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# INVERSION OF HOROSPHERICAL INTEGRAL TRANSFORM ON REAL SEMISIMPLE LIE GROUPS

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Abstract. There exists the wonderful integral transform on complex semisimple Lie groups, which assigns to a function on the group the set of its integrals over "generalized horospheres" — some specific submanifolds of the Lie group. The local inversion formula for this integral transform, discovered in 50's for $\mathrm{SL}(n;\mathbb{C})$ by Gel'fand and Graev, made it possible to decompose the regular representation on $\mathrm{SL}(n;\mathbb{C})$ into irreducible ones.

In case of real semisimple Lie group the situation becomes more complicated, and usually there is no reasonable analogous integral transform at all. Nevertheless, in the present paper we succeed to define the integral transforms on the Lorentz group and some other real semisimple Lie groups, which are in a sense analogous to the integration over "horospheres". We obtain the inversion formulas for these integral transforms.

## 0. Introduction

We consider the Lorentz group $\mathrm{SO}(1, 2n-1)$ — the group of linear transformations of $\mathbb{R}^{2n}$ preserving the indefinite bilinear form of the signature $(+1; -1; ...; -1)$. We fix once and forever the coordinates in $\mathbb{R}^{2n}$ in which our bilinear form looks like $\langle\vec{a}, \vec{b}\rangle = a^1b^1 - a^2b^2 - ...a^{2n}b^{2n}$.

Let's regard some isotropic vector $\vec{a} \in \mathbb{R}^{2n}$, $|\vec{a}| = 0$ and its orthogonal complement $\alpha^{2n-1}$ (in the sense of our scalar multiplication), $\alpha^{2n-1} \perp \vec{a}$. The hypersurface $\alpha^{2n-1}$ can be defined also as a tangent space to the isotropic cone at the point $\vec{a}$. We consider the group of those transformations, which act trivially on the vector $\vec{a}$ and on all vectors of the quotient $\alpha^{2n-1}/a$, where $a$ is a line, spanned by vector $\vec{a}$. It would be convenient for some reasons to use

$$\vec{a} = (1, \underbrace{0, 0, ..., 0}_{n-1}, 1, \underbrace{0, 0, ..., 0}_{n-1})$$

as the isotropic vector. Then, in matrix realization the subgroup $Z$ is the subgroup of matrices of the form

$$\tag{0.1} \begin{pmatrix} 1+\frac{1}{2}\sum(u^i)^2 & u^2\cdots u^n & -\frac{1}{2}\sum(u^i)^2 & u^{n+2}\cdots u^{2n} \\ u^2 & \begin{matrix} 1 & \cdots & 0 \end{matrix} & -u^2 & \begin{matrix} 0 & \cdots & 0 \end{matrix} \\ \vdots & \begin{matrix} \vdots & \ddots & \vdots \end{matrix} & \vdots & \begin{matrix} \vdots & \ddots & \vdots \end{matrix} \\ u^n & \begin{matrix} 0 & \cdots & 1 \end{matrix} & -u^n & \begin{matrix} 0 & \cdots & 0 \end{matrix} \\ \frac{1}{2}\sum(u^i)^2 & u^2\cdots u^n & 1-\frac{1}{2}\sum(u^i)^2 & u^{n+2}\cdots u^{2n} \\ u^{n+2} & \begin{matrix} 0 & \cdots & 0 \end{matrix} & -u^{n+2} & \begin{matrix} 1 & \cdots & 0 \end{matrix} \\ \vdots & \begin{matrix} \vdots & \ddots & \vdots \end{matrix} & \vdots & \begin{matrix} \vdots & \ddots & \vdots \end{matrix} \\ u^{2n} & \begin{matrix} 0 & \cdots & 0 \end{matrix} & -u^{2n} & \begin{matrix} 0 & \cdots & 1 \end{matrix} \end{pmatrix}$$

*Remark* 0.1. It would be convenient for us to enumerate coordinates $u^i$ at this subgroup by indices $\hat{1}, 2, ..., n, \widehat{n+1}, n+2, ..., 2n$, omitting indices 1 and $n+1$.

So the subgroup $Z$ is a $(2n-2)$-dimensional subgroup of $\mathrm{SO}(1, 2n-1)$. In a sense, $Z$ is naturally analogous to the subgroup of upper triangular matrices with ones at the diagonal in $SL(n, \mathbb{C})$.

We now consider the following family of submanifolds in $\mathrm{SO}(1, 2n-1)$ : all subgroups $l_0 Z l_0^{-1}$ conjugate to $Z$ and also all the cosets $l l_0 Z l_0^{-1}$ of these subgroups. Here $l, l_0 \in \mathrm{SO}(1, 2n-1)$. We denote by $H_Z(\mathrm{SO}(1, 2n-1))$ the family of submanifolds (which we shall usually call "horospheres") thus constructed.

We fix the invariant measure $d\mu$ on $Z$; it induces the measures on the surfaces $l l_0 Z l_0^{-1}$. Now we can define the integral transform from functions on the group $\mathrm{SO}(1, 2n-1)$ to functions on the manifold $H_Z(\mathrm{SO}(1, 2n-1))$. For any $z_{l,l_0} \in H_Z(SO(1; 2n-1))$, where the point $z_{l,l_0}$ corresponds to coset $Z_{l,l_0} = l l_0 Z l_0^{-1}$ we define

$$\hat{f}(z_{l,l_0}) = \int_{Z_{l,l_0}} f \, d\mu \tag{0.2}$$

Here $\hat{f}$ is the function now defined on the manifold $H_Z(SO(1; 2n-1))$.

Our goal is to investigate how one can reconstruct the original function $f$, knowing its integrals over $Z_{l,l_0}$, i.e., to reconstruct a function $f$ at the Lie group $SO(1; 2n-1)$ (or some other real semisimple Lie group), knowing $\hat{f}$. The present paper is devoted to this very problem.

The inversion formula, obtained in the present paper is a bit unusual in the following sense. The space of horospheres $H_Z(\mathrm{SO}(1, 2n-1))$ where the function $\hat{f}$ is defined has the same dimension as the original group $\mathrm{SO}(1, 2n-1)$. So we have a transform from $\frac{n(n-1)}{2}$-dimensional manifold to the $\frac{n(n-1)}{2}$-dimensional one. Nevertheless, due to the inversion formula, obtained in the present paper, we need only restriction of $\hat{f}$ on some subspace of horospheres of codimension $(n-2)$ to reconstruct a value $f(l)$ of original function at a point $l$. There is no contradiction here - for reconstruction of $f$ at different points $l_1 \neq l_2 \in \mathrm{SO}(1, 2n-1)$ we use different subspaces of horospheres.

## 1. Reduction to a Radon Type Transform

Let's construct the family $H_Z(\mathrm{SO}(1, 2n-1))$ of horospheres a bit more carefully. First of all, let us consider the subfamily of the horospheres passing through the common point of $\mathrm{SO}(1, 2n-1)$, say through the identity element $e$. It is easy to see, that it is the following family of horospheres in $\mathrm{SO}(1, 2n-1)$: the family $G_Z(\mathrm{SO}(1, 2n-1)) = \left\{ l Z l^{-1} \right\}_{l \in SO(1;2n-1)}$ of all subgroups in $\mathrm{SO}(1, 2n-1)$, conjugate to $Z$. One provides the family of horospheres with transitive action of $\mathrm{SO}(1, 2n-1)$ as follows: an element $l \in \mathrm{SO}(1, 2n-1)$ acts on $l_0 Z l_0^{-1}$ as $(l l_0) Z (l l_0)^{-1}$. Hence $G_Z(\mathrm{SO}(1, 2n-1))$ is a homogeneous space. As a stability subgroup of a "point" $Z \in G_Z(\mathrm{SO}(1, 2n-1))$ one obtains the normalizer $N(Z) \subseteq \mathrm{SO}(1, 2n-1)$ of $Z$. Consequently $G_Z(\mathrm{SO}(1, 2n-1))$ is diffeomorphic to $\mathrm{SO}(1, 2n-1)/N(Z)$.

The normalizer $N(Z) \subset \mathrm{SO}(1, 2n-1)$ is the subgroup of transformations, preserving the isotropic line $\langle \vec{a} \rangle_{\mathbb{R}}$ (see more details in Section 4). Hence the homogeneous space can be identified with the space of isotropic lines, i.e., with the space of line components of isotropic cone, i.e., with $(2n-2)$-dimensional sphere $\mathrm{SO}(1,2n-1)/N(Z) = \mathrm{S}^{2n-2}$.

We have provided our subfamily of horospheres with a structure of a homogeneous space $\mathrm{SO}(1,2n-1)/N(Z)$. A coset $l \cdot N(Z) \in \mathrm{SO}(1,2n-1)/N(Z)$ corresponds to the horosphere $lZl^{-1} \in G_Z(\mathrm{SO}(1,2n-1))$ in this realization.

We now replenish our family of horospheres in $\mathrm{SO}(1,2n-1)$. A group $l_0 Z l_0^{-1}$ acts naturally on $\mathrm{SO}(1,2n-1)$; the group $\mathrm{SO}(1,2n-1)$ is stratified into diffeomorphic orbits under this action. The space of orbits is the homogeneous space $\mathrm{SO}(1,2n-1)/l_0 Z l_0^{-1} \approx \mathrm{SO}(1,2n-1)/Z$. For any $l_0 Z l_0^{-1} \in G_Z(\mathrm{SO}(1,2n-1))$ we supplement the family of horospheres by collection of orbits of $l_0 Z l_0^{-1}$. We denote by $H_Z(\mathrm{SO}(1,2n-1))$ the obtained family of horospheres $\left\{l l_0 Z l_0^{-1}\right\}_{l,l_0 \in SO(1;2n-1)}$.

In the case when $L$ is a complex semi-simple Lie group and $Z$ is Borel subgroup the space $H_Z(L)$ has been studied in[1,2] where it has been called "a space of horospheres". The integral transform corresponding to this family of horospheres (see below) is extremely useful in decomposition of the regular representation. The articles[1,2] are devoted in fact to the latter problem.

By its construction the family of horospheres $H_Z(\mathrm{SO}(1,2n-1))$ possesses a structure of a bundle $\mathrm{SO}(1,2n-1)/Z \to H_Z(\mathrm{SO}(1,2n-1)) \to G_Z(\mathrm{SO}(1,2n-1))$ with a fiber $\mathrm{SO}(1,2n-1)/Z$ and a base $G_Z(\mathrm{SO}(1,2n-1)) = \mathrm{SO}(1,2n-1)/N(Z) = \mathrm{S}^{2n-2}$.

We consider now the space of incident pairs of the form (a point $l$ of $\mathrm{SO}(1,2n-1)$, a horosphere $\beta$ from $H_Z(\mathrm{SO}(1,2n-1))$ containing $l$). It is easy to see that $\beta$ can be presented as $l l_0 Z l_0^{-1}$ for some $l_0$, where the subgroup $l_0 Z l_0^{-1}$ is uniquely defined. Hence an incident pair $(l,\beta)$ uniquely defines a point $\left(l \times l_0 Z l_0^{-1}\right) \in \mathrm{SO}(1,2n-1) \times G_Z(\mathrm{SO}(1,2n-1))$ of the product $\mathrm{SO}(1,2n-1) \times G_Z(\mathrm{SO}(1,2n-1))$, where $\beta = l l_0 Z l_0^{-1}$. The correspondence is invertible: a point $\left(l \times l_0 Z l_0^{-1}\right) \in \mathrm{SO}(1,2n-1) \times G_Z(\mathrm{SO}(1,2n-1))$ uniquely determines an incident pair $(l,\beta)$, $\beta = l l_0 Z l_0^{-1}$. Hence the space of incident pairs is diffeomorphic to the product $\mathrm{SO}(1,2n-1) \times G_Z(\mathrm{SO}(1,2n-1))$.

