to matrix A.

We recall that

$$\log^-(x) := \begin{cases} 0 & \text{when } x > 1 \\ \log(x) & \text{when } 0 < x \le 1 \end{cases}$$

$$\log^+(x) := \begin{cases} \log(x) & \text{when } x \ge 1 \\ 0 & \text{when } 0 < x < 1. \end{cases}$$

We will need several elementary facts concerning dual cocycles.

LEMMA 3.1. *Consider a measurable cocycle A with the values in $GL(m)$. If the function $\log^- |\det(A)|$ is integrable, then the dual cocycle is measurable.*

Proof. First note that for square $m \times m$ matrices A, the norms

$$\|A\|_1 = m \cdot \max_{1 \le i,j \le m} |A_{ij}| \quad \text{and} \quad \|A\|_2 = (\text{maximal eigenvalue of } {}^{t}AA)^{1/2}$$

are equivalent. From now on we will use the first norm. Now

$$\begin{aligned}
\log^+ \|{}^{t}A^{-1}\| &= \log^+ \max_{1 \le i,j \le m} |{}^{t}A^{-1}_{ij}| + \log m \\
&\le \log^+ \left(|(\det A)^{-1}| \cdot (m-1)! \cdot \left(\max_{1 \le i,j \le m} |A_{ij}| \right)^{m-1} \right) + \log m \\
&\le -\log^- |\det A| + \log(m!) + (m-1) \log^+ \left(\max_{1 \le i,j \le m} |A_{ij}| \right) \\
&\le -\log^- |\det A| + \log(m!) + (m-1) \log^+ \|A\|.
\end{aligned}$$

□

COROLLARY 3.1. *A cocycle dual to a measurable cocycle with the values in $SL(m)$ is always measurable.*

In this section it would be convenient for us to enumerate Lyapunov exponents in ascending order.

LEMMA 3.2. *Suppose that a cocycle A and its dual cocycle ${}^{t}A^{-1}$ are both measurable. Let $\theta_1(A) \le \theta_2(A) \le \cdots \le \theta_m(A)$ be Lyapunov exponents of the cocycle A. Then the Lyapunov exponents of the dual cocycle are as follows:*

$$\begin{aligned} \theta_1({}^{t}A^{-1}) &= -\theta_m(A) \\ \theta_2({}^{t}A^{-1}) &= -\theta_{m-1}(A) \\ \cdots &= \cdots \\ \theta_m({}^{t}A^{-1}) &= -\theta_1(A). \end{aligned}$$

Note that we do not require ergodicity of the corresponding space, so the Lyapunov exponents may depend on the point in the nonergodic case.

Proof.

$$\begin{aligned} \theta_i(A) &= \lim_{k\to+\infty} \frac{1}{2k} \log(i\text{th eigenvalue of } ({}^{t}A^{(k)}) \cdot A^{(k)}) \\ \theta_i({}^{t}A^{-1}) &= \lim_{k\to+\infty} \frac{1}{2k} \log(i\text{th eigenvalue of } (A^{(k)})^{-1} \cdot ({}^{t}A^{(k)})^{-1}) \\ &= \lim_{k\to+\infty} \frac{1}{2k} \log(((m-i+1)\text{th eigenvalue of } {}^{t}A^{(k)} \cdot A^{(k)})^{-1}) \\ &= -\lim_{k\to+\infty} \frac{1}{2k} \log((m-i+1)\text{th eigenvalue of } ({}^{t}A^{(k)}) \cdot A^{(k)}) \\ &= -\theta_{m-i+1}(A). \end{aligned}$$

When speaking about the 'ith eigenvalue' of a symmetric matrix, we assume the eigenvalues are enumerated in the same (ascending or descending) order. □

3.1. *Lyapunov exponents for the annihilator of a subspace.* Consider a map $g : Y \to Y$ preserving a probability measure on the space Y, and consider a measurable cocycle $A(y)$, $y \in Y$, with the values in the group $GL(m)$. Suppose that the dual cocycle is also measurable, and that a point y is a regular point for the cocycle A in the sense of the multiplicative ergodic theorem. Let $K \subset \mathbb{R}^m_{(y)}$ be some linear subspace. We will improve Lemma 3.2 by establishing the relation between subcollections of Lyapunov exponents of the cocycles A and ${}^{t}A^{-1}$ corresponding to the subspaces K and $\mathrm{Ann}(K)$ respectively. First we consider a special case.

Let

$$\theta_{(1)}(A) < \theta_{(2)}(A) < \cdots < \theta_{(q)}(A)$$

be the Lyapunov exponents of the cocycle A at the point y now repeated without multiplicities.

For any vector $v \in \mathbb{R}^m_{(y)}$ and any covector $h \in (\mathbb{R}^m_{(y)})^*$ denote

$$\theta(v) := \lim_{k\to\infty} \frac{1}{k} \log \|A^{(k)} v\|, \quad \theta^*(h) := \lim_{k\to\infty} \frac{1}{k} \log \|({}^{t}A^{(k)})^{-1} h\|.$$

The assumption that y is a regular point means that for any nonzero v and h the limits above exist; moreover, the value $\theta(v)$ equals one of the $\theta_{(i)}$, where $1 \le i \le q$. Lemma 3.2 implies that for any covector h the value $\theta^*(h)$ equals one of the $-\theta_{(i)}$, where $1 \le i \le q$.

Let

$$\mathcal{F} = (0 = F_0 \subset F_1 \subset \cdots \subset F_q = \mathbb{R}^m)$$

be the flag of linear subspaces in $\mathbb{R}^m_{(y)}$ furnished by the cocycle A. Here $F_i = \{v \in \mathbb{R}^m \mid \theta(v) \le \theta_{(i)}\}$, for $1 \le i \le q$. We will need the following notation:

$$d(\mathcal{F}) := (d_1(\mathcal{F}), \ldots, d_q(\mathcal{F})), \quad \text{where } d_i(\mathcal{F}) := \dim F_i - \dim F_{i-1}, \text{ for } 1 \le i \le q. \tag{3.1}$$

Obviously, $d_i(\mathcal{F})$ is exactly the multiplicity of the Lyapunov exponent $\theta_{(i)}$. Let

$$\mathcal{H} = (0 = H_0 \subset H_1 \subset \cdots \subset H_q = (\mathbb{R}^m)^*)$$

be the flag corresponding to the dual cocycle. Lemma 3.2 guarantees that it really has the same number of components. Moreover, we have the following lemma.

