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ASYMPTOTIC FLAG OF AN ORIENTABLE MEASURED FOLIATION ON A SURFACE

ANTON ZORICH

SISSA (ISAS), via Beirut 2–4, 34013, Trieste, ITALY

zorich@tsmi19.sissa.it

ABSTRACT

We state several conjectures on asymptotic "spectral properties" of transformation operators involved in Rauzy induction for a generic interval exchange transformation. Modulo these conjectures we get a very precise approximation for dynamics of leaves of a generic orientable measured foliation on a surface. The main object, which we get is a flag of subspaces in the first (co)homology group of the surface of dimensions $1, \ldots, g$, where g is a genus of the surface. This flag of subspaces generalizes asymtotic cycle; in particular the smallest subspace is spanned by the asymtotic cycle. Presumably this flag of subspaces provides a new invariant of foliation.

We illustrate the conjectures by treating a specific example, which comes from a model of electron dynamics on a Fermi-surface suggested by I.Dynnikov.

Authors belief in validity of conjectures proclaimed is strongly supported by numerous computer experiments, which gave affirmative results.

1. Introduction

It is well known, that leaves of a generic orientable measured foliation on a surface M_g^2 of genus g wind around the surface along one and the same cycle from the first homology group $H_1(M_g^2, \mathbb{R})$ of the surface, which is called asymptotic cycle[22]. In a sense asymptotic cycle gives the first term of approximation of dynamics of leaves. Here we study other terms of approximation. Computer experiments show, that taking the next term of approximation we get a two-dimensional subspace in $H_1(M_g^2, \mathbb{R})$, i.e., with a good precision leaves deviate from asymtotic cycle not arbitrary, but inside one and the same two-dimensional subspace in the first homology. Taking further steps $n = 3, ..., g$ of approximation we get subspaces of dimension k for the k-th step; collection of the subspaces generates a flag of subspaces in the first homology group. The largest, g-dimensional subspace, gives a Lagrangian subspace in $2g$-dimensional symplectic space $H_1(M_g^2, \mathbb{R})$, with the intersection form considered

This paper is in final form and no version of it will be submitted for publication elsewhere as a symplectic form. We stop at level g since in a sense at this level we get the best possible approximation — it looks like the error can be in a sense uniformly bounded.

Having a measured foliation generated by a generic closed 1-form on a surface, one can consider interval exchange transformation induced by the first return map on a closed transversal. This interval exchange transformation would be minimal and uniquely ergodic, provided we started from a generic closed 1-form. Our hypothetical approximation is based on several conjectures on asymptotic "spectral properties" of transformation operators ${}^{(k)}A$ involved in Rauzy induction corresponding to this interval exchange transformation. The conjectures are stated in section 2.

In section 3 we describe behavior of trajectories modulo conjectures on asymptotic "spectral properties" of Rauzy induction.

In section 4 we list some properties of operators ${}^{(k)}A$ and suggest some speculations on possible proofs of conjectures.

In section 5 we apply general constructions to some particular case arising from an example suggested by I.Dynnikov. This example came from study of Novikov's problem on electron trajectories on Fermi-surfaces in a weak homogeneous magnetic field. Here closed 1-form under consideration is obtained as a restriction of a specific 1-form on three-dimensional torus with constant coefficients to a specific surface of genus 3 embedded into the torus. Rauzy process in this case is periodic, which simplifies the picture. Besides, unfolding the torus we can "make visible" our trajectories.

In section 6 we present several illustrations for sections of Dynnikov surface.

2. Conjectures on "spectral properties" of Rauzy induction

Consider a minimal uniquely ergodic interval exchange transformation with probability vector $(\lambda^1, \dots, \lambda^n)$ and nondegenerate permutation $\sigma \in \mathfrak{S}_n$. To settle notations we remind construction of Rauzy induction[20]. Our notations are almost the same as in[10].

Let us describe one step of Rauzy induction. Denote by $I_{i,j}$ square $n\times n$-matrix, which has only one nonzero entry, which equals one, at the (i,j) place. By E we denote identity $n\times n$-matrix. Let

$$
{}^{(1)}A = \begin{cases} E + I_{n,\sigma^{-1}(n)} & \text{if } \lambda_n > \lambda_{\sigma(n)}, \\ E + I_{\sigma^{-1}(n),n} & \text{if } \lambda_n < \lambda_{\sigma(n)} \end{cases}
$$

Let

$$
{}^{(1)}\lambda = {}^{(1)}A^{-1}\lambda
$$

Let $\sigma_{\text{dom}} = (1, 2, \dots, n)$ and $\sigma_{\text{im}} = \sigma$.

If $\lambda_n > \lambda_{\sigma(n)}$ modify σ_{im} by cyclically moving forward one step all those entries occurring after the last entry in σ_{dom}, i.e., after $\sigma_{\text{dom}}(n)$. Denote the permutation obtained by ${}^{(1)}\sigma_{\rm im}$, and let ${}^{(1)}\sigma_{\rm dom} = \sigma_{\rm dom}$ unchanged. If $\lambda_n < \lambda_{\sigma(n)}$ modify $\sigma_{\rm dom}$ by cyclically moving forward one step all those entries occurring after the last entry in $\sigma_{\rm im}$, i.e., after $\sigma_{\rm im}(n)$. Denote the permutation obtained by ${}^{(1)}\sigma_{\rm dom}$, and let ${}^{(1)}\sigma_{\rm im} = \sigma_{\rm im}$ unchanged. Let

$$ {}^{(1)}\sigma = {}^{(1)}\sigma_{\rm dom}^{-1} \cdot {}^{(1)}\sigma_{\rm im} $$

Here the product of permutations should be understood as a composition of operators, from right to left.

Vector ${}^{(1)}\sigma_{\rm dom}^{-1}\left({}^{(1)}\lambda\right)$ and permutation ${}^{(1)}\sigma$ determine a new interval exchange transformation . This interval exchange transformation is just an induction of original interval exchange transformation to subinterval $[0, 1-\eta[$, where $\eta = \min(\lambda_n, \lambda_{\sigma^{-1}(n)})$. Note, that vector ${}^{(1)}\lambda$ has L^1-norm smaller then λ; we do not renormalize it.