We have shown that a closed subgroup $Z$ of a Lie group $\mathrm{SO}(1,2n-1)$ determines the following double fibration

$$
\begin{tikzcd}
& \mathrm{SO}(1,2n-1) \times \mathrm{S}^{2n-2} \arrow[dl] \arrow[dr] & \\
\mathrm{SO}(1,2n-1) & & H_Z(\mathrm{SO}(1,2n-1))
\end{tikzcd}
\tag{1.1}
$$

where $G_Z(\mathrm{SO}(1,2n-1)) = \mathrm{SO}(1,2n-1)/N(Z) = \mathrm{S}^{2n-2}$, and the space of horospheres $H_Z(\mathrm{SO}(1,2n-1))$ is a bundle with base $G_Z(\mathrm{SO}(1,2n-1))$ and fiber $\mathrm{SO}(1,2n-1)/Z$.

In a sense, we will reconstruct the values of the original function $f$ for each point $l \in \mathrm{SO}(1,2n-1)$ separately. Let us fix some point of the group, say the identity element, and study in detail the structure of subfamily of horospheres passing close to the fixed point of $\mathrm{SO}(1,2n-1)$.

We consider a neighborhood of the identity element $e \in \mathrm{SO}(1,2n-1)$. Using the exponential mapping we choose the Lie algebra $\mathfrak{so}(1;2n-1)$ as a chart of the neighborhood. The horospheres $\alpha \in H_Z(\mathrm{SO}(1,2n-1))$, passing through $e$, are of the form $lZl^{-1}$. Hence they become linear subspaces $l\mathfrak{z}l^{-1} = \mathrm{Ad}_l \mathfrak{z}$ in our chart. It happens, that the horospheres close to a horosphere $l_0 Z l_0^{-1}$ having the form $n l_0 Z l_0^{-1} \subseteq N\left(l_0 Z l_0^{-1}\right)$, where $n$ belongs to the normalizer $N\left(l_0 Z l_0^{-1}\right)$ of $l_0 Z l_0^{-1}$

in our chart are represented by those affine subspaces of the linear space $l_0\mathfrak{n}l_0^{-1} \subseteq \mathfrak{so}(1;2n-1)$ which are parallel to the linear space $\mathrm{Ad}_{l_0}\mathfrak{z} = l_0\mathfrak{z}l_0^{-1} \subset l_0\mathfrak{n}l_0^{-1} = \mathrm{Ad}_{l_0}\mathfrak{n}$.

The last remark is due to the following obvious

**Lemma 1.1.** *Let $G$ be a Lie group with a normal Lie subgroup $K$. Consider the chart $\mathfrak{g}$ defined by the exponential mapping in a neighborhood of the identity element $e \in G$. In this chart the cosets $gK$ are represented by affine subspaces (of the linear space $\mathfrak{g}$) parallel to linear space $\mathfrak{k}$. Here $\mathfrak{k}$ and $\mathfrak{g}$ denote the Lie algebras of the groups $K$ and $G$ respectively.*

□

We have proved that the exponential (or, more precisely, the logarithmic) mapping determines local coordinates in a neighborhood of $e \in \mathrm{SO}(1,2n-1)$, in which the double fibration becomes linear. All the horospheres passing through $e$ transform into planes in these coordinates. Some horospheres passing close to $e$ transform into affine horospheres. Forgetting about the rest of the horospheres we reduce our double fibration to a linear one.

Let us study now, what form has our invariant measure on horospheres $ll_0Zl_0^{-1} \subset \mathrm{SO}(1,2n-1)$, $l \in N(Z)$, in exponential coordinates.

**Lemma 1.2.** *The measure on the linear spaces $\mathrm{Ad}_l\mathfrak{z}$ induced from the invariant measure on subgroups $lZl^{-1}$ is just a Euclidean measure.*

*Proof.* The assertion is a straightforward consequence of the Helgason formula (see[3] or[4])

$$(\exp)^*(dl) = \det\left(\frac{1-e^{-\mathrm{ad}X}}{\mathrm{ad}X}\right) dX$$

We recall that in the case of a nilpotent Lie algebra one can represent operators $\mathrm{ad}X$, $X \in \mathfrak{g}$, by upper-triangular matrices with zero diagonal. (Due to Engel theorem it can be done in some basis even simultaneously for all operators $\mathrm{ad}X$.) Hence, the matrix $(1-\frac{1}{2}\mathrm{ad}X+\frac{1}{6}\mathrm{ad}^2X-...)$ under determinant is upper-triangular with ones at the diagonal. Hence it is determinant equals one. □

So, the problem of reconstruction of the value of original function $f$ in the fixed point of $\mathrm{SO}(1,2n-1)$ is reduced to the following problem:

How one can reconstruct the value of a function at the origin of a linear space $\mathfrak{so}(1;2n-1)$, knowing its integrals over affine subspaces of the form $v+\mathrm{Ad}_l\mathfrak{z}$, where $v \in \mathrm{Ad}_l\mathfrak{n}(\mathfrak{z})$, $l \in \mathrm{SO}(1,2n-1)$? Here $\mathfrak{n}(\mathfrak{z})$ is the normalizer of Lie algebra $\mathfrak{z}$.

This reformulation makes clear, how close our problem is connected with the problem of reconstruction of Radon transform.

## 2. Inversion Formula for the Radon Transform

Inversion formula for the integral transform (0.2), we are interested in, will be based to great extent on the inversion formula for the Radon transform, obtained by I. M. Gelfand, M. I. Graev, and Z. Ya. Shapiro in[5]. That's why we recall some basic facts, concerning the Radon transform, and the inversion formula for the Radon transform in[5].

We consider $m$-dimensional affine subspace $\mathbb{R}^m$ and the family $H_s(\mathbb{R}^m)$ of all affine $s$-dimensional subspaces in the space $\mathbb{R}^m$, $0 < s < m$, where $s$ is even. We provide each subspace with a measure, for example the measure induced by an inner product in $\mathbb{R}^m$. By integrating $f$ defined in $\mathbb{R}^m$ over all $s$-dimensional subspaces we obtain a function $\widehat{f}$, now defined on the space $H_s(\mathbb{R}^m)$ of $s$-dimensional affine surfaces in $\mathbb{R}^m$. The transform $f \to \widehat{f}$ is called *the Radon transform.*

Let us find out what double fibration corresponds to Radon transform. First we note that the manifold $H_s(\mathbb{R}^m)$ of $s$-dimensional *affine* subspaces is the total space of a vector bundle over the Grassmann manifold $G_s(\mathbb{R}^m)$ of $s$-dimensional *linear* subspaces in $\mathbb{R}^m$. For we can define the projection $p : H_s(\mathbb{R}^m) \to G_s(\mathbb{R}^m)$ by assigning to every affine subspace the parallel linear subspace (i.e., the parallel space passing through the origin). What is the fiber of such bundle? It is easy to see, that the family of subspaces parallel to a given linear subspace $\gamma_0^s \subset \mathbb{R}^m$, is isomorphic to the quotient $\mathbb{R}^m/{\gamma_0}^s$. But $\gamma_0^s$ is a fiber of the tautological bundle $\gamma^s$ over the Grassmann manifold $G_s(\mathbb{R}^m)$ at the point $[{\gamma_0}^s] \in G_s(\mathbb{R}^m)$. Hence the manifold of affine subspaces in $\mathbb{R}^m$ is isomorphic to the total space $(G_s(\mathbb{R}^m), \varepsilon^m/\gamma^s)$ of the normal bundle $\varepsilon^m/\gamma^s$ over the Grassmann manifold.

$$H_s(\mathbb{R}^m) = (G_s(\mathbb{R}^m), \varepsilon^m/\gamma^s).$$

Here have we denoted by $\varepsilon^m$ trivialized $m$-dimensional vector bundle over the Grassmann manifold and $\gamma^s$ is the tautological bundle embedded in $\varepsilon^m$. We want now to demonstrate, that the space $A$ of incident pairs of the form (a point $v_0$ from $\mathbb{R}^m$, $s$-dimensional affine subspace ${\beta_0}^s$ containing the point) can be represented by the product $\mathbb{R}^m \times G_s(\mathbb{R}^m)$. By assigning a point $v_0$ to an incident pair $(v_0, {\beta_0}^s) \in A$ , we define a projection of $A$ onto $\mathbb{R}^m$. Assigning to $(v_0, {\beta_0}^s)$ the linear space ${\gamma_0}^s = -v_0 + {\beta_0}^s$ parallel to ${\beta_0}^s$ (and of course passing through the origin) we define a projection of $A$ onto $G_s(\mathbb{R}^m)$. Conversely, a point $(v_0 \times {\gamma_0}^s) \in \mathbb{R}^m \times G_s(\mathbb{R}^m)$ uniquely determines the incident pair $(v_0, {\beta_0}^s)$, where ${\beta_0}^s = v_0 + {\gamma_0}^s$ is the affine subspace passing through $v_0$ and parallel to ${\gamma_0}^s$.

It will be convenient to interpret the product $\mathbb{R}^m \times G_s(\mathbb{R}^m)$ as the total space $(G_s(\mathbb{R}^m), \varepsilon^m)$ of the trivialized bundle $\varepsilon^m$ over the Grassmann manifold $G_s(\mathbb{R}^m)$.

We are now ready to present the double fibration corresponding to the Radon transform:

$$\begin{array}{ccc} & (G_s(\mathbb{R}^m), \varepsilon^m) & \\ \swarrow & & \searrow \\ \mathbb{R}^m & & (G_s(\mathbb{R}^m), \varepsilon^m/\gamma^s) \end{array} \tag{2.1}$$

where the left arrow is the projection onto the factor $\mathbb{R}^m$, and the right one is the quotient map of the trivialized vector bundle $\varepsilon^m$ over the embedded tautological bundle $\gamma^s$.

We now present the inversion formula for the Radon transform in the form it was obtained in[5].

Dealing with integral transforms we shall assume throughout that the dimension of the surfaces over which we integrate our functions is even. Hence we consider the Radon transform in the case of affine subspaces of even dimension only.

There is a great difference between inversion formulas in odd-dimensional and even-dimensional cases: the formulas are local in the even-dimensional case and nonlocal in the odd-dimensional case. In other words, in even-dimensional case we use only those affine subspaces, which pass infinitesimaly close to a point to reconstruct the value of initial function at this point, and in odd-dimensional case we have to use also affine subspaces passing far from our point. The inversion formulae in odd-dimensional case can be found in[6] and[7].

We denote coordinates in $\mathbb{R}^m$ by $\left(v^1,...,v^m\right)$. We choose homogeneous coordinates in the Grassmann manifold $G_s\left(\mathbb{R}^m\right)$, assigning to a linear subspace (i.e., to a point of the Grassmann manifold) a collection of $s$ linearly independent vectors $\vec{V}_1,...,\vec{V}_s$ contained in the subspace, where $\vec{V}_i=\left(V_i^1,...,V_i^m\right)$. More precisely, a frame $\vec{V}_1,...,\vec{V}_s$ defines a point of noncompact Stiefel manifold. And the Grassmann manifold is obtained by identification of linearly equivalent frames, i.e., by taking the quotient of the Stiefel manifold by the natural action of the group $\mathrm{GL}\left(s,\mathbb{R}\right)$.

We choose coordinates $(v^1,...,v^m;\vec{V}_1,...,\vec{V}_s)$ in the manifold of incident pairs $\mathbb{R}^m\times G_s\left(\mathbb{R}^m\right)=\left(G_s\left(\mathbb{R}^m\right),\varepsilon^m\right)$ with the same assumptions as above. Finally we use the same coordinates $\left(v^1,...,v^m,\vec{V}_1,...,\vec{V}_s\right)$ in $H_s\left(\mathbb{R}^m\right)=\left(G_s\left(\mathbb{R}^m\right),\varepsilon^m/\gamma^s\right)$ tacitly assuming identification $\left(\vec{v};\vec{V}_1,...,\vec{V}_s\right)\sim\left(\vec{v}+\left(u^1\vec{V}_1+...+u^s\vec{V}_s\right);\vec{V}_1,...,\vec{V}_s\right)$.

