Appendix

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Contents

Introduction

In the study of the moduli space of plane branches with a single characteristic pair $(\overline{\beta}_0, \overline{\beta}_1)$, an important point (cf. Ch. VI of the Course, where $(\overline{\beta}_0, \overline{\beta}_1)$ is denoted (n, m)) is that every such branch is the deformation of the branch C_0 : $Y^{\overline{\beta}_0} - X^{\overline{\beta}_1} = 0$. As a result, one obtains a surjection

$$m : \mathbb{C}^N \longrightarrow M$$

where C^N is the base of the *miniversal equisingular deformation* of C_0 (cf. ch. VI, Eqs. Defr. C_0 of the Course), M is the moduli space of Zariski corresponding to $(\overline{\beta}_0, \overline{\beta}_1)$, and m satisfies the property that its restriction to every neighborhood of $\mathbf{0}$ is surjective (cf. ch. VI, §2.4).

Now a general theorem (cf. Addendum 2.7, the “efficiency of miniversal deformations”) justifies the naming of $\mathbb{C}^N$ as the space of local moduli of C_0. As a result, we are in the particularly simple situation where a global moduli space is the quotient of a local moduli space. This allows us to apply general theorems about the miniversal deformation, in particular the Product Decomposition Theorem (Addendum 2.1), to study the mapping m and therefore M (compare with Ch. VI, §2.5 - 2.7). One should think of the Product Decomposition Theorem as an implicit and imprecise, but general, variant of Zariski's theorems (cf. Ch. IV §3 of the Course) that identify those terms in the Puiseux development of a branch that can be eliminated without changing the analytic type.

The initial purpose of this Appendix was to clarify the role played by the Theorems of Efficiency and Product Decomposition in Zariski's Course. However, their discussion is delayed until the third chapter (Addendum) because I thought it worthwhile to construct first in Chapter I a type of local moduli space, for which the moduli space of a branch (plane or not) with given semigroup Γ is always a quotient. Then, in Chapter II, I apply this construction to study the moduli space of any branch. (We should recall that two plane branches are equisingular if and only if they have the same semigroup (cf. Ch II of the Course), in which case, it seems reasonable to say that two non planar branches are equisingular if they have the same semigroup.) It seems to me that this discussion helps shed light on the constructions of Ch. VI of the Course, in particular, how the two Theorems, cited above, are actually used. Moreover, one obtains in this way a *natural compactification* of the moduli space of a plane branch with given characteristic.

The fundamental point here is that the role played by C_0 : $Y^{\overline{\beta}_0} - X^{\overline{\beta}_1} = 0$ in Ch. VI of Zariski's Course is now played by the monomial curve $C^\Gamma \subset \mathbb{C}^{g+1}$, given parametrically by $u_i = t^{\overline{\beta}_i}$ $(0 \leq i \leq g)$, where the $\overline{\beta}_i$ determine a minimal set of generators of the semigroup Γ, associated to a given equisingularity class (cf. Ch. II of the Course). We establish this point and deduce some additional consequences in Chapter I below. In Chapter II, I use the fact that two plane branches are equisingular iff they have the same semigroup Γ to show that Zariski's moduli space is an open dense subset of a quasi-compact quotient of the local moduli space D_Γ of C^Γ. In particular, in §3, which motivated the entire text, I use the fact that D_Γ is smooth to show how the generic component of the moduli space can be defined by a semicontinuity argument. Unfortunately, this approach does not yield any information about the dimension of this generic component. Finally, I show how this work relates to that of Pinkham [Pi], which I have found useful.

Numerous conversations with M. Merle have been very helpful, as well as, of course, the inspiration I drew from Zariski's Course itself. I hope that this text will eventually prove useful in calculating the dimension of the generic component of moduli spaces[1]

References such as (VI, 3.2) refer to Zariski's Course, ch. VI, §3, No. 2, and those like [Pi] refer to the Bibliography at the end of this Appendix. Finally, (App., ch. I, 2.11) is a reference to chapter I of the Appendix.

I have tried to make this text reasonably self-contained. This has made it somewhat longer than it should perhaps be.

[1]Added in translation: This has been the case: see Pierrette Cassou-Noguès *Courbes de semi-groupe donné* Rev. Mat. Univ. Complut. Madrid, **4** No. 1 (1991) 13-44.

Chapter I
The monomial curve C^{Γ} and its deformations

1. The monomial curve C^{Γ}

1.1. Let $\mathcal{O}(=\mathcal{O}_{C,0})$ be the analytic algebra of the germ of an analytically irreducible curve (i.e. of a branch) $(C,\mathbf{0})\subset(\mathbb{C}^k,\mathbf{0})$. The normalization morphism of $\mathcal{O}$ can be understood as an injection $i:\mathcal{O}\hookrightarrow\mathbb{C}\{t\}$, and is interpreted geometrically as a parametrization of C in a neighborhood of $\mathbf{0}$. The $t-$adic valuation of $\mathbb{C}\{t\}$ is denoted ν, and $\Gamma\subset\mathbb{Z}_+$ denotes the associated semigroup:

$$\Gamma=\{\nu(\xi):\xi\in\mathcal{O}-\{0\}\ \}.$$

I will also say that the branch $(C,\mathbf{0})$ has the semigroup Γ.

Since $\mathcal{O}$ is an analytic algebra, $\mathbb{C}\{t\}$ is an $\mathcal{O}-$module of finite type that is contained in the fraction field of $\mathcal{O}$, as in the case of plane branches (II, §1), and there exists a smallest integer c, the conductor, such that $t^c\cdot\mathbb{C}\{t\}\subset\mathcal{O}$. As a result, any integer in $[c,\infty)$ must belong to Γ, which implies that $\mathbb{Z}_+-\Gamma$ is finite. (In order to verify this, one can also show that a "sufficiently general" linear projection $\mathbb{C}^k\to\mathbb{C}^2$ maps C onto a plane branch $(C',\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ whose algebra $\mathcal{O}'$ is a subalgebra of $\mathcal{O}$ that has the same fraction field, and then apply (II. §2) to C'.)

I will denote the maximal ideal of $\mathcal{O}$ by $\mathcal{M}$, and the ideal of elements of $\mathcal{O}$ with $(t)-$valuation $\geq i$ by $\overline{\mathcal{M}}^i$:

$$\overline{\mathcal{M}}^i=\{\xi\in\mathcal{O}:\nu(\xi)\geq i\}.$$

As a result, one obtains a filtration of $\mathcal{O}$:

$$\mathcal{O}=\overline{\mathcal{M}}^0\supsetneq\overline{\mathcal{M}}^1\supseteq\cdots\supseteq\overline{\mathcal{M}}^i\supseteq\overline{\mathcal{M}}^{i+1}\supseteq\cdots.$$

1.1.1. I recall that a filtration $\mathcal{F}$ of a ring $\mathcal{O}$ is a decreasing sequence of additive subgroups of $\mathcal{O}$:

$$\mathcal{F}:\ \mathcal{O}=\mathcal{F}_0\supseteq\mathcal{F}_1\supseteq\cdots\supseteq\mathcal{F}_i\supset\mathcal{F}_{i+1}\supseteq\cdots$$

satisfying $\mathcal{F}_i{\cdot}\mathcal{F}_j\subset\mathcal{F}_{i+j}$. If $\bigcap_{i\in\mathbb{Z}_+}\mathcal{F}_i=\{0\}$ then the filtration is said to be separable. One also will assume $\mathcal{F}_1\neq\mathcal{O}$.

One then observes that each $\mathcal{F}_i$ is an ideal of $\mathcal{O}$ since $\mathcal{F}_0=\mathcal{O}$. If $\xi\in\mathcal{F}_i-\mathcal{F}_{i+1}$, then ξ has order i, and one writes $\nu(\xi)=i$. In this event, $in_{\mathcal{F}}(\xi)$ (the initial form of ξ relative to $\mathcal{F}$) denotes the image of ξ in $\mathcal{F}_i/\mathcal{F}_{i+1}$. Moreover, multiplication in $\mathcal{O}$ induces on $gr_{\mathcal{F}}\mathcal{O}=\bigoplus_{i\in\mathbb{Z}_+}\mathcal{F}_i/\mathcal{F}_{i+1}$ a graded algebra structure - "the graded algebra associated to the filtered ring $(\mathcal{O},\mathcal{F})$" - in which $in_{\mathcal{F}}(\xi)$ is an element whose degree equals the order of ξ. (Here, I have assumed familiarity with [Bourbaki, Alg. Comm., ch. III, §§1, 2] and especially [ibid., §2 no. 4].)

1.1.2. Since $\mathbb{Z}_+ - \Gamma$ is finite, one can find a minimal generating set $(\overline{\beta}_0, \dots, \overline{\beta}_g)$ ("minimal" means here that no $\overline{\beta}_i$ belongs to the semigroup generated by the others) such that $\overline{\beta}_0 < \dots < \overline{\beta}_g$, and $gcd\,(\overline{\beta}_0, \dots, \overline{\beta}_g) = 1$. Moreover, this generating set is uniquely determined by Γ. (I am using here the notations of (II, §3) except for $\overline{\beta}_0$ which Zariski denotes n.)

To fix notations I recall that $\overline{\beta}_i$ is defined inductively as the smallest nonzero element of Γ that does not belong to the semigroup generated by $\overline{\beta}_0, \dots, \overline{\beta}_{i-1}$ (which will be denoted $< \overline{\beta}_0, \dots, \overline{\beta}_{i-1} >$ in the following). In addition, one defines the two sequences of integers: e_i by the formula $e_0 = \overline{\beta}_0, e_i = (e_{i-1}, \overline{\beta}_i), i \geq 1$, and n_i by the formula $e_{i-1} = n_i e_i$. One therefore concludes:

$$\overline{\beta}_0 = n_1 \cdots n_g.$$

1.2. Let $\Gamma =< \overline{\beta}_0, \dots, \overline{\beta}_g >$ be a semigroup such that $\mathbb{Z}_+ - \Gamma$ is finite, i.e. $gcd\,(\overline{\beta}_0, \dots, \overline{\beta}_g) = 1$. Let $C^\Gamma \subset \mathbb{C}^{g+1}$ be the affine curve defined via the parametrization

$$C^\Gamma : \ u_i = t^{\overline{\beta}_i} \quad 0 \leq i \leq g.$$

The following elementary observations can then be made:

1.2.1. *The affine algebra* $\mathbb{C}[C^\Gamma]$ *of* C^Γ *is the image in* $\mathbb{C}[t]$ *of the morphism* $\varphi : \mathbb{C}[u_0, \dots, u_g] \to \mathbb{C}[t]$ *defined by* $\varphi(u_i) = t^{\overline{\beta}_i}$. In other words,
$\mathbb{C}[C^\Gamma] = \mathbb{C}[\{t^h : h \in \Gamma\}] = \mathbb{C}[t^{\overline{\beta}_0}, \dots, t^{\overline{\beta}_g}] \subset \mathbb{C}[t]$ is, in a natural way, a graded subalgebra of $\mathbb{C}[t]$.

