BONNESEN-TYPE INEQUALITIES IN ALGEBRAIC GEOMETRY, I: INTRODUCTION TO THE PROBLEM

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Introduction

Let $K \subset \mathbf{R}^2$ be a compact convex domain bounded by a curve of length L. The area S of K is subjected to the isoperimetric inequality $L^2 - 4\pi S \geq 0$, which implies that among all such domains with a given perimeter L, the disk maximizes the area. The most constructive proof of the fact that the disk is the only domain with this property is in my opinion that given by T. Bonnesen in ([2], p. 69): Bonnesen shows that if we consider the greatest radius r of a disk contained in K, and the smallest radius R of a disk containing K, we have the inequality

$$L^2 - 4\pi S \geq \pi^2(R-r)^2 \tag{1}$$

and this settles immediately the equality case of the isoperimetric inequality. The proof of (1) in fact contains the proof of stronger inequalities (see [2], p. 60, and compare with p. 123 of L. A. Santaló's magnificent book [14], and R. Osserman's beautiful paper [12]):

$$\frac{L + \sqrt{L^2 - 4\pi S}}{2\pi} \geq R \geq r \geq \frac{L - \sqrt{L^2 - 4\pi S}}{2\pi} . \tag{2}$$

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Our aim is to investigate the generalization of these inequalities (2) to compact convex domains in $\mathbf{R}^d$ for $d \geq 2$ in a much broader context.

More precisely, let d be an integer, $d \geq 2$, and let K_1 and K_2 be two compact convex subsets of $\mathbf{R}^d$. Following Minkowski (see [2], p. 105, [$2\frac{1}{2}$], p. 60) one defines the *mixed volumes* of K_1 and K_2 as the coefficients in the following expression for the volume of the Minkowski sum $\nu_1 K_1 + \nu_2 K_2$ of the homothetics $\nu_1 K_1$, $\nu_2 K_2$ of K_1 and K_2, as ν_1 and ν_2 range over the nonnegative real numbers $\mathbf{R}_+$. (By definition $\nu_1 K_1 + \nu_2 K_2 = \{y_1 + y_2, y_1 \in \nu_1 K_1, y_2 \in \nu_2 K_2\}$.) One has then the expression (see [$2\frac{1}{2}$], p. 40, [2], p. 106),

$$\mathrm{Vol}(\nu_1 K_1 + \nu_2 K_2) = \sum_{i=0}^{d} \binom{d}{i} v_i \nu_1^i \nu_2^{d-i},$$

and the $v_i \in \mathbf{R}$, sometimes written $v(K_1^{[i]}, K_2^{[d-i]})$ $(0 \leq i \leq d)$, are the mixed volumes of K_1 and K_2. Note that $v_0 = \mathrm{Vol}(K_2)$ and $v_d = \mathrm{Vol}(K_1)$, and that if $K_1 = K_2$ up to translation, then $v_0 = \cdots = v_d$.

In the special case where $K_1 = \mathbf{B}$, the unit ball in $\mathbf{R}^d$, one finds that $d \cdot v_1$ is equal to $\mathrm{Vol}(\partial K_2)$, where $\mathrm{Vol}(\partial K_2)$ is the $(d-1)$-dimensional volume. Taking in particular $d = 2$, we find that

$$\mathrm{Vol}(\nu_1 \mathbf{B} + \nu_2 K) = S\nu_2^2 + L\nu_1\nu_2 + \pi\nu_1^2 \qquad (\text{for } \nu_1 \geq 0, \nu_2 \geq 0).$$

Going back to K_1, K_2 in $\mathbf{R}^d$ we define two real numbers, the inradius $r(K_2; K_1)$ and the outradius $R(K_2; K_1)$ of K_2 with respect to K_1, by

$$r(K_2; K_1) = \mathrm{Sup}\{ r \in \mathbf{R}_+ \mid rK_1 \subseteq K_2, \text{ up to translation}\}$$

$$R(K_2; K_1) = \mathrm{Inf}\{ R \in \mathbf{R}_+ \mid K_2 \subseteq RK_1, \text{ up to translation}\}.$$

In the case where $d = 2$ and $K_1 = \mathbf{B}$ we find r and R of the inequalities (1) and (2).

PROBLEM A. Given two compact convex subsets K_1 and K_2 of $\mathbf{R}^d$, give bounds for $r(K_2; K_1)$ and $R(K_2; K_1)$ in terms of the mixed volumes v_i $(0 \leq i \leq d)$ of K_1 and K_2. These bounds should imply that when $v_0 = \cdots = v_d$, then $r(K_2; K_1) = R(K_2; K_1) = 1$, and therefore $K_1 = K_2$ up to translation.

Note that the inequalities (2) answer that problem for $d = 2$ and $K_1 = \mathbf{B}$. There is an answer for arbitrary $d \geq 2$ and $K_1 = \mathbf{B}$, due to Hadwiger and Dinghas (see [12], IIIC). There is also a splendid proof by A.D. Alexandrov, of the fact that the equalities $v_0 = \cdots = v_d$ imply that $K_1 = K_2$, up to translation. I give below a proof of the answer, originally due to Flanders [$6\frac{1}{2}$], to Problem A for $d = 2$, but my main interest, and the motivation for this paper, lies in the problems suggested in algebraic geometry by Problem A when one "embeds" a part of the theory of convex sets into algebraic geometry as explained below. In other words, I am interested in the problem in algebraic geometry of which Problem A is an avatar!