Let $f=f\left(v^1,...,v^m\right)$ be a compactly supported function in $\mathbb{R}^m$. The image $\widehat{f}$ under the Radon transform has the form

$$\hat{f}\left(\vec{V}_1,...,\vec{V}_s,v^1,...,v^m\right):=\int_{\mathbb{R}^s} f\left(\vec{v}+u^1\vec{V}_1+\cdots+u^s\vec{V}_s\right)du^1...du^s.$$

Sometimes we shall omit the indices and write down the latter relation in the form

$$\hat{f}\left(V,v\right):=\int_{\mathbb{R}^s} f\left(v+uV\right)d^s u. \tag{2.2}$$

We present the inversion formulae for the Radon transform in the form due to I. M. Gel'fand, S. G. Graev, and Z. Ya. Shapiro (see[5]). In the paper cited one can find the detailed proof of the inversion formula and a lot of additional information concerning, for example, the case of odd $s$. (Recall that in the present paper we deal only with the case of even-dimensional surfaces.) There is a different (from[5] or from the present paper) technique of inversion of integral transforms due to S.Helgason. One can find an exposition of this technique in[4] or[7].

One can also find a construction of inversion formula, by means of "extension to superdomain" (by analogy to "extension to complex domain")(see[8]).

We are finally in a position to present the inversion formula for the Radon transform. We write down the inversion formula in the coordinates defined above. We choose a point $v_0\in\mathbb{R}^m$ at which we want to reconstruct the value of original function $f\left(v\right)$. The collection of affine surfaces passing through $v_0$, produces the Grassmann manifold. We define an $s$-form on this Grassmann manifold as follows (see[5])

$$\omega\left(V,dV\right):=dV_1^{\alpha_1}\ldots dV_s^{\alpha_s}\left.\frac{\partial^s\widehat{f}\left(v,V\right)}{\partial v^{\alpha_1}\ldots\partial v^{\alpha_s}}\right|_{v=v_0} \tag{2.3}$$

We regard differentials $dV_i^{\alpha_i}$ as anticommuting (super) variables, and therefore we omit the usual wedges $\wedge$ denoting a form $dV_1^{\alpha_1}\wedge\cdots\wedge dV_s^{\alpha_s}$ by $dV_1^{\alpha_1}\ldots dV_s^{\alpha_s}$. Besides, the same as in the rest of the paper, we are tacitly assuming in (2.3) summation over all suitable pairs of the same indices.

The differential form $\omega\left(V,dV\right)$ depends on the coordinates $V_i^{a_i}$ and their differentials. Hence, precisely speaking, it is defined on the Stiefel manifold $V_s\left(\mathbb{R}^m\right)$ of $s$-frames in $\mathbb{R}^m$. But it is easy to check, that the form $\omega$ can in fact be lowered to the Grassmann manifold via the natural projection $V_s\left(\mathbb{R}^m\right)\to G_s\left(\mathbb{R}^m\right)$ (see[5]). Moreover, it turns out to be a closed form on the Grassmann manifold with cohomology class $[\omega]\in H^s\left(G_s\left(\mathbb{R}^m\right);\mathbb{R}\right)$ proportional, with coefficient $const\cdot f\left(v_0\right)$ to the Euler characteristic class $\chi(\gamma) \in H^s(G_s(\mathbb{R}^m); \mathbb{Z})$ of the tautological bundle $\gamma^s$ over $G_s(\mathbb{R}^m)$. We consider an arbitrary cycle $c \in H_s(G_s(\mathbb{R}^m); \mathbb{Z})$ in the Grassmann manifold. We denote by $\langle \chi(\gamma); c\rangle$ the value of the cohomology class $e(\gamma)$ with respect to $c$. One has the following relation (see[5]):

$$\langle \chi(\gamma); c\rangle \cdot f(v_0) = \frac{(-1)^{s/2}}{2(2\pi)^s} \int_c \omega(V, dV) \tag{2.4}$$

In the case, when $\langle \chi(\gamma); c\rangle \neq 0$ formula (2.4) provides us with inversion formula for the Radon transform.

This inversion formula will be used extensively in the present paper.

## 3. Generalization of the Inversion Formula for the Radon Transform

We recall the modification of the inversion formula (2.4) for a case, when the family of affine subspaces is not so rich as in the Radon problem (see[9]).

In the Radon transform we are able to use any $s$-dimensional linear subspace in $\mathbb{R}^m$ (which, taken together generate the Grassmann manifold $G_s(\mathbb{R}^m)$). Besides, together with every linear subspace we consider all parallel affine subspaces. Let us throw out some of the linear subspaces passing through the origin. We assume that the rest of the linear subspaces produce an $n$-dimensional submanifold $M^n \hookrightarrow G_s(\mathbb{R}^m)$ in the Grassmann manifold $G_s(\mathbb{R}^m)$. As above, together with every linear subspace $[\gamma_0^s] \in M^n$ we consider all parallel affine subspaces $\beta_0^s = \vec{v}_0 + \gamma_0^s$, $\vec{v}_o \in \mathbb{R}^m$. What manifold of affine subspaces have we obtained?

We denote by $\nu^s$ the tautological $s$-dimensional vector bundle over $M^n$; the fiber over a point $[\gamma_0^s] \in M^n$ is again $\gamma_0^s$ now considered as a linear subspace. The bundle $\nu^s$ is just the restriction of the tautological bundle $\gamma^s$ over $G_s(\mathbb{R}^m)$ to the manifold $M^n \hookrightarrow G_s(\mathbb{R}^m)$, i.e., $\nu^s = \gamma^s|_{M^n}$.

The family of affine subspaces as defined above is isomorphic to the total space $(M^n, \varepsilon^m/\nu^s)$ of the quotient of the trivialized bundle $\varepsilon^m$ (with the base $M^n$) by the subbundle $\nu^s$.

Hence we replace the double fibration (2.1) by the diagram

$$\begin{tikzcd} & (M^n, \varepsilon^m) \arrow[dl] \arrow[dr] & \\ \mathbb{R}^m & & (M^n, \varepsilon^m/\nu^s) \end{tikzcd} \tag{3.1}$$

We denote coordinates on $M^n$ by $x$. A linear space over a point $x \in M^n$ is determined by an $s$-frame $\vec{V}_1(x), ..., \vec{V}_s(x)$, where $\vec{V}_i(x) \in \mathbb{R}^m$. We provide our affine subspaces with a linear measure as above. Hence the integral transform corresponding to our family of affine subspaces $(M^n, \varepsilon^m/\nu^s) \subset H_s(\mathbb{R}^m)$ looks like (cf. (2.2)):

$$\hat{f}\left(x^1, ..., x^n, v^1, ..., v^m\right) := \hat{f}\left(\vec{V}_1(x), ..., \vec{V}_s(x), v^1, ..., v^m\right)$$

$$= \int_{\mathbb{R}^s} f\left(\vec{v} + u^1\vec{V}_1(x) + \cdots + u^s\vec{V}_s(x)\right) du^1 ... du^s. \tag{3.2}$$

As in the Radon problem we consider the form

$$\omega(x,dx) := d(V_1^{\alpha_1}(x))\dots d(V_s^{\alpha_s}(x))\frac{\partial \hat{f}^s(x,v)}{\partial v^{\alpha_1}\dots\partial v^{\alpha_s}}\bigg|_{v=v_0} \tag{3.3}$$

on the manifold $M$ of surfaces containing the origin. It is easy to see that if we restrict the form $\omega\,(V,dV)$, as defined in (2.3), from the Grassmann manifold to the submanifold $M \subset G_s\,(\mathbb{R}^m)$, we shall obtain exactly the form $\omega\,(x,dx)$ , given by (3.3). Hence if we can choose a cycle $c$ used in (2.4) within $M$, then we are able to use the same inversion formula (2.4). The latter condition means that the closed form $\omega\,(V,dV)$ does not become exact upon restriction to $M$. Recall that cohomology class $[\omega]$ of $\omega\,(V,dV)$ is proportional to the Euler characteristic class of the tautological bundle $\gamma^s$ over $G_s\,(\mathbb{R}^m)$, and the bundle $\nu^s$ is the restriction of the bundle $\gamma^s$ to the manifold $M \subset G_s\,(\mathbb{R}^m)$. Hence the cohomology class of the restriction $\omega\,(x,dx)$ of $\omega\,(V,dV)$ to the submanifold $M \subset G_s\,(\mathbb{R}^m)$ is proportional to the Euler characteristic class of the bundle $\nu^s$ over $M$. Finally note that we can consider an arbitrary (smooth) mapping $M \to G_s\,(\mathbb{R}^m)$ instead of the embedding $M \hookrightarrow G_s\,(\mathbb{R}^m)$.

We have proved that

*If the Euler characteristic class $\chi\,(\nu)$ of a bundle $\nu$ over the manifold $M$ is nontrivial, then the integral transform corresponding to the diagram* (3.1) *is invertible. Inversion formula can be written the same as in* (2.4)*, with substitution of $\omega(V,dV)$ (see* (2.3)*) by $\omega(x,dx)$ (see* (3.3)*).*

We have made some progress in solving the problem formulated at the beginning of the section. We are already able to consider families of affine subspaces in $\mathbb{R}^m$ containing only a part of all subspaces intersecting the origin, namely a manifold $M^n$, where $M^n$ is "less" than the Grassmann manifold $G_s\,(\mathbb{R}^m)$. Yet, together with any subspace $[\gamma_0] \in M^n$, containing the origin we use all affine $s$-dimensional subspaces in $\mathbb{R}^m$ parallel to $\gamma_0$. Let us try to give up the latter condition and continue decreasing the stock of available surfaces. We assume that in addition to the bundle $\nu^s$ (defined as above) one more vector bundle $\rho^r$ over the manifold $M^n$ is given. (Here $r$ is a dimension of the new bundle.) We require, that the bundle $\nu^s$ is embedded into $\rho^r$, and the bundle $\rho^r$ is embedded in the trivialized bundle $\varepsilon^m$ over $M^n$, $\nu^s \subseteq \rho^r \subseteq \varepsilon^m$.

Let us consider the family $M \subset G_s\,(\mathbb{R}^m)$ of linear subspaces passing through the origin in $\mathbb{R}^m$. Together with any linear subspace $\nu^s_{(x)}$, $x \in M$ from the family $M \subset G_s\,(\mathbb{R}^m)$ we consider all affine subspaces of the form $v_0+\nu^s_{(x)}$, where $v_0 \in {\rho_{(x)}}^r$. In other words, in addition to any linear subspace $\nu^s_{(x)}$ from the family $M \subset G_s\,(\mathbb{R}^m)$ we consider all affine subspaces parallel to $\nu^s_{(x)}$ lying inside the linear subspace $\rho^s_{(x)}$ (i.e., lying in the fiber of the bundle $\rho^r$). It is easy to see that the family of affine $s$-dimensional surfaces thus constructed is the total space of the quotient bundle $(M, \rho^r/\nu^s) \subseteq (G_s\,(\mathbb{R}^m)\,, \varepsilon^m/\gamma^s) = H_s\,(\mathbb{R}^m)$ embedded in the space $H_s\,(\mathbb{R}^m)$ of all affine subspaces.

The family of $s$-dimensional affine subspaces in $\mathbb{R}^m$ as defined above furnishes the diagram of the form:

$$\begin{array}{ccccc} & & (M,\rho^r) & & \\ & \swarrow & & \searrow & \\ \mathbb{R}^m & & & & (M,\rho^r/\nu^s) \end{array} \tag{3.4}$$

where the left arrow $(M, \rho^r) \to \mathbb{R}^m$ can be interpreted as a fiberwise injective mapping of the vector bundle $\rho^r$ over $M$ to the vector bundle $\mathbb{R}^m$ over a point; and the right arrow is a quotient map as above. (We point out that by now different points $v_1 \neq v_2$ in $\mathbb{R}^m$ are essentially "unequal in rights".)