1.2.2. C^Γ *is irreducible and, more precisely, its germ (in the analytic sense) at* $\mathbf{0}$ *is analytically irreducible.* This follows from the fact that $gcd\,(\overline{\beta}_0, \dots, \overline{\beta}_g) = 1$, and the branch $(C^\Gamma, \mathbf{0})$ has Γ as its semigroup (1.1). Indeed,

$$\mathcal{O}_{C^\Gamma, \mathbf{0}} = \mathbb{C}\{t^{\overline{\beta}_0}, \dots, t^{\overline{\beta}_g}\} \hookrightarrow \mathbb{C}\{t\}$$

clearly has Γ as its semigroup. This shows that Γ is the semigroup of at least one branch.

1.2.3. *Given the analytic algebra* $\mathcal{O}$ *of a branch with semigroup* Γ, *one can use the filtration* $\overline{\mathcal{M}}^i$ *(1.1) to define the associated graded algebra, denoted* $\overline{gr}_{\mathcal{M}}\mathcal{O}$. *It then follows that* $\overline{gr}_{\mathcal{M}}\mathcal{O}$ *is isomorphic, as a graded algebra, to* $\mathbb{C}[C^\Gamma]$.

(This was first proved by Monique Lejeune-Jalabert for plane branches, see [L.T-2]. Indeed, $\overline{gr}_{\mathcal{M}}\mathcal{O}$ is a particular example of the graded objects studied systematically in [ibid.].)

Proof of (1.2.3). We first remark that $\overline{gr}_{\mathcal{M}}\mathcal{O}$ is an integral domain since $\nu(\xi \cdot \eta) = \nu(\xi) + \nu(\eta)$. Furthermore, by the definition of Γ and the $\overline{\beta}_i$, there exist elements $\xi_i \in \mathcal{O}$ $(0 \leq i \leq g)$ such that in $\mathbb{C}\{t\}$ one has:

$$\xi_i = t^{\overline{\beta}_i} + \sum_{j > \overline{\beta}_i} \gamma_{i,j} t^j \quad (\gamma_{i,j} \in \mathbb{C}).$$

The fact that the filtration $\overline{\mathcal{M}}_i$ of $\mathcal{O}$ is induced, via the inclusion $i : \mathcal{O} \hookrightarrow \mathbb{C}\{t\}$, by the filtration $(t)^i$ of $\mathbb{C}\{t\}$ gives us a morphism:

$$gr_{(t)} i : \overline{gr}_{\mathcal{M}}\mathcal{O} \longrightarrow gr_{(t)}\mathbb{C}\{t\} = \mathbb{C}[t],$$

which is immediately seen to be injective. As a result, $\overline{gr}_{\mathcal{M}}\mathcal{O}$ is a graded subring of $\mathbb{C}[t]$, which contains, evidently, each $t^{\overline{\beta}_i} = in_{\overline{\mathcal{M}}}\xi_i$. Therefore, one has an embedding:

$$j : \mathbb{C}[C^\Gamma] = \mathbb{C}[t^{\overline{\beta}_0}, \ldots, t^{\overline{\beta}_g}] \hookrightarrow \overline{gr}_{\mathcal{M}}\mathcal{O} \subset \mathbb{C}[t].$$

It remains to show that j is surjective. This follows from the fact that the $\overline{\beta}_i$ generate Γ as follows. Let $in_{\overline{\mathcal{M}}}\xi \in \overline{gr}_{\mathcal{M}}\mathcal{O}$ be the initial form of $\xi \in \mathcal{O}$. By the definition of Γ, $\nu(\xi) \in \Gamma$, and therefore $\nu(\xi) = \sum_{i=0}^{g} a_i\overline{\beta}_i$, $a_i \in \mathbb{Z}_+$. Thus, there exists $c \in \mathbb{C}^*$ such that $\nu(\xi - c\,\xi_0^{a_0}\cdots\xi_g^{a_g}) > \nu(\xi)$, which implies

$$in_{\overline{\mathcal{M}}}\xi = in_{\overline{\mathcal{M}}}\, c\,\xi_0^{a_0}\cdots\xi_g^{a_g} = c\,t^{a_0\overline{\beta}_0}\cdots t^{a_g\overline{\beta}_g} \in \mathbb{C}[t^{\overline{\beta}_0}, \ldots, t^{\overline{\beta}_g}].$$

This shows that j is surjective and finishes the proof of 1.2.3. □

1.2.4. COROLLARY [L.T-2]. *The graded algebra $\overline{gr}_{\mathcal{M}}\mathcal{O}$ of the algebra $\mathcal{O}$ of a branch $(C, \mathbf{0})$ (with respect to the filtration $\overline{\mathcal{M}}^i$) only depends upon the semigroup Γ of $(C, \mathbf{0})$ (and also determines it). In particular, two plane branches are equisingular if and only if the corresponding graded algebras are isomorphic as graded algebras.*

(Indeed, two plane branches are equisingular if and only if they have the same semigroup Γ (II, §3).)

1.3. THEOREM 1. *Every branch $(C, \mathbf{0})$ with semigroup Γ is isomorphic to the generic fiber of a one parameter complex analytic deformation of $(C^\Gamma, \mathbf{0})$.*

We remark that in general, the generic fiber of a deformation is neither complex analytic nor necessarily defined over $\mathbb{C}$, and the above statement is actually a shorthand way of stating the following: *There exists a deformation $p : (X, \mathbf{0}) \to (\mathbb{D}, 0)$ of $(C^\Gamma, \mathbf{0})$,*[2] *with a section σ, such that for any sufficiently small representative $\tilde{p}$ of the germ of p, $(\tilde{p}^{-1}(v), \sigma(v))$ is analytically isomorphic to $(C, \mathbf{0})$, for all $v \neq 0$ in the image of $\tilde{p}$.*

I will give two proofs of this assertion. The first is formal and is based upon 1.2.3. The second is elementary and shorter (§1.10). Note that Theorem 1 is a generalization of (VI, 2.1).

1.4. *(first proof)* Let $\mathcal{O} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_i \supset \mathcal{F}_{i+1} \supset \cdots$ be a filtered ring (i.e. unitary and commutative). We extend indexing by setting $\mathcal{F}_i = \mathcal{O}$ for all $i \leq 0$ (the only possibility consistent with the inclusion $\mathcal{F}_{i+1} \subset \mathcal{F}_i$), and consider the graded algebra

$$A = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i v^{-i} \subset \mathcal{O}[v, v^{-1}].$$

One can then observe the following:

1.4.1. *There exists a graded homomorphism*

$$\Phi : A \to gr_{\mathcal{F}}\mathcal{O} = \bigoplus_{i\geq 0} \mathcal{F}_i/\mathcal{F}_{i+1}$$

defined as follows:

$$\Phi(\sum_i \zeta_i v^{-i}) = \sum in_{\mathcal{F}}\zeta_i.$$

[2] that is, a germ of a flat morphism, where $\mathbb{D} = \{v \in \mathbb{C} : |v| < 1\}$, and $(p^{-1}(\mathbf{0}), \mathbf{0})$ is analytically isomorphic to $(C^\Gamma, \mathbf{0})$

Indeed, $\sum_i \zeta_i v^{-i} \in A$ implies $\zeta_i \in \mathcal{F}_i$ for each i, and $in_{\mathcal{F}}\zeta_i \in \mathcal{F}_i/\mathcal{F}_{i+1}$. Thus, Φ is well defined since $\mathcal{F}_i/\mathcal{F}_{i+1} = 0$ whenever $i < 0$.

Moreover, by homogeneity, $\Phi(\sum_i \zeta_i v^{-i}) = 0$ if and only if $in_{\mathcal{F}}(\zeta_i) = (0)$ for each i, i.e. $\zeta_i \in \mathcal{F}_{i+1}$, which, in turn, means that $\sum_i \zeta_i v^{-i-1} \in A$, thus, $\sum_i \zeta_i v^{-i} \in vA$, and conversely. As a result, $Ker\Phi = vA$. Since Φ is evidently surjective, it follows that Φ *induces an isomorphism of graded algebras*:

$$\overline{\Phi} : A/vA \xrightarrow{\sim} gr_{\mathcal{F}}\mathcal{O}.$$

1.4.2. *$A \supset \mathcal{O}[v]$, and, in fact, $\mathcal{O}[v]$ is the set of elements of degree ≤ 0 of A.*

From now on, we will consider A as an $\mathcal{O}[v]$ algebra.

1.4.3. *Every element of the $\mathcal{O}[v]$ module $A/\mathcal{O}[v]$ is annihilated by some power of v.*

Indeed, if $\alpha = \sum_{i=i_0}^{i_1} \zeta_i v^{-i}$, then $v^{i_1}\alpha \in \mathcal{O}[v]$.

1.4.4. *If the graded $\mathcal{O}-$algebra $\bigoplus_{i\geq 0} \mathcal{F}_i$ is of finite type (resp. of finite presentation), then so is the $\mathcal{O}[v]-$algebra A.*

Indeed, the hypothesis means that one can find a graded surjection of graded $\mathcal{O}-$algebras

$$\mathcal{O}[T_1, \ldots, T_N] \xrightarrow{\pi} \bigoplus_{i\geq 0} \mathcal{F}_i v^{-i}$$

(resp. whose kernel I is of finite type).

Then the morphism of $\mathcal{O}-$algebras:

$$\mathcal{O}[v, T_1, \ldots, T_N] \xrightarrow{\tilde{\pi}} A$$

defined by: $\tilde{\pi}(T_i) = \pi(T_i)$, $\tilde{\pi}(v) = v$, is also graded and surjective, which shows, in fact, that A is an $\mathcal{O}$ algebra of finite type. To verify the finite presentation, it suffices to observe that the kernel of $\tilde{\pi}$ is generated by I.