By ${}^{(k)}\lambda$, ${}^{(k)}\sigma$, ${}^{(k)}\sigma_{\rm im}$, ${}^{(k)}\sigma_{\rm dom}$ we denote the data obtained after k steps of Rauzy induction. By ${}^{(0)}\lambda = \lambda$, ${}^{(0)}\sigma = \sigma$, ${}^{(0)}\sigma_{\rm im} = \sigma$, ${}^{(0)}\sigma_{\rm dom} = (1, 2, \dots, n)$ we denote the initial data. By ${}^{(k)}A$ we denote a product of k elementary matrices corresponding to first k steps of induction, so that

$$ {}^{(0)}\lambda = {}^{(k)}A \cdot {}^{(k)}\lambda \tag{1} $$

or in coordinates

$$ {}^{(0)}\lambda^i = {}^{(k)}A^i_j \cdot {}^{(k)}\lambda^j \tag{2} $$

Recall, that having an interval exchange transformation one can construct a Riemann surface and a closed (harmonic) 1-form, which defines a measured foliation on Riemann surface (see[14] and[24]). Initial interval exchange transformation would be generated as a first return map to a specific transversal to the foliation. Denote genus of corresponding Riemann surface by g. Though value of g is determined by combinatorics of permutation σ, we referred to construction of Riemann surface to emphasize topological meaning of g, which is rather essential in this paper.

Let ${}^{(k)}x_1, \dots, {}^{(k)}x_n$ be eigenvalues of ${}^{(k)}A$ enumerated according to decreasing order of their norms: $|{}^{(k)}x_1| \geq |{}^{(k)}x_2| \geq \dots \geq |{}^{(k)}x_n|$.

We formulate propositions and conjectures below everywhere assuming k is sufficiently large, and initial vector λ is generic. We start with reminding a well-known fact, concerning the greatest eigenvalue.

Proposition 1. *The greatest eigenvalue ${}^{(k)}x_1$ is real and positive; it tends to infinity as k tends to infinity; it is much greater then norms of other eigenvalues*

$$ \lim_{k\to\infty} {}^{(k)}x_1 = +\infty $$

$$ \lim_{k\to\infty} \frac{{}^{(k)}x_i}{{}^{(k)}x_1} = 0 \quad \textit{for } i = 2, \dots, n $$

In particular ${}^{(k)}x_1$ has multiplicity one. Corresponding eigenvector ${}^{(k)}V_1$ has positive coefficients. Being normalized in L^1-norm it tends to ${}^{(0)}\lambda$.

$$\lim_{k\to\infty} {}^{(k)}V_1 = {}^{(0)}\lambda$$

Conjecture 1. *Eigenvalues ${}^{(k)}x_1, \dots, {}^{(k)}x_g$ and ${}^{(k)}x_{n-g+1}, \dots, {}^{(k)}x_n$ are all real provided k is sufficiently large.*

Conjecture 2. *Eigenvalues ${}^{(k)}x_1, \dots, {}^{(k)}x_g$ tend to infinity; their ratios $\frac{{}^{(k)}x_{i+1}}{{}^{(k)}x_i}$ for $i = 1, \dots, g-1$ tend to zero, i.e., ${}^{(k)}x_1 \gg |{}^{(k)}x_2| \gg \dots \gg |{}^{(k)}x_g| \gg 1$*

$$\lim_{k\to\infty} |{}^{(k)}x_i| = \infty \quad \text{for } i = 1, \dots, g$$

$$\lim_{k\to\infty} \frac{{}^{(k)}x_{i+1}}{x_i} = 0 \quad \text{for } i = 1, \dots, g-1$$

Conjecture 3. *Eigenvalues $x_{n-g+1}, \dots, x_n$ tend to zero; ratios $\frac{x_{i+1}}{x_i}$ for $i = n-g+1, \dots, n-1$ tend to zero, i.e., $1 \gg |x_{n-g+1}| \gg |x_{n-g+2}| \gg \dots \gg |x_n|$*

$$\lim_{k\to\infty} x_i = 0 \quad \text{for } i = n-g+1, \dots, n$$

$$\lim_{k\to\infty} \frac{x_{i+1}}{x_i} = 0 \quad \text{for } i = n-g+1, \dots, n-1$$

Conjecture 4. *Eigenvalues ${}^{(k)}x_{g+1}, \dots, {}^{(k)}x_{n-g}$ can be complex, but with probability p their absolute values are uniformly bounded by a constant $C(g,p)$, for any $p < 1$. (As a probability measure we consider a natural measure on simplex Δ^{n-1}, parametrizing λ.)*

$$|{}^{(k)}x_i| \le C(g,p) \qquad \text{for} \qquad i = g+1, \dots, n-g$$

In other words

$${}^{(k)}x_{g+1} \sim \dots \sim {}^{(k)}x_{n-g} \sim 1$$

Conjecture 5. *Pairwise products of eigenvalues ${}^{(k)}x_i {}^{(k)}x_{n-i+1}$ for $i = 1, \dots, g$ are close to 1, i.e., ${}^{(k)}x_1 {}^{(k)}x_n \sim 1; \dots; {}^{(k)}x_g {}^{(k)}x_{n-g+1} \sim 1$*

Note that $\det {}^{(k)}A = 1$, and hence $\prod_{i=1}^n {}^{(k)}x_i = 1$.

Morally we claim, that operator ${}^{(k)}A$ behaves "similar" to a high power of a symplectic operator with real eigenvalues.

Consider a flag of subspaces ${}^{(k)}\mathcal{L}^1 \subset {}^{(k)}\mathcal{L}^2 \subset \dots \subset {}^{(k)}\mathcal{L}^g$, where subspace ${}^{(k)}\mathcal{L}^i$, $1 \le i \le g$, is spanned by eigenvectors ${}^{(k)}V_1, \dots, {}^{(k)}V_i$ corresponding to "top" i eigenvalues of operator ${}^{(k)}A$. According to Conjecture 1 above, subspace ${}^{(k)}\mathcal{L}^i$, where $1 \le i \le g$, is real and has dimension i. Consider this flag as a point of corresponding flag manifold $F_{1,2,\dots,g}(\mathbb{R}^{2g})$.

Conjecture 6. *Flags ${}^{(k)}\mathcal{L}^1 \subset {}^{(k)}\mathcal{L}^2 \subset \dots \subset {}^{(k)}\mathcal{L}^g$ have a limit as $k \to \infty$ with respect to natural topology on flag manifold.*

Consider much more general problem. Let $f : M \to M$ be a transitive Anosov diffeomorphism. Let f^* be induced mapping in cohomology. It is known, that the largest by absolute value eigenvalue x_1 of f^* is real, and that $1/x_1$ is also eigenvalue of f^*; corresponding eigenvectors are called Ruelle—Sullivan classes of f, they are Poincaré dual one to the other.

Problem 1. Does f^* have any other "spectral properties" (may be under some additional assumptions on f)? Are there any generalizations of Ruelle—Sullivan classes, say, some invariant subspaces in cohomology?