LEMMA 3.3. *Flags $\mathcal{F}$ and $\mathcal{H}$ are dual to each other:*

$$H_{q-i} = \operatorname{Ann}(F_i) \quad \textit{for } 0 \le i \le q.$$

Proof. Let $v \in F_i$ and $h \in (\mathbb{R}^m)^*$ have non-vanishing pairing $\langle v, h\rangle \neq 0$. This implies that

$$\langle A^{(k)} v, ({}^{t}A^{(k)})^{-1} h\rangle = \langle v, h\rangle \neq 0. \tag{3.2}$$

Since $v \in F_i$ we have $\theta(v) \le \theta_{(i)}$. By (3.2) we see that $\theta^*(h) \ge -\theta_{(i)}$. Hence if for some covector h_0 we have $\theta^*(h_0) < -\theta_{(i)}$, then $h_0 \in \operatorname{Ann}(F_i)$. Note that Lemma 3.2 implies that for any $h_0 \in H_{q-i}$ we have $\theta^*(h_0) < -\theta_i$. Hence $H_{q-i} \subseteq \operatorname{Ann}(F_i)$. On the other hand, Lemma 3.2 implies that H_{q-i} has complimentary dimension to F_i, and hence $H_{q-i} = \operatorname{Ann}(F_i)$. □

Now consider a linear subspace $K \subset \mathbb{R}^m_{(y)}$. Denote by

$$\mathcal{K} = (0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_q = K)$$

the induced flag in K obtained as $K_i := K \bigcap F_i$. The vector $d(\mathcal{K}) = (d_1(\mathcal{K}), \ldots, d_q(\mathcal{K}))$ (see (3.1)) provides us with induced multiplicities of the Lyapunov exponents $\theta_{(1)} < \cdots < \theta_{(q)}$ on K. Consider the dual flag in $\operatorname{Ann}(K)$:

$$\begin{aligned}(0 = (\operatorname{Ann}(F_q) \cap \operatorname{Ann}(K)) &\subseteq (\operatorname{Ann}(F_{q-1}) \cap \operatorname{Ann}(K)) \subseteq \ldots \\ \cdots &\subseteq (\operatorname{Ann}(F_0) \cap \operatorname{Ann}(K)) = \operatorname{Ann}(K)).\end{aligned}$$

Combining Lemma 3.2 with Lemma 3.3, and with elementary considerations from linear algebra we get the following.

LEMMA 3.4. *A subcollection of the Lyapunov exponents corresponding to the subspace* $\operatorname{Ann}(K) \in (\mathbb{R}^m)^*_{(y)}$ *is obtained by taking Lyapunov exponents $-\theta_{(i)}$ with multiplicities $d_i(\mathcal{F}) - d_i(\mathcal{K})$, $1 \le i \le q$. In other words, the multiplicity of the Lyapunov exponent $-\theta_{(i)}$ on the annihilator* $\operatorname{Ann}(K)$ *is complementary to the multiplicity $d_i(\mathcal{K})$ of the Lyapunov* *exponents $\theta_{(i)}$ on K with respect to the total multiplicity $d_i(\mathcal{F})$ of $\theta_{(i)}$ on the whole space $\mathbb{R}^m_{(y)}$.*

The subspace of all covectors $h \in \operatorname{Ann}(K)$ such that $\theta^*(h) < -\theta_{(i)}$ coincides with $\operatorname{Ann}(F_i) \bigcap \operatorname{Ann}(K)$.

4. *Lyapunov exponents of B^{-1}, ${}^{t}B$ and ${}^{t}B|_{\operatorname{Ann}(\lambda)}$*

In [**11**] we proved the following lemma (by analogy with the proof of a similar statement in [**8**]).

LEMMA 4.1. *The non-negative function*

$$\log\left(\max_{1\leq i,j\leq m} B_{ij}(\lambda,\pi)\right)$$

on $\Delta^{m-1} \times \mathfrak{R}$ is integrable with respect to μ.

Taking into consideration that $\det B(\lambda, \pi) = 1$, we can use Corollary 3.1 to obtain the following lemma.

LEMMA 4.2. *Cocycles $B^{-1}(\lambda, \pi)$ and ${}^{t}B(\lambda, \pi)$ are measurable with respect to the measure μ.*

Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m$ be a collection of Lyapunov exponents of the cocycle $B^{-1}(\lambda, \pi)$. Recall the following theorem from [**11**].

THEOREM 2. *The middle $m - 2g$ Lyapunov exponents of the cocycle $B^{-1}(\lambda, \pi)$ are equal to zero:*

$$\theta_{g+1} = \theta_{g+2} = \cdots = \theta_{m-g} = 0.$$

The remaining $2g$ Lyapunov exponents are distributed into pairs:

$$\theta_k = -\theta_{m-k+1} \qquad \text{for } k = 1, \ldots, g.$$

Here $g = g(\mathfrak{R}) \leq m/2$ is an integer which is defined by the Rauzy class $\mathfrak{R}(\pi)$ of the permutation π. (Actually, g is the genus of corresponding surface, see [**8**].) Combining the above theorem with Lemma 3.2 we get the following.

COROLLARY 4.1. *Lyapunov exponents of the cocycles $B^{-1}(\lambda, \pi)$ and $B^T(\lambda, \pi)$ coincide.*

We will also need the following lemma, which follows immediately from a combination of general results in [**10**, §6] with Lemma 5.2.

LEMMA 4.3. *The multiplicity of the largest Lyapunov exponent $\theta_1(B^{-1})$ is equal to one;*

$$\theta_1(B^{-1}) > \theta_2(B^{-1}).$$

We recall that the cocycle $B^{-1}(\lambda, \pi)$ has a nice invariant one-dimensional subbundle corresponding to the smallest eigenvalue $-\theta_1$. The fiber of this subbundle over a point (λ, π) is just $\langle\lambda\rangle_{\mathbb{R}}$, i.e. it is spanned by the vector λ.

Consider the vector subbundle $\operatorname{Ann}(\lambda) \subset (\mathbb{R}^m)^*$. Since the one-dimensional subbundle $\langle\lambda\rangle_{\mathbb{R}}$ is invariant under the action of $(\mathcal{G}, B^{-1})$, we see that the annihilator $\operatorname{Ann}(\lambda)$ is invariant under the action of $(\mathcal{G}, {}^{t}B)$. Consider the restriction ${}^{t}B|_{\operatorname{Ann}(\lambda)}$ of the cocycle ${}^{t}B(\lambda, \pi)$ to this subbundle. Lemma 3.4 implies the following statement.