1.4.5. To every element $v_0 \in \mathcal{O}^*$ (the invertible elements of $\mathcal{O}$), one can associate a morphism : $e_{v_0} : A \to \mathcal{O}$ defined by : $e_{v_0}(\sum \zeta_i v^{-i}) = \sum \zeta_i v_o^{-i} \in \mathcal{O}$, whose kernel is $(v - v_o)A$. Since e_{v_o} is clearly surjective, this gives an isomorphism

$$\overline{e}_{v_o} : A/(v - v_o)A \xrightarrow{\sim} \mathcal{O}.$$

1.5. We now consider the particular case in which $\mathcal{O}$ contains a field k. One can then view A as a $k[v]-$algebra via the natural inclusion $k[v] \subset \mathcal{O}[v]$. Since it is clear that A is $k[v]$ torsion free, one deduces [S.G.A. I] that A is a flat $k[v]-$module, or geometrically, that the morphism

$$F : spec\, A \longrightarrow \mathbb{A}^1(k) = spec\, k[v] \quad (\text{ the affine line}),$$

corresponding to the inclusion $k[v] \subset A$, is flat. Moreover, by 1.4.5, for each $v_0 \in k^* \subset \mathbb{A}^1(k) - \{0\}$ (where 0 denotes the origin), one has an isomorphism:

$$\begin{cases} F^{-1}(v_0) \simeq spec\mathcal{O} \\ F^{-1}(0) \simeq spec\, gr_{\mathcal{F}}(\mathcal{O}) \qquad \text{by 1.4.1.} \end{cases}$$

1.6. In addition to the hypothesis of 1.5, let us also assume that $\mathcal{O}$ is a local ring with maximal ideal $\mathcal{M}$, for which k is a field of representatives (i.e. the composition $k \hookrightarrow \mathcal{O} \to \mathcal{O}/\mathcal{M}$ is the identity map on k).

Denoting by S the ideal of A generated by

$$\Big\{\sum_{i\geq 1}\mathcal{F}_i v^{-i}, \sum_{i\geq 0}\mathcal{M}v^{-i}\Big\},$$

one can verify that *the mapping* $\sigma^* : A \to k[v]$ *defined by:*

$$\sigma^*\big(\sum_{i=i_0}^{i=i_1}\zeta_i v^{-i}\big) = \sum_{\substack{i=i_0\\ i\leq 0}}^{i=i_1}\overline{\zeta}_i v^{-i}$$

(where $\overline{\zeta}_i$ *is the image of* ζ_i *in* $\mathcal{O}/\mathcal{M} = k$*) is in fact a ring homomorphism.* This follows because $\mathcal{F}_i \subset \mathcal{M}$ for each $i \geq 1$, and $\mathcal{F}_i$ is a nontrivial ideal of $\mathcal{O}$. One also verifies without any difficulty that σ^* *is surjective with kernel* S *and that the composition of morphisms:*

$$k[v] \hookrightarrow \mathcal{O}[v] \hookrightarrow A \xrightarrow{\sigma^*} k[v]$$

is the identity.

Geometrically, σ^* corresponds to a section σ of the morphism $F : spec\, A \to \mathbb{A}^1(k)$ of 1.5. We can therefore summarize the situation by the diagram

$$F : spec\, A \overset{\sigma}{\underset{}{\overset{\curvearrowleft}{\longrightarrow}}} \mathbb{A}^1(k)$$

where F is flat, and for each $v_0 \in k^* \subset \mathbb{A}^1(k) - \{0\}$, the local ring of $F^{-1}(v_o)$ at the point $\sigma(v_o)$ is $\mathcal{O}$, by 1.4.5 and the fact that $e_{v_o}(S) = \mathcal{M} \subset \mathcal{O}$ for each $v_o \in k^*$. If k is algebraically closed, then $k^* = \mathbb{A}^1(k) - \{0\}$. In fact, we can even reduce to this case by pulling back F via the base change $spec\,\overline{k}[v] \to spec\, k[v] = \mathbb{A}^1(k)$.

1.7. To summarize, we have shown the following:
Any ring R *that contains a field is the deformation of a graded ring* $gr_{\mathcal{F}}R$ *determined by any filtration* $\mathcal{F}$.

More precisely, by combining the different results of §1.4, we conclude:

1.7.1. Theorem 2. [3] *Let* $\mathcal{O}$ *be unitary commutative ring containing the field* k*. Let* $\mathcal{F} : \mathcal{O} = \mathcal{F}_0 \supset \cdots \supset \mathcal{F}_i \supset \mathcal{F}_{i+1} \supset \cdots$ *be any filtration. Then there exists a flat morphism*

$$F : specA \longrightarrow \mathbb{A}^1(k)$$

such that

$$F^{-1}(v_o) \simeq spec\,\mathcal{O}, \quad \textit{for any } v_o \in k^* \subset \mathbb{A}^1(k) - \{0\},$$
$$\textit{and} \qquad F^{-1}(0) \simeq spec\, gr_{\mathcal{F}}\mathcal{O}.$$

In addition, 1.4.4 shows us, on the one hand, that if $\mathcal{O}$ is a $k-$algebra of finite type (i.e. "geometric over k") then so too is A, while on the other hand, 1.6 gives precise conditions that insure the existence of a section of F. Finally, 1.4.3 tells us what is true when $\mathcal{O}$ does not contain a field, i.e. $A\bigotimes_{\mathcal{O}[v]}\mathcal{O}(v) \simeq \mathcal{O}(v)$.

[3] While writing this text, I learned from D. Eisenbud and S. Kleiman that this result is not new. S. Kleiman gave me the following reference : M. Gerstenhaber, *On the deformations of rings and algebras*, Ann. of Math., vol. 79 no. 1 (1964). The point of view of that article is however quite different from that taken here.

1.7.2. Remark: The reader will immediately observe that commutativity has only been used to simplify the discussion. □

1.8. In particular, if $\mathcal{O}$ is an analytic algebra, corresponding to the germ of an analytic space (W, w), and if $\mathcal{F}$ is a filtration of finite presentation in the sense of 1.4.4, the argument therein, combined with the existence of *Specan* (Séminaire Cartan 60-61, exposés de Houzel, Publ. de l'I.H.P.), shows that for a sufficiently small representative W of (W, w) one can construct a morphism of complex analytic spaces

$$Specan_{W\times\mathbb{C}}\, A \longrightarrow W \times \mathbb{C}$$

whose composition $F : Specan_{W\times\mathbb{C}}\, A \to W \times \mathbb{C} \xrightarrow{pr_2} \mathbb{C}$ admits a section σ so that the following properties are satisfied:

1. $F : (Specan_{W\times\mathbb{C}}\, A, \mathbf{0}) \overset{\sigma}{\underset{\curvearrowleft}{\longrightarrow}} (\mathbb{C}, 0)$ is flat;
2. $(F^{-1}(v_o), \sigma(v_o)) \xrightarrow{\sim} (W, w)$ for each $v_o \in \mathbb{C} - \{0\}$ (for a sufficiently small representative of the germ of F);
3. $(F^{-1}(0), \sigma(0))$ is the analytic space associated to the germ at $\mathbf{0}$ of the "affine algebraic space above W", denoted $Specan_W\, gr_{\mathcal{F}}\, \mathcal{O}$.

I recall that $gr_{\mathcal{F}}\,\mathcal{O}$ is a graded $\mathcal{O}-$algebra of finite presentation since $\mathcal{F}$ is, by hypothesis, of finite presentation, and $gr_{\mathcal{F}}\,\mathcal{O}$ is a quotient of $\bigoplus_{i\geq 0}\left(\mathcal{F}_i \bigotimes_{\mathcal{O}} \mathcal{O}/\mathcal{F}_1\right)$.

1.8.1. Remark: The extension of 1.4 to sheaves of filtered algebras allows one to construct, for example, a deformation of immersions whose generic fiber is a given immersion $Y \hookrightarrow X$, defined by an $\mathcal{O}_X-$ideal $\mathcal{I}$, and whose special fiber is $Y \hookrightarrow C_{X,Y}$, where $C_{X,Y} = Spec_{\mathcal{O}_Y}\left(\oplus_{\nu\geq 0}\mathcal{I}^\nu/\mathcal{I}^{\nu+1}\right)$ is the *normal cone of Y in X.*

1.9. *End of first proof of Theorem 1.* To complete the proof, it suffices for us to make the monomial curve C^Γ appear as the graded object associated to the analytic algebra $\mathcal{O}$ of a branch with semigroup Γ. This is precisely what is done in 1.2.3.

1.10. *Second proof of Theorem 1.*

The abstract construction now motivates us to construct a more down to earth and explicit deformation of $(C^\Gamma, \mathbf{0})$, whose generic fiber is $(C, \mathbf{0})$. It is reasonable to view this argument as the natural generalization of (VI, 2.1).

We assume $(C, \mathbf{0})$ is given parametrically in $(\mathbb{C}^k, \mathbf{0})$ by: $z_i = \varphi_i(t), \varphi_i(t) \in \mathbb{C}\{t\}$, $1 \leq i \leq k$. As a result, $\mathcal{O} = \mathcal{O}_{C,\mathbf{0}} = \mathbb{C}\{\varphi_1(t), \ldots, \varphi_k(t)\} \to \mathbb{C}\{t\}$. By a suitable coordinate change, we may also choose the parameter t so that

$$\varphi_1 = t^n,\ \varphi_i = t^{a_i} + \sum_{j>a_i} \rho_{i,j} t^j \quad 2 \leq i \leq k,$$

where n is the multiplicity of $(C, \mathbf{0})$, and each $a_i > n$.

We remark that since Γ is the semigroup of $(C, \mathbf{0})$, each $a_i \in \Gamma$, as is n, of course, and that in fact $n = \overline{\beta}_0$ in the notations of 1.1.2. Furthermore, the definition of Γ and the $\overline{\beta}_i$ (1.1.2) assures us of the existence of elements $\xi_i \in \mathcal{O}$ $(0 \leq i \leq g)$ such that:

$$\xi_i = t^{\overline{\beta}_i} + \sum_{j>\overline{\beta}_i} \gamma_{i,j} t^j,$$

and that we can, of course, choose $\xi_0 = \varphi_1 = t^n$.

