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Square Tiled Surfaces and Teichmüller Volumes of the Moduli Spaces of Abelian Differentials

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Abstract. We present an approach for counting the Teichmüller volumes of the moduli spaces of Abelian differentials on a Riemann surface of genus g. We show that the volumes can be counted by means of counting the "integer points" in the corresponding moduli space. The "integer points" are represented by *square tiled surfaces* – the flat surfaces tiled by unit squares. Such tilings have several conical singularities with $8, 12, \dots$ adjacent unit squares. Counting the leading term in the asymptotics of the number of tilings having at most N unit squares, we get the volumes of the corresponding strata of the moduli spaces.

1 Motivations

The situation in which one can tell nothing about a concrete dynamical system, while one can give a reasonable description of a generic representative of some *family* of dynamical systems is quite common. For numerous two-dimensional dynamical systems like billiards in rational polygons, or measured foliations on Riemann surfaces the corresponding family of dynamical systems is represented by some subvariety of the moduli space of Abelian differentials on a Riemann surface. Moreover, the dynamical characteristics of a generic two-dimensional dynamical system of such (*parabolic*) type are usually expressed in terms of corresponding geometric or dynamical characteristics of the moduli space.

As an illustration one can consider the study of the growth rate of the number of closed trajectories of a rational billiard, or the similar problem of the study of the number of geodesic saddle loops or geodesic saddle connections on a translation surface. Recently A. Eskin and H. Masur have proved the existence of exact quadratic asymptotics cL^2 for the number of types of simple closed geodesics and for the number of saddle connections of bounded length $l \leq L$ on a typical translation surface of a given geometrical type [1]. The constant in this quadratic asymptotics is expressed in terms of the volumes of the corresponding strata in the moduli space of Abelian differentials.

Another illustration is related to the study of the topological dynamics of a generic orientable measured foliations on a Riemann surface. It is well known that the random walk on a plane has zero mean; typical deviation from the mean after N steps is of the order $\sqrt{N} = N^{0.5}$. The power ν which is responsible for the deviation N^{ν} of a long leaf of a generic measured foliation from the ergodic mean is determined by the corresponding Lyapunov exponent $1+\nu$ of the Teichmüller geodesic flow – the natural flow on the moduli space of quadratic differentials [8]. The Kontsevich formula for the sum of relevant Lyapunov exponents involves the volumes of the corresponding strata in the moduli space of Abelian differentials.

Here I would like to present an approach for the calculation of these volumes. This approach was proposed by A. Eskin and H. Masur, and independently by M. Kontsevich and the author, about two years ago. During these two years the principal advance in the calculation of the volumes was achieved in a very recent work of A. Eskin and A. Okounkov [2]. They used a technique of representation theory of the symmetric group, which allowed them to write the generating function, and to obtain the desired explicit rational numbers for numerous low-dimensional strata. However, in some aspects the more geometric initial approach has certain advantages, and thus proves its right for existence. Here we illustrate this approach by treating the two easiest examples.

2 Translation Surfaces Versus Flat Surfaces

The only closed Riemann surface which admits a flat metric is a torus. However, a Riemann surface of arbitrary genus can be endowed with a flat metric having a finite number of cone-type singularities. Say, the surface of a cube gives an example of a flat sphere having eight cone-type singularities with the cone angle $3\pi/2$ at each of them. Consider a flat Riemann surface and puncture the singularities. We get a monodromy representation of the fundamental group of the punctured surface into the group of rotations $SO(2)$: parallel transport in the flat structure along a loop turns a vector by some angle. *Translation surfaces* are those flat Riemann surfaces which have trivial monodromy: a parallel transport of a vector along any smooth closed path on a translation surface brings a vector to itself. Note that all cone angles of a translation surface are integer multiples of 2π.

