Apparent contours from Monge to Todd

by Bernard Teissier

In this lecture I try to follow through the nineteenth century and within algebraic geometry the avatars of the idea of apparent contour. A basic question about apparent contours is to understand to what extent the collection of all the apparent contours of a given body embedded in affine or projective space determines that body, or at least some of its numerical characters. In projective algebraic Geometry it turns out that a very direct generalization of the idea of apparent contour, the concept of polar variety, together with the notion of hyperplane section, gives the basic cohomological invariants of a projective variety, its Chern classes, and provides a generalization of the connexion between the analytic and topological interpretations of the genus of a curve. What I try to describe is a part of the merging between the currents of thought originated by Poncelet and by Riemann. In the theory of convex bodies, the analogous problem of determining a convex body in $\mathbf{R}^d$ from the collection of the apparent contours of its orthogonal projections to lower dimensional spaces was solved in arbitrary dimension only fairly recently.

The way we perceive the shape of a smooth object $A \in \mathbf{R}^3$ with our eye is through its apparent contour, the curve on the object where the lines passing through our eye and tangent to the boundary surface $S = \partial A$ of A touches S. One may say that this fact has been at the origin of a certain quantity of experimental Mathematics since the invention of perspective. One may also speculate that the origin of the mathematical study of apparent contours is, stated in modern terms, a problem of measure of complexity; first remark that the apparent contour depends only upon the bounding surface S; it is also the apparent contour of S. Remark also that we may measure the complexity of an algebraic surface by the degree of its equation. Now the following is a natural problem: if S is an algebraic surface of degree m, how complicated is its apparent contour; what is its degree (as a space curve)? Intersecting everything with a plane containing the origin 0 (the eye) and not tangent to S reduces the problem to the following: given a nonsingular algebraic curve C of degree m in $\mathbf{R}^2$, how many of the lines tangent to C pass through a given point 0?

This question would be quite natural for a mathematician like Monge, who used extensively the concept of apparent contour, in particular in his work on fortifications, and provided what one may call "geometric algorithms" to draw apparent contours to special surfaces in descriptive geometry. According to Salmon ([14]), however, the first to study this question was Wallis, who said that the number of these tangents, for a general point 0 of the plane, was at most m^2.

Later Poncelet, for whom the natural setting of this problem was complex projective geometry, considered, following Monge, the *polar curve* (the terminology is his): Let

$$f(X, Y, X) = 0,$$

where f is a homogeneous polynomial of degree m, be an equation for C. The points of C where the tangent goes though the point of $\mathbf{P}^2(\mathbf{C})$ with coordinates (ξ, η, ζ) are on C and on the curve of degree $m-1$ with equation

$$P_{(\xi,\eta,\zeta)}(f) = \xi\frac{\partial f}{\partial X} + \eta\frac{\partial f}{\partial Y} + \zeta\frac{\partial f}{\partial Z} = 0$$

obtained by *polarizing* the polynomial f with respect to the point (ξ, η, ζ). If C is non singular, the points we seek are all the intersection points of C and $P_{(\xi,\eta,\zeta)}(C)$. By Bézout's theorem, the number of these points counted with multiplicity is $m(m-1)$, for every point (ξ, η, ζ), so the number of real tangents to the real part of C passing through a point $0 \in \mathbf{R}^2$ is at most that number.

Monge and Poncelet almost never wrote equations; their arguments were "synthetic", and a bit difficult to follow for modern geometers. The first to write equations in our style are Bobillier ([1]) and Plücker ([8], [9]). They used, perhaps for the first time, the projective coordinates of a point of projective space. They also considered the family of polar curves as parametrized linearly by the coordinates (ξ, η, ζ), and began the study of what is one of the first examples of a *linear system* of curves of degree > 3 with base points (when the curve has singularities, all the polar curves go through the singular points).

In fact a number of important ideas are introduced at that time. First and foremost is the use of the complex projective plane as a natural setting for geometry, an idea due mostly to Poncelet. Second, the idea of the group of projective transformations, of course not stated so explicitely or in these terms, but certainly present in the background of the work of Monge and Poncelet. Then there is projective duality, very dear to Poncelet and viewed perhaps as an extra transformation of the projective plane.

