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ON HYPERPLANE SECTIONS OF PERIODIC SURFACES

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ABSTRACT. We study sections of a periodic two-dimensional surface $\hat{M}^2_g$, $g \geq 2$, in $\mathbb{R}^n$ by a family of parallel hyperplanes. Assuming induced map on homology $H_1(M^2_g;\mathbb{R}) \to H_1(T^n;\mathbb{R})$, where $T^n = \mathbb{R}^n/\mathbb{Z}^n$, of the corresponding underlying surface M^2_g is a monomorphism, and assuming $n \geq 4g-3$, we show that generically each unbounded intersection line γ follows one of a finite number of asymptotic directions; moreover, the deviation from the corresponding straight line is bounded by l^α, where l is the length of a piece of the intersection line, and the power α is strictly less than 1. We show that $1+\alpha$ is the value of the second Lyapunov exponent of the Teichmüller geodesic flow on the corresponding stratum of squares of holomorphic differentials.

We prove an even more precise result for a periodic surface induced as the universal Abelian cover of the image of the Abel-Jacobi map of a surface M^2_g, $g \geq 2$, endowed with a generic complex structure. In this case *all* sections by parallel hyperplanes in $\mathbb{R}^{2g}$ generically follow one and the same direction. The deviation is bounded by $l^{\alpha(g)}$, where the power $\alpha(g)$ depends only on the genus of the surface: $1+\alpha(g)$ is the value of the second Lyapunov exponent of the Teichmüller geodesic flow on the principal stratum of squares of holomorphic differentials on a surface of genus g.

1. Introduction

The study of plane sections of periodic surfaces in three-dimensional space was initiated by S.P.Novikov in connection with problems of electron transport, see [10], [11], [12]. S.P.Novikov conjectured that the nonclosed intersection lines as a rule follow one and the same asymptotic direction. The conjecture was proved in [19] for an open dense set of directions of hyperplanes, and recently the validity of the conjecture was proved for a generic choice of the surface and the family of parallel hyperplanes [4]. It was noticed in [19] that under certain assumptions on the direction of a hyperplane, the deviation of any noncompact intersection line from the asymptotic direction is uniformly bounded by a constant which does not depend on the choice of the intersection line. According to [4], this is true in the generic situation as well.

The three-dimensional situation is rather rigid however, as can be shown by certain elementary topological arguments. A hyperplane section of a periodic surface in $\mathbb{R}^n$ has much more flexibility, and three-dimensional topological arguments are not applicable anymore. However, here one can use tools from dynamics. In particular, using the results of H.Masur [9] and W.Veech [16], it is easy to prove that generically (in this paper we always use the notion "generic" in the measure-theoretical sense) the unbounded hyperplane sections of periodic surfaces follow one of several asymptotic directions. But now the deviation from this asymptotic

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direction is not bounded by a constant anymore. This paper is a first step in the description of the deviation.

We give the exact upper bound for the deviation in the generic situation. It turns out that the deviation is bounded by some power α, $0 \le \alpha < 1$, of the length of a piece of intersection line. The power α may obtain only a finite number of values for a fixed genus of the underlying compact surface. These values are expressed in terms of the second Lyapunov exponent of the Teichmüller geodesic flow on the corresponding strata in the space of quadratic differentials.

Presumably, the picture of the deviation is much more precise, see [20] for the corresponding conjecture.

To describe the hyperplane sections of a periodic surface is the same as to describe the dynamics of leaves of the corresponding orientable measured foliation on the closed orientable underling surface. It is convenient to study the topological dynamics of leaves of a measured foliation using the first return map to a transverse interval. This first return map is an interval exchange transformation. We use the ergodic properties of interval exchange transformations, and then we translate them into the language of measured foliations.

2. Formulation of results

2.1. **Topological dynamics of leaves of an orientable measured foliation on a closed orientable surface.** Consider a smooth closed orientable surface M_g^2 of genus $g \ge 2$. Choose a smooth nondegenerate Riemannian metric $g_{ij}(x)$ on M_g^2. For any two points $p_0, p_1 \in M_g^2$, denote by ρ_{p_0,p_1} the shortest geodesic on M_g^2 joining them.

Consider a closed differential 1-form ω on the surface M_g^2. Throughout this section we shall always assume that ω has isolated critical points only; moreover we shall assume that ω has neither minima, nor maxima, only saddles. Such closed 1-form determines an orientable measured foliation with saddle point singularities. In this subsection we shall allow ω to have multiple saddles, but we shall assume that ω does not have saddle connections, in particular it does not have separatrix loops.

Take some leaf γ of ω and choose a compact connected piece γ_{p_0,p_1} of it, where p_0 and p_1 are the starting point and the endpoint with respect to a fixed orientation of the leaves. Let l be the length of γ_{p_0,p_1} By

$$c_{p_0}(l) := \left[\gamma_{p_0,p_1} \cup \rho_{p_1,p_0}\right] \in H_1(M_g^2;\mathbb{R}) \tag{2.1}$$

we shall denote the homology class of the closed loop obtained by completion of the path γ_{p_0,p_1} from p_0 to p_1 along the leaf γ with the path ρ_{p_1,p_0} along the shortest geodesic.

Define a *critical leaf* as the union of two neighboring separatrix rays corresponding to the same saddle. We do not exclude critical leaves from our considerations, for example we shall consider cycles $c_{p_0}(l)$ corresponding to critical leaves γ, whether the saddle occurs between the points p_0 and p_1 or not.

Theorem 1. *Under the assumptions formulated above, for almost all orientable measured foliations with prescribed types of saddles on a closed orientable surface* M_g^2 *the following properties are valid.*

For any leaf γ, and any point $p_0 \in \gamma$

$$\lim_{l\to\infty} \frac{1}{l} c_{p_0}(l) = c$$

where the nonzero cycle $c \in H_1(M_g^2;\mathbb{R})$ is proportional to the cycle $D[\omega]$ Poincaré-dual to the cohomology class of the closed 1-form ω determining the foliation.