We now consider the family of curves in $\mathbb{C}^{g+k}$, parametrised by v, as follows:

$$C_v : \begin{cases} Z_0 = t^n \\ Z_1 = t^{\overline{\beta}_1} + \sum_{j>\overline{\beta}_1} \gamma_{1,j}\, v^{j-\overline{\beta}_1}\, t^j \\ \vdots \quad \vdots \\ Z_g = t^{\overline{\beta}_g} + \sum_{j>\overline{\beta}_g} \gamma_{g,j}\, v^{j-\overline{\beta}_g}\, t^j \\ Z_{g+1} = t^{a_2} + \sum_{j>a_2} \rho_{2,j}\, v^{j-a_2}\, t^j \\ \vdots \quad \vdots \\ Z_{g+k-1} = t^{a_k} + \sum_{j>a_k} \rho_{k,j}\, v^{j-a_k}\, t^j\,. \end{cases}$$

It remains for us to verify that $(C_0, \mathbf{0}) \simeq (C^\Gamma, \mathbf{0})$, and that for $v \neq 0$, $(C_v, \mathbf{0}) \simeq (C, \mathbf{0})$. Now, if $v = v_o \neq 0$, one can consider the curve C'_{v_o}, deduced from C_{v_o} by the isomorphism : $t = v_o t'$, $Z_i = v_o^{-\overline{\beta}_i} Z'_i$ if $1 \leq i \leq g$, and $Z_i = v_o^{-a_i} Z'_i$ if $g+1 \leq i \leq g+k-1$. It follows that the algebra of C'_{v_o} is $\mathbb{C}\{t^n, \varphi_2(t), \ldots, \varphi_k(t), \xi_1(t), \ldots, \xi_g(t)\}$, which equals $\mathbb{C}\{t^n, \varphi_2(t), \ldots, \varphi_k(t)\} = \mathcal{O}$, since $\xi_i \in \mathcal{O}$ $(1 \leq i \leq g)$. Or equivalently, as one might prefer to say it, ξ_i is a series in $t^n, \varphi_2(t), \ldots, \varphi_k(t)$, and the coordinate change in the Z_i induces an isomorphism of $(C'_{v_o}, \mathbf{0})$ onto $(C, \mathbf{0})$. Thus, for $v_o \neq 0$, $(C_{v_o}, \mathbf{0}) \simeq (C, \mathbf{0})$.

In addition, for $v_o = 0$, $(C_0, \mathbf{0})$ is parametrized by: $Z_i = t^{\overline{\beta}_i}$ $(1 \leq i \leq g)$, and $Z_i = t^{a_i}$ $(g+1 \leq i \leq g+k-1)$. Since $a_i \in \Gamma$, which is generated by the $\overline{\beta}_j$, we know that $a_i = \sum_{j=0}^{g} b_{i,j}\overline{\beta}_j$, $b_{i,j} \in \mathbb{Z}_+$. This implies that $(C_0, \mathbf{0})$ is contained in each of the nonsingular hypersurfaces $Z_i - Z_0^{b_{i,0}} \cdots Z_g^{b_{i,g}} = 0$ $(g+1 \leq i \leq g+k-1)$. The corresponding coordinate change $Z_i = Z'_i + Z_0^{b_{i,0}} \cdots Z_g^{b_{i,g}}$ then gives us the isomorphism of $(C_0, \mathbf{0})$ onto $(C^\Gamma, \mathbf{0})$.

This completes the second proof of Theorem 1 (1.3). □

1.11. Remark: Up to isomorphism, one can always assume that no exponent $j \geq c$, ($c =$ conductor, cf. 1.1) appears in the power series expansion of the ξ_i or φ_i. Therefore, 1.10 actually gives us, for free, a deformation of *affine curves* that connects an affine curve, whose germ at $\mathbf{0}$ is isomorphic to the given branch $(C, \mathbf{0})$, to the affine curve C^Γ. □

2. The miniversal constant semigroup deformation of C^Γ

(Starting with this section, I allow myself free use of the notations and resultats about miniversal deformations that are recalled in the Addendum.)

2.1. In the case where Γ is the semigroup of a plane branch (and more generally when C^Γ is a complete intersection), I will now show that there exists a (germ of a) nonsingular subspace $(D_\Gamma, \mathbf{0})$ inside the base $(S, \mathbf{0})$ of the miniversal deformation

$$G : (X, C^\Gamma) \longrightarrow (S, \mathbf{0})$$

such that the deformation obtained by restricting G to D_Γ (i.e. $X_\Gamma = X \times_S D_\Gamma$)

$$\begin{array}{ccc} (X_\Gamma, C^\Gamma) & \xrightarrow{\text{inclusion}} & (X, C^\Gamma) \\ {\scriptstyle G_\Gamma}\Big\downarrow & & {\scriptstyle G}\Big\downarrow \\ (D_\Gamma, \mathbf{0}) & \xrightarrow{\text{inclusion}} & (S, \mathbf{0}) \end{array}$$

is miniversal for the deformations of C^Γ with reduced base each of whose fibers is irreducible with semigroup Γ. I will refer to these as "miniversal constant semigroup deformations". According to 1.2.2, among these deformations we find, in particular, those that were constructed in §1.9, 1.10.

2.2. Proposition [Lejeune-Jalabert] (*not published, see* [L.T-2]) . *If Γ is the semigroup of a plane branch, then the affine curve $C^\Gamma \subset \mathbb{C}^{g+1}$ is a complete intersection, and therefore, so too is the branch $(C^\Gamma, \mathbf{0})$.*

The proof uses:

2.2.1. Lemma (Azevedo [Az], Merle [Me]). *If $\Gamma =< \overline{\beta}_0, \ldots, \overline{\beta}_g >$ is the semigroup of a plane branch, one has (in the notations of 1.1.2)*

$$n_i \overline{\beta}_i \in < \overline{\beta}_0, \ldots, \overline{\beta}_{i-1} > \qquad 1 \leq i \leq g.$$

Proof. Using the notations and resultats of (II, 3.11), we have:

$$(*) \qquad \overline{\beta}_q = n_{q-1} \cdot \overline{\beta}_{q-1} - \beta_{q-1} + \beta_q \qquad (1 \leq q \leq g)$$

where the β_q are the Puiseux characteristic exponents of the plane branch. By the definition of the integers m_q (II, 3.11), this formula can be written in this way:

$$\overline{\beta}_q = n_{q-1} \cdot \overline{\beta}_{q-1} - m_{q-1} \cdot e_{q-1} + m_q \cdot e_q,$$

from which it follows that

$$n_q \cdot \overline{\beta}_q = n_q \cdot n_{q-1} \cdot \overline{\beta}_{q-1} + (m_q - n_q \cdot m_{q-1}) e_{q-1}.$$

We now observe that the interpretation of the $\beta_q = \frac{m_q}{n_1 \cdots n_q} \cdot \overline{\beta}_0$ as characteristic exponents implies $\beta_{q-1} < \beta_q$, i.e. $m_q - n_q \cdot m_{q-1} > 0$.

Writing the above formula as

$$n_q \cdot \overline{\beta}_q = (n_q - 1) \cdot n_{q-1} \cdot \overline{\beta}_{q-1} + n_{q-1} \cdot \overline{\beta}_{q-1} + (m_q - n_q \cdot m_{q-1}) e_{q-1},$$

one sees that to verify $n_q \cdot \overline{\beta}_q \in < \overline{\beta}_0, \ldots, \overline{\beta}_{q-1} >$, it suffices to verify the same inclusion for $n_{q-1} \cdot \overline{\beta}_{q-1} + h \cdot e_{q-1}$ for any $h \in \mathbb{Z}_+$, which we will now do by induction on q. For $q = 1$, it follows from the equation (and, by convention, $n_0 = 1$) $\overline{\beta}_0 + h \cdot e_0 = (h+1) \cdot \overline{\beta}_0$, which is evidently in $< \overline{\beta}_0 >$. Furthermore, by the definitions and (*):

$$e_{q-1} = (e_{q-2}, \overline{\beta}_{q-1}) = (e_{q-2}, m_{q-1} \cdot e_{q-1}),$$

which implies for any $h \in \mathbb{Z}_+$:

$$h \cdot e_{q-1} = \lambda \cdot m_{q-1} \cdot e_{q-1} + \mu \cdot e_{q-2} \qquad (\lambda, \mu \in \mathbb{Z}).$$

Since $n_{q-1} \cdot e_{q-1} = e_{q-2}$, one can choose λ such that $0 > \lambda > -n_{q-1}$. One then rewrites (*) for $q-1$ as:

$$\begin{aligned} n_{q-1} \cdot \overline{\beta}_{q-1} + h \cdot e_{q-1} &= (n_{q-1} + \lambda)\overline{\beta}_{q-1} - \lambda\big(n_{q-2} \cdot \overline{\beta}_{q-2} - m_{q-2} \cdot e_{q-2} + m_{q-1} \cdot e_{q-1}\big) \\ &\quad + \lambda \cdot m_{q-1} \cdot e_{q-1} + \mu \cdot e_{q-2} \\ &= (n_{q-1} + \lambda)\overline{\beta}_{q-1} - \lambda \cdot n_{q-2} \cdot \overline{\beta}_{q-2} + (\lambda \cdot m_{q-2} + \mu) e_{q-2}. \end{aligned}$$

Now, our choice of λ implies, on the one hand, $n_{q-1} + \lambda > 0$ and $-\lambda > 0$ on the other. Moreover, $\lambda m_{q-2} + \mu > 0$ since $n_{q-1}(\lambda m_{q-2} + \mu) = \lambda \cdot n_{q-1} \cdot m_{q-2} + \mu \cdot m_{q-1} > \lambda \cdot m_{q-1} + \mu \cdot n_{q-1} = h \geq 0$ (by the hypothesis on h), using again that $m_{q-1} > n_{q-1} \cdot m_{q-2}$ and $\lambda < 0$.

In this way, we see that each coefficient on the right side of the second equation above is positive. Now, our induction hypothesis:

$n_{q-2} \cdot \overline{\beta}_{q-2} + h' \cdot e_{q-2} \in < \overline{\beta}_0, \ldots, \overline{\beta}_{q-2} >$ for any $h' \in \mathbb{Z}_+$,

implies that

$$-\lambda \cdot n_{q-2} \cdot \overline{\beta}_{q-2} + (\lambda \cdot m_{q-2} + \mu) e_{q-2} \in < \overline{\beta}_0, \ldots, \overline{\beta}_{q-2} >$$

since $-\lambda$ and $\lambda \cdot m_{q-2} + \mu$ are positive. Finally, since $n_{q-1} + \lambda > 0$, it is now clear that

$$n_{q-1} \cdot \overline{\beta}_{q-1} + h \cdot e_{q-1} \in < \overline{\beta}_0, \ldots, \overline{\beta}_{q-1} >,$$

which completes the proof of 2.2.1. □

2.2.2. Remark: We have used the hypothesis that Γ is the semigroup of a plane branch in the formula (*) as well as the inequality $\beta_q > \beta_{q-1}$. The semigroup $\Gamma = < \overline{\beta}_0, \ldots, \overline{\beta}_g >$ of a plane branch therefore satisfies, by 2.2.1:

(1) $$n_q \cdot \overline{\beta}_q \in < \overline{\beta}_0, \ldots, \overline{\beta}_{q-1} >$$

and

(2) $$n_q \cdot \overline{\beta}_q < \overline{\beta}_{q+1} \qquad (0 \leq q \leq g-1)$$

(which follow immediately from (*) and $\beta_{q-1} < \beta_q$). □

I will prove below, as a corollary of Theorem 3 (cf. §3, 2.1), that properties (1), (2) in fact characterize the semigroup of plane branches, by explicitly constructing a plane branch with semigroup Γ, given that Γ satisfies (1) and (2).