The principal new fact contained in this paper is that Problem A can be deemed to be a rather special case of a problem of algebraic geometry: to find sufficient numerical conditions for an invertible sheaf to have sections. This fact is established by associating to integral polyhedra K_1 and K_2 in $\mathbf{R}^d$ algebraic varieties of dimension d with two invertible sheaves L_1 and L_2, the properties of which reflect very well the properties of K_1 and K_2. Using approximation of arbitrary compact convex sets by integral ones, we obtain a dictionary which translates problems on compact convex subsets of $\mathbf{R}^d$ into problems on invertible sheaves on some very special algebraic varieties of dimension d (Demazure varieties, or torus embeddings). This dictionary was used in [19] to show that the basic inequalities of the isoperimetric problem in dimension $d \geq 2$, namely the Minkowski-Alexandrov-Fenchel inequalities between the mixed volumes of two compact convex subset of $\mathbf{R}^d$:

(3) $$v_{i-1}^2 - v_i v_{i-2} \geq 0 \qquad (2 \leq i \leq d)$$

were consequences of the *Hodge index theorem* in the theory of algebraic surfaces. In this paper we show that our viewpoint is operative also for Problem A by giving a proof of the solution for $d = 2$. This proof relies on an easy (nowadays) but rather deep result on the geometry of invertible sheaves on a projective algebraic surface.

A point of interest is that once again this is a quite general result, the validity of which is not restricted to the Demazure surfaces which we encounter when starting from convex sets.

On the way, we recall in §1 the construction of our Note [19] and make precise the general inequalities of algebraic geometry which imply the inequalities (3). Thus this paper, although it is a continuation of [19], can be read without prerequisites. In this connection, after the Note [19] went to press, I learned that Mr. A. G. Hovanski, of Moscow, has independently given a construction analogous to that of §§1 and 2 of that Note, and, inspired like myself by my inductive proof in [16], has given in a Note [8] a sketch of a proof of the Alexandrov-Fenchel inequalities which is very similar to that of [19]. I also point to a recent paper [13] of D. Rees and R. Y. Sharp showing that the "Minkowski-type inequalities" for multiplicities of [16] were valid for arbitrary noetherian local rings. The range of validity of this type of inequality is therefore large.

I should like to thank Tadao Oda, whose paper [11] influenced the construction in §1 (or [19]) and Professor Rolf Schneider who, by the material he has sent to me, has helped me very much to study the beautiful theory of mixed volumes.

§1. *Alexandrov-Fenchel type inequalities in algebraic geometry*

1.1. The construction (after [4], [9], [19]). Let d be an integer, $d \geq 2$, and let $\mathcal{K}$ (resp. $\mathcal{K}_M$) be the set of compact convex subsets of $\mathbf{R}^d$ (resp. of those which are the convex hull of a finite number of points in the integral lattice $M \simeq \mathbf{Z}^d \subset \mathbf{R}^d$).

Given $K_1, \cdots, K_r$ in $\mathcal{K}$, each K_i has a support function $H_i : \mathbf{R}^{d^*} \to \mathbf{R}$ defined by:

If the K_i are in $\mathcal{K}_M$, the H_i are piecewise-linear and it is not hard to see that there exists a decomposition $\Sigma = (\sigma_\alpha)_{\alpha \in A}$ of $\mathbf{R}^{d^*}$ into rational convex polyhedral cones σ_α such that:

1) Each face of a σ_α is a σ_β for some $\beta \in A$.
2) $\sigma_\alpha \cap \sigma_\beta$ $(\alpha, \beta \in A)$ is a face of σ_α and of σ_β.
3) For each i, $1 \leq i \leq r$, H_i is linear on σ_α for all $\alpha \in A$.

Now to each σ_α, associate its convex-dual $\check{\sigma}_\alpha = \{x \in \mathbf{R}^d \mid u(x) \geq 0, \forall u \in \sigma_\alpha\}$. The subset $\check{\sigma}_\alpha \cap M$ of $\mathbf{R}^d$ is a submonoid of M, and hence one can define for any given field, say the field $\mathbf{C}$ of complex numbers, the "algebra of the monoid $\check{\sigma}_\alpha \cap M$," which is the subalgebra $\mathbf{C}[\check{\sigma}_\alpha \cap M]$ of the algebra $\mathbf{C}[M] \simeq \mathbf{C}[X_1, X_1^{-1}, \cdots, X_d, X_d^{-1}]$ generated by those monomials which have their exponents in $\check{\sigma}_\alpha \cap M$.