One may also think of translation surface as of a surface glued from domains of the Euclidean plane using parallel translations. Consider the Euclidean plane as a complex plane $\mathbb{C}$ with a complex coordinate z. The translation surface with punctured singularities gets the canonical complex structure. It is easy to check that this complex structure extends to the surface without punctures. Moreover, by construction we can chose coordinate charts such that all the gluing rules are just translations: $z' = z + \text{const}$. Consider a holomorphic 1-form $dz, dz', \ldots$ in every such coordinate chart. Since on the

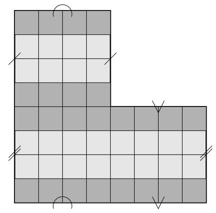

Fig. 1. Gluing the corresponding segments of the boundary by vertical and horizontal translations we get a translation surface. In this case it has genus $g = 2$ and a single singular point of the flat metric with a cone angle 6π.

overlaps $dz = dz'$, we get a globally defined holomorphic 1-form ω on the Riemann surface. Consider a singular point of the initial flat metric with a cone angle $2(n+1)\pi$. One can check that the form ω has zero of order n at this point, i.e. ω can be represented as $\omega = w^n dw$ in a local coordinate w.

Conversely, any holomorphic 1-form ω (or, what is the same, any *Abelian differential*) on a Riemann surface determines the structure of a translation surface. To see this structure it is sufficient to chose the coordinate charts on the surface (with punctured zeros of ω) in which ω has the form $\omega = dz$ in a local coordinate. Since $dz = dz'$ on the overlaps of the charts, we see that $z = z' + \text{const}$.

Note that any two Abelian differentials ω_1, ω_2 corresponding to the same translation structure differ by a constant factor $\omega_1 = e^{i\varphi}\omega_2$. To fix this factor it is sufficient to chose a vertical direction in the flat structure of the translation surface.

We see that studying families of translation surfaces we are essentially studying the *moduli spaces of Abelian differentials.*

3 Moduli Spaces of Abelian Differentials

Consider a Riemann surface M_g^2 of genus g. An Abelian differential ω is a holomorphic 1-form $\omega = \omega(z)dz$ on M_g^2. An Abelian differential has $2g-2$ zeros (counting multiplicities) on M_g^2.

Consider the moduli space of pairs (M_g^2, ω), where M_g^2 is a Riemann surface of genus g and ω is an Abelian differential on it. The *moduli space of Abelian differentials* on a Riemann surface of genus g is naturally stratified by degrees of zeros of Abelian differentials. Abelian differentials in the principal stratum have $2g-2$ simple zeros; Abelian differentials in the stratum of complex codimension one have one zero of multiplicity 2 and $2g-4$ simple zeros, etc. We denote the strata of Abelian differentials by $\mathcal{H}(d_1, \dots, d_n)$ indicating the degrees $d_1, \dots, d_k$ of zeros, where $d_1 + \dots + d_k = 2g-2$.

Consider an Abelian differential ω having zeros of orders d_i at the points $P_1, \dots, P_k$ of the Riemann surface M_g^2. Consider the period map from a neighborhood of $(\omega) \in \mathcal{H}(d_1, \dots, d_n)$ into $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$, i.e. into the first cohomology group of the Riemann surface M_g^2 relative to the subset $\{P_1, \dots, P_n\} \subset M_g^2$. We assign to an Abelian differential ω an element $[\omega]$ of the relative cohomology group $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$ by integrating ω along closed paths and along paths connecting points P_i. Locally the period map gives a one-to-one correspondence between $\mathcal{H}(d_1, \dots, d_n)$ and an open domain in the vector space $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$. Moreover, the gluing rules for these linear coordinate charts on $\mathcal{H}(d_1, \dots, d_n)$ correspond to those automorphisms of the cohomology $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$ which are induced by the diffeomorphisms of M_g^2.

Note that the cohomology group with complex coefficients contains a lattice $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\,\mathbb{Z}) \subset H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$ invariant under these automorphisms. An Abelian differential $(\omega) \in \mathcal{H}(d_1, \dots, d_n)$ represents an *integer point of the moduli space* if its image under the period map belongs to the lattice.

