"Pseudoperiodic Topology",
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HOW DO THE LEAVES OF A CLOSED 1-FORM WIND AROUND A SURFACE

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ABSTRACT. We start with a mini-survey on some problems of pseudoperiodic topology.

In the main part of the paper we consider analogs of irrational winding lines on a torus for arbitrary Riemann surfaces. These analogs are leaves of foliations defined by closed differential 1-forms. We study asymptotic topological dynamics of the winding lines. We take long pieces of leaves of the foliation and consider the behavior of cycles obtained by joining the endpoints of each piece by short segments.

We prove that generically there is a flag of subspaces $V_1 \subset V_2 \subseteq \cdots \subseteq V_g \subseteq V \subset H_1(M_g^2;\mathbb{R})$ in the first homology group with the following properties. The 1-dimensional subspace V_1 is spanned by the asymptotic cycle. Deviation of a cycle representing a long piece of leaf from the subspace V_j is of order $l^{\nu_{j+1}}$, $j = 1,\dots,g-1$, where l is the length of corresponding piece of leaf. The bound is uniform with respect to choice of leaf and position of the piece of leaf on it. The deviation of any leaf from the subspace V is uniformly bounded by a constant. "Universal constants" $0 \le \nu_j < 1$ are represented in terms of Lyapunov exponents of the Teichmüller geodesic flow on the corresponding moduli space of Abelian differentials.

This statement is a corollary of an analogous statement for interval exchange transformations.

STRUCTURE OF THE PAPER

In the first part of the paper we present a mini-survey on some problems of pseudoperiodic topology. It is independent from the remaining part of the paper. In section 2 we consider foliations on Riemann surfaces defined by closed 1-forms. We show why the interesting topological dynamics of such foliations can be represented by a class of 1-forms obtained as real parts of Abelian differentials. In section 3 we formulate the principal results. In section 4 we reformulate the problem and the principal results in the language of interval exchange transformations. Then we prove the main theorem using the properties of a discrete analog of the Teichmüller geodesic flow on the space of interval exchange transformations. In Appendix A we discuss irreducibility of the corresponding cocycle. In Appendix B we prove irreducibility for some particular case.

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Contents

1. Overview of some problems of pseudoperiodic topology

It is difficult to define what exactly is a *"pseudoperiodic"* or *"quasiperiodic"* topology. However the following problem definitely belongs to the subject.

Consider an embedding of a closed compact manifold M^m into a torus T^n. Consider the embedding of the induced *periodic* manifold $\hat{M}^m$ into the universal cover $\mathbb{R}^n$ over T^n.

$$\begin{array}{ccc} \hat{M}^m & \longrightarrow & \mathbb{R}^n \\ \Big\downarrow & & \Big\downarrow P \\ M^m & \xrightarrow[i]{} & T^n \end{array}$$

Consider now an affine subspace $A^l \in \mathbb{R}^n$.

General Problem. *Describe topology of the*
a) intersection $\hat{M}^m \cap A^l$*;*
b) complement $\hat{M}^m \backslash A^l$*.*

Taking various values of parameters m, n, l, and some specific embeddings $M^m \hookrightarrow T^n$ one gets different problems of pseudoperiodic topology. The current paper mostly deals with the case when $m = 2$, n is large enough, and $l = n - 1$. In this case the manifold M^m is just a Riemann surface, n "large enough" means $n \geq 2g$, or sometimes $n \geq 2g + 3$, where g is the genus of the surface. In other words we study hyperplane sections of periodic surfaces in $\mathbb{R}^n$. Before going into details concerning this particular case we present some outline of what is known (at least to the author) about other combinations of parameters m, n, l, and what problems are hidden in the general formulation above.

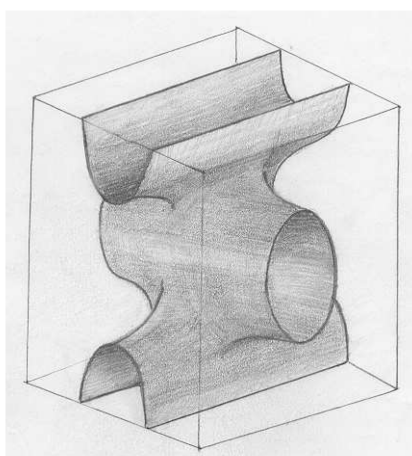

Figure 1. Embedding of a Riemann surface of genus 2 into a torus T^3

Topology of intersection of a periodic manifold with an affine subspace

1.1. **Hyperplane sections of periodic submanifolds.** The special case when $l = n - 1$, i.e., when we consider hyperplane sections of periodic manifolds is exactly the study of *levels* of closed differential 1-forms on a closed manifold M^m. In the several paragraphs below we present the construction due to V.I.Arnold [8] identifying the two problems. Morally, the idea is in analogy of diagram (1) to the corresponding diagram for the Abel–Jacobi map.

Locally a closed differential 1-form ω can be represented as a differential of a function. The function is defined up to a constant, so locally the level hypersurfaces of the function are well-defined. Thus a closed differential 1-form defines a codimension-one foliation (in general with singularities) on the manifold. Speaking about a *leaf* of the foliation we shall usually assume that it is connected. Thus any two points of a leaf could be joined by a path γ such that the restriction $\omega|_\gamma$ vanishes, $\omega|_\gamma \equiv 0$. We shall include in a single class, which will be called a *level* of the 1-form, all the leaves which can be joined by a path in the manifold M^m along which the integral of the form ω is equal to zero.

Consider a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that the hyperplane under consideration is its level surface:

$$A^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid L(x) = a_1x_1 + \cdots + a_nx_n = \text{const}\}.$$

Consider the *exact* 1-form dL on $\mathbb{R}^n$. It is easy to see that dL is induced from a *closed* differential 1-form λ on the torus, $dL = P^*\lambda$ (see (1)). Considering the torus as a unit cube with the identified opposite sides we get the following coordinate representation of the closed 1-form λ on the torus $\lambda = a_1\,dx_1 + \cdots + a_n\,dx_n$. Consider now the induced closed 1-form $\omega = i^*\lambda$ on the manifold M^m (see (1)). Consider the foliation on M^m by leaves of the closed 1-form ω. By construction any hyperplane section of $\hat{M}^m$ is projected to a level of ω. Under certain assumptions (say, when all a_i are independent over rationals) the projection would provide diffeomorphism of connected components.

Vise versa, having an arbitrary closed differential 1-form ω on a closed manifold M^m we can always pull back ω from a linear form on a torus T^n under some embedding $f : M^m \hookrightarrow T^n$. The embedding is constructed as follows. Note that we can always represent ω as a linear combination of *integer* 1-forms $\omega = a_1\alpha_1 + \cdots + a_k\alpha_k$, where all periods of every closed 1-form α_i are integer, and the coefficients a_i are just real numbers linearly independent over rationals. Every itneger closed 1-form α on M^m determines a mapping of M^m onto a circle S^1:

$$f_\alpha : M^m \to S^1, \quad \text{where} \quad f_\alpha : x \mapsto \int_{x_0}^{x} \alpha \mod \mathbb{Z}.$$

Consider also an embedding $f_0 : M^m \hookrightarrow T^{2m+1}$ such that the image of the first homology is trivial. (To construct f_0 one can just embed M^m into a "large cube" in $\mathbb{R}^{2m+1}$ using Whitney Theorem.) Taking a direct product $f_{\alpha_1} \times \cdots \times f_{\alpha_k} \times f_0$ of these maps we get an embedding $f : M^m \hookrightarrow T^n$, where $n = k + 2m + 1$, and $T^n = S^1 \times \cdots \times S^1 \times T^{2m+1}$. The form ω is induced from the linear closed 1-form $\lambda = a_1dx_1 + \cdots a_kdx_k$ on T^n, $\omega = f^*\lambda$. By construction every connected component of any level of ω on M^n is isomorphic to corresponding connected component of corresponding hyperplane section of $\hat{M}^m \subset \mathbb{R}^n$.

The study of topology of levels of a closed 1-form on a closed manifold was initiated by S.P.Novikov in [41], [42], [43]. In particular he gave a sketch of description of a *quasiperiodic structure* of such manifold. This structure depends on the *degree of irrationality* of the form, that is on the number

$$d = \dim_{\mathbb{Q}} \langle p_1, \ldots, p_k \rangle - 1$$

where p_i form a basis of periods of ω. (Sometimes degree of irrationality is defined as $d+1$.)

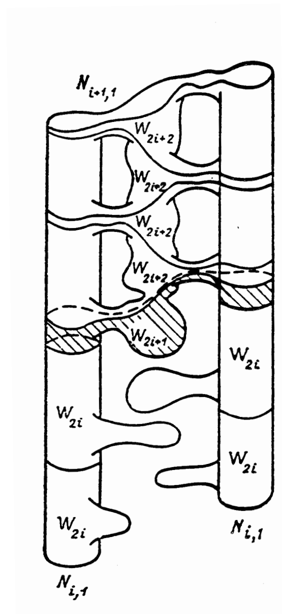

FIGURE 2. Quasiperiodic manifold of degree irationality 1

For $d = 1$ the quasiperiodic structure was described in the author's paper [68] under additional assumptions that ω is a deformation of a rational 1-form. For arbitrary d the problem was solved by Le Tu [34]; see also the paper of L.Alaniya [1]. Morally, the quasiperiodic structure of a level of a closed 1-form of degree of irrationality d is similar to a $\mathbb{Z}^d$-periodic structure. But instead of having a single topological pattern with a (singular) boundary represented by d pairs of "faces", there is a finite collection of possible patterns, with some compatibility conditions for the "faces". Note that the degree of irrationality d has nothing to do with dimension $m-1$ of patterns (here m is dimension of the manifold M^m).

The quasiperiodic structure is morally described by the following picture. Consider a finite collection of completely irrational parallel affine hyperplanes in $\mathbb{R}^d$, where $\mathbb{R}^d$ is provided with a cubic lattice $\mathbb{Z}^d$. Take a vector of lattice, and rescale it with an irrational factor. Consider all translations of our family of hyperplanes by the integer multiples $k\vec{t}$, $k \in \mathbb{Z}$ of the resulting vector $\vec{t}$. The lattice $\mathbb{Z}^d$ is sliced now by a periodic (with irrational period $\vec{t}$) family of parallel hyperplanes. Patterns corresponding to the cubes inside each slice bounded by two neighboring hyperplanes are the same; passing from one slice to the other we change the pattern; patterns corresponding to slices obtained by translation by $\vec{t}$ are the same. See [34], [68] for precise description.

1.2. **Homology theory of periodic manifolds.** Another collection of problems arising in this area is related to homology theory of (quasi)periodic or even $\mathbb{Z}$-periodic manifolds. The corresponding homology theory of periodic manifolds, and analog of Morse Theory for closed 1-forms was constructed by S.P.Novikov in [42]. The corresponding complexes are now defined over larger rings, say, for a $\mathbb{Z}$-periodic manifold the complex is defined over Laurent power series in one variable. In [23] M.Farber proved sharpness of the Morse—Novikov inequalities under certain restrictions on topology of the manifold. Further results were obtained by A.Pajitnov [48], [47], J.-C.Sikorav [54], et al. For more references we address the reader to the paper of A.Pajitnov in this volume. In this paper A.Pajitnov studies the properties of Novikov's complex under some natural restrictions, and proves some beautiful analytic properties of the complex (say, he proves that Laurent power series corresponding to *incidence coefficients* are represented by rational functions).

1.3. **Plane sections of periodic surfaces in $\mathbb{R}^3$.** It turns out, that the extreme case of $m = l = 2$, $n = 3$, that is the study of plane sections of periodic surfaces in $\mathbb{R}^3$, has immediate applications in solid state physics. In this case the hyperplane sections are of dimension one, so topologically they are trivial: every nonsingular connected component of the section is diffeomorphic to either a circle or to a line. What is important here is the way in which the open components (the lines) are embedded into $\mathbb{R}^3$.

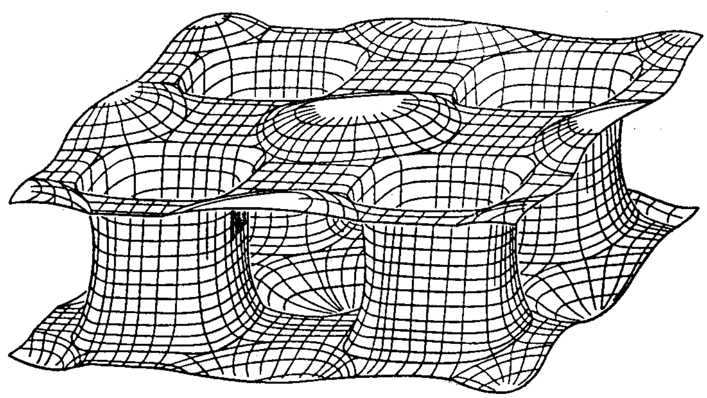

FIGURE 3. Fermi surface of tin (reproduced from [35] which cites [2] as a source). The corresponding surface $M_g^2 \hookrightarrow T^3$ has genus 3.

A periodic surface might be interpreted as a Fermi-surface of some metal in the inverse lattice, and a plane — as a plane orthogonal to a constant magnetic field. Thus the plane sections give us electron trajectories in metal in inverse lattice in the presence of a homogeneous magnetic field. This problem was formulated by S.P.Novikov in [42], where he conjectured that a typical open trajectory follows a straight line (see also papers [44] and [45] of S.P.Novikov, and paper [46] of S.P.Novikov and A.Ya.Maltsev for developments of this subject).

There are two natural approaches to this problem. We can fix a periodic surface, and consider a family of perturbations of a hyperplane, or we can fix a direction of hyperplanes and consider a family of perturbations of a periodic surface. Using the

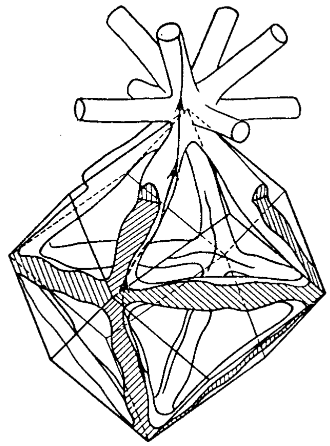

FIGURE 4. Fermi surface of iron (reproduced from [35] which cites [64] as a source).

first approach the author proved in [67] Novikov's conjecture for an open dense set of directions of hyperplanes.

Here is a more precise formulation of the result. Fix a generic periodic surface. If the direction of a hyperplane is a sufficiently small perturbation of a rational direction, then every unbounded component of any nonsingular section goes along a straight line with a bounded deviation from it. (Actually, the fact that the trajectory really *follows* a straight line from $+\infty$ to $-\infty$, i.e., that it does not "come back", was specified by I.Dynnikov [17]).

A comprehensive study of the problem was performed by I.Dynnikov in [17], [18], [20]–[22] (we address the reader to [20] for the "state of the art" in this subject). In particular, using the second approach I.Dynnikov proved in [18] the following statement. Let the periodic surface be a level surface of a periodic Morse function in $\mathbb{R}^3$; let a and b be the minimum and the maximum of this function. Fix a generic direction of a family of parallel hyperplanes. There is an interval $[c, d]$, $a < c \leq d < b$, such that for any level surface corresponding to the value outside of $[c, d]$ all connected components of the plane sections are closed. If $c < d$, then all unbounded components of plane sections of the remaining level surfaces go along straight lines with bounded deviations from them. However I.Dynnikov, proved that the situation when $c = d$ is possible, and, moreover, that for this particular level surface the behavior of the plane sections might be much more complicated.

An example of a trajectory having nontypical behavior was constructed already in 1982 by S.Tzarev [57], but his example corresponds to a rather particular situation. I.Dynnikov elaborated a highly nontrivial construction producing numerous examples of nontypical behavior of trajectories (see [20] for the description of such examples, see also [69] for some numerical simulations of Dynnikov's examples). Thus the following problem is still open:

Problem 1. *Consider a closed orientable Riemann surface embedded into T^3; consider the corresponding periodic surface in $\mathbb{R}^3$. Consider the set of directions of those hyperplanes which give "nontypical" nonsingular unbounded components of* *intersections with the periodic surface. (Here "nontypical" are those open components which are not bounded perturbations of straight lines). Is it true that this set has measure zero in the space $\mathbb{R}P^2$ of all possible directions? Describe the structure of this set. What can be said about Hausdorf dimension of this set?*

Computer simulations performed by I.Dynnikov show, that even in the particular case when a periodic surface is defined as a level surface of a trigonometric polynomial of three variables, the problem seems to be quite nontrivial. The most recent paper of I.Dynnikov [22] is closely related to this problem — it treats the geometry of the *stability zones* in the set of directions.

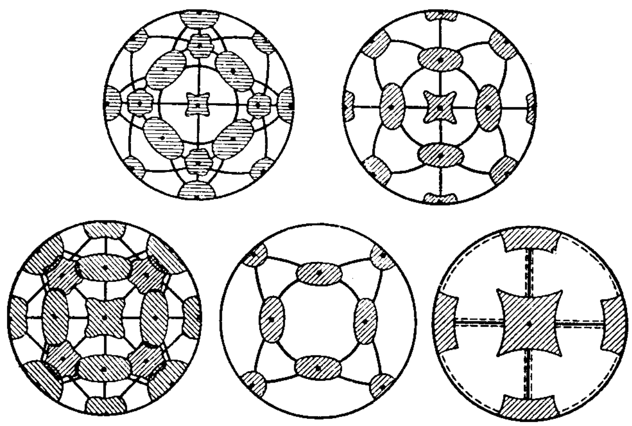

FIGURE 5. Stereographic projection of the magnetic field directions (shaded regions and continuous curves) which give rise to open trajectories for some Fermi-surfaces (experimental results in [35]).

1.4. **Hyperplane sections of periodic surfaces in $\mathbb{R}^n$.** The case of $m = 2$, $l = n - 1$ and n large enough, say $n \geq 2g + 3$, corresponds to the study of plane sections of periodic surfaces in $\mathbb{R}^n$. As it was shown in section 1.1 this problem is in some sense equivalent to the study of behavior of leaves of foliations defined by closed 1-forms on a Riemann surface.