PROPOSITION 5. *The restriction* ${}^{t}B(\lambda,\pi)|_{\mathrm{Ann}(\lambda)}$ *of the cocycle* ${}^{t}B(\lambda,\pi)$ *to the annihilator* $\mathrm{Ann}(\lambda)$ *has the following Lyapunov exponents:*

$$\theta_2(B^{-1}) \geq \theta_3(B^{-1}) \geq \cdots \geq -\theta_2(B^{-1}) \geq -\theta_1(B^{-1}).$$

(The collection is obtained from the collection of Lyapunov exponent of the cocycle $B^{-1}(\lambda,\pi)$ *by omitting the largest Lyapunov exponent* $\theta_1(B^{-1})$*.)*

Now we can easily prove Proposition 3. Consider the covectors $e_i - \lambda_i e_0$, $1 \leq i \leq m$, where $e_0 = e_1 + \cdots + e_m$. We have seen (see (2.10)) that $e_i - \lambda_i e_0 \in \mathrm{Ann}(\lambda)$ for all $1 \leq i \leq m$. Hence for all $1 \leq i \leq m$ we get

$$\lim_{k\to+\infty} \frac{1}{k} \log \|{}^{t}B^{(k)}(\lambda,\pi)\cdot(e_i - \lambda_i e_0)\| = \theta_q,$$

where $q \geq 2$ may depend on i and on the point (λ,π). We recall that the Oseledetc theorem [**6**] guarantees the existence of the above limits for almost all $(\lambda,\pi) \in \Delta^{m-1}\times\mathfrak{R}$. Note that since the whole collection of covectors generates $\mathrm{Ann}(\lambda)$ there is at least one (actually at least two) indices i, *a priori* depending on the point (λ,π), for which $q = q(i,\lambda,\pi) = 2$. (Presumably $q = 2$ for all indices i for almost all points (λ,π).)

Proposition 3 is proved. □

5. *Why the columns of* $B^{(k)}$ *are asymptotically well-distributed*

In this section we prove Proposition 2. The title of this section is motivated by the paper [**3**] where a somewhat similar question is solved for the Rauzy induction.

Fix the Rauzy class $\mathfrak{R} = \mathfrak{R}(\pi)$ of an irreducible permutation $\pi \in \mathfrak{S}^0_m$. From now on we will denote the Lyapunov exponents $\theta_i(B^{-1})$ of the cocycle B^{-1} on $\Delta^{m-1}\times\mathfrak{R}$ by $\theta_i(\mathfrak{R})$, or sometimes just by θ_i.

LEMMA 5.1. *For almost all* $(\lambda,\pi) \in \Delta^{m-1}\times\mathfrak{R}$

$$\lim_{k\to+\infty} \frac{1}{k} \log \left(\max_{1\leq j\leq m} \|B^{(k)}(\lambda,\pi)\cdot e_j\| \right) = \theta_1(\mathfrak{R}).$$

Proof.

$$\begin{aligned} \lim_{k\to+\infty} \frac{1}{k} \log \left(\max_{1\leq j\leq m} \|B^{(k)}(\lambda,\pi)\cdot e_j\| \right) &= \lim_{k\to+\infty} \frac{1}{k} \log \left(\max_{1\leq i,j\leq m} |B^{(k)}_{ij}(\lambda,\pi)| \right) \\ &= \lim_{k\to+\infty} \frac{1}{k} \log \|{}^{t}B^{(k)}(\lambda,\pi)\| = \theta_1(\mathfrak{R}), \end{aligned}$$

where the last equality is due to Corollary 4.1 □

COROLLARY 5.1. *For almost all* $(\lambda,\pi) \in \Delta^{m-1}\times\mathfrak{R}$

$$\limsup_{k\to+\infty} \frac{1}{k} \log \|B^{(k)}(\lambda,\pi)\cdot e_j\| \leq \theta_1(\mathfrak{R}) \quad \text{for } 1 \leq j \leq m.$$

Consider the subset

$$V_l := \{(\lambda,\pi) \in \Delta^{m-1}\times\mathfrak{R} \mid B^{(l)}_{ij}(\lambda,\pi) > 0 \text{ for all } 1 \leq i,j \leq m\}. \tag{5.1}$$

Using finiteness of the measure μ we get the trivial proof of the following well known fact.

LEMMA 5.2. *For almost all points* (λ, π) *the matrix* $B^{(l)}(\lambda, \pi)$ *becomes strictly positive after a sufficiently large number* l *of iterations of* $\mathcal{G}$*:*

$$\lim_{l\to+\infty} \mu(V_l) = 1.$$

Proof. One can easily choose a set $\tilde{V}$ of positive measure such that for some constant l_0 and for any $(\lambda, \pi) \in \tilde{V}$ the matrix $B^{(l_0)}(\lambda, \pi)$ is strictly positive. Let

$$W_l := \tilde{V} \cup \mathcal{G}^{-1}(\tilde{V}) \cup \cdots \cup \mathcal{G}^{-l}(\tilde{V}).$$

The ergodicity of $\mathcal{G}$ implies

$$\lim_{l\to+\infty} \mu(W_l) = 1.$$

By the definition of $\tilde{V}$ we have the inclusion $W_l \subseteq V_{l+l_0}$. □

PROPOSITION 6. *For any* $\delta > 0$ *and for almost all* $(\lambda, \pi) \in \Delta^{m-1} \times \mathfrak{R}$

$$\liminf_{k\to+\infty} \frac{1}{k} \log \|B^{(k)}(\lambda, \pi) \cdot e_j\| \geq \theta_1(\mathfrak{R}) - \delta \text{ for all } 1 \leq j \leq m.$$

Proof. Choose positive $\varepsilon < 1$. Choose $l = l(\varepsilon)$ such that $\mu(V_l) \geq 1 - \varepsilon/2$, where the set V_l is defined by (5.1). Consider the set

$$\begin{aligned} Z = Z(\varepsilon) \quad := \quad & \left\{(\lambda, \pi) \mid \lim_{k\to+\infty} \frac{1}{k} \log \max_{1\leq j\leq m} \|B^{(k)} e_j\| = \theta_1(\mathfrak{R})\right\} \\ & \bigcap \left\{(\lambda, \pi) \mid \lim_{n\to+\infty} \frac{1}{n} \sum_{q=0}^{n-1} \chi_{V_l}(\mathcal{G}^{(q)}(\lambda, \pi)) = \mu(V_l)\right\}. \end{aligned} \tag{5.2}$$

By Lemma 5.1 the first set in the definition above is the set of full measure; due to the ergodicity of $\mathcal{G}$ the second set is also the set of full measure. Hence $\mu(Z) = 1$. We will prove that for any point of the set $Z(\varepsilon)$, where $\varepsilon = \delta/2(1 + \theta_1)$, the requirements of Proposition 6 are valid.

Fix $(\lambda, \pi) \in Z$. By the construction of Z there exists $N_1 = N_1(\lambda, \pi, \varepsilon)$ such that for any $n \geq N_1$ the inequality

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi_{V_l}(\mathcal{G}^{(k)}(\lambda, \pi)) \geq \mu(V_l) - \frac{\varepsilon}{2} \geq 1 - \varepsilon$$

is valid. Hence for any $n \geq N_1 + l$ one can find $k(n)$ satisfying $(n-l)(1-\varepsilon) \leq k(n) \leq n-l$ such that $\mathcal{G}^{(k(n))}(\lambda, \pi) \in V_l$.