Consider a linear volume element in $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$ and normalize it so that the volume of a unit cube of the lattice is equal to one. Since the changes of coordinates on $\mathcal{H}(d_1, \dots, d_n)$ are linear transformations preserving the lattice, the volume element in the vector space $H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{C})$ induces the volume element $d\mu$ on $\mathcal{H}(d_1, \dots, d_n)$.

Consider an Abelian differential ω on a Riemann surface M_g^2; let A_i, B_i be its periods. The area $S(\omega)$ of M_g^2 measured in the flat structure determined by ω equals

$$\text{Area in the flat metric} = S(\omega) = \frac{i}{2}\int_{M_g^2} \omega \wedge \bar{\omega} = \frac{i}{2}\sum_{i=1}^{g}(A_i \bar{B}_i - \bar{A}_i B_i)\,.$$

We get a homogeneous real-valued function on the moduli space of Abelian differentials:

$$S : \mathcal{H}(d_1, \dots, d_n) \to \mathbb{R} \qquad S(t\omega) = |t|^2 S(\omega), \quad t \in \mathbb{C}\,.$$

Consider a “unit sphere”, or rather a “unit hyperboloid” $\mathcal{H}_1(d_1, \dots, d_n) \subset \mathcal{H}(d_1, \dots, d_n)$ defined as $S(\omega) = 1$. The volume element $d\mu$ induces a volume element $d\mu_1 := \dfrac{d\mu}{dS}$ on this real hypersurface $\mathcal{H}_1(d_1, \dots, d_n)$.

Theorem (H. Masur; W. Veech) *The volumes of the strata of Abelian differentials $\mathcal{H}_1(d_1, \dots, d_n)$ with respect to the measure $d\mu_1$ are finite.*

4 Counting Volume by Means of Counting Integer Points

How can one evaluate the surface area of a complicated body in $\mathbb{R}^n$? One of the approaches is as follows: make a homothety with a huge coefficient r

and count the number of the integer points inside the image of the body. The asymptotics of this number is $v(r) \sim \mathrm{Vol}(r) = cr^n$. The desired surface area equals

$$\frac{d\,\mathrm{Vol}(r)}{dr}\Big|_{r=1} = n \cdot c\,.$$

In other words, to compute the surface area of the body it is sufficient to know the coefficient in the leading term of the asymptotics of the number of integer points which got inside the stretched body.

The same approach can be applied to the calculation of the volumes of the strata $\mathcal{H}_1(d_1, \dots, d_n)$, but now we have to count the *integer points* $\omega_0 \in \mathcal{H}(d_1, \dots, d_n)$, such that the area $S(\omega_0)$ is bounded by some huge number N, playing the role of the radius r. The only difference with the previous case is that $S(\omega_0)$ is a homogeneous function of degree 2, so counting the hypersurface area by derivation of the volume one has to use the additional factor 2.

Let us study more attentively the *integer points*. Having an "integer" Abelian differential $[\omega_0] \in H^1(M_g^2, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\,\mathbb{Z})$ we can define a map $f_{\omega_0} : M_g^2 \to T^2 = \mathbb{C}/(\mathbb{Z} \oplus i\,\mathbb{Z})$ by

$$f_{\omega_0} : P \mapsto \Big(\int_{P_1}^{P} \omega_0 \Big) \bmod \mathbb{Z} \oplus i\,\mathbb{Z}\,.$$

It is easy to see that f_{ω_0} is a ramified covering; moreover, it has exactly n ramification points, and the ramification points are exactly the zeros $P_1, \dots, P_n$ of ω_0. Consider the flat torus T^2 as a unit square with the identified opposite sides. The covering $f_{\omega_0} : M_g^2 \to T^2$ endows the Riemann surface M_g^2 with a tiling by unit squares. The tiling represents a standard square lattice except for the vertices $P_1, \dots, P_n$ where we have correspondingly $4(d_1 + 1), \dots, 4(d_n + 1)$ squares adjacent to a vertex. Note that all the unit squares are provided with the following additional structure: we know exactly which edge is top, bottom, right, and left; adjacency of the squares respects this structure in a natural way. We shall call a flat surface with such tiling a *square tiled surface*. We see that the problem of counting the volume of $\mathcal{H}_1(d_1, \dots, d_n)$ is equivalent to the following:

Problem: *How many different square tiled surfaces of the given topological type one can construct using at most $N \gg 1$ squares?*

Note that some strata are not connected, while for some applications it is important to know the volume of each connected component separately. In this case the topological data is described not only by the singularity type $(d_1, \dots, d_n)$, but also by the parameters fixing the connected component. When all d_i are even, this additional parameter is the *spin-structure* defined by ω. The strata $\mathcal{H}(g-1, g-1)$, and $\mathcal{H}(2g-2)$ have also *hyperelliptic* components (see [5] for the classification of connected components of the strata of Abelian differentials).

5 Two Examples of Computation

To give an idea of the computation we treat here the strata $\mathcal{H}(\emptyset)$ and $\mathcal{H}(2)$.

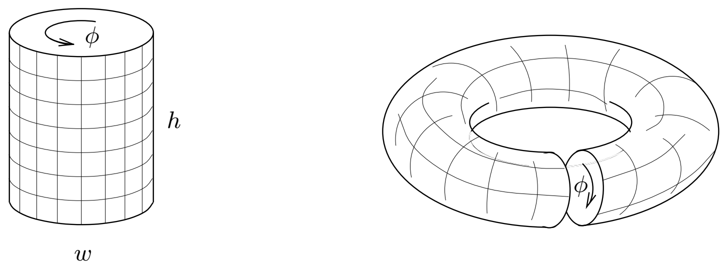

Fig. 2. A square tiled torus glued from a square tiled cylinder with the twist ϕ

We start with the case of the torus. In this case our square tiled surface has no singularities at all: we have a flat torus tiled with the unit squares in a regular way. Cutting our flat torus by a waist curve we get a cylinder with a waist curve w and a height h. The number of squares in the tiling equals $w \cdot h$. We can reglue the torus from the cylinder with some integer twist ϕ. Making an appropriate Dehn twist along the waist curve we can reduce the value of the twist ϕ to one of the values $0, 1, \dots, w-1$. In other words, fixing the integer perimeter w and height h of a cylinder we get w nondiffeomorphic square tiled tori.

Thus the number of square tiled tori constructed by using at most N squares is about

$$\sum_{\substack{w,h\in\mathbb{N}\\ w\cdot h\le N}} w = \sum_{\substack{w,h\in\mathbb{N}\\ w\le \frac{N}{h}}} w \approx \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\cdot\zeta(2) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

Actually, some of the tori present in the first sum are equivalent by a diffeomorphism, so we are counting them twice, or even several times. But this correction does not affect the leading term, so we simply neglect it.

Taking the derivative $2\cdot \frac{d}{dN}\big|_{N=1}$ we finally get

$$\mathrm{Vol}(\mathcal{H}_1(\emptyset)) = \frac{\pi^2}{3}.$$

We proceed with the stratum $\mathcal{H}(2)$. A square tiled surface of this type has a single conical point with a cone angle 6π (see an example of such surface in Fig. 1). Choose the squares adjacent to this point. Consider those horizontal sides of them which are adjacent to the critical point. They define six horizontal *separatrix rays* of our critical point.

Note, that the sides of the squares are oriented. Gluing the squares together we respect this orientation. Among the six horizontal separatrix rays three rays are *incoming* and three rays are *outgoing* (see Fig. 1).

Following the line defined by a separatrix ray we always follow the horizontal sides of the squares of the tiling. Since the number of the squares is finite every separatrix line is closed. Thus our separatrix rays are organized in three *separatrix loops*. An outgoing ray returns at the incoming one. All possible ways to arrange six separatrix rays into three separatrix loops are presented at Fig. 3.

Exercise 1. Which of these diagrams is realized by the square tiled surface from Fig. 1?