We can glue up the affine varieties $\operatorname{Spec} \mathbf{C}[\check{\sigma}_\alpha \cap M]$ along the $\operatorname{Spec} \mathbf{C}[\widecheck{\sigma_\alpha \cap \sigma_\beta} \cap M]$ to obtain a compact algebraic variety $X = X(\Sigma)$, the Demazure variety associated to the decomposition Σ. (For all this, see [4], §4 and [9].) X is a normal, integral and rational variety of dimension d. The field of fractions of each $\mathbf{C}[\check{\sigma}_\alpha \cap M]$ is $\mathbf{C}(M)$ and we are going to recall how to associate to each support function H_i a line bundle L_i on X: For each $\alpha \in A$ and i, $1 \leq i \leq r$, consider the sub $\mathbf{C}[\check{\sigma}_\alpha \cap M]$-module of $\mathbf{C}(M)$ generated by $\{m \in M \mid u(m) \geq H_i(u)$ for all $u \in \sigma_\alpha\}$ and denote it by $L_{i,\alpha}$. Since H_i is linear on σ_α, there exists $m_{i,\alpha} \in M$ such that $H_i(u) = u(m_{i,\alpha})$ on σ_α, and therefore $L_{i,\alpha}$ is generated by $m_{i,\alpha}$. Now clearly the $L_{i,\alpha}$ glue up together into an invertible sheaf of fractional ideals L_i, and a basis of $H^0(X, L_i)$ is given by $\{m \in M \mid u(m) \geq H_i(u)$ for all $u \in \mathbf{R}^{d^*}\}$ which is exactly $K_i \cap M$ in view of the convexity of K_i. Furthermore, it is proved in ([9], pp. 42-44) that for all invertible sheaves L_i obtained in this way from $K_i \in \mathcal{K}_M$, not only is L_i generated by its global sections (because necessarily $m_{i,\alpha} \in K_i \cap M$) but also we have

$$H^j(X, L_i) = 0 \quad \text{for} \quad j \geq 1 .$$

We recall also that there is an operation on $\mathcal{K}$, called the Minkowski sum: $K_1 + K_2$ is by definition $\{k_1+k_2;\ k_1 \in K_1, k_2 \in K_2\}$; a special case of it is the translation $K+m$ by the vector $\vec{Om}$. The result is independent of the choice of origin $O \in \mathbf{R}^d$, up to translation. The support function of $K_1 + K_2$ is $H_1 + H_2$ and if we took K_1, K_2 in $\mathcal{K}_M$, and Σ as above, then $H_1 + H_2$ is again linear in each σ_α, hence we can associate to $K_1 + K_2$ a line bundle on X, which is nothing but the tensor product $L_1 \otimes L_2$. In particular to the homothetic $\nu \cdot K_i$ for $\nu \in N$ we associate L_i^ν, where L_i is associated to K_i as above. (Note that if K is a point, e.g., $O \in \mathbf{R}^d$, then $L \simeq \mathcal{O}_X$, and therefore more generally the sheaf associated to $K_i + m$, for $m \in M$, is isomorphic to L_i.)

1.2. Given $K_1, \cdots, K_r$ in $\mathcal{K}$, consider for all $\nu_i \geq 0$ the homothetic convex sets $\nu_i K_i$ and their Minkowski sum $\nu_1 \cdot K_1 + \cdots + \nu_r \cdot K_r = \{x_1 + \cdots + x_r \in \mathbf{R}^d / x_1 \in \nu_1 K_1, \cdots, x_r \in \nu_r \cdot K_r\}$. By a result of Minkowski-Steiner (cf. [2], [$10\frac{1}{2}$]) the volume of this set has an expression in terms of the ν_i which is a homogeneous polynomial of degree d:

$$\mathrm{Vol}(\nu_1 \cdot K_1 + \cdots + \nu_r \cdot K_r) = \sum_{\substack{\alpha \in \mathbf{N}^r \\ |\alpha| = d}} \frac{d!}{\alpha!} v_\alpha \nu_1^{\alpha_1} \cdots \nu_r^{\alpha_r}.$$

The v_α which are defined by this expression and sometimes written $v(K_1^{[\alpha_1]}, \cdots, K_r^{[\alpha_r]})$ are called the mixed volumes of the K_i, and they have the following elementary properties:

1) All the v_α are left unchanged by translations on the K_i (indeed, translations on the K_i induce translations on the $\sum \nu_i K_i$).

2) The v_α are increasing functions of the K_i, i.e., if $K_i \subset K_i'$ (up to translation) we have:

$$v(K_1^{[\alpha_1]}, \cdots, K_i^{[\alpha_i]}, \cdots, K_r^{[\alpha_r]}) \leq v(K_1^{[\alpha_1]}, \cdots, K_i'^{[\alpha_i]}, \cdots, K_r^{[\alpha_r]})$$

and if $\alpha_i = 0$, we have equality.

In particular, all the v_α are ≥ 0.

3) For positive λ_i, we have

$$v((\lambda_1 K_1)^{[\alpha_1]}, \cdots, (\lambda_r K_r)^{[\alpha_r]}) = \lambda_1^{\alpha_1}, \cdots, \lambda_r^{\alpha_r} v(K_1^{[\alpha_1]}, \cdots, K_r^{[\alpha_r]}).$$

4) Giving $\mathcal{K}$ its natural Hausdorff topology (cf. [10^{bis}]) we have that the v_α are continuous functions on $\mathcal{K}^r$.