We write the integral transform, corresponding to our diagram the same as above:

$$f\,(x, v) := \int_{\mathbb{R}^s} f\,(v + uV(x))\, d^s u. \qquad v \in \rho^r_{(x)}, \tag{3.5}$$

where the frame $\vec{V}_1\,(x) \ldots \vec{V}_s\,(x)$ determines linear subspace $\nu^s_{(x)} \subset \mathbb{R}^m$ at $x \in M$. (We emphasize, that vectors $\vec{V}_i$ are $m$-dimensional.) We should now keep in mind that $v$ is not arbitrary, but belongs to $\rho^r\,(x)$. Suppose the manifold $M$ is $s$-dimensional, this is the most important case for applications below. Suppose the Euler class of $\nu^s$ to be nontrivial. What prevents us from using the inversion formula as above? The partial derivatives $\partial \widehat{f}\,(v, x)\,/\partial v^\alpha$ are not well-defined now (cf. (3.3))! We only possess the derivatives $\partial_{\vec{V}} f\,(v, x)$ along vectors $\vec{V} \in \rho^r_{(x)}$. Note that the partial derivatives $\partial \widehat{f}/\partial v^\alpha$ in (3.3) appear in contraction with differentials $d\,(V_i^\alpha\,(x))$. We transform the form in (3.3) as follows:

$$\begin{aligned}
\omega(x, dx) &= d(V_1^{\alpha_1}(x)) \ldots d(V_s^{\alpha_s}(x)) \left.\frac{\partial \hat{f}^s(x, v)}{\partial v^{\alpha_1} \ldots \partial v^{\alpha_s}}\right|_{v=0} \\
&= \left( dx^{\mu_1} \frac{\partial V_1^{\alpha_1}}{\partial x^{\mu_1}} \right) \ldots \left( dx^{\mu_s} \frac{\partial V_s^{\alpha_s}}{\partial x^{\mu_s}} \right) \left.\frac{\partial \hat{f}^s(x, v)}{\partial v^{\alpha_1} \ldots \partial v^{\alpha_s}}\right|_{v=0} \\
&= dx^{\mu_1} ... dx^{\mu_s} \left.\left( \left( \frac{\partial V_1^{\alpha_1}}{\partial x^{\mu_1}} \frac{\partial}{\partial v^{\alpha_1}} \right) \ldots \left( \frac{\partial V_s^{\alpha_s}}{\partial x^{\mu_s}} \frac{\partial}{\partial v^{\alpha_s}} \right) \hat{f} \right)\right|_{v=0} \\
&= \left.\left( \sum_{\mu_1 < \ldots < \mu_s} dx^{\mu_1} ... dx^{\mu_s} \cdot \operatorname{sgn}(\mu_1, ..., \mu_s) \partial_{\vec{V}_{1;\mu_1}} \ldots \partial_{\vec{V}_{s;\mu_s}} \hat{f} \right)\right|_{v=0} \\
&= dx^1 ... dx^s \left.\left(\det \begin{pmatrix} \partial_{\vec{V}_{1;1}} \ldots \partial_{\vec{V}_{1;s}} \\ \\ \partial_{\vec{V}_{s;1}} \ldots \partial_{\vec{V}_{s;s}} \end{pmatrix} \widehat{f}\right)\right|_{v=0}
\end{aligned} \tag{3.6}$$

where $\vec{V}_{i;\mu} = \frac{\partial \vec{V}_i}{\partial x^\mu}$.

Hence, if all the vectors $\vec{V}_{i;\mu}(x)$ belong to the fiber $\rho_{(x)}$, then the operator (3.6) acts along the fibers of the bundle $\rho$, and we can use operator (3.6) to obtain inversion formula. So, we have proved the following theorem.

**Theorem 3.1.** *Let us consider the integral transform* (3.5) *corresponding to the diagram* (3.4)*. Then, if the Euler characteristic class $\chi(\nu^s)$ of the vector bundle $\nu^s$ is nontrivial, and if for some basis $\vec{V}_i(x) \in \mathbb{R}^m$ in fibers $\nu^s_{(x)} \in \mathbb{R}^m$ of $\nu^s$, and for some coordinates $(x^\mu)$ at the base $M^s$ the relations $\partial \vec{V}_i(x)/\partial x^\mu \in \rho^r_{(x)}$ are valid, then the following inversion formula is valid:*

$$f(0) = \frac{(-1)^{s/2}}{2\,(2\pi)^s\,\chi} \int\limits_{M^s} dx^1 ... dx^s \left.\left(\det \begin{pmatrix} \partial_{\vec{V}_{1;1}} \ldots \partial_{\vec{V}_{1;s}} \\ \\ \partial_{\vec{V}_{s;1}} \ldots \partial_{\vec{V}_{s;s}} \end{pmatrix} \widehat{f}(x, v)\right)\right|_{v=0} \tag{3.7}$$

*Here $\chi = \langle \chi(\nu^s), M^s \rangle$ is the value of the Euler characteristic class of the bundle $\nu^s$ at the fundamental cycle.* $\square$

We recall that the Theorem above is a particular case of more general Theorem 2.1 in[9].

## 4. Inversion of horospherical transform for the Lorentz group

Let's return to our problem. We want to investigate, how to reconstruct the value of the function $f$ on the group $\mathrm{SO}(1, 2n-1)$, knowing all integrals of $f$ over submanifolds $ll_0 Z l_0^{-1}{}_{l,l_0 \in SO(1;2n-1)}$. Here $Z$ is the subgroup of pseudoorthogonal transformations, defined as (0.1).

We are now ready to outline the scheme of solution of this problem. In a sense, we will solve the problem for each point $l \in \mathrm{SO}(1, 2n-1)$ separately. Let us find the value of $f$ at the point $l$. In the manifold $H_Z(\mathrm{SO}(1, 2n-1))$ we choose the submanifold $C_l$ of the surfaces, passing through the chosen point. Then we enlarge this submanifold by choosing some "ribbon" along $C_l$. By construction our "ribbon" is provided with the structure of total space of vector bundle, where $C_l$ is its zero section. We confine $\hat{f}$ — the image of the original function $f$ with respect to our transform — to this "ribbon". Then we define some fiberwise differential operator on this vector bundle, which applied to $\hat{f}$ provides us with differential form of the type (3.6) on the zero section $C_l$ of our bundle. And then we construct the inversion formula of the type (3.7). Roughly speaking it is as follows: to obtain the value $f(l)$ one should act with the constructed fiberwise differential operator to $\hat{f}$, and then integrate the result over the zero section $C_l$ of the vector bundle. The result of the integration is the value $f(l)$.

The curious thing is that the dimension of the submanifold of horospheres, which we use for reconstruction of $f(l)$, i.e., of our "ribbon" around $C_l$, is less than the dimension of our group $\mathrm{SO}(1, 2n-1)$. Though the space of horospheres has the same dimension as $\mathrm{SO}(1, 2n-1)$, we obtain our "ribbon" spreading the manifold $C_l$ only in some specific directions. But there is no contradiction — for different points we use different submanifolds, which taken together cover all the space of horospheres.

Now we will fulfill the scheduled program. Let us find the value of the original function, say, at the origin of the group, $l = e$. For other points of $\mathrm{SO}(1, 2n-1)$ all the constructions are completely analogous.

As we have already mentioned, the subfamily $\{l^{-1} Z l\}_{l \in SO(1;2n-1)}$ of horospheres passing through the identity element $e \in \mathrm{SO}(1, 2n-1)$ is the homogeneous space $\mathrm{SO}(1, 2n-1)/N(Z) = \mathrm{S}^{2n-2}$, where $N(Z)$ is the normalizer of $Z$, and $\mathrm{S}^{2n-2}$ is the $2n-2$-dimensional sphere. (Submanifolds of horospheres, passing through other points of our group $\mathrm{SO}(1, 2n-1)$ are diffeomorphic to this one.) So $C_e = \mathrm{S}^{2n-2}$. Now we choose the "ribbon" surrounding $C_e$ as follows. Let us use the exponential coordinates in the neighborhood of the identity of $\mathrm{SO}(1, 2n-1)$. Together with any point of $C_e$, represented in these coordinates by the horosphere $l_0 \mathfrak{z} l_0^{-1}$, we also consider all the points (in the space of horospheres), which are represented by affine planes of the form $v + \mathrm{Ad}_{l_0} \mathfrak{z}$, where $v \in \mathrm{Ad}_{l_0} \mathfrak{n}(\mathfrak{z})$ (see Lemma 1.1). So, the "ribbon neighborhood" around our submanifold $C_e \subset H_Z(\mathrm{SO}(1, 2n-1))$ is constructed. We would like to stress the difference between the neighborhood just constructed with usual tubular neighborhood. Our "ribbon neighborhood" is thinner — it has codimension $n-2$, while tubular neighborhood has codimension 0. The "ribbon neighborhood" thus obtained has a natural structure of a total space of the vector bundle $\mathrm{Ad}\mathfrak{n}(\mathfrak{z})/\mathrm{Ad}\mathfrak{z}$. Here $\mathrm{Ad}\mathfrak{z} \subset \mathrm{Ad}\mathfrak{n}(\mathfrak{z}) \subset \mathfrak{so}(1;2n-1)$ are linear subbundles of the trivialized linear bundle over $C_e = \{l\mathfrak{z}l^{-1}\}_{l\in SO(1;2n-1)}$, with the fibers $\mathrm{Ad}_l\mathfrak{z}$ and $\mathrm{Ad}_l\mathfrak{n}(\mathfrak{z})$ correspondingly over a point $[l\mathfrak{z}l^{-1}] \in C_e$. So our integral transform is reduced to the integral transform, corresponding to the following diagram of the type (3.4)

$$\begin{tikzcd} & \left(\mathrm{S}^{2n-2},\, \mathrm{Ad}\mathfrak{n}(\mathfrak{z})\right) \arrow[dl] \arrow[dr] & \\ \mathrm{SO}(1,2n-1) & & \left(\mathrm{S}^{2n-2},\, \mathrm{Ad}\mathfrak{n}(\mathfrak{z})/\mathrm{Ad}\mathfrak{z}\right) \end{tikzcd}$$

To use the inversion formula obtained in Theorem 3.1, it is sufficient to verify, that

(i) The Euler class of the bundle $\mathrm{Ad}\mathfrak{z}$ over $\mathrm{SO}(1,2n-1)/N(Z) = \mathrm{S}^{2n-2}$ is not equal to zero;

(ii) For some basis $\vec{V}_i(x) \in \mathrm{Ad}_x\mathfrak{z}$ in some coordinates $(x)$ at $C_e = \mathrm{S}^{2n-2} = \mathrm{SO}(1,2n-1)/N(Z)$ the relations $\partial\vec{V}_i/\partial x_j \in \mathrm{Ad}_x\mathfrak{n}(\mathfrak{z})$ are valid.

**Lemma 4.1.** *The vector bundle* $\mathrm{Ad}\mathfrak{z}$ *is isomorphic to the cotangent bundle to* $\mathrm{SO}(1,2n-1)/N(Z) = \mathrm{S}^{2n-2}$*, and hence*

$$\langle \chi(\mathrm{Ad}\mathfrak{z}),\, [\mathrm{S}^{2n-2}]\rangle = \chi(\mathrm{S}^{2n-2}) = 2$$

*Here by* $\chi$ *we denoted Euler characteristic class as well as Euler characteristic.*

One can find a proof of completely analogous lemma in[9]. □

Before proving (ii) we choose the convenient coordinates at our sphere $\mathrm{S}^{2n-2} = \mathrm{SO}(1,2n-1)/N(Z) = C_e$.

*Remark* 4.1. In[9] we used the term "tangent". Though we are really interested only in the class of isomorphism of the corresponding real vector bundle, and tangent and cotangent bundles over smooth paracompact base are isomorphic, still it is much better to say "cotangent" bundle.

We recall, that the subgroup $Z$ was chosen as a subgroup, which acts trivially on some isotropic vector $\vec{a}$ as well as acts trivially on the quotient space $\alpha^{2n-1}/a$, where $\alpha^{2n-1} \perp \vec{a}$ is a hyperplane tangent to the isotropic cone at a point $veca$, and $a$ is the line spanned by $\vec{a}$. In our matrix realization (0.1) we used the isotropic vector $(1,0,...,0,1,0,...,0)$. It is easy to see, that the normalizer $N(Z)$ of $Z$ is the subgroup, which transforms the hyperplane $\alpha^{2n-1}$ into itself (it is not the same as "acts trivially"!), and hence $N(Z)$ transforms the isotropic line $\langle\vec{e}_0 + \vec{e}_n\rangle_{\mathbb{R}}$ into itself. Moreover, $N(Z)$ coincides with stabilizer of the isotropic line $\langle\vec{e}_0 + \vec{e}_n\rangle_{\mathbb{R}}$, with respect to the natural action of $\mathrm{SO}(1,2n-1)$ to the family of isotropic lines.