A line in the projective space $\mathbf{P}^2$ is by definition a point in the dual projective space $\check{\mathbf{P}}^2$.

Poncelet saw that given a nondegenerate conic Q, since the polar curve of Q with respect to any point is a line, we get an isomorphism between $\mathbf{P}^2$ and its dual $\check{\mathbf{P}}^2$; he insisted that duality was defined with respect to a non degenerate conic. This insistence was partly a reply to Gergonne's idea of a sort of metamathematical principle of duality. We shall, however, refrain from identifying in this way, as was done at that time, $\mathbf{P}^2$ and $\check{\mathbf{P}}^2$.

The collection of points of $\check{\mathbf{P}}^2$ corresponding to the lines in $\mathbf{P}^2$ tangent to an algebraic curve C is an algebraic curve $\check{C} \subset \check{\mathbf{P}}^2$. A point x in $\mathbf{P}^2$ corresponds to a line $\check{x}$ in $\check{\mathbf{P}}^2$;

each point of this line represents a line in $\mathbf{P}^2$ which contains x, and the lines through x tangent to C correspond to the intersection points in $\check{\mathbf{P}}^2$ of the curve $\check{C}$ and the line $\check{x}$. So the class of the curve C, defined as the number of lines tangent to C at non singular points and passing through a given general point of $\mathbf{P}^2$, is the degree $\check{m}$ of $\check{C}$; as we saw above, it is equal to $m(m-1)$ if C has no singularities. It is geometrically obvious that $\check{\check{C}} = C$; this is called biduality (it is wrong if we do geometry over a field of positive characteristic). If the curve $\check{C}$ had no singularities as well, the computation of degrees would give $m(m-1)(m^2-m-1) = m$, which holds only for $m = 2$. So if $m > 2$ the dual of a non singular curve has singularities ; for a general non singular curve, double points (a.k.a. nodes) corresponding to double tangents of C and cusps corresponding to its inflexion points.

To understand biduality better, it becomes important to find the class of a projective plane curve with singularities, at least when these singularities are the simplest: nodes and cusps. This was done by Plücker and the formula for a curve with δ nodes and κ cusps is

$$\check{m} = m(m-1) - 2\delta - 3\kappa$$

One said that" a node decreases the class by two, and a cusp by three"
This is perhaps the first example of a search of numerical invariants of singularities.

In fact, as I mentionned earlier, from the beginning the theory was extended to surfaces in $\mathbf{P}^3$; given such a surface S defined by the homogeneous polynomial $F(x, y, z, t) = 0$, a point $(\xi : \eta : \zeta : \tau) \in \mathbf{P}^3$ defines a *polar surface* $P_{(\xi:\eta:\zeta:\tau)}(S)$; the intersection of S and $P(S)$ is the apparent contour of S from the point $(\xi : \eta : \zeta : \tau)$. Now the intersection of the polar surfaces of two points lying on a line ℓ is a curve, called the *polar curve* of S with respect to ℓ. The dual surface $\check{S} \subset \check{\mathbf{P}}^3$ is the closure of the locus of points in $\check{\mathbf{P}}^3$ corresponding to the tangent planes to S at non singular points of S. The projective dual of ℓ is a line $\check{\ell}$ in $\check{\mathbf{P}}^3$, and the number of points of intersection of $\check{\ell}$ with $\check{S}$ is the number of hyperplanes tangent to S at non singular points and containing ℓ. If ℓ is a general line, the points of intersection of $\check{\ell}$ with $\check{S}$ are all simple.

This is a modernized account of what was known about the apparent contours in the middle of the last century.

To summarize, the problem of estimating the complexity of the apparent contour, together with the concept of duality intoduced by Poncelet, led to the construction of the dual curve of a given algebraic curve and the computation of its degree for curves and surfaces with simple singularities. The equation of the dual curve or surface are called "tangential equations" of the original curve or surface.