By the construction of the set Z one can find $N_2 = N_2(\lambda, \pi, \varepsilon)$ such that for any $k \geq N_2$ the following inequality is valid:

$$\left|\frac{1}{k} \log \max_{1\leq j\leq m} \|B^{(k)}(\lambda, \pi) e_j\| - \theta_1\right| \leq \varepsilon. \tag{5.3}$$

Hence choosing $N_3 = l + \max(N_2/(1 - \varepsilon), N_1)$ one guarantees for $n \geq N_3$ the existence of $k(n)$ satisfying $(n - l)(1 - \varepsilon) \leq k(n) \leq n - l$ such that $\mathcal{G}^{(k(n))}(\lambda, \pi) \in V_l$, and such that for $k(n)$ inequality (5.3) is valid.

Note that

$$B^{(n)}(\lambda,\pi) = B^{(k(n))}(\lambda,\pi)\cdot B^{(n-k(n))}(\mathcal{G}^{(k(n))}(\lambda,\pi)).$$

By the construction of $k(n)$ we have $\mathcal{G}^{(k(n))}(\lambda,\pi) \in V_l$ and $n-k(n) \geq l$. Hence the matrix $B^{(n-k(n))}(\mathcal{G}^{(k(n))}(\lambda,\pi))$ is a strictly positive integer matrix. Hence

$$\begin{aligned}\min_{1\le j\le m} \log\|B^{(n)}(\lambda,\pi)\cdot e_j\| &\geq \max_{1\le j\le m} \log\|B^{(k(n))}(\lambda,\pi)\|\\ &\geq (\theta_1-\varepsilon)k(n) \geq (\theta_1-\varepsilon)(n-l)(1-\varepsilon).\end{aligned}$$

Hence for $n \geq N_3$ we get

$$\min_{1\le j\le m} \frac{1}{n}\log\|B^{(n)}(\lambda,\pi)\cdot e_j\| \geq \frac{1}{n}(\theta_1-\varepsilon)(n-l)(1-\varepsilon) \geq \theta_1-\varepsilon-\theta_1\varepsilon-\frac{\theta_1 l}{n}.$$

Choosing $\varepsilon = \delta/2(1+\theta_1)$, and $N_4 = N_4(\lambda,\pi,\delta) = \max(N_3, 2\theta_1 l/\delta)$ we conclude that for the pair (λ,π) the desired inequality is valid for all $n \geq N_4$. □

Consider the set

$$\tilde{Z} = \bigcap_{n=1}^{+\infty} Z(\varepsilon_n) \quad \text{where } \varepsilon_n = \frac{1}{2n(1+\theta_1)}$$

(see (5.2) for the definition of $Z(\varepsilon)$). Then $\mu(\tilde{Z}) = 1$. Consider the intersection of $\tilde{Z}$ with the set of full measure from Corollary 5.1. For any point of this resulting set of full measure we have

$$\lim_{k\to+\infty} \frac{1}{k}\log\|B^{(k)}\cdot e_j\| = \theta_1$$

for any $1 \leq j \leq m$. Proposition 2 is proved. □

6. *Bound for deviation*

6.1. *Lower bound.* Let $x = 0$ be the left endpoint of the unit interval X.

PROPOSITION 7. *For almost all interval exchange transformations $T(\lambda,\pi)$, $(\lambda,\pi) \in \Delta\times\mathfrak{R}$, the following relation is valid:*

$$\max_{1\le i\le m}\limsup_{n\to+\infty} \frac{\log|\chi_i(0,n)-\lambda_i n|}{\log n} \geq \frac{\theta_2}{\theta_1}.$$

We will need several lemmas. Denote by $v_1 \in \mathbb{R}^m$ the vector $(1,0,\ldots,0)$.

LEMMA 6.1. *For almost all $(\lambda_0,\pi_0) \in \Delta\times\mathfrak{R}$ one can find a collection of integers $0 = l_1 < \cdots < l_m$ such that the vectors $v_1, B^{(l_2)}(\lambda_0,\pi_0)\cdot v_1,\ldots,B^{(l_m)}(\lambda_0,\pi_0)\cdot v_1$ are linearly independent.*

Proof. Assume any m distinct elements from the sequence

$$v_1,\, B(\lambda_0,\pi_0)\cdot v_1,\, B^{(2)}(\lambda_0,\pi_0)\cdot v_1,\ldots$$

are linearly dependent. It means that for any $l > 0$ all vectors $v_1, Bv_1,\ldots,B^{(l)}v_1$ belong to some fixed linear subspace $\mathcal{L} \subset \mathbb{R}^m$ of nontrivial codimension. Let it be spanned by the first k_0 vectors of our sequence. Note that all these vectors have integer components, and hence the subspace $\mathcal{L}$ is integer.

We may assume that all components of the vector λ_0 are rationally independent (which is a generic situation). Hence it cannot belong to the integer linear subspace $\mathcal{L}$ of nontrivial codimension, which means that the distance from the point λ_0 to the subspace $\mathcal{L}$ is positive.

Note that the map $v \mapsto B^{(k)}v/\|B^{(k)}v\|$ maps a standard simplex to a small simplex containing the point λ_0. For uniquely ergodic interval exchange transformations (λ_0, π_0), the size of this small simplex tends to zero as k grows; see [**8**] or [**3**]. Hence the distance between $B^{(k)}v_1/\|B^{(k)}v_1\|$ and λ_0 becomes arbitrarily small, and hence $B^{(k)}v_1$ is outside of the subspace $\mathcal{L}$ for sufficiently large values of k. This means that our assumption that $\mathcal{L}$ has nontrivial codimension leads to a contradiction. □

We will also need the following trivial lemma.

LEMMA 6.2. *Consider a basis of vectors $w_1, \ldots, w_m$ in $\mathbb{R}^m$. There is a positive constant $c > 0$ (depending on the basis) such that for any linear function $f \in \mathbb{R}^{m*}$ the following bound is valid:*

$$\max_{1\le i\le m} |\langle f, w_i\rangle| \ge c\|f\|.$$

Now we can prove the following.

LEMMA 6.3. *For almost all $(\lambda, \pi) \in \Delta \times \mathfrak{R}$ the following relation is valid:*

$$\max_{1\le i\le m} \limsup_{k\to+\infty} \frac{1}{k} \log |\langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1\rangle| = \theta_2.$$

We recall that a covector e_i has zero components except for the ith one which is equal to 1. We define a covector e_0 as $e_0 = (1, 1, \ldots, 1)$, and a vector v_1 as $v_1 = (1, 0, 0, \ldots, 0)$.