Any conjugate subgroup $l_0Zl_0^{-1}$ acts trivially on hyperplane $l_0 \cdot \alpha^{2n-1}$ orthogonal to isotropic line $l_0 \cdot \langle\vec{e}_0 + \vec{e}_n\rangle_{\mathbb{R}}$. So the family of subgroups $l_0Zl_0^{-1}$ is in one-to-one correspondence with the family of isotropic lines, which is isomorphic to the quotient space $\mathrm{SO}(1,2n-1)/N(Z)$. The set of all those $l \in \mathrm{SO}(1,2n-1)$, that $lZl^{-1} = l_0Zl_0^{-1}$, where $l_0$ is fixed, coincides with the coset $l_0N(Z)$, i.e., with the family of transformations, that transform one isotropic line (say, $\langle\vec{e}_0 + \vec{e}_n\rangle_{\mathbb{R}}$ ) to the other (say, to the $l_0 \cdot \langle\vec{e}_0 + \vec{e}_n\rangle_{\mathbb{R}}$).

In our matrix representation (c.f. (4.4)) the Lie subalgebra $\mathfrak{n}(\mathfrak{z})$ of the normalizer $N(Z)$ is the subalgebra of matrices of the form

$$
\left(\begin{array}{cccc}
0 & u^2 \cdots u^n & a & u^{n+2} \cdots u^{2n} \\
u^2 & & -u^2 & \\
\vdots & \mathrm{A} & \vdots & \mathrm{B} \\
u^n & & -u^n & \\
a & u^2 \cdots u^n & 0 & u^{n+2} \cdots u^{2n} \\
u^{n+2} & & -u^{n+2} & \\
\vdots & \text{-}\mathrm{B}^{\mathrm{T}} & \vdots & \mathrm{D} \\
u^{2n} & & -u^{2n} &
\end{array}\right) \tag{4.1}
$$

where $A, B, D$ are arbitrary $(n-1) \times (n-1)$-matrices, such that $A = -A^T$, and $D = -D^T$ (see Remark 0.1, concerning the numeration of coordinates $u$).

So we have to choose coordinates on $\mathrm{SO}(1, 2n-1)/N(Z)$. The natural way to choose coordinates on the quotient space is to choose some submanifold in $\mathrm{SO}(1, 2n-1)$ of complementary dimension (with respect to $N(Z)$) transversal to as many cosets $l \cdot N(Z)$ as possible. As an example of such submanifold we can use the group $Z_- = \exp(\mathfrak{z}_-)$, where elements of the group $Z_-$ are represented by the matrices of the form

$$
\left(\begin{array}{cccccccc}
1 + \frac{1}{2}\sum(x^i)^2 & x^2 \cdots x^n & & & \frac{1}{2}\sum(x^i)^2 & x^{n+2} \cdots x^{2n} & & \\
x^2 & 1 & \cdots & 0 & x^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x^n & 0 & \cdots & 1 & x^n & 0 & \cdots & 0 \\
-\frac{1}{2}\sum(x^i)^2 & -x^2 \cdots - x^n & & & 1 - \frac{1}{2}\sum(x^i)^2 & -x^{n+2} \cdots - x^{2n} & & \\
x^{n+2} & 0 & \cdots & 0 & x^{n+2} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x^{2n} & 0 & \cdots & 0 & x^{2n} & 0 & \cdots & 1
\end{array}\right) \tag{4.2}
$$

and elements of the Lie algebra $\mathfrak{z}_-$ are represented by matrices of the form

$$
\vec{X} = \sum_{i=1}^{n} x^i \vec{W}_i =
\left(\begin{array}{cccccccc}
0 & x^2 \cdots x^n & & & 0 & x^{n+2} \cdots x^{2n} & & \\
x^2 & 0 & \cdots & 0 & x^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x^n & 0 & \cdots & 0 & x^n & 0 & \cdots & 0 \\
0 & -x^2 \cdots - x^n & & & 0 & -x^{n+2} \cdots - x^{2n} & & \\
x^{n+2} & 0 & \cdots & 0 & x^{n+2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x^{2n} & 0 & \cdots & 0 & x^{2n} & 0 & \cdots & 0
\end{array}\right) \tag{4.3}
$$

(see Remark 0.1, concerning as well the numeration of coordinates $x$).

**Lemma 4.2.** *Different elements $l_1 \neq l_2 \in Z_-$ correspond to different horospheres $l_1 Z l_1^{-1} \neq l_2 Z l_2^{-1}$. There exists only one subgroup $lZl^{-1}$, which can not be represented as $l_0 Z l_0^{-1}$ for $l_0 \in Z_-$. In other words, the composition of the maps $Z_- \hookrightarrow \mathrm{SO}(1, 2n-1) \to \mathrm{SO}(1, 2n-1)/N(Z)$ is embedding, and completion to the* *image of this composition* $(\mathrm{SO}(1,2n-1)/N(Z)) \setminus Im(Z_-)$ *consists of the only one point.*

*Proof.* Isotropic lines can be parametrized by points of the sphere, obtained as intersection of isotropic cone with the hypersurface $v^0 = 1$, i.e., by isotropic vectors of the form $(1,*,...,*)$. Let's consider the action of matrix $X = \exp(x)$ of the form (4.3) on the isotropic line spanned by the vector

$$\vec{e}_0 + \vec{e}_n = (1,0,...,0,1,0,...,0)$$

It acts by the following way:

$$X \cdot (\vec{e}_0 + \vec{e}_1) = (1 + \sum (x^i)^2, 2x^1, ..., 2x^{n-1}, 1 - \sum (x^i)^2, 2x^n, ..., 2x^{2n-2})$$

$$\sim (1, \frac{2x^1}{1+\sum(x^i)^2}, ..., \frac{2x^{n-1}}{1+\sum(x^i)^2}, \frac{1-\sum(x^i)^2}{1+\sum(x^i)^2}, \frac{2x^n}{1+\sum(x^i)^2}, ..., \frac{2x^{2n-2}}{1+\sum(x^i)^2})$$

It is easy to see, that different $X$ remove the isotropic line, spanned by $\vec{e}_0 + \vec{e}_1$ to different isotropic lines. The only line, which can not be obtained by the action of $X$ at $\vec{e}_0 + \vec{e}_1$, is the line, spanned by the vector

$$\vec{e}_0 - \vec{e}_1 = (1,0,...,0,-1,0,...,0)$$

We have shown, that the coordinates $(x)$ are the stereographic coordinates at the sphere $\mathrm{S}^{2n-2}$. Let us now recall, that the points of the sphere $\mathrm{S}^{2n-2} = \mathrm{SO}(1,2n-1)/N = C_e$ parametrize the isotropic lines, and hence parametrize the various submanifolds $lZl^{-1}$, passing through the identity of $\mathrm{SO}(1,2n-1)$. Lemma 4.2 is proved. □

We use the basis $\vec{V}_2, ...\vec{V}_n, \vec{V}_{n+2}, ..., \vec{V}_{2n}$ at the subalgebra $\mathfrak{z} \subset \mathfrak{so}(1;2n-1)$ (See Remark 0.1, concerning the numeration of coordinates $u^i$.)

$$u^i \vec{V}_i = \begin{pmatrix} 0 & u^2 \cdots u^n & & & 0 & u^{n+2} \cdots u^{2n} & & \\ u^2 & 1 & \cdots & 0 & -u^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u^n & 0 & \cdots & 1 & -u^n & 0 & \cdots & 0 \\ 0 & u^2 \cdots u^n & & & 0 & u^{n+2} \cdots - u^{2n} & & \\ u^{n+2} & 0 & \cdots & 0 & -u^{n+2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u^{2n} & 0 & \cdots & 0 & -u^{2n} & 0 & \cdots & 1 \end{pmatrix} \tag{4.4}$$

Vectors $V_i(x)$ of our frame in the linear space $\mathrm{Ad}_{\exp(x)}\mathfrak{z}$, $x \in \mathfrak{z}_-$ are represented by

$$\mathrm{Ad}_{\exp(x)} V_i = \exp(\mathrm{ad}_x) V_i = V_i + \mathrm{ad}_x V_i + \frac{1}{2}\mathrm{ad}_x^2 V_i$$

The fact, that the expansion contains only these terms can be verified directly, using the matrix representation (4.3) for elements of the algebra $\mathfrak{z}_-$.

We are now ready to compute the operators $\partial_{V_{i;j}}$, where $V_{i;j} = \frac{\partial V_i}{\partial x^j}$

$$\begin{aligned} V_{i;j} &= \frac{\partial}{\partial x^j}\left(V_i + x^k\left[W_k V_i\right] + \frac{1}{2}x^{k_1}x^{k_2}\left[W_{k_1}\left[W_{k_2}V_i\right]\right]\right) \\ &= \left[W_j V_i\right] + \frac{1}{2}x^{k_2}\left[W_j\left[W_{k_2}V_i\right]\right] + \frac{1}{2}x^{k_1}\left[W_{k_1}\left[W_j V_i\right]\right] \\ &= \left[W_j V_i\right] + \left[x^k W_k\left[W_j V_i\right]\right] \\ &= \left[W_j V_i\right] + \mathrm{ad}_x\left[W_j V_i\right] \end{aligned} \tag{4.5}$$

We used the relation

$$\left[W_j\left[W_k V_i\right]\right] = \left[W_k\left[W_j V_i\right]\right]$$

due to commutativity of algebra $\mathfrak{z}_-$.

We denote by $I_{k,l}$ the $2n \times 2n$-matrix with the only one nonzero entry at the place $(k,l)$ being equal to 1. It is easy to verify, that the following relations are valid for the matrix realizations of $\mathfrak{so}(1;2n-1)$ and $\mathfrak{z}$ chosen as above:

$$\begin{aligned} [W_j V_i] &= 2(I_{j,i} - I_{i,j}) \qquad \text{for} \quad i \neq j && (4.6) \\ \mathrm{ad}_x[W_j V_i] &= 2(x^j W_j - x^i W_i) \qquad \text{for} \quad i \neq j \\ [W_i V_i] &= -2(I_{1,n+1} + I_{n+1,1}) && (4.7) \\ \mathrm{ad}_x[W_i V_i] &= -2x^k W_k \end{aligned}$$

Hence we have proved the following

**Lemma 4.3.** *The following explicit formulas for vectors $\vec{V}_{i;j}(x)$ are valid in our matrix realization:*

$$\vec{V}_{i;j}(x) \stackrel{def}{=} \frac{\partial \vec{V}_i}{\partial x^j} = (1 + \mathrm{ad}_x)[W_j V_i] = 2(I_{j,i} - I_{i,j}) + 2(x^j \vec{W}_j - x^i \vec{W}_i) \quad i \neq j$$

$$\vec{V}_{i;i}(x) \stackrel{def}{=} \frac{\partial \vec{V}_i}{\partial x^i} = (1 + \mathrm{ad}_x)[W_i V_i] = -2(I_{1,n+1} + I_{n+1,1}) - 2x^k \vec{W}_k$$

Indexes $i,j,k$ as usual enroll the set $\{\hat{1}, 2, ..., n, \widehat{n+1}, n+2, ..., 2n\}$.