2.2.3. Completing the proof of 2.2. Lemma 2.2.1 allows us to write for any i, $1 \leq i \leq g$:

$$n_i \cdot \overline{\beta}_i = \ell_o^{(i)} \cdot \overline{\beta}_0 + \cdots + \ell_{i-1}^{(i)} \cdot \overline{\beta}_{i-1}, \qquad \ell_j^{(i)} \in \mathbb{Z}_+ .$$

This implies that our monomial curve C^Γ parametrized by $u_i = t^{\overline{\beta}_i}$ $(0 \leq i \leq g)$ satisfies the following equations:

$$f_i = u_i^{n_i} - u_0^{\ell_0^{(i)}} \cdot u_1^{\ell_1^{(i)}} \cdots u_{i-1}^{\ell_{i-1}^{(i)}} = 0 \quad (1 \leq i \leq g).$$

Moreover, since the sequence $\overline{\beta}_i$ is increasing, $n_i < \sum_{j=0}^{i-1} \ell_j^{(i)}$ $(1 \leq i \leq g)$, and therefore, the term of lowest degree (in the usual sense) of f_i is $u_i^{n_i}$.

Let $I \subset \mathbb{C}\{u_0, \ldots, u_g\}$ be the ideal generated by the f_i $(1 \leq i \leq g)$, let C' be the subspace of $(\mathbb{C}^{g+1}, \mathbf{0})$ defined by I, and set $\mathcal{O}' = \mathcal{O}_{C',\mathbf{0}} = \mathbb{C}\{u_0, \ldots, u_g\}/I$. We have $(C^\Gamma, \mathbf{0}) \subset (C', \mathbf{0})$. Thus, the multiplicity of C' at $\mathbf{0}$ is at least that of C^Γ, which equals $\overline{\beta}_0$. Furthermore, Theorem 2 yields a deformation whose "generic fiber" is $\mathcal{O}'$ and special fiber $Spec\, gr_{\mathcal{M}}\, \mathcal{O}$ is the tangent cone to C' at $\mathbf{0}$, where $gr_{\mathcal{M}}\, \mathcal{O}'$ is the associated graded algebra for the filtration of $\mathcal{O}'$ by the powers of its maximal ideal. Since $(u_1^{n_1}, \ldots, u_g^{n_g})$ forms a regular sequence that defines an irreducible subspace, a direct calculation shows that $gr_{\mathcal{M}}\, \mathcal{O}' = \mathbb{C}[U_0, \ldots, U_g]/(U_1^{n_1}, \ldots, U_g^{n_g})$, where $U_i = in_{\mathcal{M}}\, u_i$ denotes the initial form of u_i in the graded algebra associated to the filtration of $\mathbb{C}\{u_0, \ldots, u_g\}$ by the powers of its maximal ideal.

Additionally, the multiplicity of C' (as well as its dimension, of course) equals that of its tangent cone. This shows that C' is a complete intersection branch with multiplicity $n_1 \cdots n_g = \overline{\beta}_0$. Indeed, by Theorem 2, C' is a flat deformation of its tangent cone, $Spec\,(gr_{\mathcal{M}}\, \mathcal{O}')$ (the u_0 axis counted $n_1 \cdots n_g$ times), which has multiplicity $\overline{\beta}_0 = n_1 \cdots n_g$, and which is a complete intersection, a property preserved by a flat deformation. Moreover, it contains C^Γ, which is irreducible and also has multiplicity $\overline{\beta}_0$. From this, one deduces that $C^\Gamma = C'$, and thus, C^Γ is a complete intersection, which completes the proof of 2.2. □

2.3. Corollary. *Any branch $(C, \mathbf{0})$, whose semigroup Γ equals the semigroup of a plane branch, is a complete intersection.*

Proof. By Theorem 1, $(C, \mathbf{0})$ is a flat deformation of $(C^\Gamma, \mathbf{0})$, which is a complete intersection by 2.2. The assertion now follows because a flat deformation preserves the property of being a complete intersection (cf. [Tj], [S]). □

2.4. Remarks:

2.4.1. The proof of 2.3 and the final part of the proof of 2.2 exemplify an application of Theorem 2 of the following type: if one is able to find a filtration $\mathcal{F}$ of a ring $\mathcal{O}$ such that $gr_{\mathcal{F}}\, \mathcal{O}$ satisfies a property that is stable under flat deformation (for example: regularity, locally complete intersection, Cohen-Macauley, isolatedness of singularities, reduced...), then $\mathcal{O}$ itself also possesses this property.

2.4.2. I do not know if the semigroups of complete intersection branches satisfy, in general, the property of Lemma 2.2.1 (cf. 2.2.2). □

In the following, I will often impose the condition that Γ is the semigroup of a plane branch, since this is the context in which we are working here, but I will only use the fact that C^Γ is a complete intersection, which is, a priori, a less restrictive hypothesis (a posteriori also, cf. §3, 3.2.3).

2.5. Corollary (of 2.2). *(cf. Pinkham [Pi] chap. I, 2.3) If Γ is the semigroup of a plane branch, one can construct a miniversal deformation $G : (X, C^\Gamma) \to (S, \mathbf{0})$ of C^Γ (as an affine curve) with $\mathbb{C}^*$ action that is compatible with those on X and S (i.e. G is equivariant relative to these actions in the sense that $G(\lambda \cdot x) = \lambda \cdot G(x)\ \forall \lambda \in \mathbb{C}^*$). The induced $\mathbb{C}^*$ action on $C^\Gamma \simeq G^{-1}(\mathbf{0})$ is the natural $\mathbb{C}^*$ action on C^Γ (given by $t \to \lambda \cdot t,\ \lambda \in \mathbb{C}^*$). Moreover, S is smooth because C^Γ is a complete intersection ([Tj], [S]).*

Proof. We begin by first recalling the following:

2.5.1. The affine algebra $\mathbb{C}[C^\Gamma]$ of $C^\Gamma \subset \mathbb{C}^{g+1}$ is isomorphic, as a graded algebra, to (notations of 2.2.3) $\mathbb{C}[u_0, \ldots, u_g]/(f_1, \ldots, f_g)$ with the quotient grading, induced from that of $\mathbb{C}[u_0, \ldots, u_g]$ where $deg\, u_i = \overline{\beta}_i$. The quotient grading exists because $f_i = u_i^{n_i} - u_0^{\ell_0^{(i)}} \cdots u_{i-1}^{\ell_{i-1}^{(i)}}$ is homogeneous of degree $n_i \cdot \overline{\beta}_i$ with respect to this grading on $\mathbb{C}[u_0, \ldots, u_g]$ (cf. 2.2.3). The ideal generated by the f_i is therefore homogeneous so that the grading passes through to the quotient. I also want to emphasize here that this quotient grading on $\mathbb{C}[u_0,\dots,u_g]/(f_1,\dots,f_g)=\mathbb{C}[C^\Gamma]\hookrightarrow \mathbb{C}[t]$ coincides with that induced by the natural grading on $\mathbb{C}[t]$ (cf. 1.2.1).

2.5.2. Additionally, one knows that since C^Γ is a complete intersection (cf. Addendum, [S], [Tj]), the miniversal deformation G of C^Γ can be described as the restriction of the natural projection $\mathbb{C}^{g+1}\times\mathbb{C}^\tau \to \mathbb{C}^\tau$ to the subspace $X\subset \mathbb{C}^{g+1}\times\mathbb{C}^\tau$ defined, in the coordinates $u_0,\dots,u_g$ on $\mathbb{C}^{g+1}$ and $w_1,\dots,w_\tau$ on $\mathbb{C}^\tau$, by the ideal generated in $\mathbb{C}[u_0,\dots,u_g,w_1,\dots,w_\tau]$ by $(F_1,\dots,F_g)$ where

$$F_i = f_i(u_0,\dots,u_i)+\sum_{j=1}^{\tau} w_j\cdot s_{ij}(u_0,\dots,u_g) \qquad 1\le i\le g.$$

Here, the vectors $\mathbf{s}_j=(s_{1,j},\dots,s_{g,j})\in\mathbb{C}[u_0,\dots,u_g]^g$ satisfy the property that their images (via the map $\nu:\ \mathbb{C}[u_0,\dots,u_g]^g\to\mathbb{C}[C^\Gamma]^g$, induced by the canonical surjection $\mathbb{C}[u_0,\dots,u_g]\to\mathbb{C}[u_0,\dots,u_g]/(f_1,\dots,f_g)\simeq\mathbb{C}[C^\Gamma]$) in the quotient $\mathbb{C}[C^\Gamma]^g/\mathcal{N}$ form a basis as $\mathbb{C}$ vector space, where $\mathcal{N}$ is the submodule of $\mathbb{C}[C^\Gamma]^g$ generated by the $g+1$ vectors

$$\boldsymbol{\gamma}_0=\nu\big(\partial f_1/\partial u_0,\dots,\partial f_g/\partial u_0\big),\dots,\boldsymbol{\gamma}_g=\nu\big(\partial f_1/\partial u_g,\dots,\partial f_g/\partial u_g\big)\,.$$

(Note that the $\mathbb{C}-$dimension of $\mathbb{C}[C^\Gamma]^g/\mathcal{N}=T^1_{C^\Gamma}$ is finite because C^Γ has an isolated singularity).

One can endow $\mathbb{C}[u_0,\dots,u_g]^g$ (resp. $\mathbb{C}[C^\Gamma]^g$) with a graded $\mathbb{C}[u_0,\dots,u_g]$ (resp. $\mathbb{C}[C^\Gamma]$) module structure so that the $\boldsymbol{\gamma}_k$ (resp. their images in $\mathbb{C}[C^\Gamma]^g$) are homogeneous relative to the grading defined by $deg\, u_i=\overline{\beta}_i$ (resp. the image of u_i in $\mathbb{C}[C^\Gamma]$ has degree $\overline{\beta}_i$). Indeed, a vector $\boldsymbol{\varphi}=(\varphi_1,\dots,\varphi_g)$ is homogeneous of degree μ if for each i, $1\le i\le g$, φ_i is homogeneous of degree $n_i\overline{\beta}_i+\mu$. (This is equivalent to writing $\mathbb{C}[u_0,\dots,u_g]^g=\bigoplus_{i=1}^g\mathbb{C}[u_0,\dots,u_g]](n_i\overline{\beta}_i)$, in the classical notation of EGA, ch. II, 2.1, Publ. Math. I.H.E.S. No. 8). In this way, $\boldsymbol{\gamma}_k$ becomes homogeneous of degree $-\overline{\beta}_k$ because $deg\,\partial f_i/\partial u_k=n_i\overline{\beta}_i-\overline{\beta}_k$ by Euler's identity.