3. Hypothetical behavior of leaves of orientable measured foliation

Having an interval exchange transformation one can associate to it a Riemann surface and a holomorphic 1-form (see[14] and[24]), which determines a measured foliation on the surface. By construction we have a specific transversal to the foliation; first return map to this transversal induces initial interval exchange transformation. We may assume, that we started from orientable measured foliation, and then choosing a transversal got interval exchange transformation; in any case, what we are interested in is homological behavior of leaves of corresponding measured foliation.

Recall, that one can associate to each subinterval under exchange a cycle in the first homology group of a surface. The cycle N_i, corresponding to subinterval X_i is represented by the following closed pass on our surface M_g^2: we start at the left endpoint of the interval X (i.e., at the left endpoint of our transversal), and go to the right along transversal till we get to some point $x \in X_i$ inside subinterval X_i. Then we follow (in positive direction) leaf of foliation starting at the point x till we hit our transversal for the first time at the point $T(x)$, where T is our interval exchange transformation. Then we go to the left along interval X till we come back to its left endpoint.

Choose some basis $c_1, \dots, c_m$ in the first homology group of M_g^2 with real coefficients. In fact we do not care, whether it is a basis in absolute or relative homology, so we do not want to specify dimension m precisely. It would be convenient to organize our cycles in a $n \times m$-dimensional matrix N as follows: row number i of matrix N is just our cycle N_i represented in components $N_i^1, \dots, N_i^m$ with respect to the basis $c_1, \dots, c_n$.

Let us trace how Rauzy induction affects the cycles N_i. Denote the cycles obtained after k steps of Rauzy induction by ${}^{(k)}N_i$. (Note, that ordering of the subintervals, and hence of the cycles, is determined by permutation ${}^{(k)}\sigma_{\mathrm{dom}}$.) We use initial basis $c_1, \dots, c_m$ in homology to decompose cycles ${}^{(k)}N_i$ in components. It is easy to see, that

$$ {}^{(k)}N = {}^{(k)}A^T \cdot {}^{(0)}N \tag{3} $$

or in coordinates

$$ {}^{(k)}N^q_j = {}^{(0)}N^q_i \cdot {}^{(k)}A^i_j \tag{4} $$

where index q enumerates components of cycles, and indices i and j enumerate cycles.

Remark 1. We would like to emphasize, that according to transformation rule Eq. 4 columns of matrix N are transformed as covariant objects with respect to linear transformation defined by matrix ${}^{(k)}A^T$, while vector ${}^{(k)}\lambda$ of lengths ${}^{(k)}\lambda^i$ of subintervals is transformed as a contravariant object with respect to the same linear transformation (c.f. Eq. 1 and Eq. 2). In other words, if we consider Eq. 4 as an action of a linear operator ${}^{(k)}R$ with matrix ${}^{(k)}A^T$ on covariant objects, then Eq. 1 and Eq. 2 define an action of adjoint operator on contravariant objects.

Probably we had to choose operator ${}^{(k)}R$ with matrix ${}^{(k)}R = {}^{(k)}A^T$ as a starting object in our presentation, otherwise "unexpected transposition" leads to some confusion. On the other hand these would lead to contradiction with existing notations in[10] and other papers.

Matrix ${}^{(k)}A^T$ of our transformation has the same collection ${}^{(k)}x_1, \dots, {}^{(k)}x_n$ of eigennumbers as ${}^{(k)}A$. According to Conjecture 2 eigennumbers ${}^{(k)}x_1, \dots, {}^{(k)}x_g$ are all distinct. Denote corresponding eigenvectors by ${}^{(k)}W_1, \dots, {}^{(k)}W_g$. We have a natural projection to one-dimensional subspaces spanned by these eigenvectors.

Consider eigen(co)vector ${}^{(k)}V_i$, where $1 \le i \le g$, of adjoint operator (having matrix ${}^{(k)}A^{-1}$) corresponding to eigennumber $\frac{1}{{}^{(k)}x_i}$. Note, that it coincides with eigen(co)vector of inverse to adjoint operator (having matrix ${}^{(k)}A$) corresponding to eigennumber ${}^{(k)}x_i$. Normalize our eigenvectors so that under a natural pairing (of covariant and contravariant objects) we get

$$ \langle {}^{(k)}W_i, {}^{(k)}V_i \rangle = 1 \quad \text{for any } i = 1, \dots, g \tag{5} $$

Let us use Eq. 5 to rewrite Eq. 3 and Eq. 4 for the columns ${}^{(k)}N^q$, $q = 1, \dots, m$ of matrix ${}^{(k)}N$.

$$ {}^{(k)}N^q = {}^{(k)}x_1 \langle {}^{(0)}N^q, {}^{(k)}V_1 \rangle \cdot {}^{(k)}W_1 + \cdots + {}^{(k)}x_g \langle {}^{(0)}N^q, {}^{(k)}V_g \rangle \cdot {}^{(k)}W_g + O(1) \tag{6} $$

We remind, that according to Conjectures 3 and 4 the tail in Eq. 6 is small with respect to the leading terms, since projections to eigenvectors ${}^{(k)}W_{n-g+1}, \dots, {}^{(k)}W_n$ would be multiplied by corresponding eigennumbers ${}^{(k)}x_{n-g+1}, \dots, {}^{(k)}x_g$, which tend to zero, while projections to the "middle" eigenvectors ${}^{(k)}W_{g+1}, \dots, {}^{(k)}W_{n-g}$ would be multiplied by eigennumbers, which presumably remain bounded.

Consider the following cycles ${}^{(k)}Z_1, \dots, {}^{(k)}Z_g$ in the first homology group (same where cycles N_i live):

$$ {}^{(k)}Z_i = \langle {}^{(0)}N^1, {}^{(k)}V_i \rangle c_1 + \cdots + \langle {}^{(0)}N^m, {}^{(k)}V_i \rangle c_m \tag{7} $$

We are interested, actually, in the rows of matrix ${}^{(k)}N$, representing cycles in the first homology group of our surface. Combining equation Eq. 6 with definition Eq. 7 we obtain

$$ {}^{(k)}N_i = {}^{(k)}x_1 {}^{(k)}W_1^i \cdot {}^{(k)}Z_1 + \cdots + {}^{(k)}x_g {}^{(k)}W_g^i \cdot {}^{(k)}Z_g + O(1) \tag{8} $$