This current of thought, rather dominant in "pure " geometry in France and Germany, seems to have been for a while fairly independant of the other developments of the time, including the work of Gauss and Riemann. Think that the second edition of Poncelet's magnum opus "Traité des propriétés projectives des figures" was published in 1863. I will not describe the work of Riemann, but quote Dieudonné in [4]:

" *Riemann's Memoir on abelian functions avoided the geometric language, and it is only in 1863 that Roch and Clebsch begin to link Riemann's results with the projective Geometry of plane algebraic curves. Their first successes quickly attract followers, and around 1870 an* *active scool of algebraic Geometry develops, around Brill and Max Noether in Germany, Smith and Cayley in England, Halphen in France, Zeuthen in Denmark, and the first generation of the Italian geometers, Cremona, C. Segre and Bertini. The main theme of their researches will be the mutual adaptation of the algebraic Geometry of the beginning of the century and the new ideas of Riemann."*

Now if we follow the thread of apparent contours, we have to move to England where Cayley and H.J.S. Smith continue to think about the general Plücker formulas for plane curves and for surfaces, and the diminution of class that a singular point imposes on a surface. Cayley proposed the idea that each singular point of a plane curve should be "equivalent" to certain numbers of cusps and nodes, as far as Plücker formulas are concerned. This caused H.J.S. Smith to write a very interesting paper containing in particular results on the contact of the polar curve with the given one near one of its singular points. It is amusing to note also that the famous "rational double points" which were so carefully studied in the 1960's and 70's appear in Salmon indexed not by their "Coxeter number" which comes from their connexion with simple Lie groups, and is equal to their "Milnor number" which is a topological invariant, but by the *diminution of class* which their presence on a projective surface would impose. For example the modern E_8 is Salmon's U_{10}.

Salmon himself made extensive computations of the numerical characters of the dual surface of a general algebraic surface of degree m lying in $\mathbf{P}^3$.

In the meantime, the identification of compact Riemann surfaces with non singular projective algebraic curves and their topological classification by the genus , as well as the birational classification by the same genus had been perfected, and by the end of the last century, the *ordre du jour* was the extension of this classification to projective surfaces. So it is not very surprising that the next important step for us comes from the search of invariants of projective surfaces.

In the theory of algebraic curves, an important formula states that given an algebraic map $f\colon C \to C'$ between algebraic curves, which is of degree $\deg f = d$ (meaning that for a general point $c' \in C'$, $f^{-1}(c')$ consists of d points, and is ramified at the points $x_i \in C,\ 1 \leq i \leq r$, which means that near x_i, in suitable local coordinates on C and C', the map f is of the form $t \mapsto t^{e_i+1}$ with $e_i \in \mathbf{N},\ \ e_i \geq 1$. The integer e_i is the *ramification index* of f at x_i. Then we have the Riemann-Hurwitz formula relating the genus of C and the genus of C' via d and the ramification indices:

$$2g(C) - 2 = d(2g(C') - 2) + \sum_i e_i$$

If we apply this formula to the case $C' = \mathbf{P}^1$, knowing that any compact algebraic curve is a finite ramified covering of $\mathbf{P}^1$, we find that we can calculate the genus of C from any linear system of points made of the fibers of a map $C \to \mathbf{P}^1$) if we know its degree and its singularities: we get

$$2g(C) = 2 - 2d + \sum e_i$$

The ramification points x_i can be computed as the so-called *jacobian divisor* of the linear system, which consists of the singular points, properly counted, of the singular members of the linear system. In particular if C is a plane curve and the linear system is the system of its plane sections by lines through a general point $x = (\xi : \eta : \zeta)$ of $\mathbf{P}^2$, the map f is the projection from C to $\mathbf{P}^1$ from x; its degree is the degree m of C and its ramification points are exactly the points where the line from x is tangent to C. Since x is general, these are simple tangency points, so the e_i are equal to 1, and their number is equal to the class $\check{m}$ of C; the formula gives

$$2g(C) - 2 = -2m + \check{m} \ ,$$

thus giving for the genus an expression linear in the degree and the class.