Consider now the union of those squares which are adjacent to the separatrix loops (on Fig. 1 they have dark color). We get a square tiled surface $\dot{M} \subset M_g^2$ with holes. Since the complement $M_g^2 \setminus \dot{M}$ does not contain any singular point, it is formed from several square tiled cylinders. Thus the boundary of $\dot{M}$ is decomposed into pairs of circles of the same length. In every pair there is one "top" and one "bottom" boundary component.

Looking at the diagrams presented in Fig. 3 we see, that the left diagram defines a surface $\dot{M}$ with a single pair of boundary components. The middle diagram defines a surface $\dot{M}$ with two pairs of boundary components. The right diagram defines a surface $\dot{M}$ with a single "top" boundary component, and with three "bottom" boundary components. Since each "top" boundary component must be attached to a "bottom" boundary component by a cylinder, this diagram is not realizable by a square tiled surface.

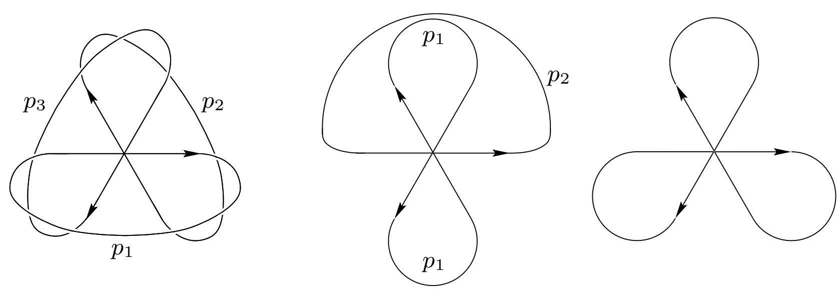

Fig. 3. The separatrix diagrams represent from left to right a square tiled surface glued from a) one cylinder; b) two cylinders; c) the diagram is not realizable by a square tiled surface.

Consider those square tiled surfaces from $\mathcal{H}(2)$ which correspond to the left diagram from Fig. 3. In this case $\dot{M}$ has one "top" and one "bottom" boundary component, so our surface is glued from a single cylinder. The waist curve of the cylinder is of length $w = p_1 + p_2 + p_3$, where p_1, p_2, p_3 are the lengths of the separatrix loops. Denote the height of the cylinder by h_1. Note, that similarly to the case of torus, there is one more integer parameter determining our square tiled surface: the twist ϕ of the cylinder. It has an integer value in the interval $[1, p_1+p_2+p_3]$. Thus the number of surfaces of this type of area bounded by N is asymptotically equivalent to the sum

$$\frac{1}{3} \sum_{\substack{p_1,p_2,p_3,h\in\mathbb{N}\\(p_1+p_2+p_3)h\le N}} (p_1+p_2+p_3)\,.$$

The coefficient $1/3$ compensates the arbitrariness of the choice of enumeration of p_1, p_2, p_3 preserving the cyclic ordering. We can regroup the entries in the sum above having the same length w of the waist curve of the cylinder. The number of ordered partitions of a large integer w in the sum of three positive integers $w = p_1+p_2+p_3$ equals approximately $w^2/2$. Thus we can rewrite the sum above as follows:

$$\begin{aligned}
\frac{1}{3} \sum_{\substack{p_1,p_2,p_3,h\\(p_1+p_2+p_3)h\le N}} (p_1+p_2+p_3) &\approx \frac{1}{3}\sum_{\substack{w,h\\w\cdot h\le N}} w\cdot\frac{w^2}{2} = \frac{1}{6}\sum_{\substack{w,h\\w\le \frac{N}{h}}} w^3\\
&\approx \frac{1}{6}\sum_{h\in\mathbb{N}}\frac{1}{4}\cdot\left(\frac{N}{h}\right)^4 = \frac{N^4}{24}\cdot\sum_{h\in\mathbb{N}}\frac{1}{h^4}\\
&= \frac{N^4}{24}\cdot\zeta(4) = \frac{N^4}{24}\cdot\frac{\pi^4}{90}\,.
\end{aligned}$$

Consider a surface $\dot{M}$ corresponding to the middle diagram from Fig. 3. It has two "top" and two "bottom" boundary components. Thus, topologically, we can glue in a pair of cylinders in two different ways. However, to have a flat structure on the resulting surface we need to have equal lengths of "top" and "bottom" boundary components. These lengths are determined by the lengths of the corresponding separatrix loops. It is easy to check that one of the two possible gluings of cylinders is forbidden: it implies that one of the separatrix loops has zero length, and hence the surface is degenerate.