For any $\phi \in Ann(D[\omega]) \subset H^1(M_g^2;\mathbb{R})$, $\phi \neq 0$ any leaf γ, and any point $p_0 \in \gamma$, we have

$$\limsup_{l\to\infty} \frac{\log|\langle\phi, c_{p_0}(l)\rangle|}{\log l} \leq \alpha < 1$$

The limits above converge uniformly with respect to γ and $p_0 \in \gamma$, i.e., for any $\varepsilon > 0$, any $\phi \in Ann(D[\omega]) \subset H^1(M_g^2;\mathbb{R})$, any leaf γ, and any point $p_0 \in \gamma$ there exists $L(\varepsilon)$ such that

$$|\langle\phi, c_{p_0}(l)\rangle| \leq l^{\alpha+\varepsilon}$$

as $l > L(\varepsilon)$, where $L(\varepsilon)$ does not depend neither on γ nor on p_0.

Remark 1. Speaking of "almost all" orientable measured foliations, we keep in mind the following construction. Let ω be a closed 1-form on M_g^2 without minima or maxima, and without saddle connections. Consider sufficiently small smooth perturbations of ω preserving the types of the saddles of ω. Consider the corresponding neighborhood $U([\omega])$ of the cohomology class of ω in the space of relative cohomology of M_g^2 with respect to the set of saddles $U([\omega]) \subset H^1(M_g^2, \{saddles\};\mathbb{R})$. Then for Lebesgue almost all $[\omega'] \in U([\omega])$ and for any small perturbation ω' of ω representing the cohomology class $[\omega']$, the statement of the Theorem is valid. The space $H^1(M_g^2, \{saddles\};\mathbb{R})$ plays the role of a local coordinate chart in the space of orientable measured foliations with prescribed singularities.

Remark 2. Actually for any $\phi \in Ann(D[\omega]) \subset H^1(M_g^2;\mathbb{R})$, outside of a linear subspace of nontrivial codimension in $Ann(D[\omega])$, for any leaf γ, and for any point $p_0 \in \gamma$, the equality

$$\limsup_{l\to\infty} \frac{\log|\langle\phi, c_{p_0}(l)\rangle|}{\log l} = \alpha$$

is valid. To avoid overloading of the paper by machinery of ergodic theory, we leave the proof of this fact for a separate paper.

2.2. **The Lyapunov exponents of the Teichmüller geodesic flow.** It turns out that the values of the powers α are the "universal constants" determined by the dynamics of the Teichmüller geodesic flow. We shall recall only some basic notions concerning the Teichmüller geodesic flow. One can find a detailed presentation of the subject in the original papers [5], [9], [17], [7], [18].

Recall that the moduli space $\mathcal{Q}_\}$ of holomorphic quadratic differentials is the cotangent bundle over the moduli space $\mathcal{M}_\}$ of complex structures on a closed surface of genus g. The Teichmüller geodesic flow is a flow on $\mathcal{Q}_\}$.

There is a subspace $\mathcal{S}_\} \subset \mathcal{Q}_\}$ of squares of holomorphic differentials in the moduli space $\mathcal{Q}_\}$. This subspace is invariant under the Teichmüller geodesic flow. The other natural subspaces in $\mathcal{Q}_\}$ invariant under the action of the flow are the subspaces of holomorphic quadratic differentials having the same numbers of zeros of orders $1, 2, \ldots$. Thus the moduli space $\mathcal{Q}_\}$ is stratified according to possible values of the *symbol* $\sigma(q) = (n(\cdot), \varepsilon)$ of the holomorphic quadratic differential q. Here $n(d)$, $d = 1, 2, \ldots$, is the number of zeros of q of order d, and $\varepsilon = +1$ or -1

as quadratic differential q is or is not a square of a holomorphic differential. Recall that the total number of zeros of a holomorphic quadratic differential taken with their multiplicities satisfies $\sum d \cdot n(d) = 4g-4$, where g is the genus of the surface. Thus the number of strata in $\mathcal{Q}_{\}}$ is finite.

We shall denote by $\mathcal{QD}_{\}}$ the moduli subspace of holomorphic quadratic differentials with area one.This subspace is also preserved by the Teichmüller geodesic flow. The stratification of $\mathcal{Q}_{\}}$ described above induces a stratification of $\mathcal{QD}_{\}}$ invariant under the action of the Teichmüller geodesic flow.

It is proved in the papers [9], [17], [7], [18] that the Teichmüller geodesic flow is ergodic on each connected component of each stratum in the space $\mathcal{QD}_{\}}$. The corresponding invariant measures are finite and absolutely continuous.

In our current study we are interested only in the strata of squares of holomorphic differentials. It was noticed in [18] that in general such strata may have several connected components. Still the number of connected components is always finite (and presumably rather small). One of the possible ways to enumerate all connected components of a given stratum is to enumerate them by means of the *extended Rauzy classes*, see [18].

We are mostly interested in the principal stratum of squares of holomorphic differentials. This stratum consists of the squares of generic holomorphic differentials that have only simple zeros. This stratum is connected.

We use a parametrization of the Teichmüller geodesic flow, where the leading Lyapunov exponent equals 2. It is known that the second Lyapunov exponent is strictly less than the first one; we denote it by $1+\nu_2$, where $1 \leq 1+\nu_2 < 2$.

Consider a closed differential form ω under assumptions preceding Theorem 1. It is easy to endow the surface M_g^2 with a complex structure for which the form ω is the real part of a holomorphic differential. Let q be the square of this holomorphic differential. Consider the connected component of the stratum in $\mathcal{QD}_{\}}$ containing q. Though q is not uniquely determined by the form ω, the connected component is uniquely determined. Let $1+\nu_2$ be the second Lyapunov exponent of the Teichmüller geodesic flow on this component.

Theorem 2. *The power α in Theorem 1 coincides with ν_2, where $1+\nu_2$ is the second Lyapunov exponent of the Teichmüller geodesic flow on the corresponding connected component of the corresponding stratum in the moduli space $\mathcal{QD}_{\}}$ of quadratic differentials.*

We reserve the notation $\alpha(g) = \nu_2(g)$ for the second Lyapunov exponent $1+\nu_2$ of the Teichmüller geodesic flow on the principal stratum of squares of holomorphic differential on a surface of genus g.