One can then choose the vectors $\mathbf{s}_j\in\mathbb{C}[u_0,\dots,u_g]^g$ to be homogeneous with respect to this grading since they are characterized solely by the fact that their images form a basis of a finite dimensional vector space. Finally, one can, in this way, endow the algebra $\mathbb{C}[u_0,\dots,u_g,w_1,\dots,w_\tau]$ with the unique grading for which $deg\, u_i=\overline{\beta}_i$, and F_i is homogeneous of degree $n_i\overline{\beta}_i$ by setting:

$$deg\, w_j=-deg\,\mathbf{s}_j.$$

Indeed, by the definition of $deg\,\mathbf{s}_j$, one then has:

$$F_i=f_i+\sum_j w_j\, s_{ij}$$

and

$$deg\, w_j s_{ij}=deg\, w_j+deg\, s_{ij}=n_i\overline{\beta}_i-deg\, s_{ij}+deg\, s_{ij}=n_i\overline{\beta}_i.$$

This permits us to define a $\mathbb{C}^*$ action on $\mathbb{C}^{g+1}$ and $\mathbb{C}^\tau$ (via $u_i\rightsquigarrow\lambda^{\overline{\beta}_i}\cdot u_i\,;\ w_j\rightsquigarrow\lambda^{deg\, w_j}\cdot w_j,\ \lambda\in\mathbb{C}^*$), which respects the subspace $X\subset\mathbb{C}^{g+1}\times\mathbb{C}^\tau$ defined by the ideal $(F_1,\dots,F_g)$, since this ideal is homogeneous by the definition of $deg\, w_j$. As a result, we therefore have an action of $\mathbb{C}^*$ on $\mathbb{C}^\tau$ and $X\subset\mathbb{C}^{g+1}\times\mathbb{C}^t$, and it is easy to verify that $G:X\to\mathbb{C}^\tau$, the restriction to X of the projection, is equivariant. This completes the proof of 2.5. □

2.6. Remarks:

2.6.1. Strictly speaking, the miniversal deformation of C^Γ is only the germ of G on C^Γ, or more precisely, if one insists, one should say that the germ of G at $\mathbf{0}$ is the miniversal deformation of the branch $(C^\Gamma, \mathbf{0})$. We have constructed an algebraic representative of these miniversal deformations in 2.5.2 (cf. [Pi]).

2.6.2. In [Pi], Pinkham shows the existence of an miniversal equivariant deformation even for a non complete intersection.

2.7. Proposition (Pinkham [Pi], chap. IV, 14.9). *If C^Γ is Gorenstein, and therefore, in particular, if Γ is the semigroup of a plane curve, the dimension τ of the base of the miniversal deformation of C^Γ is:*

$$\tau = 2\delta(\Gamma) \quad \textit{where} \quad \delta(\Gamma) = \#(\mathbb{Z}_+ - \Gamma).$$

Besides the fact that this formula helps in finding the vectors $\mathbf{s}_j$, this result will be useful to us later on.

2.8. Remark ([Pi], chap. IV, 12.5)**:** In the construction of 2.5, none of the vectors $\mathbf{s}_j$ can be chosen to have degree 0. As a result, $deg\, w_j \in \mathbb{Z}-\{0\}$ $1 \le j \le \tau$.□

Proof of 2.7. One needs to show that any vector $\mathbf{s} \in \mathbb{C}[C^\Gamma]^g$, homogeneous of degree 0, belongs to $\mathcal{N}$. Since $\mathbb{C}[C^\Gamma]^g/\mathcal{N}$ is finite dimensional, if we view $\mathbb{C}[C^\Gamma]^g$ as a $\mathbb{C}[C^\Gamma]$ submodule of $\mathbb{C}[t]^g$ (which we certainly can due to the inclusion $\mathbb{C}[C^\Gamma] \subset \mathbb{C}[t]$), then we see that for any such $\mathbf{s}$, there exists $\mu \in \mathbb{Z}_+$ such that $t^\mu \cdot \mathbf{s} \in \mathcal{N}$ (it suffices to take $\mu \ge c$, the conductor, since then $t^\mu \in \mathbb{C}[C^\Gamma]$). Now, $deg\, \mathbf{s} = 0$ means that if $\mathbf{s} = (s_1, \dots, s_g)$, then $deg\, s_i = n_i\overline{\beta}_i$, and therefore, for each i, $1 \le i \le g$, $s_i = \alpha_i \cdot t^{n_i\overline{\beta}_i}$ for some $\alpha_i \in \mathbb{C}^*$. The fact that $t^\mu \mathbf{s} \in \mathcal{N}$ is then written as follows for $1 \le i \le g$:

$$\alpha_i \cdot t^{\mu + n_i\overline{\beta}_i} = \sum_{j=0}^{g} a_{ij} \left(\overline{\partial f_i/\partial u_j}\right),$$

where $\left(\overline{\partial f_i/\partial u_j}\right)$ is the image of $\partial f_i/\partial u_j$ in $\mathbb{C}[C^\Gamma]$ (2.5.1, 2.5.2). Now, in $\mathbb{C}[t]$, $\left(\overline{\partial f_i/\partial u_j}\right)$ is homogeneous of degree $n_i\overline{\beta}_i - \overline{\beta}_j$ and must therefore equal $\beta_{ij} \cdot t^{n_i\overline{\beta}_i - \overline{\beta}_j}$ for some $\beta_{ij} \in \mathbb{C}^*$. By homogeneity, it then follows that $\alpha_{ij} = \gamma_{ij} \cdot t^{\mu + \overline{\beta}_j}$ for some $\gamma_{ij} \in \mathbb{C}^*$. Thus, dividing by t^μ we conclude

$$s_i = \alpha_i \cdot t^{n_i\overline{\beta}_i} = \sum_{j=0}^{g} \gamma_{ij} \cdot t^{\overline{\beta}_j} \left(\overline{\partial f_i/\partial u_j}\right).$$

But since, in fact, $t^{\overline{\beta}_j} \in \mathbb{C}[C^\Gamma]$, the same equation must be valid in $\mathbb{C}[C^\Gamma]$, which shows that $\mathbf{s} \in \mathcal{N}$. □

2.9. Corollary. *Fixing once and for all a choice for $\mathbf{s}_j$ that is homogeneous, and thus an equivariant miniversal equivariant deformation of C^Γ as in 2.5, we can then define a partition of $\{1, \dots, \tau\}$:*

$$\{1, \dots, \tau\} = P_+ \coprod P_-,$$

where $P_+ = \{j \in \{1, \dots, \tau\} : deg\, w_j > 0\}$, $P_- = \{j \in \{1, \dots, \tau\} : deg\, w_j < 0\}$. Setting $\tau_\pm = \#P_\pm$, then gives a decomposition $\mathbb{C}^\tau \simeq \mathbb{C}^{\tau_-} \times \mathbb{C}^{\tau_+}$.

2.10. THEOREM 3. *In the setting of 2.5, 2.9, the deformation of* C^Γ, *obtained by applying the base change* $\mathbb{C}^{\tau_-} \times \{\mathbf{0}\} \hookrightarrow \mathbb{C}^\tau$ *to the miniversal deformation* $G : (X, C^\Gamma) \to (\mathbb{C}^\tau, \mathbf{0})$, *is a miniversal constant semigroup deformation of* C^Γ, *i.e. is miniversal for the deformations of* C^Γ *with reduced base such that each fiber has a singular point with semigroup* Γ. (*This remains valid whether we deform* C^Γ *as an affine curve or branch* $(C^\Gamma, \mathbf{0})$.) *Thus, we have the following commutative diagram:*

$$\begin{array}{ccc} (X_\Gamma, C^\Gamma) & \xrightarrow{\text{inclusion}} & (X, C^\Gamma) \\ {\scriptstyle G_\Gamma}\downarrow & & {\scriptstyle G}\downarrow \\ (D_\Gamma, \mathbf{0}) =_{\text{def}} (\mathbb{C}^{\tau_-}, \mathbf{0}) & \xrightarrow{\text{inclusion}} & (\mathbb{C}^\tau, \mathbf{0}) =_{\text{def}} (S, \mathbf{0}). \end{array}$$

(*By writing* D_Γ *for* $\mathbb{C}^{\tau_-} \times \{\mathbf{0}\}$, *this is the assertion 2.1*).
Moreover, there exists a section Σ_Γ *of* G_Γ *such that* $G_\Gamma\big|_{X_\Gamma - \Sigma_\Gamma(D_\Gamma)}$ *is nonsingular* (*i.e.* Σ_Γ *picks out the unique singular point of each fiber, which must be a singular point with semigroup* Γ).

2.11. PROOF. We first show that any flat deformation $H : (Z, C^\Gamma) \to (Y, \mathbf{0})$ of C^Γ with reduced base, each of whose fibers (for a sufficiently small representative of H) is irreducible at a singular point with semigroup Γ, satisfies the property that in the following commutative diagram, whose existence is due to the versality of G,

$$\begin{array}{ccc} (Z, C^\Gamma) & \longrightarrow & (X, C^\Gamma) \\ {\scriptstyle H}\downarrow & & {\scriptstyle G}\downarrow \\ (Y, \mathbf{0}) & \xrightarrow{\varphi} & (S, \mathbf{0}), \end{array}$$

one has $\varphi(Y) \subset D_\Gamma$.