1.3. Now given invertible sheaves $L_1, \cdots, L_r$ on an algebraic variety X of dimension d, Snapper (see [10], Chap. I, §1) has shown that there is a polynomial expression for the coherent Euler-characteristic of $L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}$, as follows: for $\nu_1, \cdots, \nu_r$ in $\mathbf{Z}$, we have

$$\chi(X, L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}) = \sum_{\substack{\alpha \in \mathbf{N}^r \\ |\alpha| = d}} \frac{1}{\alpha!} s_\alpha \nu_1^{\alpha_1} \cdots \nu_r^{\alpha_r} \tag{1.3.1}$$

$$+ \text{ polynomial of degree } \leq d-1$$

and this expression defines integers s_α which we may call the *mixed degrees* of the invertible sheaves L_i (sometimes written $s_\alpha = \deg(L_1^{[\alpha_1]}, \cdots, L_r^{[\alpha_r]})$).

If we start with $K_1, \cdots, K_r$ in $\mathcal{K}_M$, and associate to them an "eventail" Σ, X and the L_i as above, we saw that the groups $H^j(X, L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r})$ are 0 for $j \geq 1$ and $\nu_i \geq 0$ and hence we have, for $\nu_i \geq 0$

$$\chi(X, L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}) = h^0(X, L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}) = \#(M \cap (\nu_1 K_1 + \cdots + \nu_r K_r))$$

as we saw, and therefore letting the ν_i tend to $+\infty$ and remembering that for any $K \in \mathcal{K}$, we have

$$\mathrm{Vol}(K) = \lim_{\nu \to +\infty} \nu^{-d} \cdot \#(M \cap \nu \cdot K)$$

we deduce that *the mixed degrees of the* L_i *are linked to the mixed volumes of the* K_i *by the equality*:

$$s_\alpha = d!\, v_\alpha . \qquad (1.3.2)$$

1.4. Now let us fix an integer t, $2 \le t \le d$, and set $v_i = v_{(i,t-i,1,\cdots,1)}$ and $s_i = s_{(i,t-i,1,\cdots,1)}$ where the sequence $1,\cdots,1$ has $d-t$ terms. In the note [19] we showed how to use the Hodge index theorem to prove the quadratic inequalities

$$s_i \cdot s_{i-2} \le (s_{i-1})^2 \qquad (2 \le i \le t) \qquad (1.4.1)$$

which imply the *Alexandrov-Fenchel inequalities*

$$v_i \cdot v_{i-2} \le (v_{i-1})^2 \text{ for } K_1,\cdots,K_r \in \mathcal{K}_M \text{ (by 1.3.2)} \qquad (1.4.2)$$

which in turn imply the same Alexandrov-Fenchel inequalities for compact convex sets $K_1,\cdots,K_r$ in $\mathcal{K}$, by using the continuity of the mixed volumes, their homogeneity, and a standard approximation procedure using the following:

1.4.2.1. FACT. For $K \in \mathcal{K}$, let us denote by $[K] \in \mathcal{K}_M$ the convex hull of $M \cap K$, where M is the integral lattice of $\mathbf{R}^d$. Then, for any $K \in \mathcal{K}$ we have for large ν:

$$0 \le \mathrm{Vol}(K - \tfrac{1}{\nu}[\nu \cdot K]) < \frac{c(K)}{\nu} \text{ where } c(K) \text{ is a constant.}$$

1.4.3. We emphasize that the result in algebraic geometry is much more general than what is needed for the Alexandrov-Fenchel inequalities:

Let us define the degree of an invertible sheaf on a complete algebraic variety X of dimension d by the equality (see [10], Ch. I)

$$\chi(X, L^\nu) = \deg L \cdot \frac{\nu^d}{d!} + \text{polynomial of degree} \le d-1 \text{ in } \nu .$$

Then we may define the mixed degrees s_α of $L_1,\cdots,L_r$ by the equality

$$\deg(L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}) = \sum_{\substack{\alpha \in \mathbf{N}^r \\ |\alpha| = d}} \frac{d!}{\alpha!} s_\alpha \nu_1^{\alpha_1} \cdots \nu_r^{\alpha_r}$$

as a glance at 1.3.1 will show.

Then, choosing an integer t, $0 \le t \le d$ as in 1.4 and with the same notations, we have:

1.4.3.1. PROPOSITION. *Let* X *be a proper integral algebraic variety and let* $L_1,\cdots,L_r$ *be invertible sheaves on* X, *generated by their sections, with* $\deg L_i > 0$ $(1 \le i \le r)$.

Then we have

1) $s_\alpha \ge 0$
2) *For each* i, $2 \le i \le t$, *we have the Alexandrov-Fenchel-type inequalities*

$$s_{i-1}^2 - s_i s_{i-2} \ge 0 .$$

COROLLARY. *Take now* $r = 2$; *then*

$$\deg L_1^{\nu_1} \otimes L_2^{\nu_2} = \sum_{i=0}^{d} \binom{d}{i} s_i \nu_1^i \nu_2^{d-i} \qquad 1.4.3.2$$

and the inequalities (2) above imply easily (see [16]) *the inequalities*

$$\underline{s_i^d \ge s_0^{d-i} \cdot s_d^i}, \qquad 0 \le i \le d . \qquad 1.4.3.3$$

(The analogue of the classical isoperimetric inequality $\mathrm{Vol}(\partial K)^d \ge d^d\, \mathrm{Vol}(\mathbf{B})\, \mathrm{Vol}(K)^{d-1}$ is the case $i = 1$.) In turn 1.4.3.3 implies, in view of 1.4.3.2

$$\deg(L_1 \otimes L_2)^{1/d} \ge (\deg L_1)^{1/d} + (\deg L_2)^{1/d} \qquad 1.4.3.4$$

(compare [16]): this is the analogue of the Brünn-Minkowski inequality (see [2], [5]) and at least if L_1 is ample, say, we have equality if and only if there exist positive integers a, b such that L_1^a and L_2^b are numerically equivalent (see [3], exp. XIII, and compare [17]).