In the Appendix, we present the list of all connected components of all strata of squares of holomorphic differentials for surfaces of genera 2 and 3. We also present the corresponding approximate values of α obtained by computer experiments.

2.3. **Hyperplane sections of periodic surfaces.** Consider a smooth surface $\hat{M}_g^2 \subset \mathbb{R}^n$ smoothly embedded in $\mathbb{R}^n$. We assume that the space $\mathbb{R}^n$ is supplied with the natural cubic lattice, and that the surface $\hat{M}_g^2$ is periodic, i.e., invariant under translations over the lattice $\mathbb{Z}^n$. We assume that the quotient $M_g^2 = \hat{M}_g^2/\mathbb{Z}^n$ is a smooth closed connected orientable surface of genus $g \geq 2$, and that the induced mapping on homology $H_1(M_g^2;\mathbb{R}) \to H_1(T^n;\mathbb{R})$, where $T^n = \mathbb{R}^n/\mathbb{Z}^n$, is a monomorphism. Moreover, we assume that $n \geq 4g-3$. Finally we assume that we deal with a generic embedding satisfying the conditions above.

Now consider a generic family of parallel hyperplanes in $\mathbb{R}^n$. and the induced family of hyperplane sections of $\hat{M}^2_g$. The surface $\hat{M}^2_g$ splits into a finite number of periodic components of two types. The components of the first type are filled with closed intersection lines, i.e., filled with intersection lines diffeomorphic to a circle. The components of the other type are filled with nonclosed (i.e., diffeomorphic to $\mathbb{R}$) intersection lines. We are interested only in the nonclosed intersection lines.

Choose orientation on the intersection lines. Take a nonclosed intersection line γ on $\hat{M}^2_g$, and fix a point p_0 on it. We assume that $\mathbb{R}^n$ is provided with a Euclidean structure; it induces the Riemannian metric on $\hat{M}^2_g$, and hence a parametrization l on γ. Consider a compact connected piece γ_{p_0,p_1} of γ, where p_0 and p_1 are the starting point and the endpoint respectively for the chosen orientation of the intersection lines. Let l be the length of γ_{p_0,p_1}, i.e., $p_1 = p(l)$. By $\vec{v}_{p_0}(l) \in \mathbb{R}^n$ we denote the vector from p_0 to $p(l) = p_1$.

Under the quotient by $\mathbb{Z}^n$, the foliation by hyperplane sections obtained on $\hat{M}^2_g$ projects to the foliation on M^2_g. The components of $\hat{M}^2_g$ filled with compact intersection lines project to components on M^2_g filled with closed leaves. Assuming we are in a generic situation, these components are diffeomorphic either to discs or to cylinders. The components on $\hat{M}^2_g$ filled with noncompact intersection lines project to the minimal components $M_i \subseteq M^2_g$, $i = 1, \ldots, q$, $q \leq g$, where the leaves are everywhere dense. These components are diffeomorphic to surfaces $M^2_{g_i}$, $i = 1, \ldots, q$, with holes. The sum of the genera $g_1 + \cdots + g_q$ equals g. Note that the topology of this decomposition is stable under small perturbations of a generic direction of the family of hyperplanes.

Theorem 3. *Consider a generic periodic surface $\hat{M}^2_g \subset \mathbb{R}^n$ such that the induced map $H_1(M^2_g;\mathbb{R}) \to H_1(T^n;\mathbb{R})$ is a monomorphism, and $n \geq 4g-3$. Consider the sections of $\hat{M}^2_g$ by a family of parallel hyperplanes. For almost all directions of the hyperplanes, the following properties are valid.*

The number q of periodic components filled with nonclosed intersection lines satisfies $1 \leq q \leq g$.

Each periodic component of $\hat{M}^2_g$ filled with nonclosed intersection lines determines an asymptotic direction $\vec{a}_i$ such that for any nonclosed intersection line γ in this component and any point $p_0 \in \gamma$ we have

$$\lim_{l\to\infty} \frac{1}{l}\vec{v}_{p_0}(l) = \vec{a}_i, \quad 1 \leq i \leq q$$

Consider the straight line $\mathcal{L}_{\surd'}$ passing through the point p_0 in the direction $\vec{a}_i$. For all nonclosed intersection lines γ in the periodic component and all points $p_0 \in \gamma$ we have the inequality

$$\limsup_{l\to\infty} \frac{\log\left(dist(p(l), \mathcal{L}_{\surd'})\right)}{\log \updownarrow} \leq \alpha(g_i) < 1$$

The limits above converge uniformly with respect to γ and $p_0 \in \gamma$. Here $1 + \alpha(g)$ is the value of the second Lyapunov exponent of the Teichmüller geodesic flow on the principal stratum of squares of holomorphic differentials on the surface of genus g, *and g_i is the genus of the corresponding minimal component of the quotient surface $M_g^2/\mathbb{Z}^n$.*

Remark 3. For every nonclosed intersection line γ and any point $p \in \gamma$, we actually have the equality

$$\limsup_{l\to\infty} \frac{\log\left(dist(p(l),\mathcal{L}_{\surd'})\right)}{\log \updownarrow} = \alpha(g_i) \ ,$$

see Remark 2.

2.4. **Abel-Jacobi periodic surfaces.** Now let us assume that the surface M_g^2 is provided with a complex structure. Consider the Abel-Jacobi map $A : M_g^2 \to J(M_g^2)$. We may consider $J(M_g^2)$ as a real $2g$-dimensional torus. Note that the map A is an embedding. Thus we obtain a periodic surface $\hat{M}_g^2$ in $\mathbb{R}^{2g}$ induced in the universal abelian cover of $J(M_g^2)$. For these particular periodic surfaces the following theorem is valid.