To do this, we first note that since $(D_\Gamma, \mathbf{0}) = (\mathbb{C}^{\tau_-} \times \{\mathbf{0}\}, \mathbf{0}) \subset (\mathbb{C}^{\tau_-} \times \mathbb{C}^{\tau_+}, \mathbf{0}) = (S, \mathbf{0})$ is a (germ of) a closed analytic subspace, it suffices to prove this assertion when $Y = \mathbb{D} = \{v \in \mathbb{C} : |v| < 1\}$. Indeed, the deformations obtained from H by base changes $(\mathbb{D}, \mathbf{0}) \to (Y, \mathbf{0})$ have constant semigroup since H does. Thus, since D_Γ is closed in S, to show that $\varphi(Y) \subset D_\Gamma$, it suffices to show that any composed morphism $(\mathbb{D}, \mathbf{0}) \to (Y, \mathbf{0}) \to (S, \mathbf{0})$ has its image in D_Γ. This follows because if $\varphi^{-1}(D_\Gamma) \neq Y$ then one can find a path $\kappa : (\mathbb{D}, \mathbf{0}) \to (Y, \mathbf{0})$ such that $\kappa(\mathbb{D} - \{\mathbf{0}\}) \subset Y - \varphi^{-1}(D_\Gamma)$ (for sufficiently small representatives of G, H, κ...).

Now, if $\mathcal{O}$ is the analytic algebra of a branch $(C, \mathbf{0})$ and $\overline{\mathcal{O}}(\simeq \mathbb{C}\{t\})$ its normalisation (1.1), one can define an integer $\delta(C, \mathbf{0}) = \delta(\mathcal{O}) = dim_{\mathbb{C}}\, \overline{\mathcal{O}}/\mathcal{O}$ that is an analytic invariant of $(C, \mathbf{0})$.

2.11.1. LEMMA. $\delta(C, \mathbf{0}) = \delta(\Gamma)$ $(= \#(\mathbb{Z}_+ - \Gamma))$, *where* Γ *is the semigroup of* $(C, \mathbf{0})$.

PROOF. Let $i_1, \ldots, i_\delta$ denote the elements of $\mathbb{Z}_+ - \Gamma$. Identifying $\overline{\mathcal{O}}$ with $\mathbb{C}\{t\}$, one first notes that $t^{i_1}, \ldots, t^{i_\delta}$ are linearly independent modulo $\mathcal{O}$ because for any choice of $\alpha_k \in \mathbb{C}$, $\nu\left(\sum_{k=1}^{\delta} \alpha_k t^{i_k}\right) = i_{k_0}$ where $k_0 = \min\{k : \alpha_k \neq 0\}$. Thus, since $i_{k_0} \notin \Gamma$, $\sum_{k=1}^{\delta} \alpha_k t^{i_k} \in \mathcal{O} \implies \alpha_k = 0,\ 1 \leq k \leq \delta$.

Furthermore, if $\varphi(t) \in \mathbb{C}\{t\} - \mathcal{O}$ is such that $\nu(\varphi) \in \Gamma$, one can find an element $\eta_0 \in \mathcal{O}$ such that $\nu(\varphi - \eta_0) > \nu(\varphi)$; if $\nu(\varphi - \eta_0) \in \Gamma$ one can then find $\eta_1 \in \mathcal{O}$ such that $\nu(\varphi - \eta_0 - \eta_1) > \nu(\varphi - \eta_0) > \nu(\varphi)$. As a result, since $\mathbb{Z}_+ - \Gamma$ is finite, after a finite number of steps, one has either constructed $\zeta_0 = \sum \eta_i \in \mathcal{O}$ such that $\nu(\varphi - \zeta) \notin \Gamma$, or one has exceeded the conductor c of Γ and $\varphi - \zeta_0 \in \mathcal{O}$, from which it follows that $\varphi \in \mathcal{O}$, in contradiction to our hypothesis.

Therefore, $\nu(\varphi - \zeta_0) = i_k \notin \Gamma$. One can then find $\alpha_k \in \mathbb{C}^*$ such that $\nu(\varphi - \zeta_0 - \alpha_k t^{i_k}) > \nu(\varphi - \zeta_0)$. As above, one can find $\zeta_1 \in \mathcal{O}$ such that $\nu(\varphi - \zeta_0 - \zeta_1 - \alpha_k t^{i_k}) \notin \Gamma$, and since the valuation strictly increases at each step, one inductively constructs in this way a finite sequence of elements $\zeta_i \in \mathcal{O}$, and $\alpha_k \in \mathbb{C}$ such that

$$\nu\Big(\varphi - \sum \zeta_i - \sum_{k=1}^{\delta} \alpha_k t^{i_k}\Big) > c, \qquad c = \text{conductor of } \Gamma \ (1.1)\,.$$

Thus,

$$\varphi = \sum_{k=1}^{\delta} \alpha_k t^{i_k} \mod \mathcal{O},$$

which is what we needed to show in order to prove that the images of the t^{i_k}, $1 \leq k \leq \delta$, generate $\overline{\mathcal{O}}/\mathcal{O}$. Since we also saw at the beginning of the argument that they were $\mathbb{C}$ independent, this completes the proof of 2.11.1. □

2.11.2. Application. *Any deformation $H : (Z, C^\Gamma) \to (Y, \mathbf{0})$ with constant semigroup is also "δ constant" in the sense that each fiber Z_y contains a singular point $z(y)$ such that $\delta(Z_y, z(y)) = \delta(\Gamma) = \delta(C^\Gamma, \mathbf{0})$.*

I can now cite the following result.

2.11.3. Theorem ([T-3], théorème 1′). *Among all the deformations with base $\mathbb{D}$ of a reduced affine curve $C \subset \mathbb{C}^N$ (resp. of a germ of reduced analytic curve $(C, \mathbf{0}) \subset (\mathbb{C}^N, \mathbf{0})$), those that are obtained by a deformation of the parametrisation, i.e. from a morphism $\overline{C} \to \mathbb{C}^N$ whose image is C (where $\overline{C}$ is the (necessarily nonsingular) normalisation of C), are characterized by the condition that the sum of δ invariants of each fiber is constant.*

2.11.4. In the case that interests me here, that of germs of irreducible curves, and more precisely, of C^Γ, this implies the following. Let $H : (Z, C^\Gamma) \to (\mathbb{D}, \mathbf{0})$ be a deformation of C^Γ such that for any sufficiently small representative, $y \to \sum_i \delta(Z_y, z_i(y))$ is constant ($= \delta(C^\Gamma, \mathbf{0}) = \delta(\Gamma)$), where $z_i(y)$ are the singular points of Z_y. Then H is obtained by a deformation of the parametrisation $u_i = t^{\overline{\beta}_i}$ for $C^\Gamma \subset \mathbb{C}^{g+1}$. This means that H is isomorphic to the restriction of the projection $\mathbb{C}^{g+1} \times \mathbb{D} \to \mathbb{D}$ to some $Z' \subset \mathbb{C}^{g+1} \times \mathbb{D}$, where Z' is the image of a morphism $\mathbb{C} \times \mathbb{D} \to \mathbb{C}^{g+1} \times \mathbb{D}$ (a "deformation of a parametrisation") defined as follows:

$$(\text{H.A})' \quad \begin{cases} u_i = u_i(t, v) = t^{\overline{\beta}_i} + \sum_j a_{ij}(v) \cdot t^j & 0 \leq i \leq g,\ a_{ij}(v) \in (v) \cdot \mathbb{C}\{v\} \\ v = v\,. \end{cases}$$

Moreover, by ([T-3], §2), the sum of the δ invariants of the fibers is an upper semi-continuous function of v (in the analytic sense). Therefore, if each fiber Z_v possesses a singular point with semigroup Γ, it must then have the same δ invariant as the special fiber $(C^\Gamma, \mathbf{0})$, and is, necessarily, the unique singular point of Z_v. Moreover, in this case (loc.cit.), the critical locus of a sufficiently small representative of $H : (Z, C^\Gamma) \to (\mathbb{D}, 0)$ is analytically isomorphic to $\mathbb{D}$ in a neighborhood of 0.

In other words, the set of points of Z that are singular in their fiber is the image of a section $\sigma;\mathbb{D}\to Z$ of H. One can then assume, by a coordinate change of the variables u_i depending upon v, that the unique singular point of the fiber curve Z_v is the origin $u_i=0$ $(0\le i\le g)$ of $\mathbb{C}^{g+1}(=\mathbb{C}^{g+1}\times\{v\})$.

As a result, one may assume in the parametrisation (H.A)′ of H that each $a_{i0}=0$ $(0\le i\le g)$.

Even better, *after an additional suitable coordinate change, we may assume, for each* $i, 0\le i\le g$, *that* $a_{ij}(v)=0$ *for each* $j\le\overline{\beta}_i$. This is justified by using the following three observations:

i) we may assume that the deformation leaves unchanged the semigroup, i.e. each curve parametrized by $u_i=u_i(t,v_0)$, $|v_0|<\varepsilon$, has Γ as its semigroup;

ii) the valuation in t of $u_i(t,v_0)$ is at most $\overline{\beta}_i$ for $|v_0|$ sufficiently small;

iii) the $\overline{\beta}_i$ form a minimal generating set of Γ.

From these facts, it follows that $\nu(u_i(t,v_0))=\overline{\beta}_i,$ $(0\le i\le g)$ can be achieved by a coordinate change of the u_i that depends upon v. As a result, $a_{ij}(v)=0$ for $j<\overline{\beta}_i$ for each i. Finally, by means of a second coordinate change of the u_i of the form $u_i=u_i'+a_{i\overline{\beta}_i}(v)\cdot t^{\overline{\beta}_i}$, one may assume $a_{i\overline{\beta}_i}(v)=0$. Thus, we can write

$$\text{(H.A)}\quad\begin{cases} u_i=u_i(t,v)=t^{\overline{\beta}_i}+\sum_{j>\overline{\beta}_i} a_{ij}(v)\cdot t^j \qquad 0\le i\le g,\ a_{ij}(v)\in(v)\cdot\mathbb{C}\{v\}\\ v=v\,. \end{cases}$$

2.11.5. We next recall the fact that our deformation H comes from a miniversal deformation G via a base change. By the definition of G, this means that one can describe H as the restriction of the projection $\mathbb{C}^{g+1}\times\mathbb{D}\to\mathbb{D}$ to a subspace X^h defined by the ideal $(F_1^h,\dots,F_g^h)\subset\mathbb{C}[v][u_0,\dots,u_g]$ where

$$\text{(H.B)}\qquad F_i^h=f_i(u_0,\dots,u_i)+\sum_{j=1}^{\tau} w_j(v)\cdot s_{ij}(u_0,\dots,u_g)\qquad(1\le i\le g)$$

and the $w_j(v)\in\mathbb{C}\{v\}$ $(1\le j\le\tau)$ specify the base change map $h:(\mathbb{D},0)\to(\mathbb{C}^{\tau},\mathbf{0})$. Since H is a deformation with constant semigroup, what we then want to prove is that $w_j(v)=0$ if $j\in P_+$ (2.9).