**Corollary 4.4.** *All the vectors $\vec{V}_{i;j}(x)$ belong to the subalgebra $\mathrm{Ad}_{\exp(x)}\mathfrak{n}(\mathfrak{z})$:*

$$\vec{V}_{i;j} \in \mathrm{Ad}_{\exp(x)}\mathfrak{n}(\mathfrak{z})$$

*Proof.* We have already proved (4.5) that $\vec{V}_{i;j} = (1+\mathrm{ad}_x)[W_j V_i]$. It is easy to check, that in our case $(1 + \mathrm{ad}_x)[W_j V_i] = \exp(\mathrm{ad}_x)[W_j V_i]$. Since $\exp(\mathrm{ad}_x) = \mathrm{Ad}_{\exp(x)}$, to verify, that some vector $\mathrm{Ad}_{\exp(x)}[W_j V_i] \in \mathrm{Ad}_{\exp(x)}\mathfrak{n}(\mathfrak{z})$ it is sufficient to show, that $[W_j V_i] \in \mathfrak{n}(\mathfrak{z})$, or, what is the same, to show, that $[\mathfrak{z}_- \mathfrak{z}] \in \mathfrak{n}(\mathfrak{z})$. The last relation is easily seen by comparison the explicit formulas (4.6), (4.7) with matrix realization (4.1) of the normalizer $\mathfrak{n}(\mathfrak{z})$ of the Lie algebra $\mathfrak{z}$. □

Having proved Lemma 4.1 and Corollary 4.4 we are able to use Theorem 3.1. In our particular case the manifold $M^s$ involved in the inversion formula (3.7) is a sphere. Since our chart $(x)$ covers all the sphere except one point (see above), we can replace integration over the sphere in (3.7) by integration over the chart (x), i.e., by integration over $\mathfrak{z}_- = \mathbb{R}^{2n-2}$. We obtain the following inversion formula:

**Theorem 4.1.** *The following inversion formula is valid:*

$$f(e) = \frac{(-1)^{n-1}}{4(2\pi)^{2n-2}} \int\limits_{\mathbb{R}^{2n-2}} \left( \det \|\partial_{\vec{V}_{i;j}(x)}\| \, \hat{f}(x,v) \right) \Bigg|_{v=0} dx^2 \dots dx^n \, dx^{n+2} \dots dx^{2n}$$

*where vectors* $\vec{V}_{i;j}(x) = (1 + \mathrm{ad}_x)[W_j V_i]$ *are evaluated explicitly by Lemma* 4.3. $\square$

We recall, that the vectors $\vec{V}_{i;j}$ live in the space of variables $(v)$.

Quite analogous formula is valid for the other points $l \in \mathrm{SO}(1, 2n-1)$ under suitable choice of coordinates at $C_l = \mathrm{S}^{2n-2}$ and at "ribbon neighborhood" along $C_l$.

As an illustration, we shall evaluate the operators $\det ||\partial_{\vec{V}_{i;j}(x)}||$ for $n = 2$ and $n = 3$.

When $n = 2$ one has

$$V_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad V_4 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$W_2 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad W_4 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$V_{2;2}(x) = V_{4;4}(x) = -2 \left(\begin{array}{cc|cc} 0 & x_2 & 1 & x_4 \\ x_2 & 0 & x_2 & 0 \\ \hline 1 & -x_2 & 0 & -x_4 \\ x_4 & 0 & x_4 & 0 \end{array}\right) = -2\mathrm{Ad}_x H = -2H(x)$$

$$V_{2;4}(x) = -V_{4;2}(x) = 2 \left(\begin{array}{cc|cc} 0 & -x_2 & 0 & x_4 \\ -x_2 & 0 & -x_2 & -1 \\ \hline 0 & x_2 & 0 & -x_4 \\ x_4 & 1 & x_4 & 0 \end{array}\right) = 2\mathrm{Ad}_x H_2 = 2H_2(x)$$

Here $H = H(0) = 2 \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad H_2 = H_2(0) = 2 \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

Hence our determinant is as follows:

$$\det ||\partial_{\vec{V}_{i;j}(x)}|| = \partial_{\vec{V}_{2;2}(x)} \partial_{\vec{V}_{4;4}(x)} - \partial_{\vec{V}_{2;4}(x)} \partial_{\vec{V}_{4;2}(x)} = 4 \left( \partial^2_{\vec{H}(x)} + \partial^2_{\vec{H}_2(x)} \right)$$

Note, that $H$ and $H_2$ are the generators of the Cartan subalgebra in $\mathfrak{so}(1;3)$. Recall the Lie algebras isomorphism $\mathfrak{so}(1;3) \cong \mathfrak{sl}(2;\mathbb{C})$. So it is not surprising, that essentially, in the case $n = 2$ we obtained the particular case of inversion formula in[1] for $\mathrm{SL}(2,\mathbb{C})$ for the integrals over horospheres in $\mathrm{SL}(2,\mathbb{C})$.

To avoid huge formula we will evaluate the determinant $\det ||\partial_{\vec{V}_{i;j}(x)}||$ in the case $n = 3$ only at $x = 0$. To obtain the formula for arbitrary $x$ one should just substitute the explicit expression for $\vec{V}_{i;j}(x)$, evaluated in Lemma 4.3. We define

$$H = I_{1,4} + I_{4,1}$$

$$E_{k,l} = -E_{l,k} = I_{k,l} - I_{l,k}, \quad \text{where} \quad k < l$$

Then

$$\det \|\partial_{\vec{V}_{i;j}(x)}\| = \det \begin{pmatrix} \partial_{\vec{V}_{2;2}} & \partial_{\vec{V}_{2;3}} & \partial_{\vec{V}_{2;5}} & \partial_{\vec{V}_{2;6}} \\ \partial_{\vec{V}_{3;2}} & \partial_{\vec{V}_{3;3}} & \partial_{\vec{V}_{3;5}} & \partial_{\vec{V}_{3;6}} \\ \partial_{\vec{V}_{5;2}} & \partial_{\vec{V}_{5;3}} & \partial_{\vec{V}_{5;5}} & \partial_{\vec{V}_{5;6}} \\ \partial_{\vec{V}_{6;2}} & \partial_{\vec{V}_{6;3}} & \partial_{\vec{V}_{6;5}} & \partial_{\vec{V}_{6;6}} \end{pmatrix}$$

$$= 16 \cdot \det \begin{pmatrix} \partial_{-H} & \partial_{-E_{2;3}} & \partial_{-E_{2;5}} & \partial_{-E_{2;6}} \\ \partial_{-E_{3;2}} & \partial_{-H} & \partial_{-E_{3;5}} & \partial_{-E_{3;6}} \\ \partial_{-E_{5;2}} & \partial_{-E_{5;3}} & \partial_{-H} & \partial_{-E_{5;6}} \\ \partial_{-E_{6;2}} & \partial_{-E_{6;3}} & \partial_{-E_{6;5}} & \partial_{-H} \end{pmatrix}$$

$$= 16 \cdot \partial_H^2(\partial_H^2 + \partial_{E_{2,3}}^2 + \partial_{E_{2,5}}^2 + \partial_{E_{2,6}}^2 + \partial_{E_{3,5}}^2 + \partial_{E_{3,6}}^2 + \partial_{E_{5,6}}^2)$$
$$+(\partial_{E_{2,3}}\partial_{E_{5,6}} + \partial_{E_{2,5}}\partial_{E_{3,6}} - \partial_{E_{2,6}}\partial_{E_{3,5}})^2$$

## 5. Horospherical Integral Transform for the Other Real Lie Groups

In completion we would like to look at our problem from a more general point of view, and also, to point out the analogous integral transforms and corresponding inversion formulas for the other real semisimple Lie groups.

Let $L$ be real semisimple Lie group, $\mathfrak{l}$ its Lie algebra. We would like to figure out, what nilpotent subgroups in a real semisimple Lie group $L$ may be regarded as analogs of horospheres. We recall, that in $\mathrm{SL}(n;\mathbb{C})$ horospheres are constructed by means of the subgroup of upper-triangular matrices with ones on diagonal, and in $\mathrm{SO}(1, 2n-1)$ we used subgroup $Z$ to construct horospheres.

We recall some well known facts, concerning the structure of real semisimple Lie algebras, see, e.g.,[10,11,12]. Let $\mathfrak{l} = \mathfrak{k} + \mathfrak{p}$ be Cartan decomposition of $\mathfrak{l}$. Let $\mathfrak{h} = \mathfrak{h}_+ + \mathfrak{h}_-$ be Cartan subalgebra, chosen in accordance with Cartan decomposition: $\mathfrak{h}_+ = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_- = \mathfrak{h} \cap \mathfrak{p}$, where $\mathfrak{h}_-$ is a maximal commutative subalgebra in $\mathfrak{p}$. Let $\mathfrak{l}^{\mathbb{C}}$ be the complexification of $\mathfrak{l}$, and $\sigma$ — corresponding conjugation in $\mathfrak{l}^{\mathbb{C}}$. We denote by $R \subset \mathfrak{h}^{\mathbb{C}^*}$ the root system , corresponding to Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of algebra $\mathfrak{l}^{\mathbb{C}}$, and by $B$ the system of simple roots. One has the following antiinvolution on $\mathfrak{h}^{\mathbb{C}^*}$:

$$(\sigma^*\xi)(x) := \overline{\xi(\sigma x)} \quad \text{where } x \in \mathfrak{h}^{\mathbb{C}}\ ,\ \xi \in \mathfrak{h}^{\mathbb{C}^*} \tag{5.1}$$

it is easy to see, that $\sigma^*(R) = R$, and $\sigma\mathfrak{l}_\nu = \mathfrak{l}_{\sigma^*\nu}$, where $\nu \in R$. We use the following notation: $B_0 = \{\nu \in B,\, \sigma^*\nu = -\nu\}$ $\quad B_1 = B \setminus B_0$.

One can choose the collection of simple roots so that the set $R_+(B_1)$ would be invariant under $\sigma^*$. Then one can check that one can uniquely assign to every simple root $\alpha \in B_1$ a simple root $\beta \in B_1$, such that $\sigma^*\alpha - \beta = \sum_{\gamma\in B_0} k_\gamma\gamma$ where $k_\gamma \geq 0$. Defining $\tilde\sigma$ by $\tilde\sigma\,\alpha = \beta$ we obtain an involution of subsystem $B_1$.

The simple roots from $B_0$ are said to be "black", and ones from $B_1$ — to be "white". it is convenient to determine real semisimple Lie group by its Satake scheme — Dynkin diagram, having the roots from $B_0$ being painted in black and the roots from $B_1$ being painted in white. Besides this one should link by arrows the pairs of white roots, transposed by the involution $\tilde{\sigma}$.

Let now $\Sigma \subset B_1$ be a subsystem of white roots, invariant under involution $\tilde{\sigma}$. The empty subsystem is also acceptable. We consider the set of positive roots, which can not be represented by linear combinations of roots from $B_0 \cup \Sigma$, i.e., we consider the set of roots $R_+(B) \setminus \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_+}$. Let

$$\mathfrak{s}^{\mathbb{C}}(\Sigma) = \sum \mathfrak{l}_\nu \ , \quad \text{where } \ \nu \in R_+(B) \setminus \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_+}$$

It is easy to see, that the set of roots $R_+(B) \setminus \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_+}$ is closed, i.e., for any pair $\nu_1, \nu_2 \in R_+(B) \setminus \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_+}$ one has $\nu_1 + \nu_2 \in R_+(B) \setminus \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_+}$. Hence $\mathfrak{s}^{\mathbb{C}}(\Sigma)$ is subalgebra in $\mathfrak{l}^{\mathbb{C}}$. It is easy to see, that $\mathfrak{s}^{\mathbb{C}}(\Sigma)$ is nilpotent. The root system $R_+(B) \setminus \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_+}$ is invariant under involution. Hence subalgebra $\mathfrak{s}^{\mathbb{C}}(\Sigma)$ is the complexification of the algebra $\mathfrak{s}(\Sigma) = \mathfrak{s}^{\mathbb{C}}(\Sigma) \cap \mathfrak{l}$. The normalizer of $\mathfrak{s}^{\mathbb{C}}(\Sigma)$ is the parabolic subalgebra

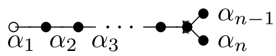
$$\mathfrak{p}^{\mathbb{C}}(\Sigma) = \mathfrak{h}^{\mathbb{C}} + \sum \mathfrak{l}_\alpha \ , \quad \text{where } \ \alpha \in R_+(B) \cup \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_-}$$

( We apologize for using the same letter "$\mathfrak{p}$" for the term of Cartan decomposition as well as for parabolic subalgebras. In the second case "$\mathfrak{p}$" will have an argument, e.g., $\mathfrak{p}(\Sigma)$.) The set of roots $R_+(B) \cup \langle B_0 \cup \Sigma \rangle_{\mathbb{Z}_-}$ is also invariant under $\sigma^*$, and hence the algebra $\mathfrak{p}^{\mathbb{C}}(\Sigma)$ is the complexification of the algebra $\mathfrak{p}^{\mathbb{C}}(\Sigma) \cap \mathfrak{l}$ — the normalizer of $\mathfrak{s}$. Finally we consider the subgroup $S(\Sigma) = \exp(\mathfrak{s}(\Sigma)) \subset L$. It is natural to consider the family of submanifolds $\{ll_0 S(\Sigma) l_0^{-1}\}_{l,l_0 \in L}$ as an analog of the family of horospheres in $L$.