Theorem 4. *Consider generic complex structure on a surface M_g^2 of genus g, the universal abelian cover of the Abel-Jacobi map*

$$\begin{array}{ccc}
\hat{M}_g^2 & \longrightarrow & \mathbb{R}^{2g} \\
\big\downarrow & & \big\downarrow \\
M_g^2 & \longrightarrow & T^{2g} = J(M_g^2)
\end{array}$$

and the intersection lines of periodic surface $\hat{M}_g^2$ with a family of parallel hyperplanes in $\mathbb{R}^{2g}$. Under a generic choice of direction of hyperplanes, all intersection lines follow one and the same direction:

$$\lim_{l\to\infty} \frac{1}{l}\vec{v}_{p_0}(l) = \vec{a}$$

Consider the straight line $\mathcal{L}_{\surd'}$ passing through some point $p_0 \in \gamma$ in direction $\vec{a}$. For all intersection lines γ and all points $p_0 \in \gamma$, we have

$$\limsup_{l\to\infty} \frac{\log\left(dist(p(l),\mathcal{L}_{\surd'})\right)}{\log \updownarrow} \le \alpha(g) < 1$$

The limits above converge uniformly with respect to γ and $p_0 \in \gamma$. Here $1+\alpha(g)$ is the value of the second Lyapunov exponent of the Teichmüller geodesic flow on the principal stratum of squares of holomorphic differentials on a surface of genus g.

Remark 4. For every nonclosed intersection line γ and any point $p \in \gamma$, we actually have the equality

$$\limsup_{l\to\infty} \frac{\log\left(dist(p(l),\mathcal{L}_{\surd'})\right)}{\log \updownarrow} = \alpha(g) \ ,$$

cf. Remarks 2 and 3.

The theorem above is an answer to a question of I.K.Babenko.

We prove Theorem 1 in section 3. Theorem 3 is a corollary of Theorem 1; it is proved in section 4. At the end of section 3 we also prove Theorem th:AJ, which immediately follows from Theorem 1. In the Appendix we present the list of all connected components of all strata of squares of holomorphic differentials for the surfaces of genera 2 and 3. We also present the corresponding approximate values of α obtained by computer experiments.

3. Topological dynamics of leaves of an orientable measured foliation on a closed orientable surface

In this section we prove Theorem 1. We start with several preliminary comments.

For any two points $p_0, p_1 \in M_g^2$ fix some path $\rho_{p_0,p_1} \subset M_g^2$ joining them, i.e., $\rho_{p_0,p_1} : [0;1] \to M_g^2$, is a continuous map such that $\rho_{p_0,p_1}(0) = p_1$, $\rho_{p_0,p_1}(1) = p_2$. We do not assume that ρ_{p_0,p_1} depends continuously on the parameters p_0 and p_1, but we assume that the lengths of the paths (in terms of the metric g_{ij}) are uniformly bounded:

$$\sup_{p_0,p_1 \in M_g^2} l(\rho_{p_0,p_1}) < \infty \tag{3.1}$$

In particular, by defining ρ_{p_0,p_1} as the shortest geodesic joining p_0 and p_1 we satisfy (3.1).

Consider another family ρ'_{p_0,p_1} satisfying condition (3.1). The pair ρ_{p_0,p_1}, ρ'_{p_0,p_1} defines the difference map $d_{\rho,\rho'} : M_g^2 \times M_g^2 \to H_1(M_g^2;\mathbb{Z})$. Consider a norm on $H_1(M_g^2;\mathbb{Z})$ coming from some Euclidean structure. Note that (3.1) implies the following obvious

Lemma 1. *The image of the difference map $d_{\rho,\rho'}$ is bounded in $H_1(M_g^2;\mathbb{Z})$.*

Thus if Theorem 1 is valid for any particular choice of the family ρ_{p_0,p_1}, it is valid for any other family satisfying (3.1).

Throughout this section we always consider only those closed 1-forms that give rise to uniquely ergodic foliations without saddle connections (and separatrix loops in particular).

3.1. Asymptotic cycle.

Now we can prove Theorem 1. The idea of the proof is to reduce the two-dimensional problem of dynamics of leaves of an orientable measured foliation to the one-dimensional problem of dynamics of the first return map to some transverse interval. In this subsection we prove the first part of the Theorem concerning the asymptotic cycle. In fact, part of the statement is widely known in folklore. What is new in our proof is *uniform* convergence to the asymptotic cycle for *all* leaves.

Note that a generic orientable measured foliation as described above is minimal, i.e., every leaf is dense on the surface. Thus taking a transverse interval X to the foliation, we get the first return map $T : X \to X$. The map T is an *interval exchange transformation.*

Recall the notion of interval exchange transformation. Consider an interval X, and cut it into m subintervals $X_1 \sqcup \cdots \sqcup X_m$ of lengths $\lambda_1, \ldots, \lambda_m$. Now glue the subintervals together in a different order according to some permutation $\pi \in \mathfrak{S}_\mathfrak{m}$ and preserving the orientation. We again obtain an interval X of the same length, and hence we get a mapping $T : X \to X$, which is called interval exchange transformation. Our mapping is piecewise linear, and it preserves the orientation and Lebesgue measure. It is singular at the cut points unless two consecutive intervals separated by a cut point are mapped to consecutive intervals in the image.

Define the following piecewise-constant function $c(x)$ on X with values in the first homology group $H_1(M_g^2;\mathbb{R})$. Let $x \in X_j$, $1 \le j \le m$. Consider the following closed path on M_g^2: we start at the left endpoint of the interval X and follow the interval X till we get to the point x. We continue by following the leaf γ passing through x in the positive direction until we get to the interval X for the first time; note that by definition we get to the point $T(x)$. We complete by joining the point $T(x)$ to the left endpoint of X. We define the cycle $c(x) \in H_1(M_g^2;\mathbb{R})$ to be the cycle represented by the indicated closed path:

$$c(x) := \big[\ [0;x] \cup \gamma_{x,Tx} \cup [Tx,0]\big] \in H_1(M_g^2;\mathbb{R}) \tag{3.2}$$

Note that for any j, $1 \le j \le m$, and for any two points $x_1, x_2 \in X_j$ we have $c(x_1) = c(x_2)$. We denote the corresponding cycle by c_j,

$$c_j := c(x_j) \quad \text{where } x_j \text{ is any point of } X_j \tag{3.3}$$

Recall that according to the results of [9], [16] almost all interval exchange transformations are uniquely ergodic with respect to Lebesgue measure on the interval. Assume for convenience that the length of the interval X is normalized, $|X| = 1$. Then applying the ergodic theorem to the function $c(x)$, we get

$$\lim_{k\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c(T^k(x)) = \sum_{j=1}^{m} \lambda_j c_j := c \tag{3.4}$$

for almost all $x \in X$.