The closed 1-forms considered in the previous section ($n = 3$) have some very specific properties. Now we consider arbitrary closed 1-forms (see section 1.1). The three-dimensional situation is rather rigid, as can be shown by certain elementary topological arguments. A hyperplane section of a periodic surface in $\mathbb{R}^n$ has much more flexibility, and three-dimensional topological arguments are not applicable anymore. However, here one can use tools from dynamics. In particular, using the results of H.Masur [38] and W.Veech [59], it is easy to prove that generically (in this paper we always use the notion "generic" in the measure-theoretical sense) the unbounded hyperplane sections of periodic surfaces follow one of several asymptotic directions. But now a deviation from this asymptotic direction is not bounded by a constant anymore. This paper describes this deviation. It turns out that description of the deviation can be obtained by means of dynamical characteristics (namely, Lyapunov exponents) of the Teichmüller geodesic flow

To describe the hyperplane sections of a periodic surface is the same as to describe the dynamics of leaves of the corresponding orientable measured foliation on the closed orientable underling surface. It is convenient to study the topological dynamics of leaves of a measured foliation using the first return map to a transverse interval. This first return map is an *interval exchange transformation.* We use the ergodic properties of interval exchange transformations, and then we translate them into the language of measured foliations.

* * *

In our approach we unfold the Riemann surface using the universal *abelian* cover of the surface, and we study the asymptotic behavior of leaves on the corresponding periodic surface. Actually, the study of asymptotic behavior of leaves of a foliation on a Riemann surface was initiated by A.Weil, and then developed by D.V.Anosov (see [3], [4] for complete references), N.Markley [37], and later by S.Kh.Aranson, V.Z.Grines, E.Zhuzhoma et al (see [5] for references). In this approach one lifts the leaves of a foliation on a Riemann surface to the *universal* cover, and studies their asymptotic behavior on Lobachevskii disk comparing the unfolded leaves with geodesics. Note that in this setting a wider class of foliations and curves is considered.

There is another problem concerning "irrational flow" on a Riemann surface. One can consider a Morse closed 1-form ω on a Riemann surface as a multivalued Hamiltonian, and consider corresponding Hamiltonian flow along the leaves of the foliation. This flow was considered by V.A.Arnold for the torus [8]; Ya.G.Sinai and K.M.Khanin proved in [55] that generically such flow on a torus is mixing. Recently the result was generalized by K.M.Khanin and A.Nogueira [31] for the Riemann surfaces of arbitrary genus $g \geq 2$.

Remark 1. There is a significant difference between "Hamiltonian" parametrization of leaves and one used in the present paper. We use parametrization of leaves by length in some nondegenerate Riemannian metric, or equivalent parametrizations. In particular, passing close to a zero of the closed 1-form is not distinguished in this consideration, while in Hamiltonian parametrization the motion along the leaves is logarithmically slow near simple zeros of the closed 1-form, see [8]. Moreover, in the presence of separatrix loops homologous to zero (*"traps"*) the flow parametrized by natural parameter, in some sense, "does not notice" the traps: the two currents of the flow which are splitted by zero merge again almost as if there was no splitting. In Hamiltonian parametrization those part of the flow which has to overpass the "trap" merges with the other one with a considerable delay (see Remark 1 in [8]). Morally, this delay is exactly the source of mixing property of the flow, see [55].

1.5. **Plane sections of hypersurfaces.** Another extreme case of the "General Problem" above is the case of $m = n - 1$, $l = 2$, that is the case when we study intersections of a periodic hypersurface with a 2-dimensional affine subspace. Here the sections are again 1-dimensional, so topologically nonsingular connected components are diffeomorphic to either a circle ot to a line. As an illustration we present an example suggested by V.I.Arnold, see [7]. Take a level surface of the following periodic function in $\mathbb{R}^5$:

$$f(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} \cos(2\pi\, x_i)$$

Consider the 2-dimensional linear subspace spanned by two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^5$:

$$\vec{v}_1 = \big(1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}\big)$$

$$\vec{v}_2 = \big(0, \sin\frac{2\pi}{5}, \sin\frac{4\pi}{5}, \sin\frac{6\pi}{5}, \sin\frac{8\pi}{5}\big)$$

Here is a computer simulation of the picture of level structure of f restricted to the plane:

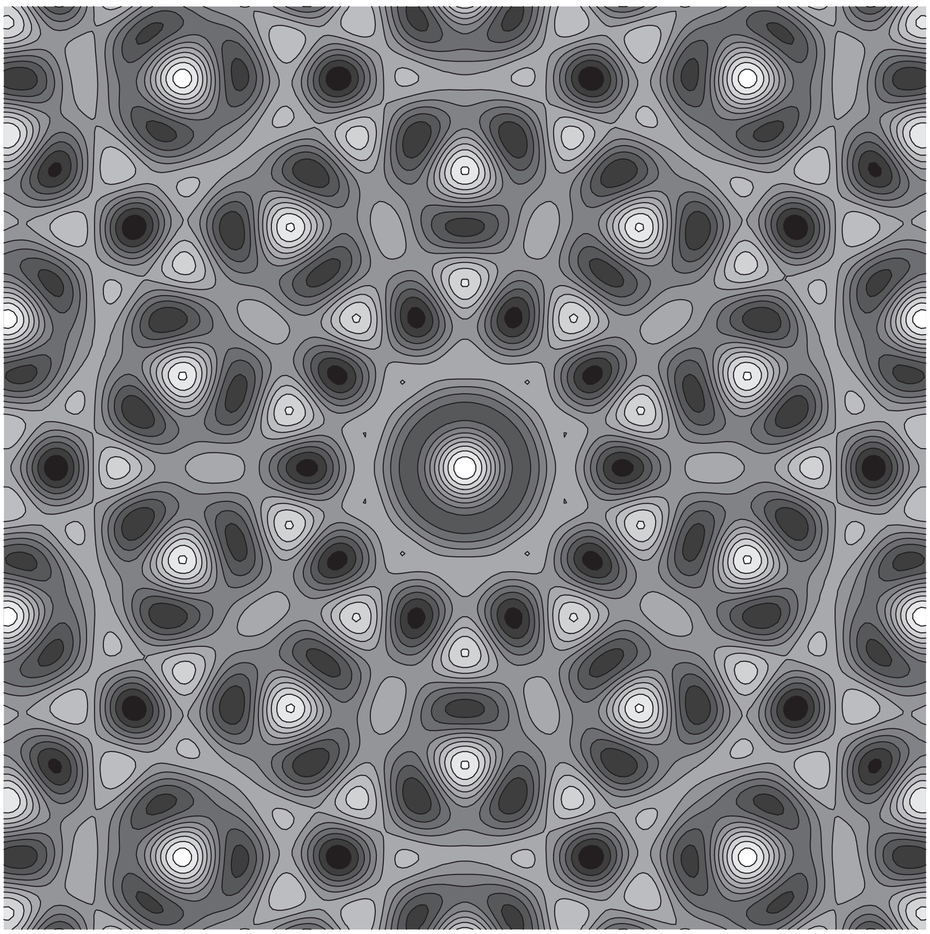

FIGURE 6. Level curves of the function $\sum_{i=1}^{5} \cos(2\pi\, x_i)$ restricted to the irrational 2-dimensional plane

The picture has an obvious rotational symmetry of order 10. It also has translational "quasisymmetry": if we choose a bounded region of the picture, then translating it by an appropriate vectors we will see infinitely many regions where the picture is almost the same. (Since the region presented at figure 6 is relatively small the quasisymmetry is not quite visible at the picture.)

When a quasisymmetry of this kind (quasicristalls) was discovered in solid states physics about fifteen years ago, it made a sensation (see [7] for a popular introduction).

Though the example above is known for more than ten years, up to my best knowledge, none of the following questions of V.I.Arnold concerning this or similar examples have found an answer.

Problem 2. *Are there any nonclosed intersection lines? Is the size (length, diameter) of the closed intersection lines uniformly bounded?*

The results of S.M.Gussein-Zade (see this volume), and of I.Dynnikov [20] seem to be relevant to this problem.

1.6. **Polyintegrable flows.** As a generalization of the previous problem one can consider a particular case of "General Problem" when parameters l, m, n obey the following relation: $m + l = n + 1$. In this case the intersection of an affine plane A^l with a periodic submanifold M^m is again one-dimensional. There is a particular case here, which was thoroughly investigated: the case when the periodic manifold M^m is a torus T^m.

One can represent an affine subspace A^l as an intersection of $m-1$ affine hyperplanes. Playing the same game as before we represent the affine hyperplanes as level hyperplanes of linear functions $L_1, \dots, L_{m-1}$ in $\mathbb{R}^n$. We lift down the differentials dL_i to the closed 1-forms λ_i on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. We pull back the closed 1-forms λ_i to the forms $\omega_i = i^*\lambda_i$ on the torus T^m under the embedding $T^m \hookrightarrow T^n$. (Note that the embedding $i : T^m \hookrightarrow T^n$ is not linear in general!) Taking the universal Abelian cover over T^m we get $m-1$ exact 1-forms dF_i on $\mathbb{R}^m$. Our intersection lines become the intersection lines of level hypersurfaces of $m-1$ pseudoperiodic functions $F_1, \dots, F_{m-1}$ on $\mathbb{R}^m$.

Vice versa, having $m-1$ pseudoperiodic functions $F_1, \dots, F_{m-1}$ on $\mathbb{R}^m$, or, what is the same a pseudoperiodic mapping $F : \mathbb{R}^m \to \mathbb{R}^{m-1}$, we can lift down the differentials dF_i to closed 1-forms ω_i on the quotient $T^m = \mathbb{R}^m/\mathbb{Z}^m$. Each closed 1-form ω_i on T^m can be represented as a sum of an exact 1-form and a linear combination of standard linear forms α_i on $T^m = \mathbb{R}^m/\mathbb{Z}^m$: that is $\omega_i = d\phi_i + \sum a_{ij}\alpha_j$. Consider the mapping $f_0 : T^m \to \mathbb{R}^{m-1}$ defined by

$$f_0 : x \mapsto (\phi_1(x), \dots, \phi_{m-1}(x))$$

We may always choose a large cube in $\mathbb{R}^{m-1}$ such that the image of f_0 is contained in this cube. Thus we may consider the mapping f_0 as a mapping into the torus T^{m-1}. Taking the direct product of the identity map $T^m \to T^m$ and the map f_0 we get an embedding $i : T^m \hookrightarrow T^{2m+1}$. Each form ω_i is a pullback of a linear form on T^{2m-1} under this embedding, $\omega_i = i^*\lambda_i$.

The curves of intersection of leaves of the closed 1-forms ω_i on T^m have natural parametrization by the leafwise 1-form $\Omega/(\omega_1 \wedge \dots \wedge \omega_{m-1})$, where Ω is the standard volume form on T^m. Thus we get a flow along this one dimensional foliation. This flow was introduced by V.I.Arnold in [9], where it was called a *polyintegrable flow.*

The study of polyintegrable flows was developed by I.Dynnikov [19], who proved that all regular unbounded fibers of a pseudoperiodic map $f : \mathbb{R}^m \to \mathbb{R}^{m-1}$ are deformations of straight lines. Dynnikov also proved the number of such unbounded regular fibers is constant and odd provided f is generic. However, it was not clear whether there are any examples, when this number is different from 1, that is when a fiber has several unbounded components. Such examples and further results on polyintegrable flows were obtained by D.Panov in [50]. For more information on polyintegrable flows see the survey of D.Panov in this volume.

A one-dimensional foliation defined by $m-1$ closed 1-forms on a closed manifold M^m seems to be quite a curious object. Up to my best knowledge it was never studied for any manifolds different from T^m. It seems to be a reasonable generalization of a measured foliation on a Riemann surface, where the closed 1-forms provide a sort of transverse measure. The foliation can be parametrized similar to the case of torus: the parametrization is given by a generic volume form Ω on M^m "divided" by the wedge product of $m-1$ 1-forms under consideration. When we fix the embedding $M^m \hookrightarrow T^n$ we can use a generic linear m-form on T^n to define m-form Ω on M^m. This parametrization of leaves enables us to consider corresponding flow along the foliation. Similar to the case of T^n the flow seems to resemble a flow of an incompressible fluid, see [9].

None of the natural questions like decomposition of the foliation into components sharing the same dynamical properties, topological dynamics of leaves, ergodic properties of the corresponding flow, etc have been ever considered.

1.7. **General case.** The only result known to the author for arbitrary m, n, l is the Theorem of S.M.Gussein-Zade [25] concerning density of topological invariants (see the paper of S.M.Gussein-Zade in this volume).

Topology of complement of a periodic manifold to an affine subspace

1.8. **Unbounded components of a complement of a periodic manifold to a hyperplane.** Let an embedding $M^m \hookrightarrow T^n$ be analytic. As usual we assume that M^m is connected. Let an affine hyperplane A^{n-1} be completely irrational. V.I.Arnold proved in [8] that under these assumptions the complement $\hat{M}^m \backslash A^{n-1}$ contains exactly two unbounded components — one in each of the two halfspaces defined by $A^{n-1} \subset \mathbb{R}^n$.

Up to my best knowledge this is the only result obtained for a complement of a periodic submanifold to an affine subspace. The question, whether the condition of analyticity of the embedding $\hat{M}^m \to \mathbb{R}^n$ is essential, is still open. Some advances in generalization of Arnold's Theorem for smooth embeddings were obtained by Yu.Chekanov [13].

The theorem of Arnold might be thought of as a theorem on the *essential* H_0-*homology* of the complement $\hat{M}^m \backslash A^{n-1}$, whatever "essential homology" means.

Problem 3 (V.I.Arnold)**.** *Formulate and prove generalization of the theorem for the "essential* H^{n-l-1}*-homology" of the complement* $\hat{M}^m \backslash A^l$*.*

Here a proper definition of "essential homology" is part of the problem.

2. Introduction

2.1. **Decomposition of a measured foliation into minimal and periodic components.** Consider a closed 1-form ω on a Riemann surface M_g^2. We assume that ω is a *Morse form*, that is all zeros of ω are nondegenerate. This means the following. Consider a small neighborhood of a zero of ω; let $\omega = df$ in this neighborhood. A zero is called nondegenerate, if it is a nondegenerate critical point of f. Since f is defined up to a constant, the notion is well-defined. A Morse 1-form on a Riemann surface may have the same critical points as a function: minima, maxima, and simple saddles.

If we fix an embedding $M_g^2 \hookrightarrow T^n$, and consider closed 1-forms coming from all linear 1-forms on T^n (see section 1.1) then almost all induced 1-forms would be of Morse type. (Throughout this section we use the notion of "almost all" 1-forms in the same sense without referring each time to Lebesgue measure on the linear space of linear 1-forms on $\mathbb{R}^n/\mathbb{Z}^n$.) A C^1-small perturbation of a Morse form is again a Morse form.

Consider foliation by leaves of a closed 1-form of Morse type on a Riemann surface. It was shown by Maier [36] that the surface can be decomposed into several components of two types: *periodic components* and *minimal components*. Periodic components are filled with closed leaves of the foliation, while every nonsingular leaf living in a minimal component is everywhere dense in it. For example standard rational foliation on a torus has single periodic component; standard irrational foliation on a torus has single minimal component. The boundaries of components are formed from critical leaves of the foliation: *separatrix loops* and *saddle connections*.

We distinguish separatrix loops and saddle connections by the following reason. A saddle connection, that is a singular leaf of the foliation joining two distinct zeros of ω disappears under almost all small deformations of the form ω. In other words almost all 1-forms do not have any saddle connections at all. The same is true for separatrix loops representing nontrivial homology cycles. On the contrary, if a separatrix loop represents a zero cycle in homology, it survives under any C^1-small deformation of ω.

Figure 7. Foliation near a separatrix loop

Let us prove the latter statement. Suppose we have a separatrix loop γ_0 such that the cycle $[\gamma_0] \in H_1(M_g^2; \mathbb{R})$ is homologous to zero $[\gamma_0] = 0$ (see figure 7). Note that a foliation defined by a closed 1-form has a transverse measure. Thus monodromy along any closed leaf is trivial, and all leaves of the foliation passing close to γ_0 on at least one side of γ_0 are closed (see figure 7). Thus we get a whole cylinder (or a punctured disk) filled with "parallel" closed leaves. All these closed leaves are homotopic to each other and to the separatrix loop γ_0. The separatrix loop γ_0 is one of two components of the boundary of the cylinder (punctured disk). Since $[\gamma] = 0$ the form ω is exact on the corresponding cylinder (disc). This implies that the leaves homologous to zero would survive under any C^1-small deformation of the initial closed 1-form. The boundary of a deformed disk contains a separatrix loop homologous to zero — the deformation of the initial separatrix loop.

Dynamics of leaves on periodic components is trivial. The interesting part of dynamics is represented by minimal components. Thus we can cut out all cylinders filled by closed leaves homologous to zero. What we get in generic situation is a collection of disjoint minimal components (see, say, [72] for justification). Every minimal component is represented by a Riemann surface with several holes formed by separatrix loops.

Let us shrink each hole to a point. The corresponding critical point disappears under this operation. We get several disjoint minimal components without any separatrix loops or saddle connections. It is easy to see that if we study the behavior of unfolded leaves in $\mathbb{R}^n$, or (what is almost the same) if we study asymptotics of homology cycles obtained by joining the ends of long pieces of leaves, then the operation of "shrinking the holes" does not change dynamics (see [72] for details). (Dynamics in *Hamiltonian* parametrization (see Remark 1) of the same foliation can be drastically changed by this operation.)

2.2. **Closed 1-forms versa harmonic 1-forms.** We claim that the closed 1-forms obtained on minimal components after the surgery described in the previous section are harmonic in some Riemannian metric. To show this we can use the following criterion of E.Calabi [12]:

Calabi Theorem. *A closed Morse 1-form ω on a closed manifold M is harmonic with respect to some Riemannian metric if and only if for every nonsingular point $x \in M$ there exists a closed path $\rho : [0;1] \to M$ through x such that $\omega(d\gamma/dt) > 0$ for any $t \in [0;1]$.*

The fact that a closed 1-form without separatrix loops and saddle connections on a Riemann surface is harmonic in some Riemannian metric was independently proved by A.Katok [27]. For Riemann surfaces an analog of Calabi Theorem was independently proved by J.Hubbard and H.Masur for the forms having arbitrary isolated singularities (see [26]).

Any harmonic 1-form ω_0 on a Riemann surface can be represented as a real part of a holomorphic 1-form in an appropriate complex structure. To see this take a 1-form $\omega_1 = *\omega_0$, where $*$ is the Hodge operator. Foliations defined by closed 1-forms ω_0 and ω_1 form a pair of transversal measured foliations. Thus they define a complex structure on the Riemann surface; the 1-form $\omega_0 + i\omega_1$ would be holomorphic in this complex structure.

The following theorem can be considered as a dual formulation of the theorems mentioned above:

Theorem 1. *An orientable measured foliation on a closed Riemann surface is a horizontal foliation of a holomorphic differential in some complex structure if and only if any cycle obtained as a union of closed paths following in the positive direction a sequence of saddle connections is not homologous to zero.*

Theorem 1 is proved in [33].