1.5. REMARK (ubiquitous inequalities). All these analogies between isoperimetric inequalities and algebraic ones first appeared (see [18]) in the case of multiplicities of primary ideals n_1, n_2 in a complex analytic algebra $\mathcal{O}$, the mixed multiplicities of n_1 and n_2 being defined by

$$e(n_1^{\nu_1} n_1^{\nu_2}) = \sum_{i=0}^{d} \binom{d}{i} e_i \nu_1^i \nu_2^{d-i} \qquad \text{where } d = \dim \mathcal{O} \geq 1 .$$

and the inequalities which were proved were: (see [16], [17])

1) $e_i \cdot e_{i-2} \geq e_{i-1}^2$

2) $e_i^d \leq e_0^{d-i} e_d^i$

3) $e(n_1 \cdot n_2)^{1/d} \leq e(n_1)^{1/d} + e(n_2)^{1/d}$.

Provided $\mathcal{O}$ is normal, there is equality in 3) if and only if there exist positive integers a and b such that n_1^a and n_2^b have the same integral closure. These results on multiplicities correspond to the negative-definiteness of the intersection matrix of the components of the exceptional divisor in a resolution of singularities of a germ of a normal surface, and the results on degrees, inspired by these, correspond, as already explained, to the Hodge Index theorem. It is interesting to note that the same year 1937 saw the publication by A.D. Alexandrov of his inequalities (cf. [1]) and the publication by Hodge of his Index theorem (cf. [7]). Furthermore, Hodge thanks DuVal for the formulation of the Index theorem, and DuVal himself had a few years before discovered the negative-definiteness of the intersection matrix (cf. [6]), a fact which is the local version of the Hodge Index theorem; and is equivalent to the following statement: Given two curves A and B on a germ of a normal surface $(S,0)$, their intersection multiplicity defined by Mumford, which is a rational number, must satisfy the inequality $(A,B)_0 \geq \frac{m_0(A)\cdot m_0(B)}{m_0(S)}$ where m_0 is the multiplicity, and equality must hold if and only if the two curves have no common tangent at 0. See [16].

Recall from the Introduction that given K_1 and K_2 in $\mathcal{K}$ we defined the inradius of K_2 with respect to K_1

$$r(K_2;K_1) = \operatorname{Sup}\{r \in \mathbf{R}_+ / rK_1 \subseteq K_2 \text{ up to translation}\} .$$

Our problem as set in the introduction is so symmetric that it is enough to study one-half of it:

2.0. PROBLEM A′. Give bounds for $r(K_2;K_1)$ in terms of the mixed volumes v_i of K_1 and K_2 defined by

$$\operatorname{Vol}(\nu_1 K_1 + \nu_2 K_2) = \sum_{i=0}^{d} \binom{d}{i} v_i \nu_1^i \nu_2^{d-i} \qquad (\nu_1, \nu_2 \in \mathbf{R}^+)$$

$$(v_d = \operatorname{Vol}(K_1), v_0 = \operatorname{Vol}(K_2)) .$$

We are going to translate this problem into algebraic geometry; we use the constructions and notations of §1. Here we have

2.1. KEY LEMMA. *Let d be an integer, $d \geq 2$, let K_1 and K_2 be in $\mathcal{K}_M$ and let the compact algebraic variety X and the line bundles L_1 and L_2 be associated to them as in paragraph 1. Then given integers a,b, $a \geq 0$, $b \geq 0$, for every integer $\nu > 0$ there is a bijection*

$$\{m \in \tfrac{1}{\nu} M \mid a\cdot K_1 + m \subseteq b\cdot K_2\} \leftrightarrows \text{basis of } H^0(X, (L_1^{-a} \otimes L_2^b)^\nu) .$$

Proof. The inclusion $a\cdot K_1 + m \subseteq b \cdot K_2$ is equivalent to the inequalities $a\cdot H_1(u) + u(m) \geq b\cdot H_2(u)$ for all $u \in \mathbf{R}^{d*}$.

That is: $u(m) \geq b\cdot H_2(u) - a\cdot H_1(u)$ and this precisely means that if we have $\nu\cdot m \in M$, then $\nu \cdot m$ corresponds to a global section of the invertible sheaf $(L_1^{-a} \otimes L_2^b)^\nu$. The converse is obtained by reading backwards, in view of the fact that $H^0(X, (L_1^{-a} \otimes L_2^b)^\nu)$ has a natural M-grading for which all non-zero homogeneous components are of dimension 1.