According to [14], the *asymptotic cycle* c in (3.4) is proportional to the Poincaré dual $D[\omega]$, where $[\omega]$ is the cohomology class of the closed 1-form ω determining the foliation.

In (3.4) we have a "discrete parametrization" of the leaf of our foliation by the number n of returns to the transversal. Let us replace the discrete parametrization by the continuous one, by the length of corresponding piece of leaf. Consider a nondegenerate Riemannian metric on the initial surface M_g^2. Define the following function $l(x)$ on X. Let γ be the leaf passing through the point $x \in X$. Consider the piece $\gamma_{x,Tx}$ of γ between the points x and $T(x)$; let

$$l(x) := l(\gamma_{x,Tx}) \tag{3.5}$$

be its length. The function $l(x)$ is bounded

$$0 < l_{min} \le l(x) \le l_{max} < \infty \qquad \text{for all } x \in X \tag{3.6}$$

By $\bar{l}$ we denote the ergodic mean of $l(x)$:

$$\lim_{k\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} l(T^k(x)) = \int_X l(x)\, dx := \bar{l} \tag{3.7}$$

Remark 5. The statement of the ergodic theorem can be slightly strengthened for a generic interval exchange transformation. It follows from [22] that the limits (3.4) and (3.7) converge for all points $x \in X$; moreover the convergence is uniform.

For any point $p \in M_g^2$ define $x(p) \in X$ as follows. We emit the leaf γ from the point p in the negative direction. We define $x(p)$ to be the first intersection of γ with the transversal X. If $p \in X$, then we define $x(p) := p \in X$. If the leaf γ hits one of the saddles before meeting the transversal X, by convention we prolong γ along the next separatrix in the clockwise direction. For each saddle choose some incoming separatrix; if p is already a saddle point, we use this separatrix to follow.

Define the path ρ_{p_0,p_1} joining two points $p_0, p_1 \in M_g^2$ as follows. We start at p_0, then we follow the leaf γ passing through p_0 in the negative direction untill we get to $x(p_0) \in X$. Then we join $x(p_0)$ with the left endpoint of X along the transversal X. Then we join the left endpoint of X with $x(p_1)$ along the transversal X. Finally we join $x(p_1)$ with p_1 along the corresponding leaf. Due to (3.6), condition (3.1) is satisfied.

Consider a point p_0 in M_g^2, a leaf γ passing through p_0, and a piece γ_{p_0,p_1} of the leaf γ of length l emitted from p_0 in the positive direction, where p_1 is its endpoint. Consider the piece $\gamma_{x(p_0),x(p_1)}$ of the same leaf bounded by the points $x(p_0)$ and $x(p_1)$; let $\tilde{l}(p_0,l)$ be its length. By construction and due to definition (2.1), we have

$$c_{p_0}(l) = c_{x(p_0)}(\tilde{l}) \tag{3.8}$$

Note that (3.6) implies

$$|\tilde{l}(p_0,l) - l| \leq l_{max} \tag{3.9}$$

Note that by construction there is some nonnegative integer $n = n(p_0,l)$ such that $x(p_1) = T^n(x(p_0))$, and

$$\tilde{l}(p_0,l) = \sum_{k=0}^{n-1} l(T^k(x(p_0))) \tag{3.10}$$

Finally note that (3.8) and definition (3.2) imply

$$c_{p_0}(l) = \sum_{k=0}^{n-1} c(T^k(x(p_0))) \tag{3.11}$$

Hence we get

$$\begin{aligned}\lim_{l\to\infty} \frac{1}{l} c_{p_0}(l) &= \lim_{l\to\infty} \frac{1}{\tilde{l}(p_0,l)} c_{p_0}(l) = \lim_{l\to\infty} \frac{n(p_0,l)}{\tilde{l}(p_0,l)} \frac{1}{n(p_0,l)} c_{p_0}(l) \\ &= \lim_{n\to\infty} \Big(\frac{1}{n}\sum_{k=0}^{n-1} l(T^k(x(p_0)))\Big)^{-1} \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} c(T^k(x(p_0)))\end{aligned} \tag{3.12}$$

Taking into consideration (3.4) and (3.7) we obtain the desired relation:

$$\lim_{l\to\infty} \frac{1}{l} c_{p_0}(l) = \frac{1}{\bar{l}} c \tag{3.13}$$

Moreover, according to Remark 5, the limit above converges uniformly for all $p_0 \in M_g^2$. We have proved the first part of Theorem 1.

3.2. **Deviation from the asymptotic cycle.** Recall the result from [22] concerning the error term for the ergodic sum of type (3.4) corresponding to generic interval exchange transformation. First we must recall several more notions concerning interval exchange transformations.

Recall that the permutation $\pi \in \mathfrak{S}_\mathfrak{m}$ is *reducible* if there exists j, $1 \leq j < m$, such that $\pi\{1,\ldots,j\} = \{1,\ldots,j\}$. Otherwise π is *irreducible.*

The space of interval exchange transformations having the same permutation $\pi \in \mathfrak{S}_\mathfrak{m}$ forms a standard simplex $\Delta^{m-1} \subset \mathbb{R}^m$, $\Delta^{m-1} = \{\lambda \in \mathbb{R}^m \mid \lambda_j \geq 0; \lambda_1 + \cdots + \lambda_m = 1\}$. Note that interval exchange transformations defined by the same permutation π and by proportional vectors $\lambda \sim \lambda'$ are conjugate, so we normalize the vector λ by setting $\lambda_1 + \cdots + \lambda_m = 1$.

Each irreducible permutation π determines the Rauzy class: the set of permutations $\mathfrak{R}(\pi) \subset \mathfrak{S}_{\mathfrak{m}}$, $\pi \in \mathfrak{R}$, see [13]. Each Rauzy class determines a collection of numbers $\theta_1 > \theta_2 \geq \cdots \geq \theta_g \geq 0$. These numbers are Lyapunov exponents of some natural multiplicative cocycle on the space of interval exchange transformations $\Delta^{m-1} \times \mathfrak{R}(\pi)$, see [21].