We are going to analyze now equation 8, which is a key equation in this section. According to Proposition 1 we have ${}^{(k)}x_1 \gg |{}^{(k)}x_i|$ for $i = 2, \ldots, n$. Hence the first term of approximation in Eq. 8 is defined by cycle ${}^{(k)}Z_1$. This means, that if we will rescale cycles ${}^{(k)}N_i$ by $1/{}^{(k)}x_1$ we get

$$ {}^{(k)}N_i = {}^{(k)}W_1^i \cdot {}^{(k)}Z_1 + o(1) \tag{9} $$

i.e., cycle ${}^{(k)}N_i$ is proportional to ${}^{(k)}Z_1$ with a coefficient of proportionality ${}^{(k)}W_1^i$ up to an error, which tends to zero as $k \to +\infty$. We would like to note that this result is based only on Proposition 1, it does not depend on conjectures, so it is quite rigorous. Still for this case we get nothing new. According to the same Proposition 1 one has

$$ \lim_{k \to +\infty} {}^{(k)}V_1 = {}^{(0)}\lambda $$

Hence Eq. 7 leads to

$$ \lim_{k \to +\infty} {}^{(k)}Z_1 = {}^{(0)}\lambda_1 \cdot {}^{(0)}N_1 + \cdots + {}^{(0)}\lambda_n \cdot {}^{(0)}N_n $$

i.e., cycle ${}^{(k)}Z_1$ tends to asymptotic cycle (see[22]).

Recall now, that according to Conjecture 2 we have ${}^{(k)}x_1 \gg \cdots \gg |{}^{(k)}x_g| \gg 1$. Hence if we take leading r terms in approximation Eq. 8, $1 \leq r \leq g$, we get

$$ {}^{(k)}N_i \approx {}^{(k)}x_1 \cdot {}^{(k)}W_1^i \cdot {}^{(k)}Z_1 + \cdots + {}^{(k)}x_r \cdot {}^{(k)}W_r^i \cdot {}^{(k)}Z_r $$

In other words up to a relatively small error all the cycles belong to a r-dimensional subspace in the first homology group spanned by cycles ${}^{(k)}Z_1, \ldots, {}^{(k)}Z_r$. Compare this r-dimensional subspace with one obtained after some other number k' of steps in Rauzy induction. New cycles ${}^{(k')}Z_1, \ldots, {}^{(k')}Z_r$ may change, since they are defined in terms of eigen(co)vectors ${}^{(k')}V_1, \ldots, {}^{(k')}V_r$, which may change. Still, according to Conjecture 6, the space ${}^{(k')}\mathcal{L}_r$ generated by eigen(co)vectors ${}^{(k')}V_1, \ldots, {}^{(k')}V_r$ is close to the space ${}^{(k)}\mathcal{L}_r$ generated by eigen(co)vectors ${}^{(k)}V_1, \ldots, {}^{(k)}V_r$ in the sense of natural topology of Grassmann manifold $G_r(\mathbb{R}^n)$. Hence (see definition Eq. 7) of cycles Z_i) subspaces generated by cycles ${}^{(k)}Z_1, \ldots, {}^{(k)}Z_r$ and ${}^{(k')}Z_1, \ldots, {}^{(k')}Z_r$ would be also close.

Denote the subspace of the space of first homology of M_g^2 with real coefficients spanned by cycles ${}^{(k)}Z_1, \ldots, {}^{(k)}Z_r$ by ${}^{(k)}\mathcal{H}^r$. We showed that Conjectures 1, 2, 3, 4, and 6 imply the following statement:

Main Conjecture. *Flags ${}^{(k)}\mathcal{H}^1 \subset {}^{(k)}\mathcal{H}^2 \subset \cdots \subset {}^{(k)}\mathcal{H}^g$ have a limit as $k \to \infty$ with respect to natural topology on flag manifold.*

We checked this statement by computer experiments with small genuses (up to genus 5) using *Mathematica* package[25]. We used random initial data, and high precision to be able to take approximately a thousand steps in Rauzy induction and compared relative differences in Plucker coordinates. Typical result for the tail of the sequence is 10^{-10} for small genuses.

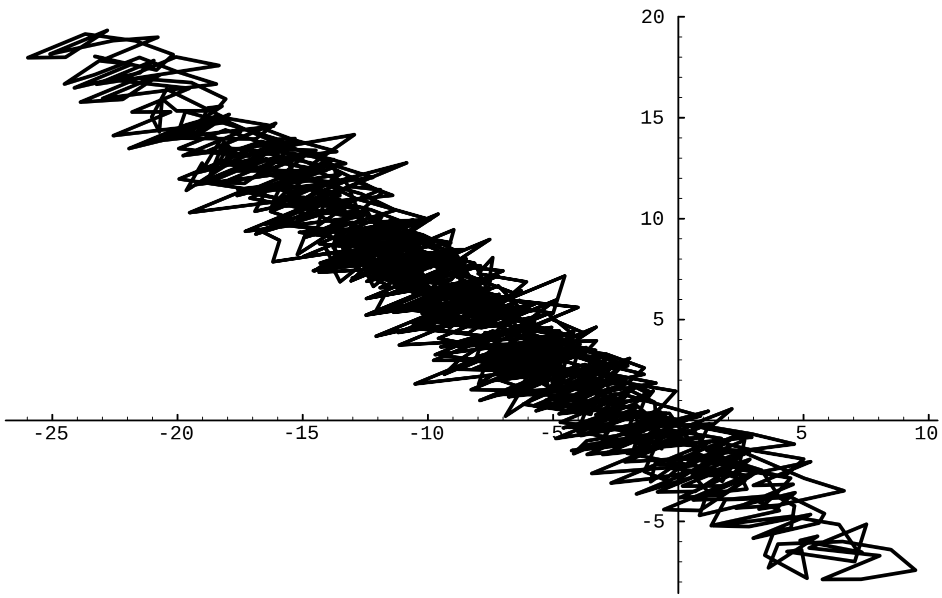

FIGURE 1. Computer simulation of "trajectory" for the case, when "asymptotic cycle" equals zero. Initial permutation $\sigma = (6,5,3,8,7,4,2,1)$ corresponds to a surface of genus 3. Number of iterations is 100.000.

The other obvious computer experiment is as follows. Chose arbitrary two dimensional vectors $N_1, \dots, N_n$, playing a role of cycles, which satisfy $\sum \lambda^i N_i = 0$. Consider a "trajectory" for some large number of iterations of interval exchange transformations. According to Main Conjecture our "trajectory" is supposed to follow a straight line with direction Z_2. This hypothetical straight line becomes already visible (see figure 1) starting with 100000 iterations for small genuses; for greater values of g and n one has to take more iterations.

4. Properties of operators ${}^{(k)}A$ and some speculations on possible proofs of conjectures

Remind some properties of operators ${}^{(k)}A$.