2.2. Let X be a complete integral algebraic variety and let L_1 and L_2 be two invertible sheaves on X, of positive degree. Define the *inradius* of L_2 with respect to L_1:

$$r(L_2; L_1) = \sup_{(a,b)\in N} \left\{ \frac{a}{b} \,\middle|\, H^0(X, L_1^{-a} \otimes L_2^{b}) \neq 0 \right\}$$

and remark that for all integers $\nu > 0$,

$$r(L_2^{\nu}; L_1^{\nu}) = r(L_2; L_1).$$

It follows immediately from the equality 1.3.2, the Key Lemma 2.1 and 1.4.2.1 that Problem A′ is a special case of the following:

2.2.1. PROBLEM B. Given two invertible sheaves of positive degree L_1 and L_2, generated by their sections, on a complete algebraic variety X, give bounds for $r(L_2; L_1)$ in terms of the mixed degrees s_i of L_1 and L_2, defined by

$$\deg(L_1^{\nu_1} \otimes L_2^{\nu_2}) = \sum_{i=0}^{d} \binom{d}{i} s_i \nu_1^i \nu_2^{d-i}$$

$$(s_d = \deg L_1,\ s_0 = \deg L_2).$$

2.3. Here we are going to study this problem only in the case where X is a projective variety, and we assume that $L_1 \otimes L_2$ is very ample. Indeed, this is sufficient for the applications to the problem on convex set which is outlined in the introduction, for the following reason: if we take for Σ the *coarsest* subdivision of R^{d^*} which makes H_1 and H_2 linear in each piece, then by the amplitude criterion given by Demazure (cf. [4], §4, and also [9], p. 48) we have precisely that $L_1 \otimes L_2$ is ample and since we may freely replace L_1 and L_2 by L_1^{ν} and L_2^{ν} $(\nu > 0)$, we may assume that $L_1 \otimes L_2$ is very ample.

§3. An example: the case d = 2 (and a proof of Bonnesen's inequality)

We shall use the following

3.1. LEMMA (see [10], p. 3.8 and [3], Exp XIII, Appendice, Lemma 7.1.2, and Exp. X). *Let S be an integral projective surface with a very ample invertible sheaf $H = \mathcal{O}_X(D)$. Let L be an invertible sheaf on X. The following conditions are equivalent:*

i) $\deg L > 0$ *and* $\deg L_D > 0$ ($L_D = L \otimes \mathcal{O}_D$, *an invertible sheaf on* D)

ii) *For sufficiently large* ν *we have*

$$\dim H^0(X, L^{\nu}) > \varepsilon \cdot \nu^2 \quad \textit{with } \varepsilon > 0.$$

(Note: Use [3], Exp. X, to check that $\deg L = c_1(L)^2$, $\deg L_D = c_1(L) \cdot c_1(H)$).

Let us apply this lemma with $L = L_1^{-a} \otimes L_2^{b}$, $a, b \in N$, where L_1 and L_2 are two invertible sheaves of positive degree such that $H = L_1 \otimes L_2$ is very ample. We have, with the notations introduced above

$$\deg L = s_0 b^2 - 2 s_1 ab + s_2 a^2$$

and using a result in ([10], Chap. I), we can compute $\deg L_D$ as the coefficient of $2\mu\nu$ in the homogeneous expression of $\deg(L^{\nu} \otimes H^{\mu})$, that is:

$$\deg(L_1^{-a\nu+\mu} \otimes L_2^{b\nu+\mu}) = s_0(b\nu+\mu)^2 + 2s_1(b\nu+\mu)(-a\nu+\mu) + s_2(-a\nu+\mu)^2$$

where the coefficient of $2\mu\nu$ is

$$(b(s_0+s_1) - a(s_1+s_2)).$$

Therefore, we have $\deg L > 0$ and $\deg L_D > 0$ if and only if $a, b \in N$ satisfy:

$$s_0 b^2 - 2s_1 ab + s_2 a^2 > 0$$

$$b(s_0+s_1) - a(s_1+s_2) > 0$$

but we can remark that

$$\frac{s_1 - \sqrt{s_1^2 - s_0 s_2}}{s_2} \leq \frac{s_0 + s_1}{s_1 + s_2}$$

and therefore, when we let a/b increase starting from -1, as long as we have

$$\frac{a}{b} < \frac{s_1 - \sqrt{s_1^2 - s_0 s_2}}{s_2}$$

which is the smallest root of the polynomial $s_0 - 2s_1 T + s_2 T^2 = 0$, we are sure that $H^0(X, (L_1^{-a} \otimes L_2^b)^\nu) \neq 0$ for large enough ν.

In other words we have just proved the inequality

$$r(L_2; L_1) \geq \frac{s_1 - \sqrt{s_1^2 - s_0 s_2}}{s_2}$$

(we note that the Hodge Index theorem has been used to get that $s_1^2 - s_0 s_2 \geq 0$).