The other gluing is realizable. In this case there is a pair of separatrix loops of equal lengths p_1 (see Fig. 3). The surface M_g^2 is glued from two cylinders: one having a waist curve p_1, and the other one having waist curve p_1+p_2. Denote the heights and twists of the corresponding cylinders by h_1, h_2 and ϕ_1, ϕ_2. The twist of the first cylinder takes value in the interval $[1, p_1]$; the twist of the second cylinder takes value in the interval $[1, p_1+p_2]$. Thus the number of surfaces of 2-cylinder type of area bounded by N is asymptotically equivalent to the sum

$$\sum_{\substack{p_1,p_2,h_1,h_2\\p_1h_1+(p_1+p_2)h_2\le N}} p_1(p_1+p_2) = \sum_{\substack{p_1,p_2,h_1,h_2\\p_1(h_1+h_2)+p_2h_2\le N}} p_1^2+p_1p_2\,.$$

For any fixed relatively small h_1, h_2 we can replace the sum with respect to p_1, p_2 by the integral. Let $x_1 := p_1 \cdot \frac{h_1+h_2}{N}$ and $x_2 := p_2 \cdot \frac{h_2}{N}$ be the new variables, where h_1, h_2 are considered as parameters. After this change of variables our sums with respect to p_1, p_2 become the integral with respect to x_1, x_2, where we integrate over the simplex $\Delta = \{x_1 + x_2 \le 1 : x_1 \ge 0; x_2 \ge 0\}$:

$$\begin{aligned}
&\sum_{h_1,h_2} \int_\Delta \left[\left(\frac{x_1 N}{h_1+h_2}\right)^2 + \left(\frac{x_1 N}{h_1+h_2}\right)\left(\frac{x_2 N}{h_2}\right)\right]\left(\frac{N}{h_1+h_2}dx_1\right)\left(\frac{N}{h_2}dx_2\right) \\
&= N^4 \left[\int_\Delta x_1^2\, dx_1 dx_2 \cdot \sum_{h_1,h_2\in\mathbb{N}} \frac{1}{h_2(h_1+h_2)^3}\right. \\
&\quad \left. + \int_\Delta x_1 x_2\, dx_1 dx_2 \cdot \sum_{h_1,h_2\in\mathbb{N}} \frac{1}{h_2^2(h_1+h_2)^2}\right] \\
&= \frac{N^4}{24}\big[2\cdot\zeta(1,3) + \zeta(2,2)\big] = \frac{N^4}{24}\left[2\cdot\frac{\zeta(4)}{4} + \frac{3\zeta(4)}{4}\right] \\
&= \frac{N^4}{24}\cdot\frac{5}{4}\cdot\frac{\pi^4}{90}.
\end{aligned}$$

Joining the impacts of the two diagrams and applying $2\frac{d}{dN}\big|_{N=1}$ we finally get

$$\mathrm{Vol}(\mathcal{H}_1(2)) = \frac{\pi^4}{120}.$$

A Volumes of Some Strata of Abelian Differentials

Computations similar to those presented above give the volumes of other strata (connected components of the strata) of Abelian differentials for small genera g.

The stratum $\mathcal{H}(4)$ has two connected components: the component $\mathcal{H}^{odd}(4)$ contains those Abelian differentials with a single zero of order 4 which have the odd spin structure; the component $\mathcal{H}^{hyp}(4)$ contains Abelian differentials with a single zero of order 4 on a hyperelliptic surface. Similarly, $\mathcal{H}^{hyp}(6)$ is the component of Abelian differentials with a single zero of order 6 on a hyperelliptic surface (see [5] for details).