The observations above show that interesting dynamics of "generic" foliations defined by closed 1-forms on Riemann surfaces is described by the closed 1-forms which are real parts of Abelian differentials in some complex structure on the Riemann surface. The moduli space of Abelian differentials is already a finite-dimensional variety (orbifold), so what we gained by this construction is that now dynamics is described by some finite-dimensional space of parameters.

2.3. **Teichmüller geodesic flow.** We remind briefly the basic facts concerning the Teichmüller geodesic flow, see [38], [62], [63]. The moduli space of holomorphic quadratic differentials might be considered as a total space of cotangent bundle over the moduli space of complex structures on a Riemann surface of genus g. Morally, the Teichmüller geodesic flow is the geodesic flow on the moduli space of quadratic differentials with respect to Teichmüller metric on the moduli space of complex structures. ("Morally" because Teichmüller metric is not Riemannian, but a Finsler metric.) More rigorously the Teichmüller geodesic flow is defined as follows. There is a natural action of $SL(2,\mathbb{R})$ on the moduli space of quadratic differentials. Action of the diagonal subgroup generates the Teichmüller geodesic flow.

The moduli space of holomorphic 1-forms (Abelian differentials) can be considered as a subvariety of the moduli space of quadratic differentials: one associates to an Abelian differential $f(z)dz$ a quadratic differential $f^2(z)(dz)^2$. The moduli space of Abelian differentials on a closed complex curve of genus g is naturally stratified by degrees of zeros of Abelian differential.

The number of zeros of an Abelian differential (counting multiplicities) on a complex curve of genus g equals $2g-2$. Thus the strata are enumerated by unordered partitions $(k_1, k_2, \dots, k_s)$, where $k_1 + \cdots + k_s = 2g-2$, and $k_i \in \mathbb{N}$. For example there are only two possibilities for $g = 2$. Here $2g-2 = 2$, so either an Abelian differential has two simple zeros which (partition $(1,1)$), or it has one single zero of degree 2 (partition (2)). We denote the strata of Abelian differentials by $\mathcal{H}(k_1, \dots, k_s)$.

There is a natural function $A : \mathcal{H}(k_1, \dots, k_s) \to \mathbb{R}_+$

$$A(\omega) = \frac{1}{2i} \cdot \int_{M_g^2} \omega \wedge \bar{\omega}$$

The subvariety of Abelian differentials, all the strata, and the function A are invariant under the action of the Teichmüller geodesic flow. It was proved by H.Masur [38] and W.Veech [59] that the Teichmüller geodesic flow is ergodic on a "unit sphere" $A = 1$ of each connected component of each stratum with respect to some natural finite measure.

The following proposition is widely known in folklore (say, it can be extracted from combination of [59], [62], and [63]; from [56]; it can be also obtained by combining results from [63] and [70]):

Proposition 1. *Consider a stratum $\mathcal{H}(k_1, \dots, k_s)$ in the moduli space of Abelian differentials. Let $g = k_1 + \cdots + k_s$ be the genus of the surface.*

The collection of Lyapunov exponents of the Teichmüller geodesic flow on a connected component of $\mathcal{H}(k_1,\dots,k_s)$ has the following form:

$$-2 < -(1+\nu_2) \le -(1+\nu_3) \le \dots \le -(1+\nu_g) \le \underbrace{-1 = \dots = -1}_{s-1} \le$$

$$-(1-\nu_g) \le \dots \le -(1-\nu_2) < 0 < (1-\nu_2) \le \dots \le (1-\nu_g)$$

$$\le \underbrace{1 = \dots = 1}_{s-1} \le (1+\nu_g) \le (1+\nu_{g-1}) \le \dots \le (1+\nu_2) < 2$$

The numbers $0 \le \nu_g \le \dots \le \nu_2 < 1$ depend only on connected component of the stratum $\mathcal{H}(k_1,\dots,k_s)$.

3. Formulation of results

Consider smooth closed orientable surface M_g^2 of genus g. Choose a smooth nondegenerate Riemannian metric $g_{ij}(x)$ on M_g^2. For any two points $P_0, P_1 \in M_g^2$ define a path $\rho(P_0,P_1) \subset M_g^2$ joining them. We do not assume that $\rho(P_0,P_1)$ depends continuously on parameters P_0 and P_1, but we will assume that the lengths of the paths (in terms of metric g_{ij}) are uniformly bounded

$$\sup_{P_0,P_1\in M_g^2} \mathit{length}(\rho(P_0,P_1)) = \text{const} < \infty \tag{1}$$

Say, we can define $\rho(P_0,P_1)$ as a shortest geodesic joining P_0 and P_1. We make another choice of such family of paths later on. Since M_g^2 is compact it is easy to see that all choices satisfying 1 are equivalent for our purposes.

Take some leaf γ of ω and choose a compact connected piece of it. Since we assume the orientation of the surface is fixed, the leaves are oriented; so let P_0 be the starting point of our piece of leaf, and let P_1 be the endpoint. Let l be the length of the piece of leaf γ bounded by the points P_0, P_1. By $c_{P_0}(l) \in H_1(M_g^2;\mathbb{R})$ we will denote the homology class of the closed loop obtained by completion of the path from P_0 to P_1 along the leaf γ with the path $\rho(P_1,P_0)$.

Theorem 2. *For almost all Abelian differentials ω in any connected component of any stratum $\mathcal{H}(k_1,\dots,k_s)$ the foliation defined by the closed 1-form $\omega_0 = \mathrm{Re}(\omega)$ has the following properties.*

There exist a flag of subspaces (depending only on ω)

$$V_1 \subset V_2 \subseteq \dots \subseteq V_g \subseteq V \subset H_1(M_g^2;\mathbb{R})$$

such that

For any leaf γ, and any point $P_0 \in \gamma$

$$\lim_{l\to\infty} \frac{c_{P_0}(l)}{l} = c$$

where nonzero asymptotic cycle $c \in H_1(M_g^2;\mathbb{R})$ is proportional to the cycle Poincaré dual to the cohomology class of ω_0. The one-dimensional subspace V_1 is spanned by c.

For any $\phi \in Ann(V_j) \subset H^1(M_g^2;\mathbb{R})$, $\phi \notin Ann(V_{j+1})$ any leaf γ, and any point $P_0 \in \gamma$

$$\limsup_{l\to\infty} \frac{\log|\langle\phi, c_{P_0}(l)\rangle|}{\log l} = \nu_{i+1} \text{ for } i = 1,\dots,g-1$$

For any $\phi \in Ann(V_g) \subset H^1(M_g^2;\mathbb{R})$, $\phi \notin Ann(V)$ *any leaf* γ, *and any point* $P_0 \in \gamma$

$$\limsup_{l\to\infty} \frac{\log|\langle\phi, c_{P_0}(l)\rangle|}{\log l} = 0$$

For any $\phi \in Ann(V) \subset H^1(M_g^2;\mathbb{R})$, $\|\phi\| = 1$ *any leaf* γ, *any point* $P_0 \in \gamma$, *and any length* l

$$|\langle\phi, c_{P_0}(l)\rangle| \le const$$

where the constant depends only on the foliation, and on the choice of the norm in the cohomology.

All the limits above converge uniformly with respect to γ *and* $P_0 \in \gamma$, *i.e., their convergence depends only on* l.

The numbers $2, 1+\nu_2, \dots, 1+\nu_g$ *are the top* g *Lyapunov exponents of the Teichmüller geodesic flow on the corresponding connected component of the stratum* $\mathcal{H}(k_1,\dots,k_s)$, *(see Proposition* 1*).*

Conjecture 1. *For any connected component of any stratum of Abelian differentials all Lyapunov exponents of the Teichmüller geodesic flow except the one corresponding to tangential direction to the flow are nonzero.*

$$\nu_g > 0$$

Conditional Theorem 3. *Conjecture* 1 *implies that subspaces* V_g *and* V *in Theorem* 2 *coincide. Moreover,* $V_g = V \subset H_1(M_g^2;\mathbb{R})$ *is a Lagrangian subspace in the homology, where the symplectic structure is determined by the intersection form.*

Conjecture 2. *For any connected component of any stratum of Abelian differentials the top* g *Lyapunov exponents of the Teichmüller geodesic flow are distinct and strictly greater than* 1

$$1 > \nu_2 > \dots > \nu_{g-1} > 0$$

In other words all Lyapunov exponents except the trivial ones occur with multiplicity one.

Conditional Theorem 4. *Conjecture* 2 *implies that the flag* $V_1 \subset V_2 \subset \dots \subset V_g = V \subset H_1(M_g^2;\mathbb{R})$ *from Theorem* 2 *is a complete flag of subspaces in the Lagrangian subspace* V_g.

4. Asymptotic flag determined by an interval exchange transformation

4.1. **Interval exchange transformations.** Recall the notion of an *interval exchange transformation*, see [28]. Consider an interval X, and cut it into m subintervals of lengths $\lambda_1,\dots,\lambda_m$. Now glue the subintervals together in another order, according to some permutation $\pi \in \mathfrak{S}_m$ and preserving the orientation. We again obtain an interval X of the same length, and hence we get a mapping $T : X \to X$, which is called an *interval exchange transformation*. Our mapping is piecewise linear, and it preserves the orientation and Lebesgue measure. It is singular at the points of cuts, unless two consecutive intervals separated by a point of cut are mapped to consecutive intervals in the image.

Remark 2. The study of interval exchanges was proposed by V.I.Arnold as an interesting problem already in early sixties (see a particular case in the section "Unsolved problems" in [6]).

An interval exchange transformation T is completely determined by a pair (λ, π), $\lambda \in \mathbb{R}^m_+$, $\pi \in \mathfrak{S}_m$. Let $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \lambda_j$, and $X_i = [\beta_{i-1}, \beta_i[$ so that $X = X_1 \sqcup \dots \sqcup X_m$. Define skew-symmetric $m{\times}m$-matrix:

$$\Omega_{ij}(\pi) = \begin{cases} 1 & \text{if } i < j \text{ and } \pi(i) > \pi(j) \\ -1 & \text{if } i > j \text{ and } \pi(i) < \pi(j) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Consider the translation vector $\tau = \Omega(\pi)\lambda$. Our interval exchange transformation T is defined as follows:

$$T(x) = x + \tau_i, \qquad \text{for } x \in X_i,\ 1 \le i \le m$$

Note, that if for some $k < m$ we have $\pi\{1, \dots, k\} = \{1, \dots, k\}$, then the map T decomposes into two interval exchange transformations. We consider only the class $\mathfrak{S}^0_m$ of *irreducible* permutations — those which have no invariant subsets of the form $\{1, \dots, k\}$, where $1 \le k < m$.

Two interval exchange transformations sharing the same permutation π and having proportional vectors of lengths of subintervals are obviously equivalent. Thus speaking about *the space of interval exchange transformations* it is natural to normalize the length of the interval to one: $\|\lambda\| = \lambda_1 + \dots + \lambda_m = 1$. We can identify the space of all interval exchange transformations with the product $\Delta^{m-1} \times \mathfrak{S}^0_m$ of the standard $(m-1)$-dimensional simplex $\Delta^{m-1} = \{\lambda \in \mathbb{R}^m_+ \mid \|\lambda\| = 1\}$ with the set $\mathfrak{S}^0_m$ of irreducible permutations.

4.2. **Interval exchange transformations versa measured foliations.** We are considering now only those measured foliations which are defined by a closed 1-form $\mathrm{Re}(\omega)$, where ω is an Abelian differential from some fixed stratum $\mathcal{H}(k_1, \dots, k_s)$. A generic foliation like this is minimal, i.e., every leaf is dense on the Riemann surface. Taking a transverse interval X to the foliation, we get the first return map $T : X \to X$. The map T is an interval exchange transformation.

Remark 3. The permutation π corresponding to the interval exchange transformation keeps all topological information concerning the Abelian differential ω, that is it uniquely determines the corresponding stratum $\mathcal{H}(k_1, \dots, k_s)$, and even the connected component of the stratum.

In section 3 we considered the following families of cycles: we choose some point P_0, then we took a piece of leaf $\gamma_{P_0,P_1} = \gamma_{P_0}(l)$ of length l passing through P_0 in positive direction, and joined the endpoints P_0 and P_1 by a shortest geodesic to obtain a cycle $c_{P_0}(l)$. Let us consider a sequence of similar cycles related with a transversal interval X.

For every $x \in X$ consider a piece of leaf which has x as a starting point, and $T^{N-1}(x)$ as an endpoint. In other words, emit a leaf γ from the point x in the positive direction and follow it till it intersects with X exactly N times (we are counting x as an intersection). Now join the endpoints of $\gamma_{x,T^{N-1}(x)}$ along X. We get a closed path; let $c_x(N)$ be the corresponding cycle in the first homology.

Morally it is clear that asymptotic properties of $c_{P_0}(l)$ as $l \to \infty$ and of $c_x(N)$ as $N \to \infty$ are the same. To prove it rigorously we have to find the relation between discrete parameter N and continuous parameter l. In other words, we need to know how many intersections with X has a piece of leaf of length l, and vice versa: we need to know what is the length of a piece of leaf obtained after N returns to the transversal X. The answer is given by ergodic theorem applied to the interval exchange transformation $T : X \to X$.

For every point P on M_g^2 define a point $x(P) \in X$ on the transversal interval X as the point of the first intersection of the leaf emitted from P in the *negative* direction with the transversal X. (The definition can be easily extended to the points P on critical leaves by some conventions, see [72].) Having a piece of leaf $\gamma_{P_0}(l)$ of length l bounded by the points P_0, P_1, consider a piece of the same leaf having $x(P_0)$ and $x(P_1)$ as a starting point and an endpoint correspondingly. Now the endpoints of this modified piece of leaf are already on the interval X. Let $N = N(P_0, l)$ be the number of intersections of $\gamma_{P_0}(l)$ with X, i.e. let $x(P_1) = T^{N-1}(x(P_0))$.

We formulate the following obvious

Lemma 1. *The difference between the cycles $c_{P_0}(l)$ and $c_x(N)$ is uniformly bounded with respect to $P_0 \in M_g^2$, and $l \in \mathbb{R}_+$:*

$$\|c_{P_0}(l) - c_x(N)\| \leq const$$

where $x = x(P_0)$; $N = N(P_0, l)$.

To compare parametrization of cycles by the length l and by the number N of returns to a transverse interval consider the following function $l(x)$ on X. Let γ be the leaf passing through the point $x \in X$. Consider the piece $\gamma_{x,Tx}$ of γ between the points x and $T(x)$; let

$$l(x) := l(\gamma_{x,Tx}) \tag{3}$$

be its length. The function $l(x)$ is continuous on every subinterval X_i, so it is bounded:

$$0 < l_{min} \leq l(x) \leq l_{max} < \infty \quad \text{for all } x \in X$$

Recall that almost all interval exchange transformations are uniquely ergodic with respect to Lebesgue measure on the interval (see [38], [59]). By $\bar{l}$ we denote the ergodic mean of $l(x)$:

$$\lim_{K\to\infty} \frac{1}{N} \sum_{K=0}^{N-1} l(T^K(x)) = \frac{1}{|X|} \int_X l(x)\, dx =: \bar{l} \tag{4}$$

Here $|X| = \int_X \omega_0$.

Remark 4. The statement of the ergodic theorem can be slightly strengthened for a generic interval exchange transformation. It follows from [71] that the limits (5) and (4) converge for all points $x \in X$; moreover the convergence is uniform.

We see now that parametrization of cycles by the length l and by the number N of returns to a transverse interval are equivalent: $l(x, N) \sim N\bar{l}$. Taking into consideration Lemma 1 we conclude that all asymptotic properties of the families of cycles $c_P(l)$ and of sequences of cycles $c_x(N)$ are the same. Let us show now how the asymptotic properties of $c_x(N)$ are determined by dynamics of the interval exchange transformation $T : X \to X$.

Consider the function $c(x) := c_x(1)$ on X with values in the first homology group $H_1(M_g^2; \mathbb{R})$. In other words, $c(x)$ is obtained as follows: we emit a leaf γ from $x \in X$ in the positive direction, wait till it hits X for the first time (by definition it hits X at $T(x)$) and we join x and $T(x)$ along X. Note that $c(x)$ is constant on every subinterval X_j; we denote the corresponding values by $c_j \in H_1(M_g^2; \mathbb{R})$.

It is easy to see that

$$c_x(N) = \sum_{K=0}^{N-1} c(T^K(x))$$

Applying ergodic theorem to the function $c(x)$, we get

$$\lim_{K\to\infty} \frac{1}{N} \sum_{K=0}^{N-1} c(T^K(x)) = \frac{1}{|X|}\sum_{j=1}^{m} \lambda_j c_j =: c \tag{5}$$

for almost all $x \in X$ (actually, for *all* $x \in X$, see Remark 4). Here $\lambda_j = |X_j|$ are the "lengths" of the subintervals under exchange measured by ω_0:

$$|X_j| = \int_{X_j} \omega$$

The ergodic mean c in (5) is called *asymptotic cycle* c (see [53]). We formulate the following elementary Lemma, which actually can be extracted from [53].

Lemma 2. *The asymptotic cycle c is proportional to the cycle Poincaré dual to cohomology class of ω_0,*

$$c = |X| \cdot D[\omega_0].$$

Proof. The cycles c_i span the homology group $H_1(M_g^2, \mathbb{R})$ (see [72]). Thus it is sufficient to show that for every i, $1 \le i \le m$ the cycle $|X| \cdot c$ has a proper intersection number with c_i. It is easy to see that the intersection index of the cycles c_i, c_j is given by corresponding entry of the matrix Ω (see (2)): $c_i \circ c_j = \Omega_{ij}$. Thus

$$c_i \circ (|X| c) = c_i \circ (\sum_{j=1}^{m} \lambda_j c_j) = \sum_{j=1}^{m} \lambda_j c_i \circ c_j = \sum_{j=1}^{m} \lambda_j \Omega_{ij} = \tau_i = \int_{c_i} \omega_0$$

□

Remark 5. The Lemma above gives us the coefficient of proportionality between the asymtotic cycle in Theorem 2 and $D[\omega_0]$:

$$\lim_{l\to\infty} \frac{c_{P_0}(l)}{l} = \lim_{N\to\infty} \frac{c_{x(P_0)}(l(N))}{\bar{l}N} = \frac{D[\omega_0]}{\bar{l}|X|}$$

Let the length l be measured in the flat metric determined by the Abelian differential ω. The area of the Riemann surface measured in this flat metric equals $\bar{l}|X|$. Suppose that M_g^2 has unit area in this metric, i.e., suppose that ω is normalized by $(1/2i)\int \omega \wedge \bar{\omega} = 1$. Then we have

$$\lim_{l\to\infty} \frac{c_{P_0}(l)}{l} = D[\omega_0]$$

This calculation shows that as soon as we get any information concerning asymptotic dynamics of the interval exchange transformation $T : X \to X$ we immediately get information on the asymptotic dynamics of the foliation.