Given an interval exchange transformation T corresponding to a pair (λ, σ), $\lambda \in \mathbb{R}^n_+$, $\sigma \in \mathfrak{S}_n$, set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \lambda_j$, and $X_i = [\beta_{i-1}, \beta_i[$. Define skew-symmetric

$n \times n$-matrix $S(\sigma)$ as follows:

$$S(\sigma)_{ij} = \begin{cases} 1 & \text{if } i < j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j) \\ -1 & \text{if } i > j \text{ and } \sigma^{-1}(i) < \sigma^{-1}(j) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Consider a translation vector

$$\tau = S(\sigma)\lambda \tag{11}$$

Our interval exchange transformation T is defined as follows:

$$T(x) = x + \tau_i, \qquad \text{for } x \in X_i,\ 1 \le i \le n$$

To each permutation $\pi \in S^n$ we assign $n \times n$-matrix which we will denote by $P(\pi)$:

$$P(\pi)_{i,j} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Our first comment is that operators ${}^{(k)}A$ preserve skew-symmetric scalar product $S(\sigma)$ in the following sense (see[15]):

$$P^T({}^{(k)}\sigma_{\text{dom}}) S({}^{(k)}\sigma) P({}^{(k)}\sigma_{\text{dom}}) = {}^{(k)}A^T \cdot S({}^{(0)}\sigma) \cdot {}^{(k)}A \tag{13}$$

In particular for those values of k, when ${}^{(k)}\sigma = {}^{(0)}\sigma$ and ${}^{(k)}\sigma_{\text{dom}} = {}^{(0)}\sigma_{\text{dom}}$, Eq. 13 simplifies as follows:

$$S({}^{(0)}\sigma) = {}^{(k)}A^T \cdot S({}^{(0)}\sigma) \cdot {}^{(k)}A \tag{14}$$

i.e., for those values of k operators ${}^{(k)}A$ preserve "degenerate symplectic form" $S({}^{(0)}\sigma)$.

The other comment concerns kernels of operators $S({}^{(k)}\sigma)$ (see Eq. 11). Recall construction of a Riemann surface and a measured foliation on the surface corresponding to a given interval exchange transformation (see[14] and[24]). Due to this construction our initial interval exchange transformation $({}^{(0)}\sigma, {}^{(0)}\lambda)$ is represented as a first return map to a transversal generated by the measured foliation. Enumerate saddles $P_1, P_2, \dots, P_s$ on our surface. Assign to each endpoint of subintervals $X_i,\ i = 1, \dots, n$ under exchange corresponding saddle. To each saddle point P assign a vector $K \in \mathbb{R}^n$ as follows:

$$K^j = \begin{cases} 1 & \text{if } P \text{ is assigned to the left endpoint of } X_j, \\ -1 & \text{if } P \text{ is assigned to the right endpoint of } X_j, \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

We got s vectors $K_1, \dots, K_s$ corresponding to saddles $P_1, \dots, P_s$.

Proposition 2. *Vectors K_i, $i = 1, \dots, s$ belong to the kernel of operator $S(\sigma)$, i.e.,*

$$S(\sigma)K_i = 0\,.$$

Kernel of operator $S(\sigma)$ has dimension $s-1$; it coincides with a linear span of vectors $K_1, \dots, K_s$.

Since a step of Rauzy induction can be considered as induction to a proper subinterval of the transversal of the first return map, we get a natural identification of saddles corresponding to interval exchange transformations $({}^{(k)}\sigma, {}^{(k)}\lambda)$. Consider vectors ${}^{(k)}K_i\,, i = 1, \dots, s$ corresponding to interval exchange transformation obtained after k steps of Rauzy induction.

Proposition 3. *Operator ${}^{(k)}A$ maps vector ${}^{(k)}K_i$ to vector ${}^{(0)}K_i$:*

$${}^{(k)}A({}^{(k)}K_i) = {}^{(0)}K_i \qquad \textit{for } i = 1, \dots, s\ .$$

Construction of a Riemann surface in[14] and[24] by given interval exchange transformation in fact provides us with a natural basis in the first relative (co)homology of the surface with respect to subset of saddle points. Recall, that a measured foliation in this construction is obtained as a foliation of leaves of a closed 1-form. Note, that values λ_i represent integrals over the basic relative 1-cycles. Note also, that values τ_i of the translation vector in Eq. 11 represent integrals of the 1-form over cycles N_i (see previous section). Consider the following terms of exact sequence of a pair (set of saddle points)$\subset$(Riemann surface M_g^2):

$$\dots \to H^0(\text{saddles}; \mathbb{R}) \to H^1(M_g^2, \{\text{saddles}\}; \mathbb{R}) \to H^1(M_g^2; \mathbb{R}) \to H^1(\text{saddles}; \mathbb{R}) = 0$$

Under identification with cohomology suggested above, Eq. 11 can be considered as a mapping from relative to absolute cohomology from the exact sequence of the pair, while the set of vectors K_i defined by Eq. 15 represents image of the mapping $H^0(\text{saddles}; \mathbb{R}) \to H^1(M_g^2, \{\text{saddles}\}; \mathbb{R})$. Moreover, under identification of our space (where vector λ lives) with the first cohomology of the surface, skew symmetric matrix $S(\sigma)$ represents intersection form on (co)homology.

Recall construction of Veech[24] of pseudo-Anosov maps related to given interval exchange transformation. (See also[14].) Having an interval exchange transformation consider a collection of rectangles, such that their bottom edges are represented by intervals under exchange. Mark proper points on the side edges of rectangles (see[24]), and define some gluing rules, so that after proper identification of sides, the union of rectangles gives a closed Riemann surface provided with a complex structure and closed differential form $Re(dz)$, which determines our measured foliation.

Rauzy induction for interval exchange transformation is naturally generalized in[24] to corresponding modification of our collection of rectangles. Under Rauzy induction the "building" of rectangles grows high and becomes more narrow. Suppose after k_0 steps of Rauzy induction we come back to initial permutation, i.e., ${}^{(k_0)}\sigma = {}^{(0)}\sigma$, and ${}^{(k_0)}\sigma_{\mathrm{dom}} = {}^{(0)}\sigma_{\mathrm{dom}}$. It is shown in[24], that under a special choice of parameters (lengths of the sides of rectangles and heights of the marked points) one can contract the resulting "building" in vertical direction in μ times and expand in horizontal direction in μ times to get initial "building" of rectangles, which produces a pseudo-Anosov diffeomorphism.

For those numbers k_0 of iterations of Rauzy induction, which give initial permutation, operators ${}^{(k_0)}A$ preserve the space $Ker(S(\sigma)) = Im(H^0(saddles;\mathbb{R})$ (see propositions 2 and 3). Moreover, they act on this space as identity mapping. Hence action of ${}^{(k_0)}A$ on the quotient space is well defined and coincides with the mapping of the first cohomology, induced by pseudo-Anosov diffeomorphism. Thus Conjectures 4 and 5 are valid at least for these specific values $k = k_0$.