By $\chi_i(x, n)$ we denote the number of visits of the trajectory $x, T(x), \ldots, T^{n-1}(x)$ to the subinterval X_j under iterations of the interval exchange transformation T. In other words

$$\chi_j(x, n) = \sum_{k=0}^{n-1} \chi_{X_j}(T^k(x))$$

where χ_{X_j} is the characteristic function of the subset $X_j \subset X$.

Denote by $\chi(x, n)$ the vector $(\chi_1(x, n), \ldots, \chi_m(x, n))$. The following theorem is a reformulation of the main result of [22].

Theorem 5. *For any irreducible permutation π of more than two elements and for any λ from a set of full Lebesgue measure in Δ_{m-1}, the following property is valid. Consider the interval exchange transformation $T(\lambda, \pi)$ defined on the unit interval X. For any linear function $f \in Ann(\lambda) \subset (\mathbb{R}^m)^*$ and any $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that for any $x \in X$, and any $n > N(\varepsilon)$ we have*

$$\frac{\log |\langle f, \chi(x, n)\rangle|}{\log n} \leq \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))} + \varepsilon$$

The number $N(\varepsilon)$ depends on ε and on the pair (λ, π), but does not depend on the point $x \in X$.

Hence the following limit exists and satisfies the following bound

$$\limsup_{n \to +\infty} \frac{\log |\langle f, \chi(x, n)\rangle|}{\log n} \leq \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))}$$

Now consider the map

$$C : \mathbb{R}^m \to H_1(M_g^2; \mathbb{R}), \quad (v_1, \ldots, v_m) \mapsto v_1 c_1 + \cdots + v_m c_m$$

Here the cycles c_i are defined by (3.3). Note that by construction the vector λ is mapped to the asymptotic cycle c (see (3.4)). Hence the adjoint map C^* maps $Ann(c)$ to $Ann(\lambda)$. Note that

$$C(\chi(x, n)) = \sum_{k=0}^{n-1} c(T^k(x))$$

Since the asymptotic cycle c is nonzero, and since it is proportional to the cycle $D[\omega]$, $c \sim D[\omega]$, Theorem 5 implies

Corollary 1. *For any irreducible permutation π of more than two elements and for any λ from a set of full Lebesgue measure in Δ_{m-1}, the following property is valid. Consider interval exchange transformation $T(\lambda, \pi)$ defined on the unit interval X. For any cocycle $\phi \in Ann(D[\omega]) \subset H^1(M_g^2; \mathbb{R})$ and any $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that for any $x \in X$, and any $n > N(\varepsilon)$ we have*

$$\frac{\log |\langle \phi, \sum_{k=0}^{n-1} c(T^k(x))\rangle|}{\log n} \leq \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))} + \varepsilon$$

The number $N(\varepsilon)$ depends on ε and on the pair (λ, π), but does not depend on the point $x \in X$.

Hence the following limit exists and satisfies the following bound

$$\limsup_{n\to+\infty} \frac{\log|\langle\phi, \sum_{k=0}^{n-1} c(T^k(x))\rangle|}{\log n} \le \frac{\theta_2(\mathfrak{R}(\pi))}{\theta_1(\mathfrak{R}(\pi))}$$

Applying the same trick as in (3.9)–(3.12), we derive the second part of Theorem 1 from Corollary 1. Theorem 1 is proved.

4. Hyperplane sections of periodic surfaces

Consider a smooth periodic surface $\hat{M}_g^2 \subset \mathbb{R}^n$, and a family of parallel hyperplanes. The family of parallel hyperplanes can be defined as the family of levels of a linear function $h \in (\mathbb{R}^n)^*$. Now consider the quotient surface, i.e., the compact orientable surface $M_g^2 \subset T^n$ embedded in the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Consider the closed differential 1-form dh on T^n. Restrict the form dh to M_g^2, setting $\omega := dh|_{M_g^2}$.

Consider the commutative diagram

$$\begin{array}{ccc} \hat{M}_g^2 & \longrightarrow & \mathbb{R}^n \\ \downarrow & & \downarrow \\ M_g^2 & \longrightarrow & T^n = \mathbb{R}^n/\mathbb{Z}^n \end{array}$$

By construction, the intersection lines of our hyperplanes with $\hat{M}_g^2$ are projected to the leaves of the foliation determined by ω on M_g^2.

Assuming that the linear function h determining the direction of the hyperplanes is generic, we may assume that the form ω has only isolated nondegenerate singularities. In particular, we may assume that all the saddles are simple. We can also assume that ω has maximal rank, i.e., all the periods of the form are rationally independent.

In this section must to consider the case when the form ω may have minima and maxima. It is easy to see that the foliation defined by the form ω generically does not have saddle connections between different saddles. However, now it may have stable separatrix loops. In this case the surface is split into several components filled with closed leaves and several minimal components filled with everywhere dense leaves. We are interested only in the components of the second type. We apply the results of section 3 to these ergodic components.

4.1. **Decomposition of the surface into components.** In this subsection we consider a decomposition of the surface into components, following [8].

A connected component of a singular leaf of ω passing through a saddle point may have a loop γ, or even two loops. The integral of ω over γ is obviously zero. Since ω is of maximal rank, it follows that the cycle γ is homologous to zero, $[\gamma] = 0$, $[\gamma] \in H_1(M_g^2, \mathbb{Z})$. Hence, when we cut M_g^2 along γ, we obtain two closed components $M_g^2 = W_1 \cup W_2$. It may happen that the restriction $\omega|_{W_1}$ or $\omega|_{W_2}$ is exact on one of the components. The component W on which the form is exact may contain other components W_i', obtained by the cuts over loops on leaves passing through the other saddles inside W. Following [1], we call a maximal component W of that type a *trap*. Since by definition ω is exact on any trap, all leaves are closed inside the trap. Each trap is homeomorphic to a disk D^2.