Note that the choice of normalization allows some arbitrariness. For example, we are not numbering the zeroes. The stratum $\mathcal{H}(1,1)$ with *numbered* zeroes is a double covering over the same stratum with *nonnumbered* zeroes, so the corresponding volume would be twice as much as one presented below.

In the normalization described above we get the following values:

Table 1. Teichmüller volumes of low-dimensional strata of Abelian differentials

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{H}_1(\emptyset)) &= 2\cdot\zeta(2) = \frac{1}{3}\cdot\pi^2\\
\mathrm{Vol}(\mathcal{H}_1(2)) &= \frac{2}{3!}\cdot\frac{9}{4}\cdot\zeta(4) = \frac{1}{120}\cdot\pi^4\\
\mathrm{Vol}(\mathcal{H}_1(1,1)) &= \frac{2}{4!}\cdot 4\cdot\zeta(4) = \frac{1}{2}\cdot\frac{1}{135}\cdot\pi^4\\
\mathrm{Vol}(\mathcal{H}_1^{hyp}(4)) &= \frac{2}{5!}\cdot\frac{135}{16}\cdot\zeta(6) = \frac{1}{6720}\cdot\pi^6\\
\mathrm{Vol}(\mathcal{H}_1^{odd}(4)) &= \frac{2}{5!}\cdot\frac{70}{3}\cdot\zeta(6) = \frac{1}{2430}\cdot\pi^6\\
\mathrm{Vol}(\mathcal{H}_1(1,3)) &= \frac{2}{6!}\cdot 128\cdot\zeta(6) = \frac{16}{42525}\cdot\pi^6\\
\mathrm{Vol}(\mathcal{H}_1^{hyp}(6)) &= \frac{2}{7!}\cdot\frac{2625}{64}\cdot\zeta(8) = \frac{1}{580608}\cdot\pi^8
\end{aligned}
$$

B Lyapunov Exponents of the Teichmüller Geodesic Flow

Consider a stratum $\mathcal{H}_1(d_1,\dots,d_n)$ of Abelian differentials on a surface of genus $g = d_1+\dots+d_n$. The Lyapunov exponents of the Teichmüller geodesic flow on a connected component of such stratum have the form

$$
\begin{aligned}
-(1+\nu_1) < -(1+\nu_2) \le -(1+\nu_3) \le \dots \le -(1+\nu_g) \le \underbrace{-1=\dots=-1}_{n-1}\\
\le -(1-\nu_g) \le \dots \le -(1-\nu_2) < 0 < (1-\nu_2) \le \dots \le (1-\nu_g)\\
\le \underbrace{1=\dots=1}_{n-1} \le (1+\nu_g) \le (1+\nu_{g-1}) \le \dots \le (1+\nu_2)\\
< (1+\nu_1)\,.
\end{aligned}
$$

The sum of the positive Lyapunov exponents in the normalization $\nu_1 = 1$ used in the formula above is equal to $2g+n-1$ (a result obtained by W. Veech).

However, it is much more difficult to get any information about the ν_i. Numerical simulations show that they are all distinct and positive. Recently G. Forni has found a proof that all ν_i, $1 \le i \le g$ are strictly positive [3]. Knowledge of the volumes of the corresponding strata enables to compute the values $\nu_1+\dots+\nu_g$ of the sums of ν_i by means of the Kontsevich formula. I present the results for some low-dimensional strata. I am proud to note that numerical simulations performed by M. Kontsevich and the author perfectly match these exact answers.

In the computations below I used, in particular, the volumes of the strata $\mathcal{H}(1^4)$, $\mathcal{H}(1^6)$, and $\mathcal{H}(1^8)$ obtained by A. Eskin and A. Okounkov.