Proof. Proposition 3 immediately implies that

$$\max_{1\le i\le m} \limsup_{k\to+\infty} \frac{1}{k} \log |\langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1\rangle| \le \theta_2. \tag{6.1}$$

According to Proposition 3 for almost all (λ, π) and for at least one index i (presumably for all indices) the equality is valid:

$$\lim_{k\to+\infty} \frac{1}{k} \log \|{}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0)\| = \theta_2. \tag{6.2}$$

Fix this index i. Note that

$$\begin{aligned}
&\langle {}^{\mathrm{t}}B^{(k+l)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1\rangle \\
&\quad= \langle {}^{\mathrm{t}}B^{(l)}(\mathcal{G}^{(k)}(\lambda, \pi)) \cdot {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1\rangle \\
&\quad= \langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), B^{(l)}(\mathcal{G}^{(k)}(\lambda, \pi)) \cdot v_1\rangle.
\end{aligned} \tag{6.3}$$

Consider any point (λ_0, π_0) for which Lemma 6.1 is valid. By construction, the vectors $w_1 := v_1$, $w_2 := B^{(l_2)}(\lambda_0, \pi_0) \cdot v_1, \ldots, w_m := B^{(l_m)}(\lambda_0, \pi_0) \cdot v_1$ are linearly independent. Let c be the positive constant from Lemma 6.2. Let $\Delta_0 \times \pi_0$ be a simplex containing the point (λ_0, π_0), and sharing the same matrices $B^{(l_r)}(\lambda', \pi_0) = B^{(l_r)}(\lambda_0, \pi_0)$ for all $(\lambda', \pi_0) \in \Delta_0 \times \pi_0$ and for all $r = 2, \dots, m$. Since $\mu(\Delta_0 \times \pi_0) > 0$, and since the map $\mathcal{G}$ is ergodic, a trajectory of almost any point (λ, π) under the action of the map $\mathcal{G}$ will visit our subsimplex infinitely many times. Let k obey $\mathcal{G}^{(k)}(\lambda, \pi) \in \Delta_0 \times \pi_0$. The relation (6.3) means that for any j, $1 \le j \le m$, we have

$$\langle {}^{\mathrm{t}}B^{(k+l_j)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1 \rangle = \langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), w_j \rangle$$

and, hence, according to Lemma 6.2 we get

$$\max_{1 \le r \le m} |\langle {}^{\mathrm{t}}B^{(k+l_r)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1 \rangle| \ge c \cdot \|{}^{\mathrm{t}}B^{(k)}(\lambda, \pi)(e_i - \lambda_i e_0)\|.$$

Since all l_r, $1 \le r \le m$, and c are fixed, while k gets arbitrary large, the latter relation in combination with (6.1) and (6.2) completes the proof of the lemma. □

Now everything is prepared to prove Proposition 7. We will use very specific 'times' $n = n_1^{(k)}(\lambda, \pi)$, see (2.3), to prove the proposition. According to (2.5) and (2.9)

$$\chi_i(X, 0, T, n_1^{(k)}) - \lambda_i n_1^{(k)} = \langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1 \rangle.$$

Recall that according to (2.3) we have $n_1^{(k)}(\lambda, \pi) = \|B^{(k)}(\lambda, \pi) e_1\|$. We get

$$\begin{aligned}
&\max_{1 \le i \le m} \limsup_{n \to +\infty} \frac{\log |\chi_i(X, 0, T, n) - \lambda_i n|}{\log n} \\
&\ge \max_{1 \le i \le m} \limsup_{k \to +\infty} \frac{\log |\chi_i(X, 0, T, n_1^{(k)}) - \lambda_i n_1^{(k)}|}{\log n_1^{(k)}} \\
&= \max_{1 \le i \le m} \limsup_{k \to +\infty} \frac{\log |\langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1 \rangle|}{\log \|B^{(k)}(\lambda, \pi) \cdot v_1\|} \\
&= \max_{1 \le i \le m} \limsup_{k \to +\infty} \frac{1}{k} \log |\langle {}^{\mathrm{t}}B^{(k)}(\lambda, \pi) \cdot (e_i - \lambda_i e_0), v_1 \rangle| \\
&\quad \cdot \left(\lim_{k \to +\infty} \frac{1}{k} \log \|B^{(k)}(\lambda, \pi) \cdot v_1\| \right)^{-1} = \frac{\theta_2}{\theta_1}.
\end{aligned}$$

In the last two equalities above we used Proposition 2 and Lemma 6.3. Proposition 7 is proved. □

Note that any point x of the form $T^n(0)$, $n \in \mathbb{Z}$, would have the same property.

6.2. *Choice of proper recurrence times.* From now on we will always assume that $m \ge 3$. This implies that $\theta_2 \ge 0$, which immediately follows from Theorem 2.

For proper recurrence times we will choose something like arithmetic progression with an additional property determined by the following proposition.

PROPOSITION 8. *For almost all* $(\lambda, \pi) \in \Delta \times \mathfrak{R}$ *and for any* $\varepsilon > 0$, $\delta > 0$, $r \in \mathbb{N}$, *there exists* $N = N((\lambda, \pi), \varepsilon, \delta, r)$ *such that for any* $n > N$ *one can choose the sequence of* *integers* $0 = n_0 < n_1 < \cdots < n_r = n$ *with the following property. For any* $1 \le l \le r$,

$$\max_{1\le j\le m}\left|\frac{1}{n_l - n_{l-1}}\log\|B^{(n_l-n_{l-1})}(\lambda^{(n_{l-1})},\pi^{(n_{l-1})})\cdot e_j\| - \theta_1(\mathfrak{R})\right| \le \varepsilon$$

$$\max_{1\le i\le m}\frac{1}{n_l - n_{l-1}}\log\left\|B^{(n_l-n_{l-1})}(\lambda^{(n_{l-1})},\pi^{(n_{l-1})})\cdot\left(e_i - \frac{\lambda_i^{(n_{l-1})}}{\|\lambda^{(n_{l-1})}\|}e_0\right)\right\| \le \theta_2(\mathfrak{R}) + \varepsilon.$$

Moreover, for any $0 \le l \le r$

$$\left|\frac{n_l}{n} - \frac{l}{r}\right| \le \delta.$$

Proof. First let us construct a set of full measure for which we then prove the proposition.