4.3. **Euclidean algorithm as a renormalization procedure.** We need to study very long pieces of leaves of the foliation. In order to make a leaf γ wind for a very long time before the first return to X we should choose very short transversal X, this makes the cycles c_j corresponding to the interval exchange transformation on X very long. To implement this idea we consider certain procedure of shortening transversal X, which allows us to trace modifications of c_j while passing from transversal X to a shorter one. To give an idea of such procedure we consider it in the elementary case, when the Riemannian surface is a torus, the foliation is a standard irrational foliation, and the initial transversal X is closed. In this case the first return map $T : X \to X$ is just a rotation of a circle.

Consider rotation of a circle $T : S^1 \to S^1$ by an angle α. Let the length of the circle be normalized to one. Consider trajectory $x, Tx, T^2x, \dots$ of a point x (see figure 8). Denote the length of the arc (x, Tx) by $\lambda = \alpha/(2\pi)$.

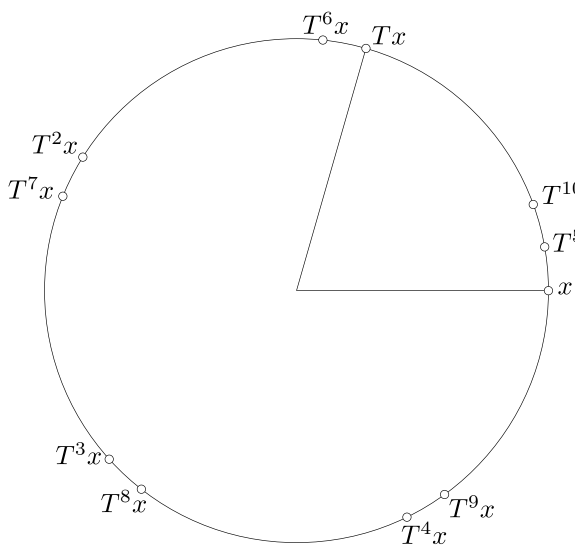

FIGURE 8.

Cutting the circle at the point x we get an interval X; the rotation of the circle generates a map of the interval X to itself which we denote by the same symbol $T : X \to X$. The map acts on X as follows: cut the unit interval X into two pieces of lengths $1 - \lambda$ and λ correspondingly; shift the left piece up to the right endpoint of the interval X and the right piece up to the left endpoint. The map thus obtained is an *interval exchange transformation* of two subintervals.

Let us study how do the points of the trajectory $x, Tx, T^2x, \dots$ accumulate near x, say to the left from x. Since we are not interested in the points which are far from x, we may look how do the points of the trajectory visit the arc $X^{(1)} = (x, Tx)$. For the rotation presented at figure 8 the first point of the trajectory which gets back to this arc is T^5x. Let $\lambda^{(1)} = \{\frac{1}{\lambda}\}$, where by $\{y\}$ we denote the fractional part of $y \in \mathbb{R}$. The length of the arc (x, T^5x) is equal to $\lambda^{(1)} \cdot \lambda$.

The next point of the trajectory $x, Tx, T^2x, \dots$ which gets to the arc $X^{(1)} = (x, Tx)$ is the point $T^{10}x$ (see figure 8); the length of the arc $(T^5x, T^{10}x)$ is the same as the length of the arc (x, T^5x). Moreover, consider the *first return map* $T^{(1)}$ of the arc $X^{(1)}$ to itself. The image of a point $x' \in X^{(1)}$ is defined as the first point of the trajectory $Tx', T^2x', \dots$ which comes back to the arc $X^{(1)}$. If we identify the endpoints of the arc $X^{(1)}$ this map gives us a new rotation of this new circle. If we consider the map $T_{(1)}$ as a map of the interval $X^{(1)}$ to itself we again obtain an interval exchange transformation of two subintervals. The length of the whole interval $X^{(1)}$ is equal to λ; the lengths of subintervals are equal to $\lambda(1-\lambda^{(1)})$ and $\lambda \cdot \lambda^{(1)}$.

Tracing the points of the trajectory $x, Tx, T^2x, \dots$ closest to x from the right we get a sequence of points at the distances λ, $\lambda^{(1)} \cdot \lambda$, $\lambda^{(2)} \cdot \lambda^{(1)} \cdot \lambda, \dots$ from x. We can also go on with the procedure of confinement to smaller and smaller arcs of corresponding lengths. This procedure gives us not only the closest points, but the full description of how do the points of the trajectory appear near the point x. The "procedure" associates to a rotation of a circle (or, what is almost the same, to an interval exchange transformation of two subintervals) a new rotation (new interval exchange transformation). If we rescale the new interval to have the length one, then starting with an exchange of two subintervals of the lengths λ and $1-\lambda$ we get an exchange of two subintervals of lengths $\{\frac{1}{\lambda}\}$ and $1-\{\frac{1}{\lambda}\}$. One can recognize Euclidean algorithm in our "confine to a smaller arc and rescale" procedure. The map

$$g : \lambda \mapsto \{\frac{1}{\lambda}\} \tag{6}$$

can be considered as a map from "the space of rotations" to itself, or what is the same, a map from "the space of interval exchange transformations of two subintervals" to itself. The map g is ergodic with respect to the invariant probability measure

$$d\mu = \frac{1}{\log 2} \cdot \frac{d\lambda}{(\lambda+1)} \tag{7}$$

which is called the *Gauss measure*.

4.4. **Renormalization procedure for interval exchange transformations.** Similar to the case of interval exchange transformation of two subintervals one can construct a renormalization procedure for the interval exchanges of m subintervals. The particular renormalization procedure which we use is based on the *Rauzy induction* [52], [59]. The rigorous definition of this procedure is presented in [70]. Below we list only its properties — those which we use in the present paper. The details and the proofs are contained in [70] and [71].

We assign to a given interval exchange transformation T corresponding to a pair (λ, π) some special subinterval $X^{(1)} \subset X$. Consider the induced map $T^{(1)} = T|_{X^{(1)}}$ of this subinterval to itself. $T^{(1)}$ is again an interval exchange transformation. Moreover, under a special choice of the subinterval $X^{(1)} \subset X$ we get an interval exchange transformation $T^{(1)}$ of the same number m of subintervals $X_1^{(1)}, \dots, X_m^{(1)}$.

For a point $x \in X_j^{(1)}$ in the "new" subinterval $X_j^{(1)}$ define

$$\begin{aligned} B_{ij} = {} & \text{number of visits of the trajectory } x, Tx, \dots, T^{l-1}(x) \\ & \text{to the "old" subinterval } X_i \text{ before the first return} \\ & T^l(x) \in X^{(1)} \text{ to the "new" subinterval } X^{(1)} \end{aligned} \tag{8}$$

We choose the subinterval $X^{(1)} \subset X$ in such a way that for any pair $1 \le i, j \le m$ the number B_{ij} is the same for all $x \in X_j^{(1)}$. Moreover, the vector of lengths $\lambda_1^{(1)}, \dots, \lambda_m^{(1)}$ of subintervals $X_1^{(1)}, \dots, X_m^{(1)}$ is expressed in terms of the vector of lengths $\lambda_1, \dots, \lambda_m$ as

$$\lambda^{(1)} = B^{-1}\lambda \tag{9}$$

Permutation $\pi^{(1)}$ corresponding to the induced interval exchange transformation $T^{(1)} : X^{(1)} \to X^{(1)}$ is always irreducible in our induction procedure, provided that the initial permutation π is irreducible, $\pi \in \mathfrak{S}_m^0$. Rescaling proportionally the interval $X^{(1)}$ to unit length we get a map

$$\mathcal{G} : \Delta^{m-1} \times \mathfrak{S}_m^0 \to \Delta^{m-1} \times \mathfrak{S}_m^0$$

on the space of interval exchange transformations. Actually the set $\mathfrak{S}_m^0$ of all irreducible permutations decomposes into subsets invariant under the map $\mathcal{G}$; these subsets are called the *Rauzy classes*, see [52], [59].

It was proved in [70] that for every Rauzy class $\mathfrak{R}$ the map $\mathcal{G}$ is ergodic on $\Delta^{m-1} \times \mathfrak{R}$ with respect to an absolutely continuous invariant probability measure μ. The matrix-valued function $B(\lambda, \pi)$ defined by (8) determines two *measurable cocycles* on $\Delta^{m-1} \times \mathfrak{R}$ with respect to this measure, i.e., $\int \log^+ \|B^{-1}\| d\mu$ and $\int \log^+ \|{}^tB\| d\mu$ are both finite. Here and below we denote by ${}^tA$ the matrix transposed to matrix A.

The map $\mathcal{G}$ is analogous to the map g defined by (6) (though they do not literally coincide for $m = 2$). Thus morally the map $\mathcal{G}$ represents a sort of multidimensional Euclidean algorithm, while the matrices $B(\lambda, \pi)$, $B\big(\mathcal{G}(\lambda, \pi)\big), \dots,$ $B\big(\mathcal{G}^{(k-1)}(\lambda, \pi)\big)$ play the role of the entries of the continued fraction expansion of a real number.

Remark 6. In matrix representation of a continued fraction expansion one has elementary matrices from the group $SL(2, \mathbb{Z})$. Numerous multidimensional generalizations of continued fraction algorithms use matrices from $SL(n, \mathbb{Z})$. In dimension 2 the groups SL and Sp coincide, so the other way to generalize the continued fraction algorithm is to use matrices from $Sp(n, \mathbb{Z})$. One can think of the map $\mathcal{G}$ as of generalization of this type.

The continued fractions are closely related to the geodesic flow on the upper half-plane (see, say [11] for a very nice exposition of this relation). Our "*Sp*-generalization" is closely related to the Teichmüller geodesic flow (which can be considered as a generalization of the geodesic flow on the upper half-plane for genera greater than 1).

Remark 7. The induction procedure and the renormalization map $\mathcal{G}$ constructed in [70] are obtained as a modification (a "speed up") of the Rauzy induction and of the corresponding map $\mathcal{T}$ in [59]. In particular ergodicity of $\mathcal{G}$ is proved closely following the original proof of W.Veech. The relation between maps $\mathcal{T}$ and $\mathcal{G}$ is similar to the relation between *additive* and *multiplicative* continued fraction algorithms in [11], in particular the invariant measure corresponding to the map $\mathcal{T}$ is infinite, which was the reason for construction of the modified map $\mathcal{G}$.

4.5. **Properties of the "continued fraction" cocycle.** . The matrix-valued function $B(\lambda,\pi)$ defined by (8) determines two *measurable cocycles* on $\Delta^{m-1}\times\mathfrak{R}$ with respect to the measure μ, i.e., $\int\log^+\|B^{-1}\|d\mu$ and $\int\log^+\|{}^tB\|d\mu$ are both finite. Here and below we denote by ${}^tA$ the matrix transposed to matrix A. These cocycles are dual to each other. They play a crucial role in our study of interval exchange transformations. In this section we remind briefly their principal properties. One can find the proofs in [70], [71].

Let

$$(\lambda^{(k)},\pi^{(k)})=\mathcal{G}^k(\lambda,\pi)$$

$$B^{(k)}(\lambda,\pi)=B(\lambda,\pi)\cdot B(\lambda^{(1)},\pi^{(1)})\cdot\dots\cdot B(\lambda^{(k-1)},\pi^{(k-1)})$$

The cocycles preserve the degenerate symplectic form (2) in the following sense:

$$\Omega(\pi)=\left({}^tB^{(k)}(\lambda,\pi)\right)^{-1}\cdot\Omega(\pi^{(k)})\cdot\left(B^{(k)}(\lambda,\pi)\right)^{-1} \tag{10}$$

As we already mentioned above, the map $\mathcal{G}$ is ergodic on $\Delta^{m-1}\times\mathfrak{R}$ with respect to absolutely continuous invariant probability measure μ analogous to the Gauss measure (7). The cocycles ${}^tB(\lambda,\pi)$ and $B^{-1}(\lambda,\pi)$ have the following spectrum of Lyapunov exponents which is the same for both cocycles:

$$\theta_1>\theta_2\geq\theta_3\geq\dots\geq\theta_g\geq\underbrace{0=\dots=0}_{m-2g}\geq-\theta_g\geq\dots\geq-\theta_3\geq-\theta_2>-\theta_1$$

Here g is an integer number determined by the Rauzy class $\mathfrak{R}$; topologically g is the genus of corresponding surface.

By

$$\mathcal{H}_1(\lambda,\pi)\subset\mathcal{H}_2(\lambda,\pi)\subseteq\dots\subseteq\mathcal{H}_g(\lambda,\pi)\subseteq\mathcal{H}(\lambda,\pi)\subset\mathbb{R}^m \tag{11}$$

we denote the corresponding flag of subspaces in $\mathbb{R}^m$ determined by the cocycle $B^{-1}(\lambda,\pi)$. These flag is defined for μ-almost all (λ,π). One has

$$\lim_{k\to\infty}\frac{\log\|\left(B^{(k)}(\lambda,\pi)\right)^{-1}v\|}{k}=-\theta_j\quad\forall v\in\mathcal{H}_j, v\notin\mathcal{H}_{j-1}$$

The flag of subspaces in $\mathbb{R}^{m*}$ corresponding to the dual cocycle ${}^tB(\lambda,\pi)$ is dual to the flag (11): for any linear function f in the annihilator of $\mathcal{H}_j$, $f\in Ann(\mathcal{H}_j)$, such that $f\notin Ann(\mathcal{H}_{j+1})$, one has

$$\lim_{k\to\infty}\frac{\log\|{}^tB^{(k)}(\lambda,\pi)f\|}{k}=\theta_{j+1}\quad\forall v\in Ann(\mathcal{H}_j), v\notin Ann(\mathcal{H}_{j+1})$$

The relation between the map $\mathcal{G}$ and the Teichmüller geodesic flow is as follows (see [59], [62], [63]). One can consider a suspension over the space of (nonnormalized) interval exchange transformations — a space of *zippered rectangles*. One can think of a "zippered rectangle" as of some way to cut initial Riemann surface provided with a flat structure determined by the initial Abelian differential into rectangular pieces. Using some restrictions on the decomposition of the Riemann surface into rectangles it is possible to define a fundamental domain in the space of zippered rectangles. This fundamental domain might be considered as a finite (ramified) covering over a connected component of a stratum $\mathcal{H}(_1, \dots, k_s)$ in the moduli space of Abelian differentials. The Teichmüller geodesic flow has very simple coordinate representation in this fundamental domain. Preimage of the subspace of interval exchange transformations of an interval of *unit* length determines a hypersurface in the fundamental domain. The Teichmüller geodesic flow determines the first return map on the hypersurface. The map $\mathcal{G}$ is the projection of this first return map. In this sense the map $\mathcal{G}$ is a discrete version of the Teichmüller geodesic flow.

Morally, this construction is quite visible already in the simplest case of genus 1, see [11] for a very clear presentation.

The Lyapunov exponents of the Teichmüller geodesic flow are expressed in terms of the Lyapunov exponents of the cocycles B^{-1} and ${}^tB$ as follows:

$$\nu_i = \frac{\theta_i}{\theta_1} \quad \text{for} \quad i = 1, \dots, g$$

The connected components of the strata $\mathcal{H}(_1, \dots, k_s)$ are in one-to-one correspondence with the *extended Rauzy classes* ([63]).

4.6. **Function counting the visits as an additive cocycle.** Having an interval exchange transformation $T : X \to X$ one can consider two different sorts of "time" related to it. The usual "additive time" enumerates the iterations of the interval exchange transformation T, or in the other words it enumerates points of a trajectory $x, Tx, \dots, T^{N-1}x$.

There is also a "multiplicative" time, which enumerates the iterates of the induction procedure $\mathcal{G}$ on the space of interval exchange transformations.

In the case when there are only two intervals any interval exchange transformation is equivalent to rotation of a circle. In this case the "additive time" corresponds to rotations, while the "multiplicative time" corresponds to Euclidean algorithm.

Let us study the interaction between these two "times".

Let $T : X \to X$ be an interval exchange transformation. Let $X = X_1 \sqcup \dots \sqcup X_m$ be corresponding partition of the interval X into subintervals under exchange. For every index i, $1 \le i \le m$, define the counting function

$$S_i(X, T, x, N) := \sum_{L=0}^{N-1} \chi_{X_i}(T^L(x))$$

where

$$\chi_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

Thus $S_i(X, T, x, N)$ is the number of visits of the trajectory $x, Tx, \dots, T^{N-1}x$ to the subinterval X_i. Usually we shall deal with the vector

$$S(X, T, x, N) := \big(S_1(X, T, x, N), \dots, S_m(X, T, x, N)\big)$$

Note that $S(X, T, x, N)$ is an *additive cocycle* with respect to the discrete "additive time":

$$S(X, T, x, N_1 + N_2) = S(X, T, x, N_1) + S(X, T, T^{N_1}x, N_2) \tag{12}$$

4.7. **Vector of visits as a multiplicative cocycle.** Assume that the interval exchange transformation $T : X \to X$ is uniquely ergodic. According to results in [38] and in [59] this is the generic situation. Consider subinterval $X^{(k)}$ obtained from the interval X after k iterations of the induction procedure $\mathcal{G}$. The *induced map* $T_{(k)} : X^{(k)} \to X^{(k)}$ is defined for $x \in X^{(k)}$ as the first return of the trajectory $x, Tx, \dots$ to the subinterval $X^{(k)}$. By construction the map $T_{(k)} : X^{(k)} \to X^{(k)}$ induced by T on $X^{(k)} \subset X$ is again an interval exchange transformation of the same number m of subintervals: $X^{(k)} = X^{(k)}_1 \sqcup \dots \sqcup X^{(k)}_m$.

Consider a finite piece of trajectory $x, Tx, \dots, T^{N-1}x$ which starts and finishes at the interval $X^{(k)}$, i.e., $x \in X^{(k)}$ and $T^N x \in X^{(k)}$. Consider consequetive visits of this piece of trajectory to the subinterval $X^{(k)}$. By construction they coincide with the trajectory $x, T_{(k)}x, T^2_{(k)}x, \dots, T^{N_{(k)}-1}_{(k)}x$, of the induced map $T_{(k)}$ where $T^{N_{(k)}}_{(k)}x = T^N x$. The counting functions $S(X, T, x, N)$ and $S(X^{(k)}, T_{(k)}, x, N_{(k)})$ corresponding to the pair of trajectories as above are related by means of the matrix $B^{(k)}$ representing the induction procedure:

$$S(X, T, x, N) = B^{(k)} \cdot S(X^{(k)}, T_{(k)}, x, N_{(k)}) \tag{13}$$

In this sense the counting function $S(X, T, x, N)$ has the *multiplicative* properties with respect to the "time" k. Here the "multiplicative time" stands for the iterations of the induction procedure $\mathcal{G}$.