Let us cut off all the traps. Since we did not lose any nontrivial cycles, we obtain a surface M'_g of genus g with several holes. The boundaries of the holes are closed loops on singular leaves of ω, i.e., separatrix loops. When $g>1$, we may still have closed leaves inside M'_g. Note that we have gotten rid of minima and maxima, all critical points of ω on M'_g are of saddle type only. Let us count the number of inner saddles, without taking into consideration the saddles that belong to the cuts (we remind the reader that we are assuming that ω does not have multiple saddles).

Lemma 2. *There are $2g-2$ inner saddles of ω on M'_g.*

Proof. Let us temporarily paste the holes in M'_g with disks. Consider a smooth extension of ω to the disks, placing a single additional critical point on each disk: maximum or minimum. Count the Euler characteristic of the closed surface of genus g thus obtained as the algebraic sum of numbers of critical points of our extended 1-form:

$$\begin{aligned} 2g-2 &= (\#\text{inner saddles}) \\ &\quad + [(\#\text{saddles on the cuts}) - (\#\text{minima} + \#\text{maxima})] \end{aligned}$$

Since by construction there is a one-to-one correspondence between saddles on the cuts and cuts, the number inside the rectangular brackets equals zero. □

Let us proceed by performing surgery on M'_g. Some of the $2g-2$ inner saddles on M'_g may have loops of singular leaves passing through them. Let us cut M'_g along all such loops. We recall, that any such loop γ is homologous to zero in M'_g. Consider the connected components $M^2_{(1)},\dots,M^2_{(q)}$ thus obtained for which ω restricted to $M^2_{(i)}$ is not exact. Each $M^2_{(i)}$ is a surface of genus $g_i \geq 1$ with several holes.

Lemma 3. *The following equation holds*

$$g_1+\cdots+g_q=g$$

Proof. To prove Lemma 3 it is sufficient to show that we can construct a basis of cycles on M^2_g that does not intersect any cuts γ_j. Since all γ_j are homologous to zero, the desired basis can be easily constructed. □

By *necks* we call maximal components N_q obtained by cutting M'_g along loops of critical leaves for which the restriction $\omega|_{N_q}$ is exact. Each neck is diffeomorphic to a cylinder with several holes. Necks are of no interest for us, since all leaves are closed on the necks.

It is easy to see (cf. Lemma 2) that the number of inner saddles on the component $M^2_{(i)}$, of genus g_i, $i=1,\dots,q$, equals $2g_i-2$. We shall call them *essential saddles.*

Recall that we assume the initial embedding $M^2_g \to T^n$ to be in general position. Consider a small perturbation $h' \in (\mathbb{R}^n)^*$ of the initial linear function h determining the closed 1-form ω. The corresponding closed 1-form ω' on M^2_g defines a new decomposition of the surface into components. The critical points of ω' are just small deformations of the initial ones. In particular, they are in one-to-one correspondence with those of ω; the new decomposition is isotopic to the one determined by ω. Consider some minimal component $M^2_{(i)}$, $1 \leq i \leq q$, and the collection of the essential saddles inside it. We get a well-defined map $U(h) \to H^1(M^2_{(i)}, \{\text{essential saddles}\}; \mathbb{R})$ in a small neighborhood $U(h) \subset (\mathbb{R}^n)^*$ of the initial linear function h.

Lemma 4. *For a generic choice of the embedding and of the linear function h, the point h is a regular point of the map $U(h) \to H^1(M^2_{(i)}, \{essential\ saddles\}; \mathbb{R})$ for every minimal component $M^2_{(i)}$.*

Proof. Since by assumption the map $H_1(M^2_g; \mathbb{R}) \to H_1(T^n; \mathbb{R})$ is a monomorphism, the map $H_1(M^2_{(i)}; \mathbb{R}) \to H_1(T^n; \mathbb{R})$ is also a monomorphism. Note that

$$\dim H^1(M^2_{(i)}, \{\text{essential saddles}\}; \mathbb{R}) = 4g_i - 3 \leq n \tag{4.1}$$

where the inequality above is valid by assumption. Thus, supposing that we have a generic embedding $M^2_g \to T^n$, and a generic linear function $h \in (\mathbb{R}^n)^*$, we see that the essential saddles are in general position. Hence the induced map

$$H_1(M^2_{(i)}, \{\text{essential saddles}\}; \mathbb{R}) \to \mathbb{R}^n \tag{4.2}$$

is a monomorphism. The adjoint map to the map above is the tangent map to the map U. Lemma 4 is proved. □

Corollary 2. *For almost all linear functions $h \in (\mathbb{R}^n)^*$ from a sufficiently small neighborhood $V(h) \subset (\mathbb{R}^n)^*$ of h, the measured foliation induced on each minimal component $M^2_{(i)}$ has all the properties listed in Theorem 1.*

Indeed, it is easy to see that the presence of holes in $M^2_{(i)}$ corresponding to the cuts over separatrix loops does not affect Theorem 1. Theorem 3 now follows from Theorem 1.

* * *

We complete this section with the proof of Theorem 4.

Proof. First note that the map $A : M^2_g \to J(M^2_g)$, where $g \geq 1$, is actually an embedding. Indeed, assume that the images of two different points P_1 and P_2 coincide. Then the image of the divisor $P_2 - P_1$ vanishes: $A(P_2 - P_1) = 0$. Hence due to Abel's theorem, one can find a meromorphic function on M^2_g with a simple zero at P_2 and a simple pole at P_1, having no other zeros and poles. But then M^2_g is equivalent to $\mathbb{C}P^1$, which leads to a contradiction.

Let $z_1, \dots, z_g$ be the standard coordinates in the complex g-dimensional torus $J(M^2_g)$. The induced forms $A^*(dz_k)$ generate a basis of holomorphic differentials on M^2_g. Consider the universal abelian cover of $J(M^2_g)$ as the real space $\mathbb{R}^{2g}$. The embedding A induces an isomorphism between the dual space $(\mathbb{R}^{2g})^*$ and the space of harmonic differentials: for every linear function $h \in (\mathbb{R}^{2g})^*$ the image $A^*(dh)$ is a harmonic differential on M^2_g. Note that for almost all $h \in (\mathbb{R}^{2g})^*$ the corresponding harmonic differential shall have $2g - 2$ nondegenerate saddles.