For any $\varepsilon > 0$ and $N \in \mathbb{N}$ consider the set

$$O_N(\varepsilon) := \left\{(\lambda,\pi)\;\middle|\; \begin{array}{ll} \max\limits_{1\le i\le m}\dfrac{1}{k}\log\|{}^{\mathrm{t}}B^{(k)}(\lambda,\pi)\cdot(e_i-\lambda_i e_0)\| \le \theta_2+\varepsilon & \text{for } k\ge N \\ \max\limits_{1\le i\le m}\left|\dfrac{1}{k}\log\|B^{(k)}(\lambda,\pi)\cdot e_j\| - \theta_1\right| \le \varepsilon & \text{for } k \ge N \end{array}\right\}.$$

By definition

$$O_N(\varepsilon) \subseteq O_{N+1}(\varepsilon)$$

and for any $\varepsilon_1 < \varepsilon_2$

$$O_N(\varepsilon_1) \subseteq O_N(\varepsilon_2).$$

By Propositions 2 and 3 for any $\varepsilon > 0$

$$\mu(O_N(\varepsilon)) \to 1 \quad \text{as } N \to +\infty.$$

In particular, for any $\varepsilon > 0$ one can find $K(\varepsilon)$ such that for any $N \ge K(\varepsilon)$ the measure $\mu(O_N(\varepsilon))$ is positive. Since the sets

$$O_\infty(\varepsilon) := \bigcup_{N=1}^{+\infty} O_N(\varepsilon)$$

have full measure, the intersection

$$O := \bigcap_{n=1}^{+\infty} O_\infty\left(\frac{1}{n}\right)$$

has full measure as well:

$$\mu(O) = 1.$$

For any $\varepsilon > 0$ and $N \ge K(\varepsilon)$ define the set

$$U_N(\varepsilon) := \left\{(\lambda,\pi)\;\middle|\; \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\chi_{O_N(\varepsilon)}(\mathcal{G}^{(k)}(\lambda,\pi)) = \mu(O_N(\varepsilon))\right\}.$$

For any $\varepsilon > 0$ and $N \ge K(\varepsilon)$ we have

$$\mu(U_N(\varepsilon)) = 1.$$

Hence the set

$$U := \bigcap_{n=1}^{+\infty} \bigcap_{k=K(1/n)}^{+\infty} U_k\left(\frac{1}{n}\right)$$

is the set of full measure

$$\mu(U) = 1.$$

From now on we consider only those pairs (λ, π) which belong to $O \cap U$, and which correspond to uniquely ergodic interval exchange transformations. We also assume that for any $k > 0$ no components of $\lambda^{(k)}$ are equal to zero. Note that we have confined ourselves to the set of full measure in $\Delta^{m-1} \times \mathfrak{R}$. We will prove Proposition 8 assuming (λ, π) is from the set thus defined.

Fix (λ, π) from our set of full measure. Choose arbitrary numbers $0 < \varepsilon < \theta_1$, $0 < \delta$, and $r \in \mathbb{N}$. Reducing δ if necessary, we can get

$$r \leq \frac{1}{4\delta}. \tag{6.4}$$

Let $q = [1/\varepsilon] + 1$. We will always denote the integer part of a number by square brackets. Since $(\lambda, \pi) \in O$ and hence $(\lambda, \pi) \in O_\infty(1/q)$ there is some $N = N(q, \lambda, \pi)$ such that $(\lambda, \pi) \in O_N(1/q)$. Since $1/q < 1/\varepsilon$, we get $O_N(1/q) \subseteq O_N(\varepsilon)$. Enlarge N if necessary to ensure

$$\mu(O_N(1/q)) \geq 1 - \delta. \tag{6.5}$$

By our choice of N we also have $N \geq K(1/q)$. Since $(\lambda, \pi) \in U$ we conclude that $(\lambda, \pi) \in U_N(1/q)$. Hence, by the definition of $U_N(1/q)$ and by (6.5) we can find $M \in \mathbb{N}$ such that for any $n \geq M$

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{O_N(\varepsilon)}(\mathcal{G}^{(k)}(\lambda, \pi)) &\geq \frac{1}{n}\sum_{k=0}^{n-1} \chi_{O_N(1/q)}(\mathcal{G}^{(k)}(\lambda, \pi)) \\ &> \mu(O_N(1/q)) - \delta = 1 - 2\delta. \end{aligned} \tag{6.6}$$

Enlarging M if necessary, we may assume $1/M \leq \delta$.

Now we need the following technical lemma.

LEMMA 6.4. *Consider a piece of trajectory* (λ, π), $\mathcal{G}(\lambda, \pi), \ldots, \mathcal{G}^{(n-1)}(\lambda, \pi)$, *where* $n > M$. *There exist numbers* $0 = n_0 < n_1 < n_2 < \cdots < n_{r-1} < n_r = n$ *such that*

$$\left|\frac{n_l}{n} - \frac{l}{r}\right| \leq \delta \quad \textit{and} \quad \mathcal{G}^{(n_l)}(\lambda, \pi) \in O_N(\varepsilon), \quad l = 0, \ldots, r-1.$$

Proof. The auxiliary number $n_0 = 0$ obviously satisfies the desired requirements since by construction $(\lambda, \pi) \in O_N(\varepsilon)$. Take $1 \leq l \leq n-1$. Take the point $(l/r)n$ on the real line $\mathbb{R}$ and consider an interval of length $2\delta n$ centered at $(l/r)n$. We claim that our interval of length $2\delta n$ is strictly inside the interval $[0, n-1]$. Indeed, by (6.4) we have $n/r - \delta n > 0$ and $(n - n/r) + \delta n < n - \delta n$. Since $n > M \geq 1/\delta$ we get $(n - n/r) + \delta n \leq n - 1$. Now, by construction (see (6.6)) we have $\sum_{k=0}^{n-1} \chi_{O_N(\varepsilon)}(\mathcal{G}^{(k)}(\lambda, \pi)) > (1 - 2\delta)n$. Hence we are able to find at least one integer point n_l inside our interval such that $\mathcal{G}^{(n_l)}(\lambda, \pi) \in O_N(\varepsilon)$. The inequalities $n_l < n_{l+1}$ for $0 \leq l \leq r-1$ now follow from (6.4). □

To complete the proof of Proposition 8 let

$$L := \max([N/\delta], M) + 1,$$

where N and M are defined above. For any $n \geq L$ choose the collection n_l as in Lemma 6.4. Note that

$$n_l - n_{l-1} \geq \frac{n}{r} - 2\delta n \geq 4\delta n - 2\delta n \geq \delta L > \delta \frac{N}{\delta} = N, \tag{6.7}$$

where $1 \leq l \leq r$. Hence by the definition of the set $O_N(\varepsilon)$ our collection $n_0 < n_1 < \cdots < n_r$ satisfies all the requirements of the proposition. To complete the proof we just adjust the notation letting $N((\lambda, \pi), \varepsilon, \delta, r) := L$. □