Of course relation above is valid only for a very special choice of a point $x \in X$ (we assumed that $x \in X_{(k)}$), and for a very special choice of "time" N (we assumed that $T^N x \in X_{(k)}$). Let us improve the formula to fit arbitrary choice of x and of N. To do that let us extend the trajectory till the extension would hit the subinterval $X_{(k)}$ for the first time. Then we will consider three parts of the extended trajectory. The first part would be the part before the first visit to the subinterval $X^{(k)}$. The third part would be the extension part. The second part would possibly overlap with the third one: it would start at the first visit to $X_{(k)}$ and it would continue till the end of the extended trajectory. By construction the second part starts and finishes at $X_{(k)}$.

More formally, assume that the interval exchange transformation $T : X \to X$ is minimal, and that the first k steps of the induction $\mathcal{G}$ are well-defined for $T : X \to X$. Let

$$\begin{aligned}
N^+(X, x, T, N, k) &:= \min_{0 \le L} L \mid T^L(x) \in X^{(k)} \\
N^-(X, x, T, N, k) &:= \min_{0 \le L} L \mid T^{N+L}(x) \in X^{(k)}
\end{aligned} \tag{14}$$

We denote

$$x^{(k)} := T^{N^+}(x)$$

Using the additive properties (12) of $S(X, T, x, N)$ with respect to "time" N we get

$$\begin{aligned}
&S(X, T, x, N) = \\
&\quad = S(X, T, x, N^+) + S(X, T, T^{N^+}x, (N - N^+ + N^-)) - S(X, T, T^N x, N^-)
\end{aligned}$$

Using (13) for the middle term we modify this relation to the following form:

$$S(X,T,x,N) = \\ = S(X,T,x,N^+) - S(X,T,T^N x,N^-) + B^{(k)} S(X^{(k)},T_{(k)},x_{(k)},N_{(k)}) \quad (15)$$

where $x_{(k)} = T^{N^+}x$, and $T_{(k)}^{N_{(k)}} x_{(k)} = T^{N+N^-}x$.

4.8. **Formulation of the main theorem in terms of interval exchanges.** Now we are ready to reformulate the main Theorem 2 in terms of interval exchange transformations.

Theorem 5. *For almost all interval exchange transformation (λ,π) corresponding to any extended Rauzy class $\mathfrak{R}$ the function $S(X,T,x,N)$ counting visits to the subintervals has the following properties.*

For any $x \in X$

$$\lim_{N\to\infty} \frac{S(X,T,x,N)}{N} = \lambda$$

The one-dimensional subspace $\mathcal{H}_1$ is spanned by λ.

For any $x \in X$ and any $f \in Ann(\mathcal{H}_i)$, $i = 1,\dots,g-1$,

$$\limsup_{N\to+\infty} \frac{\log|\langle f, S(X,T,x,N)\rangle|}{\log N} = \frac{\theta_{i+1}}{\theta_1}$$

For any $x \in X$ and any $f \in Ann(\mathcal{H}_g(\lambda,\pi))$, $\|f\| = 1$

$$|\,\langle\, f\,,\, S(X,T,x,N)\,\rangle\,| \le const$$

where the constant does not depend on f, x or N.

All the limits above converge uniformly with respect to $x \in X$.

The numbers $2, 1+\dfrac{\theta_2}{\theta_1},\dots,1+\dfrac{\theta_g}{\theta_1}$ are the top g Lyapunov exponents of the Teichmüller geodesic flow on the connected component of the stratum $\mathcal{H}(k_1,\dots,k_s)$, corresponding to the extended Rauzy class $\mathfrak{R}$.

Taking into consideration correspondence between cycles $c_x(N)$ and $c_P(l)$, and correspondence between two parametrizations (see section 4.2 we see that Theorem 2 is an immediate corollary from the theorem above.

4.9. **Upper bound.** In this subsection we will prove the following

Proposition 2. *For almost all interval exchange transformations the following upper bound is valid:*

$$\limsup_{N\to+\infty} \frac{\log|\langle f, S(X,T,x,N)\rangle|}{\log N} \le \frac{\theta_{i+1}}{\theta_1}$$

for any $f \in Ann(\mathcal{H}_i)$ and any $x \in X$.

To prove Proposition 2 we will need the following Lemma, which is part of Proposition 8 in [71].

Lemma 3. *For almost all $(\lambda,\pi) \in \Delta\times\mathfrak{R}$ and for any $\varepsilon > 0$, $\delta > 0$, $r \in \mathbb{N}$ there exists $\tilde n = \tilde n((\lambda,\pi),\varepsilon,\delta,r)$ such that for any $n \ge \tilde n$ one can choose the sequence of integers $0 = n_0 < n_1 < \cdots < n_r = n$ with the following properties: for any $1 \le l \le r$*

$$\max_{1\le j\le m} \left| \frac{\log\|B^{(n_l-n_{l-1})}(\lambda^{(n_{l-1})},\pi^{(n_{l-1})})\cdot e_j\|}{n_l - n_{l-1}} - \theta_1 \right| \le \varepsilon$$

Moreover, for any $0 \le l \le r$ the number n_l is close to the corresponding entry of arithmetic progression:

$$\left|\frac{n_l}{n} - \frac{l}{r}\right| \le \delta$$

Proof. One can find the detailed proof of this Lemma in [71] (see Proposition 8 in [71]). Here we present only the idea of the proof, which is quite elementary.

According to [71] for almost all interval exchange transformations (λ, π) the columns of the matrix $B^{(k)}(\lambda, \pi)$ have the following asymtotic growth: for any $1 \le i \le m$

$$\limsup_{k\to+\infty} \frac{\log ||B^{(k)}(\lambda,\pi)\cdot e_i||}{k} = \theta_1$$

Of course convergence is not uniform. Still for any $\varepsilon > 0$ we can choose $\tilde{n}(\varepsilon)$ large enough, so that the subset W of those interval exchange transformations where

$$\left|\frac{\log ||B^{(k)}(\lambda,\pi)\cdot e_i||}{k} - \theta_1\right| < \varepsilon \ \text{ as } k > \tilde{n}(\varepsilon)$$

will have measure arbitrary close to the complete measure. In particular, we can make the difference $1 - \mu(W)$ between two measures much less than δ.

Now consider generic interval exchange transformation (λ, π). Since the map $\mathcal{G}$ is ergodic, the trajectory $(\lambda, \pi), \mathcal{G}(\lambda, \pi), \dots, \mathcal{G}^{(k)}(\lambda, \pi)$ would have only about $(1 - \mu(W))k$ points outside of the set W for k large enough, which by the choice of W is much less than δk. Hence, any interval of the length δk placed inside the interval $[0, k] \subset \mathbb{R}$ (where n is assumed to be large enough) will contain some integers $q_1, \dots$ such that $\mathcal{G}^{q_1}(\lambda, \pi), \dots \in W$. This enables us to choose n_l as desired. Lemma 3 is proved. □

Now let us prove Proposition 2.

Proof. We start with the following definition, which would be quite useful for us:

$$n(N) := \min_{k\ge 0} k \ \text{ such that } \sum_{J=0}^{N-1} \chi_{X_{(k)}}(T^J x) \le 1 \tag{16}$$

In other words, the numbers $n(N) - 1$ is the largest possible integer n such that the trajectory $x, Tx, \dots, T^{N-1}x$ visits subinterval $X_{(n)}$ at least twice. The number $n(N)$ depends on the point $x \in X$; still, the following Lemma shows that for large N the asymptotic behavior of $n(N)$ is the same for all $x \in X$.

Lemma 4. *For almost all interval exchange transformations the number $n(N)$ obeys the following asymptotic relation:*

$$\lim_{N\to+\infty} \frac{\log N}{n(N)} = \theta_1$$

The convergence is uniform with respect to the starting point $x \in X$.

Proof. To prove Lemma it is sufficient to prove that for almost all interval exchange transformations and for arbitrary choice of $\varepsilon > 0$ the number $n(N)$ obeys the following bounds:

$$\frac{\log N - \log 2}{\theta_1 + \varepsilon} < n(N) \le \frac{\log N}{\theta_1 - \varepsilon} + 1$$

provided N is large enough.

According to [71] for almost all interval exchange transformations the following relation is valid:

$$\lim_{k\to+\infty}\frac{\log\|B^{(k)}e_j\|}{k}=\theta_1$$

for all $1\le j\le m$. By $k(\varepsilon)$ denote an integer such that for all $k\ge k(\varepsilon)$ the ratio above differs from θ_1 at most by ε. Chose any $N\ge 2\exp\big((\theta_1+\varepsilon)k(\varepsilon)\big)$

Let us first prove the lower bound. Let

$$k=\left[\frac{\log(N/2)}{\theta_1+\varepsilon}\right]$$

By the choice of N we get $k\ge k(\varepsilon)$. Hence

$$\max_{1\le j\le m}\|B^{(k)}(\lambda,\pi)\cdot e_j\|\le\exp\big((\theta_1+\varepsilon)k\big)$$

Hence

$$\max_{1\le j\le m}\|B^{(k)}(\lambda,\pi)\cdot e_j\|\le N/2$$

We remind that piece of trajectory $y,Ty,\dots$ of any point $y\in X_j^{(k)}$ before the first return to $X_j^{(k)}$ has length $\|B^{(k)}(\lambda,\pi)\cdot e_j\|$, and any point $z\in X$ belongs to corresponding piece of trajectory of some point $y(z)\in X^{(k)}$. Hence inequality above implies, that trajectory of *any* point $x\in X$ of length N will visit subinterval $X^{(k)}$ at least twice. Hence $k<n(N)$, which implies that

$$\frac{\log N-\log 2}{\theta_1+\varepsilon}<n(N)$$

The lower bound in Lemma 4 is proved.

Now let us prove the upper bound in Lemma 4. Denote

$$l=\left[\frac{\log N}{\theta_1-\varepsilon}\right]+1$$

By the choice of N we get $l\ge k(\varepsilon)$. Hence

$$\min_{1\le j\le m}\|B^{(l)}(\lambda,\pi)\cdot e_j\|\ge\exp\big((\theta_1-\varepsilon)l\big)$$

Hence

$$\min_{1\le j\le m}\|B^{(l)}(\lambda,\pi)\cdot e_j\|\ge N$$

Hence inequality above implies, that trajectory of *any* point $x\in X$ of length N will visit subinterval $X^{(l)}$ at most once. Hence $n(N)\le l$, which implies that

$$n(N)\le\frac{\log N}{\theta_1-\varepsilon}+1$$

The upper bound in Lemma 4 is proved. $\square$

Now we are ready to prove Proposition 2. Let (λ,π) satisfy conditions of Lemma 3. Let it be generic in the following sense as well: assume that for n large enough, and for any $f\in Ann(\mathcal{H}_i)$, $\|f\|=1$, we get

$$\frac{\log\|B^{(n)}(\lambda^{(n)},\pi^{(n)})\cdot f\|}{n}\le\theta_i+\varepsilon$$

Finally assume that lemma 4 is also applicable to our (λ,π). Let $n=n(N)$. Chose $n_0<n_1<\cdots<n_r=n$ as in Lemma 3.

Let us apply (15):

$$S(X,T,x,N) = S(X,T,x,N^+) - S(X,T,T^N x,N^-)+ \\ + B^{(n_1)}(\lambda,\pi)\cdot S(X^{(n_1)},T_{(n_1)},x_{(n_1)},N_{(n_1)})$$

where $x_{(n_1)} = T^{N^+}x$ is the point of the first visit of trajectory $x, Tx, T^2x, \dots$ to the subinterval $X_{(n_1)}$, and $T_{(n_1)}^{N_{(n_1)}} x_{(n_1)} = T^{N+N^-}x$ is the point of the first visit of trajectory $T^N x, T^{N+1}x, T^{N+2}x, \dots$ to the subinterval $X_{(n_1)}$.

Now let us apply the same trick to $S(X^{(n_1)},T_{(n_1)},x_{(n_1)},N_{(n_1)})$ by making n_2-n_1 steps of induction $\mathcal{G}$. Note, that the interval $\left(X^{(n_1)}\right)^{(n_2-n_1)}$ obtained from the interval $X^{(n_1)}$ by n_2-n_1 steps of induction coincides with the interval $X^{(n_2)}$ obtained directly from X by $n_1+(n_2-n_1)$ steps of induction. The same is true for the interval exchange transformation: the interval exchange transformation $T_{(n_2)}$ obtained directly from T by n_2 steps of induction coincides with the interval exchange transformation $\left(T_{(n_1)}\right)_{(n_2-n_1)}$ obtained by (n_2-n_1) steps of induction from $T_{(n_1)}$. The same is true for the point $x_{(n_2)} = \left(x_{(n_1)}\right)_{(n_2-n_1)}$ and the time $N_{(n_2)} = \left(N_{(n_1)}\right)_{(n_2-n_1)}$.

Hence

$$S(X^{(n_1)},T_{(n_1)},x_{(n_1)},N_{(n_1)}) = \\ S(X^{(n_1)},T_{(n_1)},x_{(n_1)},N^+_{(n_1)}) - S(X^{(n_1)},T_{(n_1)},T_{(n_1)}^{N_{(n_1)}}x_{(n_1)},N^-_{(n_1)})+ \\ + B^{(n_2-n_1)}(\lambda^{(n_1)},\pi^{(n_1)})\,\cdot\, S(X^{(n_2)},T_{(n_2)},x_{(n_2)},N_{(n_2)})$$

Let us continue this procedure recursively by unfolding our expression up to the level r using the "times" $n_1, n_2, \dots, n_r$. To avoid complicated subscripts and long lists of arguments let us denote the terms obtained at the step l by $S^+_{(n_{l-1})}$, $S^-_{(n_{l-1})}$, and $S_{(n_l)}$ correspondingly. In this notation the step of recursion is represented by

$$S_{(n_{l-1})} = S^+_{(n_{l-1})} - S^-_{(n_{l-1})} + B^{(n_l-n_{l-1})}(\lambda^{(n_l)},\pi^{(n_l)})\cdot S_{(n_l)} \tag{17}$$

Consider the pairing of f with $S(X,T,x,N)$. Using the unfolded expression for $S(X,T,x,N)$, and taking the absolute value of the result we get the following bound

$$\begin{aligned} \left|\langle f\,,\, S(X,T,x,N)\rangle\right| = \left|\langle f\,,\, S_{(n_0)}\rangle\right| &\le \\ \le \left|\langle f\,,\, S^+_{(n_0)} - S^-_{(n_0)}\rangle\right| &+ \\ + \left|\langle f\,,\, B^{(n_1)}(\lambda,\pi)\cdot(S^+_{(n_1)} - S^-_{(n_1)})\rangle\right| &+ \\ + \quad \dots \quad &+ \\ + \left|\langle f\,,\, B^{(n_{r-1})}(\lambda,\pi)\cdot(S^+_{(n_{r-1})} - S^-_{(n_{r-1})})\rangle\right| &+ \\ + \left|\langle f\,,\, B^{(n_r)}(\lambda,\pi)\cdot S_{(n_r)}\rangle\right| & \end{aligned}$$

which we may continue as follows

$$
\begin{aligned}
\big|\langle f\,,\,S(X,T,x,N)\rangle\big| \le &\\
&\le \big\|f\big\|\cdot\big\|S^+_{(n_0)}-S^-_{(n_0)}\big\|+\\
&\quad+\big\|{}^tB^{(n_1)}(\lambda,\pi)f\big\|\cdot\big\|S^+_{(n_1)}-S^-_{(n_1)}\big\|+\\
&\qquad\dots\qquad+\\
&\quad+\big\|{}^tB^{(n_{r-1})}(\lambda,\pi)f\big\|\cdot\big\|S^+_{(n_{r-1})}-S^-_{(n_{r-1})}\big\|+\\
&\quad+\big\|{}^tB^{(n_r)}(\lambda,\pi)f\big\|\cdot\big\|S_{(n_r)}\big\|
\end{aligned}
$$

Now note that for arbitrary choice of X, T, x, and N the norm $\|S(X,T,x,N)\|$ satisfies the following identity:

$$\|S(X,T,x,N)\| = N$$

provided we use the norm

$$\|(v_1,\dots,v_m)\| = |v_1|+\dots+|v_m|$$

In other words the norm $\|S(X,T,x,N)\|$ equals the length of corresponding piece of trajectory.

Now note that $S^+_{(n_l)}$ corresponds to a piece of trajectory of the map $T_{(l)}$ on $X^{(l)}$ which does not visit the subinterval $X^{(l+1)}$. Hence the length of this piece of trajectory is bounded from above by

$$\max_{1\le j\le m}\|B^{(n_{l+1}-n_l)}(\lambda^{(l)},\pi^{(l)})\cdot e_j\|$$

Using our particular choice of n_l and the bound in Lemma 3 we get the following bounds:

$$\|S^+_{(n_l)}\| \le \exp\big((\theta_1+\varepsilon)(n_{l+1}-n_l)\big) \le \exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)$$

Similarly

$$\|S^-_{(n_l)}\| \le \exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)$$

Since the vectors $S^+_{(n_l)}$ and $S^-_{(n_l)}$ are positive we finally get

$$\|S^+_{(n_l)}-S^-_{(n_l)}\| \le \exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)$$

for any $0\le l\le r-1$.