Let's consider some concrete

*Example* 5.1. We choose $\mathrm{SO}(1, 2n-1)$ as a group $L$. The complexification of $\mathfrak{l}$ is $\mathfrak{l}^{\mathbb{C}} = \mathfrak{so}(2n; \mathbb{C})$. The Satake scheme of $\mathfrak{so}(1; 2n-1)$ is as follows:

$D_n^{II}$




The only nontrivial choice of subset $\Sigma \subset B_1$ is $\Sigma = \emptyset$. Then

$$R_+(B) \setminus \langle B_0 + \Sigma \rangle_{\mathbb{Z}_+} = R_+(B) \setminus \langle B_0 \rangle_{\mathbb{Z}_+} =$$

$$= \{\alpha_1, \ \ \alpha_1 + \alpha_2, \ \ \ldots \ \ , \alpha_1 + \alpha_2 + \ldots + \alpha_{n-2}, \ \ \alpha_1 + \alpha_2 + \ldots + \alpha_{n-2} + \alpha_{n-1},$$
$$\alpha_1 + \alpha_2 + \ldots + \alpha_{n-2} + \alpha_n, \ \ \alpha_1 + \alpha_2 + \ldots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n,$$
$$\alpha_1 + \ldots + \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \ \ldots \ , \ \alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n\}$$
$$= \{\varepsilon_1 - \varepsilon_j, \varepsilon_1 + \varepsilon_j\}_{1 < j \leq n}$$

The corresponding nilpotent (in this particular case even abelian) subgroup $S(\Sigma)$ coincides with subgroup $Z \subset \mathrm{SO}(1, 2n-1)$ introduced in (0.1).

We are now able to use the same scheme as before. We choose an invariant measure at the nilpotent subgroup $S(\Sigma)$, and then spread it to all the other submanifolds $ll_0 S(\Sigma) l_0^{-1}$. Then we define the corresponding integral transform, assigning to a function $f$ at the Lie group $L$ the set of integrals of the function over submanifolds $ll_0 S(\Sigma) l_0^{-1}$. Thus we again obtain function $\hat{f}$ on the "space of horospheres" $H_{S(\Sigma)}(L)$. And now we will study this transform.

Originally the integral transform of this kind was considered in[1], where the family of "horospheres" in complex group $\mathrm{SL}(n;\mathbb{C})$ was constructed by nilpotent subgroup of upper-triangular matrices with ones at diagonal. We note that the corresponding tangent Lie subalgebra to such a subgroup is subalgebra spanned by all positive root vectors, i.e., this is subalgebra of the same type as our $\mathfrak{s}(\Sigma)$. The inversion formula obtained in[1] for this integral transform was the key element of decomposition of regular representation at $\mathrm{SL}(n;\mathbb{C})$ in irreducible ones.

Integral transforms, corresponding to families of horospheres, constructed by complex subgroups of the type $S(\Sigma)$ of complex semisimple Lie groups, were studied in[9]. It was proved there, that all of them are invertible; the inversion formula was presented.

A lot of results in[9] can be applied (almost without changes) to the case at hand — the case of real nilpotent subgroups of the type $S(\Sigma)$ in real semisimple Lie groups. We will use these results in great extent, addressing the reader to the paper[9], containing detailed presentation of the subject.

As in the present paper above, and as in[9] we consider the exponential coordinates in a neighborhood of a point $g$ of our real semisimple Lie group $L$. The horospheres, passing through the fixed point will be represented in this coordinates by linear subspaces $l_0\mathfrak{s}(\Sigma)l_0^{-1}$ of algebra $\mathfrak{l}$. The horospheres $ll_0\mathfrak{s}(\Sigma)l_0^{-1}$, where $l \in l_0N(S(\Sigma))l_0^{-1}$ will be represented by affine subspaces $\vec{v} + \mathrm{Ad}_{l_0}\mathfrak{s}(\Sigma)$ , $\vec{v} \in \mathrm{Ad}_{l_0}\mathfrak{n}(\mathfrak{s}(\Sigma))$ parallel to the linear subspace $\mathrm{Ad}_{l_0}\mathfrak{s}(\Sigma)$. Here $N(\Sigma) = N(S(\Sigma))$ is the normalizer of $S(\Sigma)$, $\mathfrak{n}(\Sigma) = \mathfrak{n}(\mathfrak{s}(\Sigma))$ is the corresponding Lie algebra. The same way as above, we choose in the family of all horospheres $H_{S(\Sigma)}(L)$ the subfamily of those horospheres, which are represented by affine spaces in our exponential coordinates. By construction, such submanifold of horospheres has a structure of a "ribbon" over the submanifold $G_{S(\Sigma)}(L)$ of horospheres passing through the chosen point. Moreover, it has a structure of vector quotient bundle $\mathrm{Ad}\mathfrak{n}(\mathfrak{s}(\Sigma))/\mathrm{Ad}\mathfrak{s}(\Sigma)$ over the base $G_{S(\Sigma)}(L)$. Vector bundles $\mathrm{Ad}\mathfrak{n}(\mathfrak{s}(\Sigma))$ and $\mathrm{Ad}\mathfrak{s}(\Sigma)$ have the fibers $\mathrm{Ad}_l\mathfrak{n}(\mathfrak{s}(\Sigma))$ and $\mathrm{Ad}_l\mathfrak{s}(\Sigma)$ correspondingly over a point $x = [lS(\Sigma)l^{-1}] \in G_{S(\Sigma)}(L)$. By construction our vector bundles are consecutively embedded into trivialized vector bundle with the fiber $\mathfrak{l}$ , where $\mathfrak{l}$ is the Lie algebra of the Lie group $L$.

$$\mathrm{Ad}\mathfrak{s}(\Sigma) \subset \mathrm{Ad}\mathfrak{n}(\mathfrak{s}(\Sigma)) \subset \mathfrak{l}$$

Hence we again reduced our integral transform to a relative of Radon transform, corresponding to the following diagram

$$\begin{array}{ccc} & (G_{S(\Sigma)}(L), \mathrm{Ad}\mathfrak{n}(\Sigma)) & \\ \swarrow & & \searrow \\ \mathfrak{l} & & (G_{S(\Sigma)}(L), \mathrm{Ad}\mathfrak{n}(\Sigma)/\mathrm{Ad}\mathfrak{s}(\Sigma)) \end{array}$$

it is convenient to describe our integral transform as follows:

We lift up the function from the Lie algebra $\mathfrak{l}$ to the direct product $G_{S(\Sigma)}(L) \times \mathfrak{l}$, then restrict the function to the total space of vector bundle $\big(G_{S(\Sigma)}(L), \mathrm{Ad}\mathfrak{n}(\Sigma)\big) \hookrightarrow G_{S(\Sigma)}(L) \times \mathfrak{l}$ embedded into the product, and after that we integrate the function over affine subspaces (in the fibers of the bundle), parallel to fibers of subbundle $\mathrm{Ad}\mathfrak{s}(\Sigma)$. Finally we obtain a function $\hat{f}$ on the total space of the vector bundle $\big(G_{S(\Sigma)}(L), \mathrm{Ad}\mathfrak{n}(\Sigma)/\mathrm{Ad}\mathfrak{s}(\Sigma)\big)$. We are going to find out, how one can reconstruct the value $f(0)$ of the original function at the origin of $\mathfrak{l}$.

The same as above, we are able to choose the following coordinate system at the base $G_{S(\Sigma)}(L) = L/N(\Sigma)$: as the coordinate system we use the algebra $\mathfrak{s}_-(\Sigma) =$

$\sum \mathfrak{l}_\mu$, where $\mu \in R_-(B) \setminus \langle B_0 \cup \Sigma\rangle_{\mathbb{Z}_-}$, and the exponential mapping. A point $x \in \mathfrak{s}_-(\Sigma)$ corresponds to a point $[\exp(x)S(\Sigma)\exp^{-1}(x)] \in G_{S(\Sigma)}(L)$.

A basis in the Lie algebra $\mathfrak{s}$ generated by the root vectors $E_\nu$, $\nu \in R_+(B) \setminus \langle B_0 \cup \Sigma\rangle_{\mathbb{Z}_+}$, determines a basis $\vec{V}_\nu(x) \in \mathrm{Ad}_l\mathfrak{s}(\Sigma)$ , in fibers $\mathrm{Ad}_{\exp(x)}\mathfrak{s}(\Sigma)$ of the bundle $\mathrm{Ad}\mathfrak{s}(\Sigma)$.

Let's consider a fiberwise linear operator (with constant coefficients along fibers), which acts in the fibers of the quotient bundle $\mathfrak{l}/\mathrm{Ad}\mathfrak{s}(\Sigma)$, and which has the following form:

$$\det \|\partial_{\vec{V}_{\nu;\mu}(x)}\| \tag{5.2}$$

Here we have denoted by $\vec{V}_{\nu;\mu}(x)$ vector $\partial\vec{V}_\nu(x)/\partial x^\mu \in \mathfrak{l}$, projected to the quotient bundle $\mathfrak{l}/\mathrm{Ad}_l\mathfrak{s}(\Sigma)$. In our case dimension $\dim\mathfrak{s}(\Sigma)$ of a fiber of the bundle $\mathrm{Ad}\mathfrak{s}(\Sigma)$ is equal to the dimension $\dim G_{S(\Sigma)}(L) = \dim L/N(\Sigma)$ of the base, so we really have the same number of indices $\mu$ and $\nu$.

We recall, that we are so interested in the operator (5.2), because it may provide us with an inversion formula (cf. (3.7)).

According to Theorem 2.1 in[9] one has the following sufficient conditions of invertibility:

*(i) The Euler class of the bundle* $\mathrm{Ad}\mathfrak{s}(\Sigma)$ *is nontrivial;*

*(ii) The fiberwise linear operator* (5.2)*, acting along fibers of the bundle* $\mathfrak{l}/\mathrm{Ad}\mathfrak{s}(\Sigma)$*, actually acts along fibers of the subbundle* $\mathrm{Ad}\mathfrak{n}(\Sigma)/\mathrm{Ad}\mathfrak{s}(\Sigma) \subset \mathfrak{l}/\mathrm{Ad}\mathfrak{s}(\Sigma)$.

This time we start with discussion of condition (ii). We point out, that in the case of subgroup $Z \subset \mathrm{SO}(1, 2n-1)$ we deal with in Sections 1–4, it turned out, that all the vectors $\vec{V}_{i,j}(x)$ belong to the fibers of the bundle $\mathrm{Ad}\mathfrak{n}(\mathfrak{z})$, $\vec{V}_{i,j}(x) \in \mathrm{Ad}_{\exp(x)}\mathfrak{n}(\mathfrak{z})$, and hence the condition (ii) is obviously valid. This is not true in the general case. It may happen, that some vector $\vec{V}_{\nu;\mu}(l)$ outstands from the fiber of the bundle $\mathrm{Ad}_l\mathfrak{n}(\mathfrak{z})$. Still, this doesn't mean, that the operator (5.2) does not acts along the fibers of $\mathrm{Ad}_l\mathfrak{n}(\Sigma)/\mathrm{Ad}_l\mathfrak{s}(\Sigma)$. When we will calculate the determinant, part of the terms will vanish. One terms — because some entries $\vec{V}_{\nu;\mu}$ equal 0, the other — because some entries $\vec{V}_{\nu;\mu}(l)$ belong to $\mathrm{Ad}_l\mathfrak{s}(\Sigma)$, and, hence, they turn into zero after factorization.