6.3. *Proof of Theorem 1.* Consider the interval exchange transformation $T(\lambda, \pi)$, where (λ, π) satisfies the conditions of Proposition 8. Consider arbitrary $0 < \varepsilon < \theta_1$, $0 < \delta < 1/2$, and $r \in \mathbb{N}$. Take corresponding $N = N((\lambda, \pi), \varepsilon, \delta, r)$ as in Proposition 8. Enlarge N if necessary to make it obey the following technical conditions: $N \geq 2$, and

$$\begin{aligned} &\left(1 - \exp\left(-(\theta_1 - \varepsilon)(1 - 2\delta)\frac{N}{r}\right)\right)^{-1} \\ &\leq \quad \exp\left((\theta_1 + \varepsilon)(1 + 2\delta)\frac{N}{r}\right) - \exp\left((\theta_1 - \varepsilon)(1 + 2\delta)\frac{N}{r}\right). \end{aligned} \tag{6.8}$$

Choose a piece of trajectory $x, T(x), T(T(x)), \ldots, T^{t-1}(x)$ with

$$t > [\exp((\theta_1 - \varepsilon)N)] + 1.$$

To avoid confusion, from now on we will denote 'time' related to interval exchange transformations by t, possibly with subscripts. We will reserve k, n, and N for the 'time' related to the Gauss map. Let

$$n := \left[\frac{\log t}{\theta_1 - \varepsilon}\right] + 1. \tag{6.9}$$

Since $n > N$ we can choose a collection $0 = n_0 < n_1 < n_2 < \cdots < n_{r-1} < n_r = n$ satisfying the conditions of Proposition 8. Now everything is prepared for the r-step recursion (see Proposition 1). Let

$$k_l := n_l - n_{l-1} \quad \text{for } l = 1, \ldots, r.$$

Note that by the choice of n_l (see Proposition 8) for all $1 \leq l \leq r$ we have

$$\frac{n}{r}(1 - 2\delta) \leq k_l \leq \frac{n}{r}(1 + 2\delta) =: k. \tag{6.10}$$

Define

$$(\lambda_{(0)}, \pi_{(0)}) := (\lambda, \pi) \quad X_{(0)} := X \quad x_{(0)} := x \quad T_{(0)} := T \quad t_{(0)} := t.$$

Assuming that the point $(\lambda_{(l)}, \pi_{(l)})$ and the aggregate $X_{(l)}, x_{(l)}, T_{(l)}, t_{(l)}$ are defined for $0 \leq l < r - 1$, define

$$(\lambda_{(l+1)}, \pi_{(l+1)}) := \mathcal{G}^{k_{l+1}}((\lambda_{(l)}, \pi_{(l)})).$$

Define

$$X_{(l+1)} := X_{(l)}^{(k_{l+1})},$$

i.e. we define $X_{(l+1)}$ as the (non-normalized) interval obtained from the interval $X_{(l)}$ by k_l steps of the induction procedure corresponding to k_l iterations of the map $\mathcal{G}$ starting from the point $(\lambda_{(l)}, \pi_{(l)})$. Obviously $X_{(l+1)} \subset X_{(l)}$. Next define

$$x_{(l+1)} := T^{n^+}(x_{(l)}) \quad \text{where } n^+ = n^+(X_{(l)}, x_{(l)}, T_{(l)}, t_{(l)}, k_{l+1})$$

(see (2.1) for notation). Here $x_{(l+1)}$ is the point of the first visit of the trajectory $x_{(l)}$, $T_{(l)}(x_{(l)})$, $T_{(l)}^2(x_{(l)}), \ldots$ to the subinterval $X_{(l+1)}$. Further, define

$$T_{(l+1)} := T_{(l)}|_{X_{(l)}}$$

to be the induced interval exchange transformation on the subinterval $X_{(l+1)}$. Finally, define

$$t_{(l+1)} := \sum_{i=0}^{t_{(l)}-1} \chi_{X_{(l+1)}}(T_{(l)}^i(x_{(l)})),$$

i.e. $t_{(l+1)}$ is the number of visits of the trajectory $x_{(l)}, T_{(l)}(x_{(l)}), \ldots, T_{(l)}^{t_{(l)}-1}(x_{(l)})$ to the subinterval $X_{(l+1)}$.

By the choice of n_l provided by Proposition 8 for all $0 \le l \le r-1$, we have

$$\exp((\theta_1 - \varepsilon)k_{l+1}) \le \|B^{(k_{l+1})}(\lambda_{(l)}, \pi_{(l)}) \cdot e_j\| \le \exp((\theta_1 + \varepsilon)k_{l+1}), \quad 1 \le j \le m$$
$$\max_{1\le i\le m} \left\| B^{(k_{l+1})}(\lambda_{(l)}, \pi_{(l)}) \cdot \left(e_i - \frac{(\lambda_{(l)})_i}{\|\lambda_{(l)}\|} e_0 \right) \right\| \le \exp((\theta_2 + \varepsilon)k_{l+1}). \tag{6.11}$$

Hence, according to Proposition 1 we get

$$\begin{aligned} &|E_i(X_{(l)}, x_{(l)}, T_{(l)}, n_{(l)})| \\ &\quad \le \; 2 \cdot \exp((\theta_1 + \varepsilon)k_{l+1}) \\ &\qquad + m \cdot \exp((\theta_2 + \varepsilon)k_{l+1}) \cdot |E_i(X_{(l+1)}, x_{(l+1)}, T_{(l+1)}, n_{(l+1)})|. \end{aligned} \tag{6.12}$$

Inequalities (6.11) imply

$$\begin{aligned} t_{(l+1)} &\le 1 + \frac{t_{(l)}}{\min_{x\in X_{(l+1)}} \text{ first return time to } X_{(l+1)}} \\ &= 1 + \frac{t_{(l)}}{\min_{1\le j\le m} \|B^{(k_{l+1})}(\lambda_{(l)}, \pi_{(l)}) \cdot e_j\|} \\ &\le 1 + \frac{t_{(l)}}{\exp((\theta_1 - \varepsilon)k_{l+1})}. \end{aligned} \tag{6.13}$$

To complete the preparations consider the last term in our recursion. Use the obvious bound for it and then use (6.13) recursively:

$$\begin{aligned} &\max_{1\le i\le m} |E_i(X_{(r-1)}, x_{(r-1)}, T_{(r-1)}, t_{(r-1)})| \\ &\quad \le \; t_{(r-1)} \\ &\quad \le \; 1 + \frac{t_{(r-2)}}{\exp((\theta_1 - \varepsilon)k_{r-1})} \le \cdots \end{aligned}$$

$$\leq \quad 1+\exp(-(\theta_1-\varepsilon)k_{r-1})+\exp(-(\theta_1-\varepsilon)(k_{r-1}+k_{r-2}))+\cdots$$
$$+\exp(-(\theta_1-\varepsilon)(k_{r-1}+\cdots+k_2))+\frac{t_{(0)}}{\exp((\theta_1-\varepsilon)(k_{r-1}+\cdots+k_1))}. \tag{6.14}$$