Table 2. Values of the sums $\nu_1 + \cdots + \nu_g$ for the Lyapunov exponents of the Teichmüller geodesic flow

$\mathcal{H}(2)$	$\mathcal{H}(1,1)$	$\mathcal{H}^{hyp}(4)$	$\mathcal{H}^{odd}(4)$	$\mathcal{H}(1,3)$	$\mathcal{H}^{hyp}(6)$	$\mathcal{H}(1^4)$	$\mathcal{H}(1^6)$	$\mathcal{H}(1^8)$
$\frac{4}{3}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{8}{5}$	$\frac{7}{4}$	$\frac{16}{7}$	$\frac{53}{28}$	$\frac{839}{377}$	$\frac{235761}{93428}$

C Conjectural Probability $P(n)$ of n Bands of Trajectories for a Rational Interval Exchange Transformation

For some applications a more detailed information on the distributions of "integer points" is required. The "integer points" which we used for the calculation of the volumes represent surfaces glued from several flat cylinders. One can calculate separately the "integer points" representing 1-cylinder surfaces, 2-cylinder surfaces, etc... Presumably, the corresponding proportions give the probabilities to see a given number of bands of closed trajectories for a "random" integer (or rational) interval exchange transformation.

Consider an interval exchange transformation T with intervals of integer lengths $\lambda_1, \ldots, \lambda_m$. Every orbit of T is closed. Geometrically the number of types of closed orbits can be seen as follows: constructing a suspension over our interval exchange transformation we can get a square tiled surface; the orbits of T correspond to the leaves of the vertical foliation on the surface.

Consider now the interval exchange transformations with a fixed permutation π, with rational lengths λ_i, assuming that the denominator of all these rational numbers is bounded by a large integer q. Consider all such interval exchange transformations with normalized length of the interval: $\lambda_1 + \cdots + \lambda_m \leq 1$. We conjecture that the proportions of the numbers of interval exchange transformations having $1, 2, 3, \ldots$ types of trajectories are asymptotically (as q tends to infinity) the same as the asymptotic proportions of the numbers of square tiled surfaces having $1, 2, 3, \ldots$ cylinders.

Note that while the volumes of the strata are always represented by π^{2g} with a rational coefficient, the impact corresponding to k-cylinder surfaces is represented by combinations of multiple zeta values (or, a priori, by even more complicated expressions).

As an example we consider corresponding proportions for the 770 permutations in the extended Rauzy class of the permutation (7,6,5,4,1,3,2).

As suspensions of corresponding interval exchange transformations we get surfaces from the stratum $\mathcal{H}(3,1)$. In this case a rational interval exchange transformation has 1 to 4 bands of closed trajectories; presumably the probability $P(k)$ to find exactly k bands of trajectories, $k=1,2,3,4$, is presented by the following proportions of k-cylinder square tiled surfaces in the stratum $\mathcal{H}(3,1)$:

Table 3. Relative impact $P(k)$, $k=1,2,3,4$, of the k-cylinder square tiled surfaces to the volume $\operatorname{Vol}(\mathcal{H}_1(3,1))$.

$$0.19 \approx P(1) = \frac{3\,\zeta(7)}{16\,\zeta(6)}$$

$$0.47 \approx P(2) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned} 0.30 \approx P(3) = &\frac{1}{32\,\zeta(6)}\Big(12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3) \\ &+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2) \\ &- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4) \\ &+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4) \\ &- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3) \\ &- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big) \end{aligned}$$

$$0.04 \approx P(4) = \frac{\zeta(2)}{8\,\zeta(6)}\left(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\right)$$

Remark 1. Following the topological intuition based on the Morse theory one may think that generically a square tiled surface has the critical points at distinct levels, and that it generically has the maximal possible number of cylinders. This is wrong. For example, the square tiled surfaces having critical points at distinct levels represent only $1-(2\zeta(5)-\zeta(1,4))/(2\zeta(4))$, i.e. less than 9% of all square tiled surfaces in $\mathcal{H}(1,1)$. The calculation above shows that for the stratum $\mathcal{H}(3,1)$ the square tiled surfaces having the maximal possible number of cylinders (four in this case) represent only about 4% of all square tiled surfaces.

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