It happens that the condition (ii) is always valid for real semisimple Lie groups. In fact, according to Lemma 2.4 in[9] the condition (ii) is valid for complexified algebras $\mathfrak{l}^{\mathbb{C}}$, $\mathfrak{s}^{\mathbb{C}}(\Sigma)$, $\mathfrak{s}^{\mathbb{C}}_-(\Sigma)$, $\mathfrak{n}^{\mathbb{C}}(\Sigma)$. But validity of this condition does not depend on the choice of a basis in a fiber of $\mathrm{Ad}\mathfrak{s}^{\mathbb{C}}(\Sigma)$ — whenever it is valid for one basis, it is valid for any other, in particular for the basis

$$\mathrm{Ad}_l(E_\nu + \sigma(E_\nu)) \qquad \mathrm{Ad}_l(iE_\nu - i\sigma(E_\nu))$$

where $\nu \in R_+(B) \setminus \langle B_0 \cup \Sigma\rangle_{\mathbb{Z}_+}$. Hence, the condition (ii) is valid as well for the real case, and we have proved the following theorem:

**Theorem 5.1.** *Let* $\vec{V}_\nu(x) \in \mathfrak{l}$ *be the basis in a fiber* $\mathrm{Ad}_{\exp(x)}\mathfrak{s}(\Sigma) \subset \mathfrak{l}$ *of the cotangent bundle over the base* $G_{S(\Sigma)}(L) = L/N(\Sigma)$ *of the following form:* $\vec{V}_\nu(x) = \mathrm{Ad}_{\exp(x)}E_\nu$*, where* $\nu \in R_+(B) \setminus \langle B_0 \cup \Sigma\rangle_{\mathbb{Z}_+}$*. Let* $x^\mu \in \mathfrak{s}_-(\Sigma)$*, where* $\mu \in R_-(B) \setminus \langle B_0 \cup \Sigma\rangle_{\mathbb{Z}_-}$*, be the exponential coordinates at the unique Schubert cell of highest dimension at the base* $G_{S(\Sigma)}(L) = L/N(\Sigma)$ *of our bundle. Let* $\vec{V}_{\nu;\mu}(x)$ *be projection of a vector* $\partial\vec{V}_\nu(x)/\partial x^\mu \in \mathfrak{l}$ *to the quotient space* $\mathfrak{l}/\mathrm{Ad}_{\exp(x)}\mathfrak{s}(\Sigma)$*. Then*

*The linear operator* $\det \|\partial_{\vec{V}_{\nu;\mu}(x)}\|$ *defined as a fiberwise linear operator, acting along fibers of the bundle* $\mathfrak{l}/\mathrm{Ad}\mathfrak{s}(\Sigma)$*, actually acts along fibers of the subbundle* $\mathrm{Ad}\mathfrak{n}(\mathfrak{s}(\Sigma))/\mathrm{Ad}\mathfrak{s}(\Sigma)$.

*The following relation is valid*

$$\det \|\partial_{\vec{V}_{\nu;\mu}(x)}\| = \det \|\partial_{\exp(ad(x))[E_\nu;E_\mu](x)}\|$$

Note, that the matrices under the determinants do not coincide by entries in general case.

And now, let's discuss the condition (i). We can prove literary the same as in[9], that the bundle $\mathrm{Ad}\mathfrak{s}(\Sigma)$ over $L/N(S(\Sigma))$ is isomorphic to the cotangent (see Remark 4.1) bundle. Hence, in order to find out, whether the Euler class of our vector bundle is equal to zero, one should just find out, whether the Euler characteristic of the homogeneous space $L/N(S(\Sigma))$ is equal to zero. One can extract the list of all simply-connected homogeneous spaces of our type, having nonzero Euler characteristic, from theorem 2 and table 1 in[12]. We point out, what essentially separates the complex case from the real one — this is the condition $\chi(L/N(S(\Sigma))) \neq 0$. While in complex case the Euler characteristic of quotient space of semisimple Lie group over parabolic subgroup is always nonzero, in real case this is not true in general case.

We list here those classical nonconpact real simple Lie groups, and subsets $\Sigma$ of white roots in the related Satake schemes, for which the Euler characteristic $\chi\left(G_{S(\Sigma)}(L)\right) = \chi\left(L/N(S(\Sigma))\right) \neq 0$, see[11].

$\mathrm{SL}(\boldsymbol{2n};\mathbb{R})$

$A^I_{2n-1}$ ○—○— ⋯ —○—○

Subset $\Sigma$ can be obtained by eliminating from $B$ an arbitrary subset of roots with even numbers, i.e.

$$\Sigma = B \setminus \{\nu_{2i_1}, \nu_{2i_2}, ..., \nu_{2i_k}\}$$

$$L/N(\Sigma) = F_{2i_1,2i_2,...,2i_k}(\mathbb{R}^{2n}).$$

Here $F_{2i_1,2i_2,...,2i_k}(\mathbb{R}^{2n})$ is the manifold of flags $\gamma^{2i_1} \subset \gamma^{2i_2} \subset ...\gamma^{2i_k} \subset \mathbb{R}^{2n}$, where $\gamma^j$ is linear $j$-dimensional subspace in $\mathbb{R}^{2n}$. As far as the formula

$$\chi(F_{2i_1,2i_2,...,2i_k}(\mathbb{R}^{2n})) = \frac{n!}{i_1!(i_2-i_1)!...(i_k-i_{k-1})!(n-i_k)!} \tag{5.3}$$

for Euler characteristic of flag manifold is not rather popular, we suggest a simple proof of it below.

Manifolds of real and complex flags have similar, cell complexes (contained of Schubert cells); in particular at any dimension $j$ the real flag manifold has the same number of cells as the complex one in dimension $2j$. This means, that one can evaluate the Euler characteristic (related to coefficients in $\mathbb{Z}$) of the manifold of real flags as the value of Poincare polynomial $P(t^2)$, corresponding to the manifold of complex flags, at the point $t^2 = -1$. The Poincare polynomial of the manifold of complex flags $F_{2i_1,2i_2,...,2i_k}(\mathbb{C}^{2n})$ is as follows (see section 1.3 in Chapter 4 in[13])

$$P(t^2) = \frac{\Pi_{2n}(t^2)}{\Pi_{2i_1}(t^2)\Pi_{2i_2-2i_1}(t^2)...\Pi_{2i_k-2i_{k-1}}(t^2)\Pi_{n-2i_k}(t^2)} \tag{5.4}$$

where $\Pi_s(\lambda) = (1-\lambda^s)(1-\lambda^{s-1})...(1-\lambda)$. When $t^2 = -1$ both the numerator and denominator become equal to zero, so to evaluate $P(-1)$ one has to use a decomposition over a small parameter $u$ near the point $t^2 = -1$:

$$\begin{aligned}\Pi_{2j}(-1+u) &= (1-(1-2ju))\cdot 2\cdot(1-2(j-1)u)\cdot 2\cdot ...\cdot(1-2u)\cdot 2\\ &= 2^{2j}(j!)u^j + o(u^j) \quad u\to 0\end{aligned}$$

Substituting the formula above into (5.4), and evaluating the limit while $u \to 0$ we get (5.3).

**SL$(2n+1;\mathbb{R})$**

$A^I_{2n}$ 

Subsets $\Sigma$ in the basis $B$ satisfy the following conditions: let $\{\nu_{m_1},...,\nu_{m_k}\}$ be the completion $B\setminus\Sigma$, where $1\le m_1 < m_2 < ... < m_k \le 2n$. We assume all even numbers $m_q = 2i_q$ to precede all odd numbers $m_r = 2i_r+1$. In other words, the set $m_1,...,m_k$ is of one of the following types:

$$2i_1,...,2i_k$$
$$2i_1,...,2i_{j-1},2i_j+1,2i_{j+1}+1,...,2i_k+1$$
$$2i_1+1,...,2i_k+1$$

The same as in the previous case the set $\Sigma = B\setminus\{\nu_{m_1},...,\nu_{m_k}\}$ states for the flag manifold $L/N(\Sigma) = F_{m_1,...,m_k}(\mathbb{R}^{2n})$, which has the following Euler characteristic:

$$\chi\left(F_{m_1,...,m_k}(\mathbb{R}^{2n})\right) = \frac{n!}{\left[\frac{m_1}{2}\right]!\left[\frac{m_2-m_1}{2}\right]!\,...\,\left[\frac{m_k-m_{k-1}}{2}\right]!\left[\frac{2n-m_k}{2}\right]!}$$

We have denoted by $[\,]$ the integer part of a number.

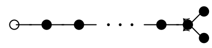
**SL$(n;\mathbb{H})$**

$A^{II}_{2n-1}$ 

In this case one can use an arbitrary subset of white roots as a subset $\Sigma$. The submanifold $L/N(\Sigma)$ thus obtained is a manifold of quaternion flags $F_{i_1,i_2,...,i_k}(\mathbb{H}^n)$ having the following Euler characteristic:

$$\chi(F_{i_1,i_2,...,i_k}(\mathbb{H}^n)) = \frac{n!}{i_1!(i_2-i_1)!...(i_k-i_{k-1})!(n-i_k)!}$$

**SO$(1, 2n-1)$**

$D^{II}_n$ 

The only possible choice here is $\Sigma = \emptyset$. this case was considered in details in Sections 1–4. We have here $L/N(\Sigma) = \mathrm{S}^{2n-2}$, and $\chi(\mathrm{S}^{2n-2}) = 2$.

Before summarizing the obtained results into a theorem we note, that one can feel free in composing the simple groups, and their subgroups $S(\Sigma)$ as listed above, into corresponding semisimple Lie groups and related subgroups.

**Theorem 5.2.** *Let $L$ and $S(\Sigma)$ be real semisimple Lie group and its nilpotent subgroup composed from the simple Lie groups and corresponding nilpotent subgroups as listed above. Then the related horospherical integral transform*

$$I : \mathcal{C}_0^\infty(L) \to \mathcal{C}^\infty\left(H_{S(\Sigma)}(L)\right)$$

*is invertible. The following local inversion formula is valid:*

$$f(l) = \frac{(-1)^{s/2}}{2\,(2\pi)^s\,\chi} \int\limits_{\mathfrak{s}_-(\Sigma)} dx^{\mu_1}...dx^{\mu_s} \left(\det \|\partial_{\exp(ad(x))[E_\nu;E_\mu]}\|\, \widehat{f}(x,v)\right)\Bigg|_{v=0}$$

*Here $x^\mu \in \mathfrak{s}_-(\Sigma)$, are the exponential coordinates at the unique Schubert cell of the highest dimension at the manifold $L/N(\Sigma) = G_{S(\Sigma)}(L) \hookrightarrow H_{S(\Sigma)}(L)$, where $G_{S(\Sigma)}(L)$ is submanifold of horospheres, passing through the point $l$. Coordinates $v$ are coordinates in specific directions transversal to submanifold $G_{S(\Sigma)}(L)$ as described above (coordinates in the fibers of "ribbon neighborhood"); $\chi\left(G_{S(\Sigma)}(L)\right)$ is the Euler characteristic of $G_{S(\Sigma)}(L)$, and $s = \dim S(\Sigma)$.*

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## References

[1] I.M.Gel'fand, M.I.Graev, *Trans. Moscow Math. Soc.*, **8** (1959).
[2] I.M.Gel'fand, M.I.Graev, N.J.Vilenkin, *Integral Geometry and Related Subjects of Representation Theory*, (Fizmatgiz, Moscow, 1962) (in Russian).
[3] S.Helgason, *Differential Geometry and Symmetric Spaces* , (Academic Press, New York, 1962).
[4] S.Helgason, *Groups and Geometrical Analysis*, (Academic Press, New York, 1984).
[5] I.M.Gel'fand, M.I.Graev, Z.Ya.Shapiro, *Functional Anal. Appl.*, **3** (1969).
[6] I.M.Gel'fand, S.G.Gindikin, M.I.Graev, *J. Soviet Math*, **18** (1982), no.1, 39-167.
[7] S.Helgason, *The Radon Transform*, (Birkhäuser, Boston, 1980).
[8] A.V.Zorich, *Russian Math. Surveys*, **42**(1987), no.4, 151-152.
[9] A.V.Zorich, *Leningrad Math. J.*, **2** (1990), No.1, 65–96
[10] Sh. Araki, *J.of Math., Osaka City University*, **13** (1962) 1–34.
[11] A.L.Oniščik, *Math. USSR — Sbornik* **3** (1967), no. 3, 373–388; and **4** (1968), no. 2, 233–240.
[12] E.B.Vinberg, A.L.Oniščik, *Seminar on Lie Groups and Algebraic Groups*, (Nauka, Moscow, 1988) (in Russian).
[13] D.B.Fuchs, *Classical Manifolds*, (Itogi Nauki i Tekhniki: Sovremennye Problemy Mat., Fundament. Napravleniya, vol.12, VINITI, Moscow, 1986, pp. 253-314) (in Russian).
[14] A.B.Goncharov, *Functional Anal. Appl.*, **23** (1989).

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