Consider the principal stratum of squares of holomorphic differentials; we recall, that this stratum is connected. Due to Fubini's theorem, we can reformulate Theorem 1 as follows. Instead of claiming the validity of the corresponding statements for "almost all measured foliations with prescribed singularities", we can say that these statements are valid for the closed 1-forms obtained as the real part of a generic holomorphic differential. In other words, we can say that given a generic complex structure and a generic harmonic differential corresponding to this complex structure, Theorem 1 is valid for the orientable measured foliation defined by the harmonic differential. Theorem 4 is proved. □

5. Appendix. Second Lyapunov exponent of the Teichmüller geodesic flow

The following proposition is known in the folklore (it can be extracted from [16], [17], and [18]; from [15], or by combining results from [18] and [21]:

Consider a connected component of some stratum of squares in the space of holomorphic quadratic differentials. All quadratic differentials in the stratum have the same number of saddles; we denote this number by s. Let g be the genus of the surface.

Proposition 1. *The collection of Lyapunov exponents of the Teichmüller geodesic flow confined to a connected component of the stratum of squares has the following form:*

$$
\begin{aligned}
&-2 < -(1+\nu_2) \le -(1+\nu_3) \le \cdots \le -(1+\nu_g) \le \underbrace{-1 = \cdots = -1}_{s-1} \le \\
&\qquad -(1-\nu_g) \le \cdots \le -(1-\nu_2) < 0 < (1-\nu_2) \le \cdots \le (1-\nu_g) \\
&\qquad\qquad \le \underbrace{1 = \cdots = 1}_{s-1} \le (1+\nu_g) \le (1+\nu_{g-1}) \le \cdots \le (1+\nu_2) < 2
\end{aligned}
$$

As we already indicated in subsection 3.2, a uniquely ergodic measured foliation determines an interval exchange transformation on a transverse segment. Consider the corresponding permutation π. Each irreducible permutation π determines the Rauzy class: a set of permutations $\mathfrak{R}(\pi) \subset \mathfrak{S}_{\mathfrak{m}}$, $\pi \in \mathfrak{R}$, see [13]. Each Rauzy class determines a collection of numbers $\theta_1 > \theta_2 \ge \cdots \ge \theta_g \ge 0$. These numbers are the Lyapunov exponents of some natural multiplicative cocycle on the space of interval exchange transformations $\Delta^{m-1} \times \mathfrak{R}(\pi)$, see [21].

An *extended Rauzy class*, see [18], is obtained as the union of several Rauzy classes. Having an orientable measured foliation, one can define it as follows. Consider the interval exchange transformations induced on all possible intervals transverse to the foliation. Chose those of them that give rise to exchanges of the minimal possible number of subintervals. Consider all the corresponding permutations. They form the extended Rauzy class.

As we already mentioned, the extended Rauzy classes are in one-to-one correspondence with the connected components of the strata of squares of holomorphic differentials in the moduli space $\mathcal{QD}_{\mathfrak{g}}$ (see [18]).

Lemma 5. *The Lyapunov exponents of the Teichmüller geodesic flow on a connected component of a stratum of squares corresponding to the extended Rauzy class $\mathfrak{R}_{\mathfrak{ex}}$ are related to the Lyapunov exponents discussed above as follows:*

$$
\nu_k = \frac{\theta_k(\mathfrak{R})}{\theta_1(\mathfrak{R})} \quad k = 2, \ldots, g
$$

where $\mathfrak{R} \subseteq \mathfrak{R}_{\mathfrak{ex}}$ is any Rauzy class from the extended Rauzy class $\mathfrak{R}_{\mathfrak{ex}}$.

Lemma 5 above is an elementary corollary of the results in [18] and in [21].

Lemma 5 allows to evaluate the approximate values of the second Lyapunov exponent of the Teichmüller geodesic flow by computer experiments (cf. results of similar computer experiments dealing with Lyapunov exponents of multidimensional continued fraction algorithms [3], [2]).

Below we present the list of all Rauzy classes with the information on θ_1 and θ_2 for all connected components of all strata of squares of holomorphic differentials corresponding to genera 2 and 3. Horizontal lines separate extended Rauzy classes. We indicate the multiplicities of zeros of the corresponding holomorphic differentials. The symbol of the quadratic differential corresponding to a holomorphic differential with zeros of orders $(d_1, \ldots, d_n)$ is $(2d_1, \ldots, 2d_n; +1)$. The induced measured foliation has n saddles with $2d_1 + 2, \ldots, 2d_n + 2$ prongs correspondingly.

For genus $g = 2$, both strata of squares of the holomorphic differentials are connected. For genus $g = 3$, the strata corresponding to symbols $(8; +1)$ and $(4, 4; +1)$ have two connected components; the other strata are connected (see also [18]).

Genus $g = 2$

Representative of Rauzy class π	Cardinality of Rauzy class $\operatorname{Card}\mathfrak{R}(\pi)$	Lyapunov exponents $\theta_1(\pi)$	$\theta_2(\pi)$	Ratio $\alpha = \dfrac{\theta_2(\pi)}{\theta_1(\pi)}$	Types of zeros
$(4,3,2,1)$	7	0.48679	0.16227	0.3333	(2)
$(5,4,3,2,1)$	15	0.37716	0.18857	0.5000	$(1,1)$

Genus $g = 3$

$(6,5,4,3,2,1)$	31	0.30830	0.18980	0.6156	(4)
$(6,5,4,2,1,3)$	134	0.30322	0.12671	0.4179	(4)
$(2,5,1,3,7,4,6)$ $(7,6,5,4,1,3,2)$	509 261	0.25580	0.13307	0.5202	$(3,1)$ $(1,3)$
$(7,6,5,4,3,2,1)$	63	0.26096	0.17961	0.6883	$(2,2)$
$(7,6,5,2,1,4,3)$	294	0.25422	0.10722	0.4218	$(2,2)$
$(3,1,6,2,4,8,5,7)$ $(8,7,6,5,3,2,1,4)$	1258 919	0.22017	0.11882	0.5397	$(1,2,1)$ $(2,1,1)$
$(9,8,3,6,5,4,7,2,1)$	1255	0.19292	0.10644	0.5517	$(1,1,1,1)$

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