In the next section we illustrate how our conjectures work for the easiest case, when Rauzy process is periodic. We hope, that in general, quasiperiodic case, the whole picture is similar.

5. Electron trajectories in Dynnikov's example

In this section we want to illustrate ideas of section 3 by treating a particular measured foliation. On the one hand the structure of Rauzy induction is very easy for this case. On the other hand this example has some independent interest since it came from the framework of S.Novikov problem on behavior of electron trajectories on a Fermi-surface in the presence of a weak homogeneous magnetic field (see[16, 17, 26], and[4]).

We remind briefly mathematical formulation of initial problem[16, 17]. Let $\hat{M}^2_g \subset \mathbb{R}^3$ be a periodic surface in $\mathbb{R}^3$, i.e., a surface invariant under translations of cubic lattice in $\mathbb{R}^3$. Consider its intersection lines with a plane $ax+by+cz = const$. What can one say about behavior of these lines? S.Novikov conjectured, that generically nonclosed curves as defined go along a straight line in the plane.

It was proved in[26], that for a fixed embedding conjecture is valid for an open dense set of directions of planes (union of neighbourhoods of rational directions). For this set of directions all curves can not deviate too far from the lines along which they go — they all belong to stripes of finite width. Paper[4] assumes that our surface is a level surface of a periodic function, and proves that for any fixed direction of a plane the same behavior of curves is valid for all but at most one level of the function. There is an example due to S.Tzarev, when Novikov's conjecture is not valid.

We need to reformulate the problem as follows. Consider a closed surface M^2_g of genus g ("Fermi-surface") embedded into a three-dimensional torus T^3. We identify torus T^3 with the space $\mathbb{R}^3$ factored over a cubic lattice. Having a closed 1-form with constant coefficients $a\,dx+b\,dy+c\,dz$ on T^3 one can confine it to the surface. One gets a closed 1-form on the surface, which generically has nondegenerate singularities. This 1-form determines a measured foliation on M_g^2. Consider universal covering $\mathbb{R}^3 \to T^3$ and induced covering $\hat{M}_g^2 \to M_g^2$. Consider leaves of induced measured foliation on the surface $\hat{M}_g^2$. By construction they can be obtained as intersection lines of $\hat{M}_g^2$ with a plane $ax + by + cz = const$.

Generically measured foliation on a surface obtained by construction above splits into several minimal components (tori with holes). For a long time it was not known whether one can get in this way a minimal foliation. We can assume, that homological class of a surface is equal to zero in the second homology of torus (the case when it is nonzero is trivial). Hence, due to a remark by J.Smillie, the image of asymptotic cycle of foliation equals zero in the first homology of torus. This means that curves in $\mathbb{R}^3$ obtained by unfolding of leaves of a minimal uniquely ergodic foliation do not have any natural asymptotic direction. Hence examples of minimal foliations in this problem could lead to quite peculiar behavior of leaves.

A family of examples of minimal measured foliations on a surface of genus 3 as required was recently constructed in[5]. One of the tools in the construction is a process similar to Rauzy induction. We treat the case, when this process is periodic. Parameters, determining the surface, and the slope of the plane are obtained as components of an eigenvector of the transformation matrix D (which is morally similar to matrix A in Rauzy induction) corresponding to a period of the process.

Remark 2. We want to make a following side remark. The space of interval exchange transformations arising from foliations determined by closed 1-forms on a surface of genus g has dimension $4g-4$. Dimension of a subspace, which comes from Dynnikov construction is $2g-1$. It follows from the construction, that there are open sets (in topology of the subspace), for which interval exchange transformation is always nonminimal, which gives an estimate for dimension of stratum of nonminimal interval exchange transformation in the space of all interval exchange transformations.

We chose a transversal on Dynnikov surface and considered interval exchange transformation induced by foliation. In this example we have a surface of genus $g = 3$, the 1-form has $2g-2 = 4$ saddles, so we have interval exchange transformation of $n = 4g-3 = 9$ intervals. One can easily evaluate cycles $N_1, \dots, N_9$ (see construction in section 3). It would be convenient for us to consider images of these cycles in $H_1(T^3;\mathbb{R})$, so we will identify cycles N_i with vectors in $\mathbb{R}^3$.

For completeness of presentation we display numerical data for this example. Under particular choice of transversal one has the following picture: interval exchange transformation has permutation

$$\sigma = (3, 8, 5, 2, 7, 4, 9, 1, 6)$$

and vector $\lambda \approx (0.558, 2.871, 1.227, 1.558, 0.700, 0.368, 2.730, 0.558, 0.141)$.

Matrix N of cycles given in natural coordinates in $H_1(T^3;\mathbb{R})$ is as follows:

$$N = \begin{pmatrix} -1 & -2 & -6 \\ -1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 2 & 5 \\ 1 & 0 & 0 \end{pmatrix} \tag{16}$$

Having such data it is easy to get computer pictures for the leaves of our foliation (unfolded in $\mathbb{R}^3$). Figure 2 illustrates a piece of curve obtained by random choice of initial point.

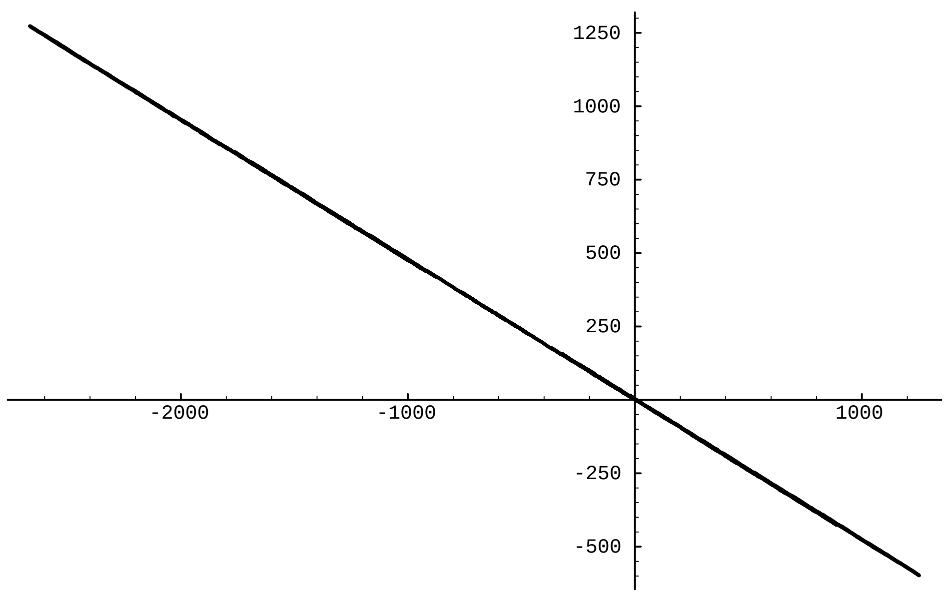

FIGURE 2. A piece of leaf after 100 000 returns to the transversal. Unit of measurement is one unit of our cubic lattice. Starting point is at the origin.