This is the first example of the relation between the "characteristic classes" (in this case only the genus) and the polar classes; in this case the curve itself, of degree m and the degree of the polar locus, or apparent contour from x, i.e. in this case the class $\check{m}$. There is a similar construction for surfaces, which gives the *Zeuthen-Segre invariant* :

Consider a linear system of curves on a non singular algebraic surface S, without base points, i.e. defined as the fibers of an algebraic map $f : S \to \mathbf{P}^1$. Let us assume that the general fiber $F = f^{-1}(t)$ (for "generic" t), which is non singular, is of genus g, and that there are σ singular fibers, each having a single ordinary double point as singularity. A computation of topological Euler-Poincaré characteristics, nowadays quite standard, gives

$$\chi(S) = 2\chi(F) + \sigma = 4 - 4g + \sigma$$

so that $Z(S) = \sigma - 4g = \chi(S) - 4$ does not depend upon the choice of the pencil of curves; it is the *Zeuthen-Segre invariant* of the surface S. We may in fact also allow pencils with base points; if there are b of these, then $Z(S) = \sigma - 4g - b$. Such pencils correspond to maps $\tilde{f} : \tilde{S} \to \mathbf{P}^1$, where $\tilde{S} \to S$ is a blowing up map determined by the structure of the base points.

Now let us take a surface S of degree m and a pencil of hyperplanes in $\mathbf{P}^3$; we can view it as the pencil of hyperplanes containing a line (a copy of $\mathbf{P}^1$) $\ell \subset \mathbf{P}^3$, and if we choose another line $\mathbf{P}^1 \subset \mathbf{P}^3$ not meeting ℓ, the map which to a point $s \in S \setminus \ell$ associates the intersection point with $\mathbf{P}^1$ of the plane determined by s and ℓ extends to a map $\tilde{S} \to \mathbf{P}^1$, where $\tilde{S} \to S$ is the blowing up of $S \cap \ell$ in S. The singular fibers correspond to planes in the pencil that are tangent to S; if ℓ is general, they are simply tangent, and their number is the class $\check{m}$ of the surface S. The Zeuthen-Segre formula gives, taking into account the fact that we have blown up m points on S, the equality $Z(S) = \chi(S) - 4 = \check{m} - 4g - m$, where now g is the genus of a general plane section of S.

We may interpret all this in two ways; first, as Zeuthen-Segre and Severi did, as creating an invariant of the algebraic surface S, which is in fact $\chi(S) - 4$, in terms of the singular curves of a linear system on S, and second as saying that, given S, if the general curve of a pencil has high genus, then there must be many degenerate fibers.

Severi and others tried to generalize this construction of invariants via the singularities of elements of linear systems, but it was Todd who in 1936 found the right formulation, which is based on the apparently less general case of linear projections. The basic idea is to generalize the notion of apparent contour, considering what Todd calls the "Polar loci" of a projective variety $X \subset \mathbf{P}^n$. Then it turns out that certain formal linear combinations of the intersections of general polar loci of X with general linear sections (of various dimensions) of X are invariants of X, i.e. do not depend upon the projective embedding of X and the choices of polar loci and linear sections.

More precisely, given a non singular $d-1$-dimensional variety X in $\mathbf{P}^{N-1}$, for a linear subspace $L \subset \mathbf{P}^{N-1}$ of dimension $N-d+k-2$, i.e. of codimension $d-k+1$, let us set

$$P_k(X;L) = \{x \in X / \dim(T_{X,x} \cap L) \geq k-1\}$$

This is the *Polar variety* of X associated to L; if L is general, it is either empty or purely of codimension k in X. We see that this construction is a direct generalization of the apparent contour. The eye 0 is replaced by the linear subspace L!

Todd shows that the following formal linear combinations of varieties

$$X_k = \sum_{j=0}^{j=k} (-1)^j \binom{d-k+j+1}{j} P_{k-j}(X;L) \cap H_j$$

where H_j is a linear subspace of codimension j, are independant of all the choices made and of the embedding of X in a projective space, provided that the $L's$ and the $H_j's$ have been chosen general enough.

The linear combination is at first sight a rather awkward object to deal with. The idea is that X_i represents a generalized variety of codimension i in X and what we should remember is that any numerical character $e(Y)$ associated to algebraic varieties Y, and which is *additive* in the sense that $e(Y_1 \cup Y_2) = e(Y_1) + e(Y_2)$ whenever Y_1 and Y_2 have the same dimension, can be extended by linearity to such a generalized variety. In particular, given a partition $i_1, \ldots, i_k$ of $d-1$, we obtain that the intersection numbers

$$(X_{i_1} . \cdots X_{i_k})$$

which are well defined since the intersection of the corresponding varieties (each one assumed to be a general representative obtained by taking general and independant linear spaces) is of dimension zero, depend only upon the structure of X as an algebraic variety.