Definition (6.9) implies that

$$\log t=\log t_{(0)}<(\theta_1-\varepsilon)n=(\theta_1-\varepsilon)(k_r+\cdots+k_1).$$

Hence, taking into consideration (6.10) and (6.8) we can continue (6.14) as follows:

$$\begin{aligned}
&\max_{1\leq i\leq m}|E_i(X_{(r-1)},x_{(r-1)},T_{(r-1)},t_{(r-1)})|\\
&\quad\leq \left(1-\exp\left(-(\theta_1-\varepsilon)(1-2\delta)\frac{n}{r}\right)\right)^{-1}+\exp((\theta_1-\varepsilon)k_r)\\
&\quad\leq \exp((\theta_1+\varepsilon)k).
\end{aligned}\tag{6.15}$$

Collecting inequalities (6.12) recurrently for $l=0,1,\ldots,r-2$, and completing with (6.15) we obtain

$$\begin{aligned}
\max_{1\leq i\leq m}|E_i(X,x,T,t)| \quad\leq\quad & 2\exp((\theta_1+\varepsilon)k_1)\\
&+m\exp((\theta_2+\varepsilon)k_1)\Bigg(2\exp((\theta_1+\varepsilon)k_2)\\
&+m\exp((\theta_2+\varepsilon)k_2)\Big(2\exp((\theta_1+\varepsilon)k_3)\\
&+\cdots\\
&+m\exp((\theta_2+\varepsilon)k_{r-2})\big(2\exp((\theta_1+\varepsilon)k_{r-1})\\
&+m\exp((\theta_2+\varepsilon)k_{r-1})\exp((\theta_1+\varepsilon)k)\big)\ldots\Big)\Bigg).
\end{aligned}$$

Multiply the last term by two. Replace all k_l by k; see (6.10). We have increased the value of the expression on the right-hand side of the inequality above. Note that we got nothing but a partial sum of the geometric progression:

$$\begin{aligned}
&\max_{1\leq i\leq m}|E_i(X,x,T,t)|\\
&\quad\leq 2\exp((\theta_1+\varepsilon)k)\cdot\Big(1+\exp((\theta_2+\varepsilon)k+\log m)\\
&\qquad+\exp(2((\theta_2+\varepsilon)k+\log m))+\cdots+\exp((r-1)((\theta_2+\varepsilon)k+\log m))\Big)\\
&\quad= 2\exp((\theta_1+\varepsilon)k)\frac{\exp(r((\theta_2+\varepsilon)k+\log m))-1}{\exp((\theta_2+\varepsilon)k+\log m)-1}.
\end{aligned}$$

Taking into consideration (6.10) and inequality $m\geq 3$, we obtain

$$\max_{1\leq i\leq m}|E_i(X,x,T,t)|\leq 2\exp((\theta_1+\varepsilon)\frac{n}{r}(1+2\delta))\cdot\exp(n(\theta_2+\varepsilon)(1+2\delta)+r\log m). \tag{6.16}$$

By definition (6.9) of n we have

$$\log t \geq (n-1)(\theta_1 - \varepsilon). \tag{6.17}$$

Combining inequalities (6.16) and (6.17) we obtain

$$\begin{aligned}
&\frac{\log \max_{1\leq i\leq m} |E_i(X, x, T, t)|}{\log t} \\
&\quad\leq \frac{\log 2 + (\theta_1+\varepsilon)\frac{n}{r}(1+2\delta) + n(\theta_2+\varepsilon)(1+2\delta) + r\log m}{(n-1)(\theta_1-\varepsilon)} \\
&\quad\leq \left(1+\frac{2}{N}\right)\left(\frac{\theta_2 + \left(\frac{\log 2}{N} + 2\delta\theta_2 + \varepsilon(1+2\delta) + \frac{1}{r}(\theta_1+\varepsilon)(1+2\delta) + \frac{1}{N} r\log m\right)}{\theta_1 - \varepsilon}\right).
\end{aligned}$$

Now note that we can choose ε, δ, and $1/r$ arbitrary small. There is a 'dangerous' term $r\log m/N$ with r in the numerator, but we can make it arbitrary small by enlarging N (and t respectively), which would not affect δ, ε, or r. Hence for any $\gamma > 0$ we can choose appropriate $\varepsilon(\gamma)$, $\delta(\gamma)$, $r(\gamma)$, and $\tilde{N}((\lambda, \pi), \gamma, \delta, \varepsilon, r)$ such that for any $t > \exp(\theta_1\tilde{N})$ we will have

$$\frac{\log \max_{1\leq i\leq m} |E_i(X, x, T, t)|}{\log t} \leq \frac{\theta_2}{\theta_1} + \gamma.$$

Note that the choice of $\varepsilon(\gamma)$, $\delta(\gamma)$, $r(\gamma)$, and of $\tilde{N}((\lambda, \pi), \gamma, \delta, \varepsilon, r)$ does not depend on the starting point $x \in X$ of the interval; the inequality above is valid for all x.

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Note added in proof. The formula for deviation presented in this paper may be improved: the maximum with respect to points $x \in X$ is unnecessary since the value of the expression

$$\limsup_{n\to+\infty} \frac{\log |\chi_i(x, n) - \lambda_i n|}{\log n}$$

is the same for all $x \in X$. I have no doubt that it does not depend on the subinterval $X_i \subset X$ as well. Although at the moment I do not have a general proof of the latter statement, I can prove it for particular Rauzy classes. Moreover, for almost all interval exchange transformations of a unit interval X, the similar formula

$$\limsup_{n\to+\infty} \frac{\log |\chi_Y(x, n) - |Y|\cdot n|}{\log n} = \frac{\theta_2}{\theta_1} < 1$$

should be valid for almost all subintervals $Y \subset X$.

The subject of this paper is closely related to the Teichmüller geodesic flow. In particular, the number $1 + \theta_2/\theta_1$ coincides with the second Lyapunov exponent of the Teichmüller geodesic flow restricted to the corresponding connected component of the corresponding stratum in the moduli space of holomorphic differentials on a surface of genus g. Presumably, the top g Lyapunov exponents of the Teichmüller geodesic flow have multiplicity one, which implies the formulae for the further terms of approximation similar to the one presented here. All these improvements will be published in a forthcoming paper.

To the best of my knowledge there are no methods for exact computation of Lyapunov exponents except for some trivial cases. However, in collaboration with M. Kontsevich we have recently discovered a beautiful formula for the sum of the first g Lyapunov exponents of the Teichmüller geodesic flow [**4**]. Besides, it is possible to find approximate values of θ_1 and θ_2 using computer calculations [**4**, **12**].

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