On the other hand, using the additivity of Euler characteristics we see that the mixed degrees are increasing functions of the sheaves, i.e., if $L_1 \subseteq L_1'$ and $L_2 \subseteq L_2'$, then we have

$$\deg(L_1^{[i]}, L_2^{[d-i]}) \leq \deg(L_1'^{[i]}, L_2'^{[d-i]}),$$

and using this and the homogeneity, we obtain immediately an upper bound for $r(L_2; L_1)$, namely

$$r(L_2; L_1) \leq \frac{s_0}{s_1} \leq \frac{s_1}{s_2}.$$

(The first inequality comes from writing that if $H^0(X, L_1^{-a} \otimes L_2^b) \neq 0$, i.e., $L_1^a \subset L_2^b$, we must have:

$$as_1 = a \deg(L_1^{[1]}, L_2^{[1]}) = \deg((L_1^a)^{[1]}, L_2^{[1]}) \leq \deg((L_2^b)^{[1]}, L_2^{[1]}) = bs_0$$

i.e., $\frac{a}{b} \leq \frac{s_0}{s_1}$.)

The second inequality is once again $s_0 s_2 \leq s_1^2$.

Finally we have proved:

3.2. PROPOSITION. *Let L_1 and L_2 be two invertible sheaves on a projective integral surface X, such that $\deg L_1 > 0$, $\deg L_2 > 0$ and $L_1 \otimes L_2$ is ample. Define the mixed degrees s_0, s_1, s_2 by*

$$\deg L_1^{\nu_1} \otimes L_2^{\nu_2} = s_2 \nu_1^2 + 2 s_1 \nu_1 \nu_2 + s_0 \nu_2^2 .$$

Then

1) $s_2 = \deg L_1$, $S_0 = \deg L_2$ *and* $s_1 = \deg(L_1^{[1]}, L_2^{[1]})$ *are positive.*

2) *The roots of the polynomial*

$$s_0 + 2s_1 T + s_2 T^2$$

are real and negative, hence the roots of the polynomial $s_0 - 2s_1 T + s_2 T^2$ *are real and positive.*

3) *We have the inequalities*

$$\frac{s_1}{s_2} \geq \frac{s_0}{s_1} \geq r(L_2; L_1) \geq \frac{s_1 - \sqrt{s_1^2 - s_0 s_2}}{s_2} .$$

A symmetric proof (considering $L = L_1^a \otimes L_2^{-b}$ and letting $\frac{a}{b}$ decrease) gives symmetric inequalities for

$$R(L_2; L_1) = \inf\left\{\frac{a}{b} / H^0(X, L_1^a \otimes L_2^{-b}) \neq 0\right\}$$

namely:

$$\frac{s_0}{s_1} \leq \frac{s_1}{s_2} \leq R(L_2; L_1) \leq \frac{s_1 + \sqrt{s_1^2 - s_0 s_2}}{s_2} .$$

Using what we have seen so far, and the easily established fact that the inradius $r(K_2; K_1)$ and the outradius $R(K_2; K_1)$ are continuous functions on the product space $\mathcal{K} \times \mathcal{K}$ endowed with the product of Hausdorff topologies, we obtain:

3.2.1 COROLLARY (Flanders $[6\frac{1}{2}]$). *Let* K_1 *and* K_2 *be two compact convex domains in* R^2, *with mixed volumes* $v_0 = \mathrm{Vol}(K_2)$, $v_1 = \mathrm{Vol}(K_2^{[1]}, K_1^{[1]})$ *and* $v_2 = \mathrm{Vol}(K_1)$. *The inradius* $r = r(K_2; K_1)$ *and the outradius* $R = R(K_2; K_1)$ *of* K_2 *with respect to* K_1 *satisfy the following inequalities*:

$$\frac{v_1 + \sqrt{v_1^2 - v_0 v_2}}{v_2} \geq R \geq \frac{v_1}{v_2} \geq \frac{v_0}{v_1} \geq r \geq \frac{v_1 - \sqrt{v_1^2 - v_0 v_2}}{v_2}$$

and by difference, we get Bonnesen's inequality

3.2.2 $$v_1^2 - v_0 v_2 \geq \frac{v_2^2}{4}(R-r)^2$$

and when we specialize to the case where K_1 *is the unit disk, we get the inequalities (1) and (2) of the introduction.*

3.3. REMARK. Going back to the case of arbitrary $d \geq 2$, one can also use the construction of §1 to obtain results on the measure of the set of translations sending K_1 into K_2, in the special case where the difference of the support functions $H_2 - H_1$ is again a convex function: in this case, thanks to the theorem of ([9], pp. 42-44) quoted in §1, we have $H^j(X, L_1^{-1} \otimes L_2) = 0$ for $j \geq 1$, and therefore

$$\chi(X, (L_1^{-1} \otimes L_2)^{\nu}) = h^0(X, (L_1^{-1} \otimes L_2)^{\nu})$$

from this equality, the Key Lemma 2.1 and approximation, we obtain that:

3.3.1. PROPOSITION (for $d \geq 1$, and K_1, K_2 in $\mathcal{K}$). *If* $H_2 - H_1$ *is convex, the measure of the set of translations sending* K_1 *into* K_2 *is given by*

$$m(K_1; K_1 \subset K_2) = \sum_{i=0}^{d} \binom{d}{i} (-1)^i v_i \quad (\textit{or } 0 \textit{ if this is } \leq 0)$$

(*compare with* [14], p. 95).