It is easy to see, that the leaf goes rather close to a straight line. Still one should not think, that our leaf just goes straight in one direction — it walks along the line to and fro many times (see section 6 for more details). We stress once more, that such behavior of the leaf can not be explained by means of asymptotic cycle which is equal to zero in the first homology of the torus.

The "straight line" behavior of leaves immediately follows from our Main Conjecture in the end of section 3. Consider images of the subspaces $\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3$ in the first homology $H_1(T^3;\mathbb{R})$ of the torus. We know, that asymptotic cycle, which spans $\mathcal{H}^1$ maps to zero. Hence the image of $\mathcal{H}^2$ is a one-dimensional subspace in $H_1(T^3;\mathbb{R})$ (unless it also maps to zero, which is not the case in our example). This one-dimensional subspace gives the direction of the line, which one sees at figure 2. One can also check, that two-dimensional image of $\mathcal{H}^3$ coincides with the plain $ax + by + cy = 0$.

Fortunately Rauzy process for interval exchange transformation in our example is so simple, that we can prove all conjectures in this particular case. After 12 steps the procedure starts to go cyclically with a period 162. Here is the list of eigennumbers of the matrix $A_{\text{cycle}} = {}^{(12)}A^{-1} \cdot {}^{(174)}A$ corresponding to a cycle in Rauzy induction: $x_1 \approx 25520$, $x_2 \approx 1260$, $x_3 \approx 20$, $x_4 = x_5 = x_6 = 1$, $x_7 \approx 0.05$, $x_8 \approx 0.0008$, $x_9 \approx 0.00004$. Taking a large power of this matrix one gets a picture as in Conjectures above.

We checked cyclic behavior of Rauzy induction in this example as follows: having initial data from Dynnikov process we got approximate initial data for interval exchange transformation with precision sufficient to be sure in first several hundred of steps. Then using computer we generated Rauzy process for our data, and got information on probable length of cycle (162) and number of starting steps (12) before going cyclically. We calculated corresponding matrices ${}^{(12)}A$ and ${}^{(174)}A$; this matrices are integer, so they were calculated precisely. Then we checked that these integer matrices obey some algebraic equation containing matrix D of period of Dynnikov process, which proved that interval exchange transformation obtained from periodic point in Dynnikov process gives periodic point (with period 162) in Rauzy induction. Unfortunately we do not see any mapping or any other direct relations between Dynnikov process and Rauzy induction, though morally they represent one and the same process (it was noticed by J.Smillie). In particular we can not prove in general, that periodic Dynnikov process generates periodic process in Rauzy induction.

Let us give some explanation of the properties of eigennumbers of matrix A_{cycle}. For simplicity take ${}^{(12)}\lambda$ and ${}^{(12)}\sigma$ as initial data. Then Rauzy process would be purely cyclic with period 162, i.e.,

$$ {}^{(162)}\sigma = {}^{(0)}\sigma \tag{17} $$

$$ {}^{(162)}\sigma_{\text{dom}} = {}^{(0)}\sigma_{\text{dom}} \tag{18} $$

$$ {}^{(0)}\lambda = {}^{(162)}A \cdot {}^{(162)}\lambda = {}^{(162)}x_1 \cdot {}^{(162)}\lambda \tag{19} $$

i.e., λ is exactly the eigenvector of ${}^{(162)}A$ corresponding to largest eigennumber ${}^{(162)}x_1$.

Consider matrix $S = {}^{(0)}S$ defined by Eq. 10. Due to Eq. 17 ${}^{(162)}S = S$, and due to Eq. 18 change of coordinates in Eq. 12 determined by ${}^{(162)}\sigma_{\text{dom}}$ is trivial — it is identity matrix. Hence in our case Eq. 13 simplifies as follows:

$$S = ({}^{(162)}A)^T \cdot S \cdot {}^{(162)}A \tag{20}$$

It means that transformation ${}^{(162)}A$ preserves three-dimensional kernel of operator S (see proposition 2). Moreover, due to Eq. 17, Eq. 18, and using proposition 3 we see, that operator ${}^{(162)}A$ acts on the space $\mathrm{Ker}S$ as identity mapping. This way we get three unity eigenvalues $x_4 = x_5 = x_6 = 1$ (cf. Conjecture 4).

We have a well-defined action of operator ${}^{(162)}A$ on the quotient space $\mathbb{R}^9/\mathrm{Ker}S$, since we factorize over invariant subspace. On the quotient space we have skew-symmetric bilinear form, which comes from skew-symmetric bilinear form on $\mathbb{R}^9$ determined by matrix S. On the quotient space our bilinear form is already nondegenerate, and according to Eq. 20 we get a symplectic operator on this six-dimensional vector space. This explains why $x_1 = 1/x_9$, $x_2 = 1/x_8$, $x_3 = 1/x_7$ (cf. Conjecture 5).

Taking powers of matrix ${}^{(162)}A$ we will get a picture of distribution of eigennumbers as in Conjectures 2 and 3.

Let us discuss behavior of flags ${}^{(k)}\mathcal{L}^1, {}^{(k)}\mathcal{L}^2, {}^{(k)}\mathcal{L}^3$. It is easy to see, that for $k_q = 162 \cdot q$ we have

$${}^{(k_1)}\mathcal{L}^i = {}^{(k_2)}\mathcal{L}^i = \cdots \stackrel{\text{def}}{=} \mathcal{L}^i \quad \text{for } i = 1, 2, 3$$

Consider some intermediate k, say, $k = 162 \cdot q + r$, where $0 < r < 162$. Then ${}^{(k)}A = {}^{(k_q)}A \cdot {}^{(r)}A$. Note, that ${}^{(r)}A$ is nondegenarate operator. Since we have a finite number of possible values for r, we can get any uniform estimates for action of ${}^{(r)}A$, so morally we can consider this operator as a "small perturbation of identity operator" with respect to "significant" operator ${}^{(k_q)}A$ (assuming k_q is sufficiently large).