In fact, Todd preferred to introduce an equivalence relation between varieties, called nowadays *rational equivalence* and instead of considering "general linear spaces" and so on considered the equivalence classes of the corresponding objects, which (by definition of rational equivalence) turn out to be independant of the choices of linear spaces and so on. This relation, however, considers as equivalent objects which are geometrically very different, while the geometry (the topology, and in fact much more) of the polar variety corresponding to a general linear space is perfectly well defined and contains a lot of information.

In any case, one of the main results of Todd is that the numbers $(X_{i_1} \cdots X_{i_k})$ depend only upon X, that they are independent invariants (there is no relation between them valid for all X's) and finally that the *arithmetic genus* of X is a linear function of these invariants. The arithmetic genus is the Euler characteristic in coherent cohomology $\chi(X, \mathcal{O}_X)$ of the sheaf of algebraic functions on X; by Serre duality (see [5]), it is equal to $(-1)^d \chi(X, \omega_X)$ where $d = \dim X$ and ω_X is the sheaf of germs of holomorphic d-forms on X. Thus, since the Euler-Poincaré characteristic of a sheaf is the stable, or computable form of the dimension of the space of global sections, the arithmetic genus is a generalization of the analytic definition of the genus of a non singular curve (dimension of the space of regular differential forms) to an arbitrary algebraic variety. The expression for the arithmetic genus is not simple; it involves what is now called the *Todd class* (see [5]) of the tangent bundle.

On the other hand, the topological Euler-Poincaré characteristic of X can be computed, in the same way we used for the Zeuthen-Segre invariant, to show the equality

$$\chi(X) = \deg X_d = \sum_{j=0}^{d} (j+1)(P_{d-j}(X).H_j)$$

where $(a.b)$ denotes the intersection number, in this case since we intersect with a linear space of complementary dimension, it is just the degree of the projective variety $P_j(X)$.

So Todd's results give a rather complete generalization of the genus formula, both in its analytic and its topological aspects. The connection between these two aspects goes much deeper and leads to the Hirzebruch-Riemann-Roch theorem. One must first realize, after Nakano, Hirzebruch, Serre, Gamkrelidze (see [5]), that the invariants X_k of Todd (or rather their cohomology classes) coincide with the *Chern classes* of the tangent bundle of X, which provides a good reason for them to be invariant!.

The Chern classes are cohomology classes expressing the obstruction to the existence of k-uples of everywhere linearly independant sections of the tangent bundle. The best known of them is the Euler class, expressing the obstruction to finding an everywhere non zero section of the tangent bundle, that is, finding an everywhere non zero vector field; it is represented by finite collections of points of X, the zeroes of a general vector field. All these finite collections of points are rationnally equivalent and their number, counted with multiplicities, is the Euler-Poincaré characteristic of X.

Then we remark that the formula for the topological Euler characteristic can be inverted to compute the degrees of the polar varieties from the topological Euler characteristics of general plane sections (of all dimensions) of X. Using the expression for the arithmetic genus which I mentionned, one expresses in the end the arithmetic genus as a linear function of the Euler-Poincaré characteristics of X, its polar varieties, and their plane sections.

Following this thread of the determination of the geometry of an object from its apparent contours, we have been led not only to the determination of such a fundamental topological invariant as the Euler-Poincaré characteristic, but also to an extension to arbitrary dimensions of the connexion between the geometry of a variety and the theory of algebraic functions on this variety discovered by Riemann. In recent years this geometric viewpoint has been extended to singular projective varieties, and to the study of an analytic space embedded in $\mathbf{C}^N$ in the vicinity of one of its singular points (see [6], [7], [18]). There is much left to understand, and in particular what form a similar theory would take for real algebraic or semi-algebraic varieties in $\mathbf{R}^N$.

Finally I remark that the analogous problem of showing that the collection of its orthogonal projections to subspaces $\mathbf{R}^i \subset \mathbf{R}^d$ determines a convex body $K \subset \mathbf{R}^d$ has been completely solved only in the last fifty years or so. The problem of doing so effectively (up to some approximation) is not yet satisfactorily solved.

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(This is only a sample)

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