On the other hand assuming n is large enough we get the following bounds:

$$\big\|{}^tB^{(n_l)}(\lambda,\pi)f\big\| \le \|f\|\exp\big((\theta_p+\varepsilon)n_l\big)\cdot$$

Finally note that the choice of n as $n(N)$ implies the following bound:

$$\big\|S_{(n_r)}\big\| = \big\|S_{(n)}\big\| \le 1$$

Collecting these bounds together we can develop the bound for $\left|\langle f, S(X,T,x,N)\rangle\right|$ as follows:

$$
\begin{aligned}
&\left|\langle f\ ,\ S(X,T,x,N)\rangle\right| \le \\
&\qquad\qquad \le \|f\|\cdot\exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)+ \\
&\qquad\qquad\quad \|f\|\exp\big((\theta_p+\varepsilon)n_1\big)\cdot\exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)+ \\
&\qquad\qquad\qquad \dots \qquad + \\
&\qquad\qquad\quad \|f\|\exp\big((\theta_p+\varepsilon)n_{r-1}\big)\cdot\exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)+ \\
&\qquad\qquad\quad +\|f\|\exp\big((\theta_p+\varepsilon)n_r\big)
\end{aligned}
$$

Note that $0\le\theta_p<\theta_1$. Multiply the last term by $\exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)$ the product thus obtained dominates all the other terms. Using Lemma 4, passing to the logarithms and taking the upper limit we get the following bound:

$$
\begin{aligned}
&\limsup_{N\to+\infty}\frac{\log\left|\langle f\ ,\ S(X,T,x,N)\rangle\right|}{\log N}\le \\
&\quad\le\limsup_{N\to+\infty}\frac{\log\left|\langle f\ ,\ S(X,T,x,N)\rangle\right|}{n(N)}\cdot\limsup_{N\to+\infty}\frac{n(N)}{\log N}\le \\
&\quad\le\limsup_{n\to+\infty}\frac{\log\left(\|f\|\cdot\exp\big((\theta_1+\varepsilon)(1+\delta)\frac{n}{r}\big)\cdot\exp\big((\theta_p+\varepsilon)n\big)\right)}{n}\cdot\frac{1}{\theta_1-\varepsilon}\le \\
&\quad\le\limsup_{n\to+\infty}\Big(\frac{\log\|f\|}{n}+\frac{(\theta_1+\varepsilon)(1+\delta)}{r}+(\theta_p+\varepsilon)\Big)\cdot\frac{1}{\theta_1-\varepsilon}= \\
&\quad=\frac{(\theta_1+\varepsilon)(1+\delta)}{r(\theta_1-\varepsilon)}+\frac{\theta_p+\varepsilon}{\theta_1-\varepsilon}
\end{aligned}
$$

Taking into account that we can choose ε and δ arbitrary small, and r arbitrary large, we get desired upper bound:

$$
\limsup_{N\to+\infty}\frac{\log\left|\langle f\ ,\ S(X,T,x,N)\rangle\right|}{\log N}\le\frac{\theta_p}{\theta_1}
$$

Proposition 2 is proved. □

4.10. **Lower bound.** In this subsection we will prove

Proposition 3. *For generic interval exchange transformation (λ,π) the following lower bound is valid:*

$$
\limsup_{N\to+\infty}\frac{\log|\,\langle\, f\ ,\ S(X,T,x,N)\,\rangle\,|}{\log N}\ge\frac{\theta_{p+1}}{\theta_1}
$$

for any $x\in X$ and any $f\in Ann(\mathcal{H}_p(\lambda,\pi))$, $f\notin Ann(\mathcal{H}_{p+1}(\lambda,\pi))$.

To prove Proposition 2 we will use the following Lemma (see Lemma 6.1 in [71]). Denote by $v_1\in\mathbb{R}^m$ the vector $(1,0,\dots,0)$.

Lemma 5. *For almost all interval exchange transformations (λ_0, π_0) one can find a collection of integers $0 = l_1 < \cdots < l_m$ (depending on the interval exchange transformation) such that the vectors $v_1, B^{(l_2)}(\lambda_0, \pi_0) \cdot v_1, \ldots, B^{(l_m)}(\lambda_0, \pi_0) \cdot v_1$ are linearly independent.*

We need the Lemma above to prove the following

Lemma 6. *For almost all interval exchange transformation (λ, π) and for any $f \in Ann(\mathcal{H}_p(\lambda, \pi))$, $f \notin Ann(\mathcal{H}_{p+1}(\lambda, \pi))$, one can find an infinite sequence of integers $t_1 < t_2 < \ldots$ with the following properties:*

1)

$$\lim_{i \to +\infty} \frac{1}{t_i} \log \left| \langle f \, , \, B^{(t_i)}(\lambda, \pi) e_1 \rangle \right| = \theta_{p+1}$$

2) There exists positive integer p (depending on the initial interval exchange transformation $T(\lambda, \pi) : X \to X$) such that for any $i = 1, 2, \ldots$

$$X^{(t_i + p)} \subseteq X_1^{(t_i)}$$

Proof. Consider generic point (λ_0, π_0) and apply Lemma 5 to it. By construction vectors $w_1 := v_1$, $w_2 := B^{(l_2)}(\lambda_0, \pi_0) \cdot v_1, \ \ldots \ , w_m := B^{(l_m)}(\lambda_0, \pi_0) \cdot v_1$ are linearly independent. Let $\Delta_0 \times \pi_0$ be a simplex containing the point (λ_0, π_0), and sharing the same matrices $B^{(l_i)}(\lambda', \pi_0) = B^{(l_i)}(\lambda_0, \pi_0)$ for all $1 \le i \le m$ and all $(\lambda', \pi_0) \in \Delta_0 \times \pi_0$.

Taking smaller subsimplex if necessary, we may find an integer $p > 0$ such that for any $(\lambda', \pi_0) \in \Delta_0 \times \pi_0$ the following relation would be valid

$$X^{(l_i + p)} \subseteq X_1^{(l_i)} \quad \text{for any } 1 \le i \le m \tag{18}$$

Since $\mu(\Delta_0 \times \pi_0) > 0$, and since the map $\mathcal{G}$ is ergodic, trajectory of almost any point (λ, π) under the action of the map $\mathcal{G}$ will visit our subsimplex infinitely many times. By $k_1, k_2, \ldots$ denote these times of visits:

$$\mathcal{G}^{(k_i)}(\lambda, \pi) \in \Delta_0 \times \pi_0$$

For each k_i we can choose l_{j_i} such that

$$\left| \langle f \, , \, B^{(k_i + l_{j_i})}(\lambda, \pi) e_1 \rangle \right| \ge c \left\| {}^t B^{(k_i)}(\lambda, \pi) f \right\| \tag{19}$$

since

$$\begin{aligned} &\left| \langle f \, , \, B^{(k_i + l_{j_i})}(\lambda, \pi) e_1 \rangle \right| = \\ &\qquad = \left| \langle {}^t B^{(k_i)}(\lambda, \pi) f \, , \, B^{(l_{j_i})}(\lambda^{(k_i)}, \pi^{(k_i)}) e_1 \rangle \right| = \\ &\qquad\qquad = \left| \langle {}^t B^{(k_i)}(\lambda, \pi) f \, , \, w_{j_i} \rangle \right| \end{aligned}$$

Let $t_i := k_i + l_{j_i}$. Note that l_{j_i} is chosen from the fixed finite collection of indices $l_1, \ldots, l_m$. Thus (19) implies that

$$\begin{aligned} &\lim_{i \to +\infty} \frac{1}{t_i} \log \left| \langle f \, , \, B^{(t_i)}(\lambda, \pi) e_1 \rangle \right| = \\ &\qquad = \lim_{i \to +\infty} \frac{1}{k_i} \left(\log c + \log \left\| {}^t B^{(k_i)}(\lambda, \pi) f \right\| \right) \cdot \lim_{i \to +\infty} \frac{k_i}{t_i} = \theta_{p+1} \end{aligned}$$

by the choice of $f \in Ann(\mathcal{H}_p)$, $f \notin Ann(\mathcal{H}_{p+1})$. Thus we proved that condition 1 is obeyed for the sequence $t_1 < t_2 < \dots$. Condition 2 follows from (18). Lemma 6 is proved. □

Now we are ready to prove Proposition 3. Assuming the interval exchange transformation (λ, π) under consideration is generic, choose the sequence of integers $t_1 < t_2 < \dots$ as in Lemma 6. We choose arbitrary $x \in X$ and fix it for the rest part of the proof.

Let

$$\limsup_{i \to +\infty} \frac{\log |\langle f, S(X, T, x, N^+_{(t_i+p)})\rangle|}{\log N^+_{(t_i+p)}} = \frac{\theta_{p+1}}{\theta_1} + \varepsilon \tag{20}$$

If $\varepsilon \geq 0$, then Proposition 3 is proved. Suppose not.

Consider the following sequence of integers:

$$N_i := \min_{J > N^+_{(t_i+p)}} J \quad \text{such that } T^J \in X^{(t_i)}$$

In other words wait till the first visit of the trajectory $x, Tx, T^2x, \dots$ to the subinterval $X^{(t_i+p)}$, which happens at the time $N^+_{(t_i+p)}$, and then wait till the first visit to the subinterval $X^{(t_i)}$, which happens at the time N_i.

We are going to prove that

$$\limsup_{i \to +\infty} \frac{\log |\langle f, S(X, T, x, N_i)\rangle|}{\log N_i} \geq \frac{\theta_{p+1}}{\theta_1}$$

which will prove Proposition 3.

Proof. Note that $T^{N^+_{(t_i+p)}} x \in X^{(t_i+p)}$ by definition 14 of N^+. By the choice of t_i we have $T^{N^+_{(t_i+p)}} x \in X_1^{(t_i)}$ in accordance with condition 2 of Lemma 6. Hence by definition of N_i we get

$$S(X, T, x, N_i) = S(X, T, x, N^+_{(t_i+p)}) + B^{(t_i)})(\lambda, \pi) \cdot e_1$$

Since

$$\limsup_{i \to +\infty} \frac{\log N^+_{(t_i+p)}}{t_i} \leq \theta_1$$

and since by assumptions $\varepsilon > 0$ in (20) the second term in the sum

$$\langle f\ ,\ S(X, T, x, N^+_{(t_i+p)})\rangle + \langle f\ ,\ B^{(t_i)})(\lambda, \pi) \cdot e_1\rangle$$

strongly dominates the first one as $i \to +\infty$ by condition 1 of Lemma 6.

Moreover, by Proposition 2 in [71] we may assume that for any $1 \leq j \leq m$, we have

$$\lim_{k \to \infty} \left| \frac{\log \|B^{(k)} e_j\|}{k} \right| = \theta_1$$

which implies that

$$\lim_{i \to +\infty} \frac{\log N_i}{t_i} = \theta_1 \tag{21}$$

Hence

$$\limsup_{i\to+\infty} \frac{\log|\langle f, S(X,T,x,N_i)\rangle|}{\log N_i} =$$
$$= \limsup_{i\to+\infty} \frac{\log|\langle f\ ,\ B^{(t_i)})(\lambda,\pi)\cdot e_1\rangle|}{t_i} \cdot \lim_{i\to+\infty} \frac{t_i}{\log N_i} = \frac{\theta_p}{\theta_1}$$

where we combined condition 1 of Lemma 6 and (21) in the latter equality. Proposition 3 is proved. □

4.11. **Uniform bound.** Finally we are going to prove

Proposition 4. *For generic interval exchange transformation (λ,π) there is a uniform bound as follows. For any $x\in X$ and any $f\in Ann(\mathcal{H}_g(\lambda,\pi))$, $\|f\|=1$*

$$|\,\langle\, f\ ,\ S(X,T,x,N)\,\rangle\,| \le const$$

where the constant does not depend on f, x or N.

We remind that

$$Ann(\mathcal{H}_g(\lambda,\pi)) = \{f\,|\,\theta(f)<0\}$$

We start with the following elementary

Lemma 7. *For almost all interval exchange transformations (λ,π) the following property is valid. For any $\varepsilon>0$ and for all sufficiently large $n\in\mathbb{N}$*

$$\|B(\lambda^{(n)},\pi^{(n)})\| \le \exp(\varepsilon n)$$

Proof. According to Proposition 1 in [70] function $\log B(\lambda,\pi)$ is integrable. Hence for almost all (λ,π)

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\|B(\lambda^{(k)},\pi^{(k)})\| = \int \log\|B\|\,d\mu$$

Convergence of the series above implies that

$$\frac{1}{n}\log\|B(\lambda^{(k)},\pi^{(k)})\| \to 0$$

which completes the proof of Lemma 7. □

Now we can prove Proposition 2.

Proof. We will again use formula (15) to unfold $S(X,T,x,N)$ as we have done already in subsection 4.9. But this time we will use not only special "multiplicative times" n_i as we have done before, but we will use all points of the trajectory (λ,π), $\mathcal{G}(\lambda,\pi)$, $\mathcal{G}^{(2)}(\lambda,\pi)$, ... We use notations from subsection 4.9, (see (17)).

$$S(X,T,x,N) =$$
$$= \left(S^+_{(0)} - S^-_{(0)}\right) + B(\lambda,\pi)\cdot\left(S^+_{(1)} - S^-_{(1)}\right) + B^{(2)}(\lambda,\pi)\cdot\left(S^+_{(2)} - S^-_{(2)}\right) + \dots$$

We prefer to use the notation as if we have an infinite sum. One should understand, that actually the sum is finite: starting with some term all the other terms in the tail are equal to zero.

Consider the pairing of f with $S(X,T,x,N)$. Using the expression above we can rewrite it as follows:

$$\begin{aligned}
&\left|\langle f\,,\,S(X,T,x,N)\rangle\right| \le \\
&\qquad\qquad \le \left|\langle f\,,\,S^+_{(0)}-S^-_{(0)}\rangle\right| + \left|\langle f\,,\,B^{(1)}(\lambda,\pi)\cdot(S^+_{(1)}-S^-_{(1)})\rangle\right| + \dots\\
&\qquad\qquad \le \left\|f\right\|\cdot\left\|S^+_{(0)}-S^-_{(0)}\right\| + \left\|{}^tB^{(1)}(\lambda,\pi)f\right\|\cdot\left\|S^+_{(1)}-S^-_{(1)}\right\| + \dots\\
&\qquad\qquad \le \sum_{k=0}^{+\infty}\left(\left\|{}^tB^{(k)}(\lambda,\pi)f\right\|\cdot\max_{1\le j\le m}\|B^{(k)}(\lambda,\pi)\cdot e_j\|\right)
\end{aligned}$$

Chose some $\varepsilon<|\theta(f)|/3$. According to Lemma 7 for k large enough

$$\log\max_{1\le j\le m}\|B^{(k)}(\lambda,\pi)\cdot e_j\|\le\varepsilon k$$

On the other hand by definition of a Lyapunov exponent for sufficiently large k one has

$$\log\left\|{}^tB^{(k)}(\lambda,\pi)f\right\|\le(\theta(f)+\varepsilon)k$$

Thus for sufficiently large k the positive series above is majorated by the converging series $\sum\exp(-\varepsilon k)$. We have got a uniform bound which does not depend neither on x nor on N, but still depends on f. It remains to note that having proved the statement for some basis in $Ann(\mathcal{H}_g(\lambda,\pi))$ we will complete the proof of Proposition 4. $\square$

Appendix A. Irreducibility of the cocycle ${}^tB(\lambda,\pi)$

Lemma 8. *If $\theta(e_j-\lambda_j e_0)<\theta_2$ on some subset of nonzero measure in $\Delta^{m-1}\times\pi$ then the same is true for almost all points of this simplex.*

Proof. Consider the subspace

$$Ann(H_{(2)}(\lambda,\pi))=\{e\,|\,\frac{1}{k}\log\|{}^tB^{(k)}e\|<\theta_2\}$$

and two dimensional plane spanned by e_0 and e_j. The condition above is equivalent to the condition that these two subspaces have nontrivial intersection on some subset of nonzero measure in $\Delta^{m-1}\times\pi$. Let

$$\dim Ann(H_{(2)}(\lambda,\pi))=d$$

We know that $d\le m-2$ (presumably $d=m-2$).

Consider a pair of vectors: one in $\Lambda^d(\mathbb{R}^m)^*$, and one in $\Lambda^2(\mathbb{R}^m)^*$, representing the subspaces $Ann(H_{(2)}(\lambda,\pi))$ and $\langle e_0;e_j\rangle_{\mathbb{R}}$. The condition that the two subspaces have nontrivial intersection is equivalent to the condition that the wedge product of corresponding vectors vanish on some subset of nonzero measure. In other words one-dimensional subspace

$$L(\lambda,\pi):=\Lambda^d\left(Ann(H_{(2)}(\lambda,\pi))\right)\subset\Lambda^d(\mathbb{R}^m)^* \tag{22}$$

obeys some constant linear relation on some subset U of nonzero measure.

Let r be the maximal possible dimension of such linear subspace $R\subset\left(\Lambda^d(\mathbb{R}^m)^*\right)^*$ that

$$L(\lambda,\pi)\subseteq Ann(R) \tag{23}$$

on some subset $W \subseteq U$ of nontrivial measure $\mu(W)$. Note that by assumption $r \geq 1$ since the vanishing of the wedge product with $\langle e_0; e_j\rangle_{\mathbb{R}}$ gives at least one linear relation.

Note also, that since R is a maximal subspace, all the linear conditions coming from the vanishing of the wedge product of $L(\lambda, \pi)$ and $\langle e_0; e_j\rangle_{\mathbb{R}}$ are contained in R. This means that if for some λ relation (23) is valid, then $\theta(e_j - \lambda_j e_0) < \theta_2$ at this point. Hence Lemma 8 would follow from the Lemma below. □

Lemma 9. *Suppose on a subset of nonzero measure in* $\Delta^{m-1} \times \pi$

$$L(\lambda, \pi) \subseteq Ann(R) \subset \Lambda^d(\mathbb{R}^m)$$

Then the same relation is true for the whole simplex $\Delta^{m-1} \times \pi$.

Proof. We may assume that for a given subset W of nonzero measure R is the maximal linear subspace such that the relation above is valid.

To prove this Lemma we will use the method due to W.Veech. First we have to remind some properties of projective linear maps.

Let A be a matrix such that $\det A = 1$, with some of the entries possibly negative. Consider projective linear map $G_A : \lambda \mapsto \frac{A\lambda}{\|A\lambda\|}$ and suppose G_A maps some compact subset $K \subset \Delta^{m-1}$ into Δ^{m-1}, $\mathrm{Im}(K) \subseteq \Delta^{m-1}$. Let J_A be Jacobian of G_A. Then according to (7.1) and (7.2) in [58]

$$\sup_{\lambda,\lambda' \in K} \frac{J_A(\lambda)}{J_A(\lambda')} \leq \sup_{\substack{\lambda,\lambda' \in K \\ 1 \leq i \leq m}} \left(\frac{\lambda_i}{\lambda'_i}\right)^m$$

Consider a subset $\Delta_\epsilon = \{\lambda \,|\, \lambda_i \geq \epsilon, \;\; i = 1, \dots, m; \;\; \sum \lambda_i = 1\}$ Then for any $K \subseteq \Delta_\epsilon$ and any matrix $A \in \mathrm{SL}(m)$ such that $A(K) \subseteq \Delta^{m-1}$ we get from the bound above, that

$$\sup_{\lambda,\lambda' \in K} \frac{J_A(\lambda)}{J_A(\lambda')} \leq \left(\frac{1}{\epsilon}\right)^m$$

Note, that this bound does not depend neither on A nor on the subset K anymore.