I propose to study the following precise form of Problem B:

4.1. PROBLEM C. Determine for which algebraic varieties X of dimension $d \geq 1$ we have that, given any two invertible sheaves L_1 and L_2 on X which satisfy the conditions:

i) L_i is generated by its global sections, $i = 1, 2$

ii) $\deg L_i > 0$, $i = 1, 2$

iii) $L_1 \otimes L_2$ is ample.

Then:

1) The roots of the polynomial $R(T) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} s_i T^i \in Z[T]$ where the s_i are defined by: $\deg L_1^{\nu_1} \otimes L_2^{\nu_2} = \sum_{i=0}^{d} \binom{d}{i} s_i \nu_1^i \nu_2^{d-i}$, all have positive real parts, say $0 < \rho_1 \leq \rho_2 \leq \cdots \leq \rho_d$.

2) The following inequality holds (notation of 2.2):

$$r(L_2; L_1) \geq \rho_1 .$$

As we have seen above, when $d \leq 2$, the answer is: all algebraic varieties. (For $d = 2$ we used the Index theorem and Lemma 3.1, which is essentially a consequence of Riemann's theorem on curves; the case $d = 1$ follows directly from Riemann's theorem.) Now we show that:

4.2. PROPOSITION. *For any dimension* d, *the class of algebraic varieties defined in Problem C contains all abelian varieties.*

This is an almost immediate consequence of a remarkable theorem of Mumford and Kempf:

THEOREM (Kempf-Mumford, see $[9\frac{1}{2}]$). *Let* L *and* M *be invertible sheaves on an abelian variety* X, *with* L *ample. Let* $P_{L,M}(n) = \chi(X, L^n \otimes M)$. *Then*:

i) All the d roots of the polynomial $P_{L,M}$ *are real*

ii) *Counting roots with multiplicities*:

$$H^k(X, M) = 0 \quad \textit{for} \quad 0 \le k < \textit{number of positive roots};$$

$$H^{d-k}(X, M) = 0 \quad \textit{for} \quad 0 \le k < \textit{number of negative roots};$$

and if 0 *is not a root, exactly one cohomology group is not zero.*

We apply this result to our problem, taking $L = L_1 \otimes L_2$ and $M = L_1^{-a} \otimes L_2^{b}$. Then, by the properties of line bundles on abelian varieties (cf., loc. cit.) and the definition of the polynomial $R(T)$, we have the equality:

$$P_{L,M}(T) = R\left(\frac{a-T}{b+T}\right) \cdot (b+T)^d .$$

Therefore, all the roots of $R(T)$ are real, and are of the form $\rho_j = \frac{a-r_j}{b+r_j}$ where $r_1, r_2, \cdots, r_d$ are the roots of $P_{L,M}(T)$. Hence we have $r_j = \frac{a-b\cdot\rho_j}{1+\rho_j}$. Since the ρ_i depend only upon L_1 and L_2, we see that taking $b = 0$ and applying the theorem gives $1 + \rho_j > 0$ for $j = 1, \cdots, d$, and taking $a = 0$ gives $\rho_j > 0$, $j = 1, \cdots, d$. Furthermore, as long as $\frac{a}{b} < \rho_1$, the smallest of the ρ_j, we have that all the r_j are < 0, and hence, applying the theorem again, $H^0(X, L_1^{-a} \otimes L_2^{b}) \ne 0$. This shows that $r(L_2; L_1) \ge \rho_1$, as desired.

4.3. REMARK. We emphasize that in general the cohomology of line bundles is much more complicated than in the case $d \le 2$ or in the case of abelian varieties, so that other methods must be used to study Problem C.

4.4. REMARKS. If a polynomial such as $R(T)$ has all its roots real, then, the coefficients s_i must satisfy the inequalities

$$s_{i-1}^2 - s_i \cdot s_{i-2} \ge 0 \qquad (2 \le i \le d)$$

as we know (compare with 1.4.3.1).

One might think that these inequalities and the positivity of the s_i suffice to imply that all the roots of the polynomial $R(T)$ have positive real parts; this is easily checked for $d \le 5$, using the Routh-Hurwitz criterion. However, it is not true for $d \ge 10$: According to a random search programmed by G. Wanner at the Math. Inst. of the University of Geneva, it seems that the inequalities imply the positivity of the real parts of the roots for $d \le 9$, but he found a counterexample with $d = 10$. I am also very grateful to Douady who suggested a construction of a counterexample by hand, and to Coray who made the first search of counterexamples on the Geneva computer, and found one of degree $d = 16$.

David Mumford has shown to me that the three-dimensional variety obtained by blowing up a point in $P^2 \times P^1$ does not belong to the class defined in Problem C. [The inequality 2) is not satisfied on some lines in the affine subspace of NS(X) consisting of classes of divisors of the form $xH + yE + K$ where H is the total transform of $P^2 \times \{a\}$, E is the exceptional divisor, and K is the total transform of $\ell \times P^1$, ℓ being a line which contains the point b, and we blow up the point (b, a).] Also, there are examples showing that if $d \ge 3$, the roots of the polynomial $R(T)$ need not be all real; the first such example was given to me by Mr. L. Brown of Purdue University: Take $K_1 = B^3 \subset R^3$ and for K_2 a very close approximation, of positive volume, of $B^2 \subset R^2 \subset R^3$.

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