More precisely we can express this idea as follows. Suppose we have a linear projection operator $P : X \to X$ on a finite-dimensional vector space X, which maps the whole space to some invariant subspace $Y \subset X$, i.e., $\mathrm{Im}(P) = Y$, and $P(Y) = Y$. Let Q be an automorphism of the vector space X. Then composition $P \cdot Q$ (first apply Q, then P) is again projection to the subspace Y, i.e. $\mathrm{Im}(P \cdot Q) = Y$, and for almost all automorphisms Q one has $(P \cdot Q)(Y) = Y$.

Morally operator ${}^{(k_q)}A$ acts as a projection P to the subspace $\mathcal{L}^i$ for $i = 1, 2, 3$ depending how many steps (1,2, or 3) of approximation we want to consider, while operator ${}^{(r)}A$ plays a role of automorphism Q. This idea can be easily formalized in our case, which implies that intermediate subspaces ${}^{(k_q+r)}\mathcal{L}^i$, where $i = 1, 2, 3$ converge to $\mathcal{L}^i$ as q tends to infinity.

6. Appendix. Sections of Dynnikov surface

This is just to present several illustrations to section 5. Consider a section of Dynnikov surface in $\mathbb{R}^3$ by a plane $ax + by + cz = const$, where coefficients a, b, c

are as in section 5. Consider a square in the (x, y) plane with a side d. Cut a parallelogram from the plane $ax + by + cz = const$ which projects to our square under projection along z-axes. A piece of section of Dynnikov surface which got into our parallelogram splits into several connected components. Take one of them. Here we present two pictures of such components for different values of d (we measure d in terms of units of our lattice). On the first picture the visible area coincides with accessible area and equals 40×40 units, i.e., $d = 40$.

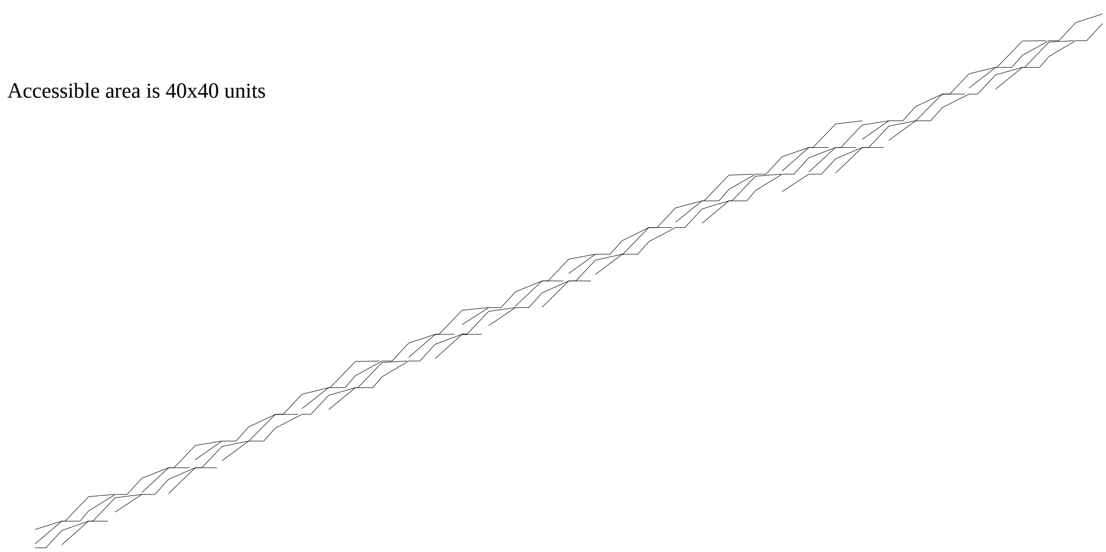

FIGURE 3. Slice of Dynnikov surface.

Problem 2. It would be rather interesting to know, how many connected components has a generic section of Dynnikov surface: one, two, finite number, or countable number?

It would not be interesting to show the whole picture for large values of d. Since our components are just unions of pieces of trajectories, we would see just a strait line for large values of d. Figure 4 demonstrates only a small part of the whole picture, as if we use a zoom. Here $d = 500$, while we see only a piece of the picture which gets inside a 50×50 square.

The picture presented is schematic — it is represented by a plane graph. The actual picture is obtained by replacement of edges of the graph by thin ribbons, and by proper conjugation of the ribbons near the vertices.

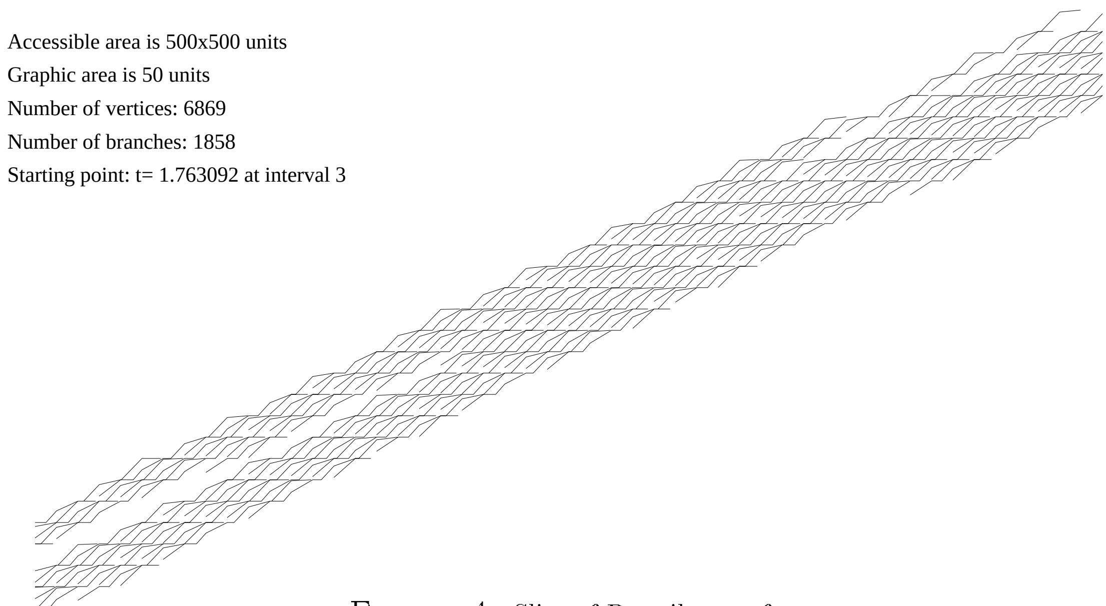

FIGURE 4. Slice of Dynnikov surface.

The second picture illustrates, that our trajectories may “wonder along the line” in a quite complicated way. Lacunas in the graph would be filled up after enlarging the size of the rectangle under consideration. But the picture shows, that trajectories have to go far enough before they come back and fill up the lacunas.

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