We remind that $\mathcal{G}^k(\lambda, \pi_0)$ is a projective linear map, i.e., there is a matrix $A = \left(B^{(k)}(\lambda, \pi)\right)^{-1}$ such that

$$\mathcal{G}^k(\lambda, \pi_0) = \left(\frac{A\lambda}{\|A\lambda\|}, \pi\right), \qquad \det A = 1$$

Consider the set $\Delta_{\mathcal{G}}(\lambda, \pi_0, k) \in \Delta^{m-1}$ of those (λ', π_0) for which $\mathcal{G}^k$ uses the same matrix A. Then $\mathcal{G}^k(\lambda, \pi_0)$ maps $\Delta_G(\lambda, \pi_0, k)$ onto one of the half-simplices $(\Delta^+(\pi), \pi)$, $(\Delta^-(\pi), \pi)$. It is known that diameters of subsimplices $\Delta_{\mathcal{G}}(\lambda, \pi_0, k)$ tend to zero as $k \to \infty$ for almost all λ. Hence up to a set of measure zero we can subdivide Δ_ϵ to subsimplices $\Delta_{\mathcal{G}}(\lambda_j, \pi_0, k)$, $\lambda_j \in \Delta_\epsilon$.

Consider now our subset W. If for some $\epsilon > 0$ we have $\mu(W \cap \Delta_\epsilon) > 0$, then, probably refining our subdivision, for any $\delta > 0$ we will find a subsimplex $\Delta_0 = \Delta_G(\lambda_0, \pi_0, k_0)$ from our subdivision such that $\mu(\bar{W} \cap \Delta_0)/\mu(\Delta_0) < \delta$. Let $(\Delta^\pm(\pi), \pi) = \mathcal{G}^{k_0}(\Delta_G(\lambda_0, \pi_0, k_0)$. Then $\mu(\mathcal{G}^{k_0}(W) \cap (\Delta^\pm(\pi))/\mu(\Delta^\pm(\pi) \geq (1 - \delta/\epsilon^m)$. Without loss of generality we may assume that $\pi = \pi_0$. Since δ can be chosen arbitrary small we can make the measure of the set $W' = \mathcal{G}^{k_0}(W) \subseteq \Delta^\pm(\pi)$ arbitrary close to the measure of $\Delta^\pm(\pi)$. In particular we may assume that W' intersects with W by a set of nonzero measure.

Note that the linear bundle $L(\lambda,\pi)$ is invariant under the action of the induced cocycle $B^{\wedge d}(\lambda,\pi)$ on the d-th exterior power $\Lambda^d(\mathbb{R}^m)^*$ of $\mathbb{R}^m$. Hence for all points of W' we get

$$L(\lambda,\pi) \subseteq Ann(R')$$

where R' is the image of the subspace R under the action of the linear map on the exterior power corresponding to the linear map A. By assumptions on maximality of R we see that R and R' coincide on $W \cup W'$, and hence they coincide everywhere. Thus for all points of W' we get

$$L(\lambda,\pi) \subseteq Ann(R)$$

Since we can make the measure of $W' \subseteq \mathcal{G}^{k_0}\Delta^{\pm}(\pi)$ arbitrary close to the measure of $\Delta^{\pm}(\pi)$ the relation above is valid for the whole $\Delta^{\pm}(\pi)$. Lemma 9 $\square$

As a byproduct of the proof we get the following

Corollary 1. *If $\theta(e_j - \lambda_j e_0) < \theta_2$ on some subset of nonzero measure in $\Delta^{\pm} \times \pi_0$ then the induced cocycle $\left(B^{-1}(\lambda,\pi)\right)^{\wedge(m-d)}$ possesses an invariant subbundle $P(\lambda,\pi) \subset \Lambda^d\mathbb{R}^m$ which is constant on every half-simplex $(\Delta^{\pm}(\pi) \times \pi)$.*

Moreover, the linear subspace $P \subset \Lambda^d\mathbb{R}^m$ corresponding to the half-simplex $\Delta^{\pm} \times \pi_0$ contains all vectors of the form

$$\underbrace{e_j \wedge e_0 \wedge \dots}_{d\ factors}$$

Proof. $\square$

A.1. **Decomposition of the representation of** $Sp(2n,\mathbb{R})$ **on** $\Lambda^2(\mathbb{R}^{2n})^*$**.** Symplectic form Ω on $\mathbb{R}^{2n}$ is an element of $\Lambda^2(\mathbb{R}^{2n})^*$. By definition of the symplectic group this element is invariant under the action of the group $Sp(2n,\mathbb{R})$. Thus we get a one-dimensional invariant subspace in $\Lambda^2(\mathbb{R}^{2n})^*$ with the trivial action of the group $Sp(2n,\mathbb{R})$ on it.

There is a natural coupling between complementary exterior powers $\Lambda^d(\mathbb{R}^{2n})^*$ and $\Lambda^{2n-d}(\mathbb{R}^{2n})^*$ compatible with the induced actions of the group $Sp(2n,\mathbb{R})$. Consider the element $\Omega^{\wedge(n-1)} \in \Lambda^{2n-2}(\mathbb{R}^{2n})^*$. This element is nontrivial, and it is invariant under the action of the group. Thus the subspace

$$Ann\Omega^{\wedge(n-1)} \subset \Lambda^2(\mathbb{R}^{2n})^*$$

is also invariant under the action of the group $Sp(2n,\mathbb{R})$. Since $\Omega \wedge \Omega^{\wedge(n-1)} = \Omega^{\wedge n}$ is just the volume element, we see that

$$\Omega \notin Ann\Omega^{\wedge(n-1)}$$

Hence the two invariant subspaces are transversal to each other.

We have got the decomposition of representation of $Sp(2n,\mathbb{R})$ on $\Lambda^2(\mathbb{R}^{2n})^*$ into direct sum of two representations

$$\Lambda^{2n-2}(\mathbb{R}^{2n})^* = \langle\Omega\rangle_{\mathbb{R}} \oplus Ann\Omega^{\wedge(n-1)} \tag{24}$$

A.2. **Decomposition of the induced cocycles on $\Lambda^2(\mathbb{R}^{2n}/Ker(\Omega))^*$ and on $\Lambda^2(\mathbb{R}^{2n}/Ker(\Omega))$.** Recall that a permutation π of m elements determines skew-symmetric $m\times m$-matrix $\Omega(\pi)$, see (2). Matrix $\Omega(\pi)$ provides us with the "degenerate symplectic form" in the fibers of $(\Delta^{m-1}\times\mathfrak{R})\times\mathbb{R}^m$. This form is preserved by the cocycle $\left(B^{(k)}\right)^{-1}$, see (10). Form Ω is in general degenerate. Since Ω is preserved by the cocycle the subbundle $Ker(\Omega)$ is invariant under the action of the cocycle; it is constant on every simplex $\Delta^{m-1}\times\pi$, where $\pi\in\mathfrak{R}$.

Recall that an interval exchange transformation (λ,π) determines a close orientable surface M_g^2 and an orientable measured foliation on it. The genus of the surface and the number and the types of the saddles are completely determined by combinatorics of the permutation. We will always consider only those permutations which correspond to the surfaces of genera $g\geq 2$.

There is natural local identification between the space $\mathbb{R}_+^m$ of interval exchange transformations with fixed permutation $\pi\in\mathfrak{S}_m^0$ and the first relative cohomology $H^1(M_g^2,\{\text{saddles}\};\mathbb{R})$ of corresponding surface M_g^2 with respect to the set of saddles of corresponding foliation.

Under local identification of the space of interval exchange transformations with relative cohomology $H^1(M_g^2,\text{saddles};\mathbb{R})$ the quotient over the subspace $Ker(\Omega)$ coincides with the absolute cohomology $H^1(M_g^2;\mathbb{R})$. The symplectic structure induced by $\Omega(\pi)$ on the quotient space coincides with the intersection form on cohomology.

Consider now the quotient cocycle. Consider the cocycle induced from the quotient cocycle on $\Lambda^2(\mathbb{R}^m/Ker(\Omega))^*$. According to the previous subsection the induced cocycle would have a pair of invariant subbundles, see (24). Each subbundle is constant on every simplex $\Delta^\pm\times\pi$.

We may apply a parallel construction to the dual cocycle as well. Now we are ready to formulate several conjectures concerning cocycles $B^{-1}(\lambda,\pi)$ and ${}^tB(\lambda,\pi)$.

A.3. **Conjectures on irreducibility of the cocycles $B^{-1}(\lambda,\pi)$ and ${}^tB(\lambda,\pi)$.** We will formulate all the statements for the cocycle ${}^tB(\lambda,\pi)$ assuming the analogous statements for the cocycle $B^{-1}(\lambda,\pi)$.

Conjecture 3. *For any Rauzy class $\mathfrak{R}$ corresponding to genus $g\geq 2$ the induced cocycle on $\Lambda^2(\mathbb{R}^m/Ker(\Omega))^*$ does not have any invariant subbundles constant on every $\Delta^\pm\times\pi$ other then those determined by decomposition (24) of representation of symplectic group into irreducible ones.*

In fact we have a stronger conjecture.

Conjecture 4. *For any Rauzy class $\mathfrak{R}$ the induced cocycle on the exterior power $\Lambda^d(\mathbb{R}^m/Ker(\Omega))^*$ for any $d=1,2,\dots,2g$ does not have any other invariant subbundles which are constant on every $\Delta^\pm\times\pi$ then those determined by corresponding decomposition of the representation of the symplectic group $Sp(2g,\mathbb{R})$ on $\Lambda^d\mathbb{R}^{2g}$ into irreducible ones.*

The most strong conjecture, which implies 4 is as follows.

Consider any permutation $\pi_0\in\mathfrak{S}_m^0$. Consider the map $\Delta^\pm\times\pi_0\to\Delta^\pm\times\pi_0$ induced from the map $\mathcal{G}$ on $\Delta^\pm\times\mathfrak{R}(\pi_0)$. Denote the corresponding cocycle induced from the cocycle $B^{-1}(\lambda,\pi)$ by $A(\lambda)$. Let G be the closure of the group generated by all matrices of the form $A^{(k)}(\lambda)$, where $k\in\mathbb{N}$ and $\lambda\in\Delta^\pm\times\pi_0$.

Conjecture 5. *The closure G of the group generated by all matrices of the form $A^{(k)}(\lambda)$ coincides with the group $Sp(2g,\mathbb{R})$.*

We have no idea how to prove Conjecture 5. On the contrary it is not that difficult to prove Conjecture 4 for any particular Rauzy class $\mathfrak{R}$. In Appendix 1 we present the proof of Conjecture 4 for the Rauzy class $\mathfrak{R}(4,3,2,1)$.

Lemma 1 immediately implies the following

Conditional Theorem 6. *For any permutation π for which Conjecture 4 is valid and for almost all λ from the simplex $\Delta^{\pm}\times\pi$ one has*

$$\theta(e_j-\lambda_j e_0)=\theta_2 \quad \text{for any } 1\le j\le m$$

Proof. Since the action of the induced cocycle on $\Omega(\pi)\in\Lambda^d(\mathbb{R}^m/Ker(\Omega))^*$ is trivial we see, that

$$\theta(\Omega(\pi),{}^tB^{\wedge 2})=0$$

On the other hand □

Appendix B. Proof of irreducibility of the cocycle for the Rauzy class $\mathfrak{R}(4,3,2,1)$

B.1. **Explicit form of geometric structures related to $\mathfrak{R}(4,3,2,1)$.** In this section we will prove validity of Conjecture 4 for the particular extended Rauzy class $\mathfrak{R}(4,3,2,1)$. This Rauzy class corresponds to an orientable measured foliation with a single 6-prongs saddle on a surface of genus 2.

In this particular case skew-symmetric bilinear form $\Omega(\pi)$ is nondegenerate, so we even do not need to consider the quotient cocycle. For $\pi=(4,3,2,1)$ matrix $\Omega(\pi)$ has the form

$$\Omega(4,3,2,1)=\begin{pmatrix} 0 & 1 & 1 & 1\\ -1 & 0 & 1 & 1\\ -1 & -1 & 0 & 1\\ -1 & -1 & -1 & 0\end{pmatrix}$$

In the basis $e_1\wedge e_2,\ e_1\wedge e_3,\ e_1\wedge e_4,\ e_2\wedge e_3,\ e_2\wedge e_4,\ e_3\wedge e_4$ in $\Lambda^2(\mathbb{R}^4)^*$ vector $\omega(4,3,2,1)$ representing skew-symmetric bilinear form $\Omega(4,3,2,1)$ has the following components:

$$\omega(4,3,2,1)=(1,1,1,1,1,1)\in\Lambda^2(\mathbb{R}^4)^*$$

The natural coupling between $\Lambda^d(\mathbb{R}^{2n})^*$ and $\Lambda^{2n-d}(\mathbb{R}^{2n})^*$ in our case determines the symmetric bilinear form $\Psi(\pi)$ on $\Lambda^2(\mathbb{R}^{2n})^*$; its value on a pair of bivectors $e_i\wedge e_j$ and $e_k\wedge e_l$ equals

$$\Psi(e_i\wedge e_j\,,\,e_k\wedge e_l)=\det e_i\wedge e_j\wedge e_k\wedge e_l \tag{25}$$

Thus in the standard coordinates in $\Lambda^2(\mathbb{R}^4)^*$ the matrix of this bilinear form equals

$$\Psi(\pi)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

The invariant subspace of codimension one

$$Ann\,(\omega(4,3,2,1))=\{\phi\mid\Psi(\phi,\omega)=0\}\subset\Lambda^2(\mathbb{R}^4)^*$$

is determined in components as

$$Ann\,(\omega(4,3,2,1)) = \{\phi \,|\, \phi_{12} - \phi_{13} + \phi_{14} + \phi_{23} - \phi_{24} + \phi_{34} = 0\}$$

We will use the basis

$$\begin{aligned} a_1 &= e_1 \wedge e_2 - e_3 \wedge e_4 \\ a_2 &= e_1 \wedge e_3 + e_3 \wedge e_4 \\ a_3 &= e_1 \wedge e_4 - e_3 \wedge e_4 \\ a_4 &= e_2 \wedge e_3 - e_3 \wedge e_4 \\ a_5 &= e_2 \wedge e_4 + e_3 \wedge e_4 \end{aligned} \tag{26}$$

in $Ann\,(\omega(4,3,2,1)) \subset \Lambda^2(\mathbb{R}^4)^*$.

B.2. **Choice of concrete trajectories.** We choose three trajectories of the map $\mathcal{G}$ more or less by chance:

The first trajectory is determined by the following path on the Rauzy graph of the Rauzy class $\mathfrak{R}(4,3,2,1)$:

$$\begin{aligned} &\{4,3,2,1\} \xrightarrow{b} \\ &\{4,1,3,2\} \xrightarrow{a} \\ &\{3,1,4,2\} \xrightarrow{b} \\ &\{3,1,4,2\} \xrightarrow{a} \{4,1,3,2\} \xrightarrow{a} \{3,1,4,2\} \xrightarrow{a} \\ &\{4,1,3,2\} \xrightarrow{b} \{4,2,1,3\} \xrightarrow{b} \\ &\{4,3,2,1\} \xrightarrow{a} \{2,4,3,1\} \xrightarrow{a} \\ &\{3,2,4,1\} \xrightarrow{b} \\ &\{3,2,4,1\} \xrightarrow{a} \\ &\{4,3,2,1\} \end{aligned}$$

Corresponding matrix ${}^tB$ equals

$${}^tB_1 = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 1 & 6 & 4 & 3 \\ 1 & 4 & 4 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

In the basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$ in $\Lambda^2(\mathbb{R}^4)^*$ corresponding matrix ${}^tB_1^{\wedge 2}$ of the induced cocycle ${}^tB^{\wedge 2}(\lambda, \pi)$ equals

$${}^tB_1^{\wedge 2} = \begin{pmatrix} 3 & 2 & 1 & 0 & -3 & -2 \\ 1 & 2 & 0 & 4 & -2 & -4 \\ -1 & -1 & 0 & -1 & 2 & 2 \\ -2 & 0 & -1 & 8 & 0 & -4 \\ -4 & -3 & -1 & -2 & 6 & 5 \\ -2 & -3 & 0 & -4 & 4 & 6 \end{pmatrix}$$

The restriction of this operator to the invariant subspace $Ann\,(\omega(4,3,2,1))$ has the following matrix in coordinates 26:

$$C_1 = \begin{pmatrix} 5 & 0 & 3 & 2 & -5 \\ 5 & -2 & 4 & 8 & -6 \\ -3 & 1 & -2 & -3 & 4 \\ 2 & -4 & 3 & 12 & -4 \\ -9 & 2 & -6 & -7 & 11 \end{pmatrix}$$

Consider the other path on the Rauzy graph of the Rauzy class $\mathfrak{R}(4,3,2,1)$:

$$\begin{aligned}
&\{4,3,2,1\} \xrightarrow{b} \\
&\{4,1,3,2\} \xrightarrow{a} \\
&\{3,1,4,2\} \xrightarrow{b} \{4,1,3,2\} \xrightarrow{b} \{3,1,4,2\} \xrightarrow{b} \{4,1,3,2\} \xrightarrow{b} \{3,1,4,2\} \xrightarrow{b} \\
&\{3,1,4,2\} \xrightarrow{a} \\
&\{4,1,3,2\} \xrightarrow{b} \\
&\{4,2,1,3\} \xrightarrow{a} \{4,3,2,1\} \xrightarrow{a} \\
&\{2,4,3,1\} \xrightarrow{b} \{3,2,4,1\} \xrightarrow{b} \\
&\{3,2,4,1\} \xrightarrow{a} \\
&\{4,3,2,1\}
\end{aligned}$$

Corresponding matrix ${}^tB$ equals

$${}^tB_2 \begin{pmatrix} 1 & 6 & 1 & 2 \\ 1 & 12 & 2 & 3 \\ 1 & 8 & 2 & 2 \\ 1 & 3 & 0 & 2 \end{pmatrix}$$

In the basis $e_1\wedge e_2$, $e_1\wedge e_3$, $e_1\wedge e_4$, $e_2\wedge e_3$, $e_2\wedge e_4$, $e_3\wedge e_4$ in $\Lambda^2(\mathbb{R}^4)^*$ corresponding matrix ${}^tB_2^{\wedge 2}$ of the induced cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$ equals

$${}^tB_2^{\wedge 2} = \begin{pmatrix} 6 & 1 & 1 & 0 & -6 & -1 \\ 2 & 1 & 0 & 4 & -4 & -2 \\ -3 & -1 & 0 & -3 & 6 & 2 \\ -4 & 0 & -1 & 8 & 0 & -2 \\ -9 & -2 & -1 & -6 & 15 & 4 \\ -5 & -2 & 0 & -6 & 10 & 4 \end{pmatrix}$$

The restriction of this operator to the invariant subspace $Ann\,(\omega(4,3,2,1))$ has the following matrix in coordinates 26:

$$C_2 = \begin{pmatrix} 7 & 0 & 2 & 1 & -7 \\ 4 & -1 & 2 & 6 & -6 \\ -5 & 1 & -2 & -5 & 8 \\ -2 & -2 & 1 & 10 & -2 \\ -13 & 2 & -5 & -10 & 19 \end{pmatrix}$$

Consider the third path on the Rauzy graph of the Rauzy class $\mathfrak{R}(4,3,2,1)$:

$$\{4,3,2,1\} \xrightarrow{b} \{4,1,3,2\} \xrightarrow{b} \{4,2,1,3\} \xrightarrow{b} \{4,3,2,1\} \xrightarrow{b}$$
$$\{4,1,3,2\} \xrightarrow{b} \{4,2,1,3\} \xrightarrow{b} \{4,3,2,1\} \xrightarrow{b}$$

Corresponding matrix ${}^tB$ equals

$$ {}^tB_2 \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

In the basis $e_1\wedge e_2$, $e_1\wedge e_3$, $e_1\wedge e_4$, $e_2\wedge e_3$, $e_2\wedge e_4$, $e_3\wedge e_4$ in $\Lambda^2(\mathbb{R}^4)^*$ corresponding matrix ${}^tB_3^{\wedge 2}$ of the induced cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$ equals

$$ {}^tB_3^{\wedge 2} = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} $$

The restriction of this operator to the invariant subspace $Ann\,(\omega(4,3,2,1))$ has the following matrix in coordinates 26:

$$ C_3 = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 2 & -1 & 4 & 2 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & -2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} $$

B.3. **Proof of irreducibility.** All the matrices under consideration are integer. In our proof of irreducibility we avoid any usage of eigenvectors and eigenvalues, restricting ourselves to integer computations only. Mostly we will use the following obvious

Lemma 10. *If operators A_1 and A_2 posses common one-dimensional invariant subspace K^1 then $K^1 \subset Ker(A_1A_2 - A_2A_1)$.*

Proof. □

Proposition 5. *Cocycles $B^{-1}(\lambda,\pi)$ and ${}^tB(\lambda,\pi)$ are irreducible on $\Delta^3 \times \mathfrak{R}$.*

Proof. First note that cocycles under consideration are conjugate. So irreducibility of one of them would imply irreducibility of the other.

Since

$$\det\left({}^tB_1{}^tB_2 - {}^tB_2{}^tB_1\right) = 4$$

Lemma 10 implies that this two operators do not have any common invariant one-dimensional subspace. Hence the cocycle ${}^tB(\lambda,\pi)$ does not have any invariant one-dimensional subspace.

Since the cocycle ${}^tB(\lambda,\pi)$ is symplectic the presence of an invariant subspace of codimension one implies the presence of a one-dimensional invariant subspace — of the annulator of the invariant subspace of codimension one with respect to the symplectic form. Thus in our case the absence of invariant one-dimensional subspaces implies the absence of three-dimensional invariant subspaces.

The absence of a two-dimensional invariant subspace of the cocycle ${}^tB(\lambda,\pi)$ is equivalent to the absence of an invariant one-dimensional subspace of the induced cocycle on the second exterior power $\Lambda^2(\mathbb{R}^4)^*$ with additional condition that the one-dimensional subspace is generated by a two-vector. Let us proceed with the induced cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$.

The one-dimensional invariant subspace generated by the vector $\omega(4,3,2,1) = (1,1,1,1,1,1)$ does not obey the Plucker relations

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$$

Hence this one-dimensional subspace is not generated by a two-vector. We are going to prove that the cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$ does not posses any other one-dimensional invariant subspaces. To prove that it is sufficient to prove that the induced cocycle on $Ann\,(\omega(4,3,2,1))$ does not posses any one-dimensional invariant subspaces.

The kernel $Ker(C_1C_3 - C_3C_1)$ is spanned by the vector

$$Ker(C_1C_3 - C_3C_1) = \langle(\frac{1}{2},2,1,\frac{1}{2},1)\rangle_{\mathbb{R}}$$

while the kernel $Ker(C_1C_3 - C_3C_1)$ is spanned by the vector

$$Ker(C_2C_3 - C_3C_2) = \langle(1,\frac{5}{2},1,\frac{1}{2},1)\rangle_{\mathbb{R}}$$

Since these two vectors do not coincide, the operators C_1, C_2, C_3 do not posses common one-dimensional subspace. In other words restriction of the induced cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$ to the subspace $Ann\,(\omega(4,3,2,1))$ does not have any invariant one-dimensional subspace. This in turn implies that the cocycle ${}^tB(\lambda,\pi)$ dos not have any invariant two-dimensional subspaces.

The induced cocycle on $Ann\,(\omega(4,3,2,1))$ preserves nondegenerate symmetric bilinear form 25. Hence invariant subspaces of dimension d and of codimension d appear only in pairs. Since we proved that this cocycle does not have any invariant one-dimensional subspaces, it does not posses any four-dimensional invariant subspaces as well.

It remains to prove that the restriction of the induced cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$ to the subspace $Ann\,(\omega(4,3,2,1))$ does not have any invariant two-dimensional subspace. This would imply the absence of any invariant three-dimensional subspaces. This would mean that there are no nontrivial invariant subspaces in $Ann\,(\omega(4,3,2,1))$ at all. This would mean that there are no other nontrivial invariant subspaces in $\Lambda^2(\mathbb{R}^4)^*$ than the one spanned by $\omega(4,3,2,1)$ and the annulator to it.

To prove that the restriction of the induced cocycle ${}^tB^{\wedge 2}(\lambda,\pi)$ to the subspace $Ann\,(\omega(4,3,2,1))$ does not have any invariant two-dimensional subspace it is sufficient to prove that the induced cocycle on the second exterior power $\Lambda^2 Ann\,(\omega(4,3,2,1))$ does not posses any invariant one-dimensional subspaces. Consider matrices D_1, D_2, D_3 corresponding to our three paths. It is easy to check that two-dimensional kernel $Ker(D_1D_2 - D_2D_1)$ is transversal to the four-dimensional kernel $Ker(D_1D_3 - D_3D_1)$. Hence according to Lemma 10 the induced cocycle on the second exterior power $\Lambda^2 Ann\,(\omega(4,3,2,1))$ does not posses any invariant one-dimensional subspaces.

It remains to prove that the induced cocycle ${}^tB^{\wedge 3}(\lambda, \pi)$ on the third exterior power $\Lambda^3(\mathbb{R}^4)^*$ does not posses any invariant subspaces. This is true due to duality between $\Lambda^3(\mathbb{R}^4)^*$ and $(\mathbb{R}^4)^*$. Proposition 5 is proved. □

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References

[1] L. A. Alaniya, *Topological structure of level surfaces of Morse 1-forms*, Russian Math. Surveys, **46** (1991), no. 3, 211–212.

[2] N. E. Alekseevskii, Yu. P. Gaidukov, I. M. Lifshits, V. G. Peschanskii, JETPh **39** (1960).

[3] D. V. Anosov, *Flows on closed surfaces and behavior of trajectories lifted to the universal covering plane*, Journ. of Dyn. and Control Syst., **1** (1995), no 1, 125-138.

[4] D. V. Anosov, *On the behavior of trajectories in the Euclidean or Lobachevskiĭ plane, covering the trajectory of flows on closed surfaces. III.*, Izvestija Ross. Akad. Nauk Ser. Mat., **59** (1995), no. 2, 63–96.

[5] S. Kh. Aranson, V. Z. Grines, E. V. Zhuzhoma, *On the geometry and topology of flows and foliations on surfaces, and the Anosov problem*, Mat. Sbornik 186 (1995), no. 8, 25–66.

[6] V. I. Arnold, *Small denominators and problems of stability of motion in classical and celestial mechanics*, Russian Math. Surveys **18** (1963) no. 6, 85–191.

[7] V. I. Arnold, Huygens and Barrow, Newton and Hooke. Pioneers in mathematical analysis and catastrophe theory from evolvents to quasicrystals. Birkhäuser Verlag, Basel, 1990. 118 pp.

[8] V. I. Arnold, *Topological and ergodic properties of closed 1-forms with rationally independent periods*, Functional Anal. Appl., **25** (1991), no. 2, 81–90.

[9] V. I. Arnold, *Poly-integrable flows*, St. Petersburg Math. J., **4** (1993), no. 6, 1103–1110.

[10] V. I. Arnold, *Remarks on the Morse theory of a divergence-free vector field, the averaging method, and the motion of a charged particle in a magnetic field*, Tr. Mat. Inst. Steklova, **216** (1997), Din. Sist. i Smezhnye Vopr., 9–19.

[11] P. Arnoux, A. Nogueira, *Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles*, Ann. scient. Éc. Norm. Sup. 4^e série, **26** (1993), 645–664.

[12] E. Calabi, *An intrinsic characterization of harmonic 1-forms, Global Analysis*, Papers in Honor of K.Kodaira, (D.C.Spencer and S.Iyanaga, ed.), 1969, pp. 101–117.

[13] Yu. Chekanov, *Private communication.*

[14] A. P. Cracknell, The Fermi surfaces of metals, Taylor & Francis LTD, 1971.

[15] A. P. Cracknell and K. C. Wong, The Fermi surface, Clarendon press, Oxford, 1973.

[16] A. Denjoy, *Le phénomène ergodique et les trajectoires sur le tore*, (French) C. R. Acad. Sci. Paris, **247** (1958), 1072–1078.

[17] I. A. Dynnikov, *A proof of S. P. Novikov's conjecture for the case of small perturbations of rational magnetic fields*, Russian Math. Surveys **47** (1992), no. 3, 172–173.

[18] I. A. Dynnikov, *A proof of the conjecture of S. P. Novikov on the semiclassical motion of an electron*, Math. Notes 53 (1993), no. 5-6, 495–501.

[19] I. A. Dynnikov, *Intersections of level surfaces of pseudoperiodic functions*, Uspekhi Mat. Nauk, **49** (1994), no. 1(295), 213–214.

[20] I. A. Dynnikov, *Semi-classical motion of the electron. Proof of Novikov's conjecture in general position and counterexamples.*, in "Solitons, Geometry, and Topology: on the Crossroad", V.M.Buchstaber and S.P.Novikov (eds.), Translations of the AMS, Ser. 2, vol. 179, AMS, Providence, RI, 1997.

[21] I. A. Dynnikov, *Surfaces in 3-Torus: Geometry of Plane Sections.*, Progress in Math., Vol. 168, Birkhäuser Verlag Basel/Switzerland 1998. pp 162–177.

[22] I. A. Dynnikov, *Geometry of stability zones in S.P.Novikov's problem on semi-classical motion of an electron.*, Russian Math. Surveys **54**, (1999), no. 1, 21–60.

[23] M. Farber, *Sharpness of the Novikov inequalities*, Functional Anal. Appl., **19** (1985), no. 1, 49–59.

[24] A. Fathi, F. Laudenbach, V. Poenaru, Travaux de Thurston sur les surfaces, *Asterisque*, **66–67** (1979).

[25] S. M. Gusein–Zade, *The number of critical points of a quasiperiodic potential*, Funct. Anal. and its Appl., **23** (1989), 129–130.

[26] J. Hubbard and H. Masur, *Quadratic differentials and measured foliations*, Acta Math., **142** (1979), 221–274.

[27] A. B. Katok, *Invariant measures of flows on oriented surfaces*, Soviet Math. Dokl., **14** (1973), 1104-1108.

[28] M. Keane, *Interval exchange transformations*, Math. Z., **141**, (1975), 25–31.

[29] S. P. Kerckhoff, *Simplicial systems for interval exchange maps and measured foliations*, Ergod. Th. & Dynam. Sys., **5** (1985), 257–271.

[30] S. Kerckhoff, H. Masur, and J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math., **124** (1986), 293–311.

[31] K. M. Khanin, A. Nogueira, *Mixing for suspension flows over interval exchange transformations*, *to appear*.

[32] M. Kontsevich and A. Zorich, *Lyapunov exponents and Hodge theory*, Preprint of IHES M/97/13, 1–16.

[33] M. Kontsevich and A. Zorich, *Connected components of the moduli spaces of holomorphic differentials with prescribed singularities*, *to appear*.

[34] Le Tu Kuok Tkhang, *Structure of level surfaces of the Morse form*, Math. Notes, **44** (1988), no. 1-2, 556–562.

[35] I. M. Lifshits, M. Ya. Azbel, M. I. Kaganov, Elektronnaia Teoriya Metallov, "Nauka", 1971; English translation: Electron Theory of Metals, Consultants Bureau, 1973.

[36] A. G. Maier, *Trajectories on closed orientable surfaces*, Math. Sbornik, **12** (54) (1943), 71–84 (in Russian).

[37] N. Markley, *The Poincare-Bendixon theorem for the Klein bottle*, Trans. AMS, **135** (1969), 159-165.

[38] H. Masur. *Interval exchange transformations and measured foliations*, Ann. of Math., **115:1** (1982), 169–200.

[39] H. Masur, *Closed trajectories for quadratic differentials with an application to billiards*, Duke Math. J., **53:2** (1986), 307–314.

[40] H. Masur, *The grouth rate of trajectories of a quadratic differential*, Ergodic Theory and Dynamical Systems, **10** (1990), 151–176.

[41] S. P. Novikov, *Multivalued functions and functionals. An analogue of the Morse theory*, Dokl. Akad. Nauk SSSR, **260** (1981), no. 1, 31–35.

[42] S. P. Novikov, *The Hamiltonian formalism and a multi-valued analogue of Morse theory*, Russian Math. Surveys, **37:5** (1982), 1–56.

[43] S. P. Novikov, *Critical points and level surfaces of multivalued functions*, Proc. of the Steklov Inst. of Math., **166** (1984), 201–209.

[44] S. P. Novikov, *Quasiperiodic structures in topology*, in "Topological methods in modern mathematics" (Stony Brook, NY, 1991), 223–233, Publish or Perish, Houston, TX, 1993.

[45] S. P. Novikov, *The semiclassical electron in a magnetic field and lattice. Some problems of low dimensional "periodic" topology*, Geometric and Functional analysis, **5**, No. 2, (1995), 434–444.

[46] S. P. Novikov, A. Ya. Maltsev, *Topological phenomena in normal metals*, Letters to JETPh, **63**, No. 10, (1996), 809–813;
`http://xxx.lpthe.jussieu.fr cond-mat/9709007`.

[47] A. Pazhitnov (Pajitnov), *An analytic proof of the real part of Novikov inequalities.*, Soviet Math. Dokl., **35** (1987), no. 2, 456 – 457.

[48] A. Pazhitnov (Pajitnov), *On the sharpness of Novikov-type inequalities for manifolds with free abelian fundamental group*, Math. USSR Sbornik, **68** (1991), 351 - 389.

[49] A. Pajitnov, *C^0-generic properties of boundary operators in Novikov Complex*, this volume.

[50] D. A. Panov, *Multicomponent pseudoperiodic mappings*, Funct. Anal. Appl. 30 (1996), no. 1, 23–29.

[51] D. A. Panov, *Pseudoperiodic mappings*, this volume.

[52] G. Rauzy. *Echanges d'intervalles et transformations induites*, Acta Arith., **34** (1979), 315–328.

[53] S. Schwartzman, *Asymptotic cycles*, Annals of Mathematics, **66** (1957), 270–284.

[54] J.-C. Sikorav, *Un probleme de disjonction par isotopie symplectique dans un fibré cotangent*, Ann. Scient. Ecole Norm. Sup., **19** (1986), 543–552.

[55] Ya. G. Sinaĭ, K. M. Khanin, *Mixing of some classes of special flows over rotations of the circle.*, Functional Anal. Appl. 26 (1992), no. 3, 155–169.

[56] J. Smillie, *Private communication.*

[57] S. P. Tzarev, *Private communication.*

[58] W. A. Veech, *Projective swiss cheeses and uniquely ergodic interval exchange transformations*, Ergodic Theory and Dynamical Systems, Vol. I, in Progress in Mathematics, Birkhauser, Boston, 1981, 113–193.

[59] W. A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals of Mathematics, **115** (1982), 201–242.

[60] W. A. Veech, *The metric theory of interval exchange transformations I. Generic spectral properties*, Amer. Journal of Math., **106** (1984), 1331–1359.

[61] W. A. Veech. *The metric theory of interval exchange transformations II. Approximation by primitive interval exchanges*, Amer. Journal of Math., **106** (1984), 1361–1387.

[62] W. A. Veech, *The Teichmüller geodesic flow*, Annals of Mathematics, **124** (1986), 441–530.

[63] W. A. Veech, Moduli spaces of quadratic differentials, Journal d'Analyse Math., **55** (1990), 117–171.

[64] S. Wakon, J. Yamashita, J.Phys. Soc. Japan, **21** (1966), 1712.

[65] A. Weil, *On systems of curves on a ring-shaped surface*, J. Indian Math. Soc., **19** (1931), no 5, 109-112.

[66] A. Weil, *Les familles de curbes sur le tore*, Mat. Sbornik, **1** (1936), 779-781.

[67] A. Zorich, *The S.P.Novikov problem on the semiclassical motion of an electron in homogeneous Magnetic Field*, Russian Math. Surveys, **39:5** (1984), 287–288.

[68] A. Zorich, *The quasiperiodic structure of level surfaces of a Morse* 1*-form that is close to a rational form — the problem of S. P. Novikov*, Math. USSR — Izvestija, **31** (1988), no. 3, 635–655.

[69] A. Zorich, *Asymptotic flag of an orientable measured foliation on a surface*, in "Geometric Study of Foliations", World Sci., 1994, 479–498.

[70] A. Zorich, *Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents*, Annales de l'Institut Fourier, **46**(2) (1996), 325–370.

[71] A. Zorich, *Deviation for interval exchange transformations*, Ergodic Theory and Dynamical Systems, **17** (1997), 1477–1499.

[72] A. Zorich, *On hyperplane sections of periodic surfaces*, in "Solitons, Geometry, and Topology: on the Crossroad", V.M.Buchstaber and S.P.Novikov (eds.), *Translations of the AMS*, Ser. 2, vol. 179, AMS, Providence, RI, 1997, 173–189.

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