2.11.6. The parametrisations (H.A) and (H.B) provide two ways of describing the algebra of Z (in the following, we will do this for the deformation of the branch $(C^{\Gamma},\mathbf{0})$ and not the "affine algebra $\mathbb{C}\{v\}[Z]$ above D"):

$$\text{(A)}\qquad \mathcal{O}_{Z,\mathbf{0}}\cong\mathbb{C}\{v\}\{u_0(t,v),\dots,u_g(t,v)\}\subset\mathbb{C}\{t,v\},$$

where the u_i are those from (H.A), or

$$\text{(B)}\qquad \mathcal{O}_{Z,\mathbf{0}}\cong\mathbb{C}\{v,u_0,\dots,u_g\}/(F_1^h,\dots,F_g^h),$$

where the F_i^h are those from (H.B).

The description in (A) gives us a filtration $\mathcal{F}$ of $\mathcal{O}_{Z,\mathbf{0}}$, that is, the restriction to $\mathcal{O}_{Z,\mathbf{0}}$ from the $(t)-$adic filtration of $\mathbb{C}\{t,v\}$, thus : $\mathcal{F}_i=(t^i)\cdot\mathbb{C}\{t,v\}\bigcap\mathcal{O}_{Z,\mathbf{0}}$.

The description in (B) gives us a filtration $\mathcal{G}$ of $\mathcal{O}_{Z,\mathbf{0}}$ induced from the canonical surjection $\mathbb{C}\{v,u_0,\dots,u_g\}\to\mathcal{O}_{Z,\mathbf{0}}$ and the filtration $\tilde{\mathcal{G}}$ of $\mathbb{C}\{v,u_0,\dots,u_g\}$ where $\tilde{\mathcal{G}}_i$ is the ideal generated by the set of monomials $\{u_0^{\alpha_0}\cdots u_g^{\alpha_g}:\sum_0^g\alpha_k\overline{\beta}_k\ge i\}$. Thus, an element $\xi\in\mathcal{O}_{Z,\mathbf{0}}$ belongs to $\mathcal{G}_i$ if and only if there exists $\tilde{\xi}\in\tilde{\mathcal{G}}_i$ whose image is ξ.

One can now make three observations.

2.11.7. $\mathcal{G}_i \subset \mathcal{F}_i$ for each i (because $u_i(t,v) \in t^{\overline{\beta}_i} \cdot \mathbb{C}\{t,v\}$). *Thus, we have a graded morphism of graded algebras:*

$$g : gr_{\mathcal{G}}\, \mathcal{O}_{Z,\mathbf{0}} \longrightarrow gr_{\mathcal{F}}\, \mathcal{O}_{Z,\mathbf{0}}.$$

2.11.8. $gr_{\mathcal{F}}\, \mathcal{O}_{Z,\mathbf{0}} = \mathbb{C}\{v\}[t^{\overline{\beta}_0}, \dots, t^{\overline{\beta}_g}]$ (by the same proof as in 1.2.3, the point being that the $t-$adic valuation of the $u_i(t,v)$ is the same for $v=0$ and $v \neq 0$).

2.11.9. Lemma. *g is a graded ($\mathbb{C}\{v\}-$) isomorphism.*

Proof. We first show the following.

2.11.10. Lemma. *Let $\mathcal{O}$ be a ring and $\mathcal{F}, \mathcal{G}$ any two filtrations of $\mathcal{O}$. We assume that for each i, $\mathcal{G}_i \subset \mathcal{F}_i$ and let $g : gr_{\mathcal{G}}\, \mathcal{O} \to gr_{\mathcal{F}}\, \mathcal{O}$ the corresponding (graded) homomorphism.*

(α) *If g is injective, then g is an isomorphism.*

(β) *If the completions of $\mathcal{O}$ with respect to $\mathcal{F}$ and $\mathcal{G}$ are the same, and if $\bigcap_i \mathcal{F}_i = (0)$, (thus $\bigcap_i \mathcal{G}_i = (0)$), then g surjective $\Longrightarrow$ g is an isomorphism.*

Proof of (α). Let $\overline{\xi} \neq 0 \in \mathcal{F}_i/\mathcal{F}_{i+1}$. We need to find $\hat{\xi} \in \mathcal{G}_i/\mathcal{G}_{i+1}$ such that $g(\hat{\xi}) = \overline{\xi}$. To do this, it suffices to choose any $\xi \in \mathcal{F}_i - \mathcal{F}_{i+1}$ whose $\mathcal{F}$ initial form is $\overline{\xi}$. Now, if $\xi \notin \mathcal{G}_i$, then $g(in_{\mathcal{G}}\xi) = 0$, contradicting the injectivity of g. Thus, $\xi \in \mathcal{G}_i$ and g must map $\hat{\xi} = in_{\mathcal{G}}\xi$ to $\overline{\xi}$, as was needed. □

Proof of (β). By hypothesis, one can assume that $\mathcal{O}$ is separated and complete for the filtration $\mathcal{F}$ since passage to the completion does not change the associated graded algebra (cf. [Bourbaki, ref. of 1.1.1]). We must show that g is injective. To this end, let $\hat{\xi} \in \mathcal{G}_i/\mathcal{G}_{i+1}$ satisfy $g(\hat{\xi}) = 0$. Since there exists $\xi \in \mathcal{G}_i$ such that $in_{\mathcal{G}}\, \xi = \hat{\xi}$, it must be the case that $\xi \in \mathcal{F}_{i+1}$. Since g is, by assumption, surjective, there exists $\eta_1 \in \mathcal{G}_{i+1}$ such that $\xi - \eta_1 \in \mathcal{F}_{i+2}$, and also $\eta_2 \in \mathcal{G}_{i+2}$ such that $\xi - \eta_1 - \eta_2 \in \mathcal{F}_{i+3}$, etc. ... By induction, one constucts a sequence of elements $\eta_k \in \mathcal{G}_{i+k} \subset \mathcal{F}_{i+k}$ such that for each ℓ, $\xi - \sum_1^{\ell} \eta_k \in \mathcal{F}_{i+\ell+1}$. Since $\mathcal{O}$ is complete for $\mathcal{F}$, we have $\sum_1^{\infty} \eta_k \in \mathcal{O}$, and since $\mathcal{O}$ is separated, $\xi = \sum_1^{\infty} \eta_k \in \mathcal{G}_{i+1}$, which verifies that $\hat{\xi} = 0$, completing the proof of (β). □

end of proof of 2.11.9. We will first show that g is surjective, i.e. any element $\xi \in \mathcal{O}_{Z,\mathbf{0}}$ with $(t)-$adic valuation i is the image of an element $a(v)u_0^{\alpha_0} \cdots u_g^{\alpha_g}$ with $\sum_0^g \alpha_k \overline{\beta}_k = i$. Let $\xi = a(v)t^i = b(t,v)t^{i+1} \in \mathcal{O}_{Z,\mathbf{0}}$ be such an element. Since i must belong to Γ, there certainly exist $\alpha_k \in \mathbb{Z}_+$ such that $i = \sum_{k=0}^g \alpha_k \overline{\beta}_k$. The form of the $u_i(t,v)$ (2.11.4) then allows us to write

$$\xi = a(v)u_0(t,v)^{\alpha_0} \cdots u_g(t,v)^{\alpha_g} \mod \mathcal{F}_{i+1},$$

which suffices to show the surjectivity of g.

We note as well that this argument shows $\mathcal{F}_i \subset \mathcal{M}^{[i/\overline{\beta}_g]}$, where $\mathcal{M} = (u_0, \dots, u_g)\mathcal{O}_{Z,\mathbf{0}}$. Moreover, since any element of $\mathcal{M}^{[i/\overline{\beta}_0]+1}$ is certainly in $\mathcal{G}_i$ we have:

$$\mathcal{M}^{[i/\overline{\beta}_0]+1} \subset \mathcal{G}_i \subset \mathcal{F}_i \subset \mathcal{M}^{[i/\overline{\beta}_g]},$$

which shows that $\mathcal{F}$ and $\mathcal{G}$ define the same topology as the $\mathcal{M}-$adic filtration of $\mathcal{O}_{Z,\mathbf{0}}$.

2.11.10 then shows that g is an isomorphism. $\square$

2.11.11. In the notations from 2.11.6, we therefore have a graded isomorphism:

$$g : \mathbb{C}\{v\}[u_0, \ldots, u_g]/in\,(F_1^h, \ldots, F_g^h) \xrightarrow{\sim} \mathbb{C}\{v\}[t^{\overline{\beta}_0}, \ldots, t^{\overline{\beta}_g}],$$

where $\mathbb{C}\{v\}[u_0, \ldots, u_g] = gr_{\tilde{\mathcal{G}}}\,\mathbb{C}\{v, u_0, \ldots, u_g\}$, and $in\,(F_1^h, \ldots, F_g^h)$ is the ideal generated by the $\tilde{\mathcal{G}}$ initial forms of elements of the ideal $(F_1^h, \ldots, F_g^h)\mathbb{C}\{v, u_0, \ldots, u_g\}$. By looking at the fibers over 0 and using the fact that the $f_i(u_0, \ldots, u_g)$ form a regular sequence, one concludes without difficulty, as in 2.3, that the $in\,F_i^h$ $(1 \leq i \leq g)$ generate $in\,(F_1^h, \ldots, F_g^h)$. Since the quotient must equal $\mathbb{C}\{v\}[t^{\overline{\beta}_0}, \ldots, t^{\overline{\beta}_g}]$, one sees that no term of degree $< n_i \cdot \overline{\beta}_i$ can appear in F_i^h. By 2.8, one can then conclude that $in\,F_i^h = f_i(u_0, \ldots, u_g)$, which precisely means, by the definition of the filtration $\tilde{\mathcal{G}}$, that $w_j(v) = 0$ if $j \in P_+$.

2.11.12. To finish the proof of Theorem 3 (2.10), we must verify that the deformation G_Γ of 2.10 is a constant semigroup deformation. Recall that G_Γ can be defined as the restriction to $X^- \subset \mathbb{C}^{g+1} \times \mathbb{C}^{\tau_-}$ of the projection $\mathbb{C}^{g+1} \times \mathbb{C}^{\tau_-} \to \mathbb{C}^{\tau_-}$, where X^- is defined by the ideal generated by the elements

$F_i^- = f_i(u_0, \ldots, u_i) + \sum_{j \in P_-} w_j \cdot s_{ij}(u_0, \ldots u_g),\ 1 \leq i \leq g,$

of the ring $\mathbb{C}\{u_0, \ldots, u_g\}[\{w_j\}_{j \in P_-}]$.

Defining the filtration $\tilde{\mathcal{G}}$ on $\mathbb{C}\{u_0, \ldots, u_g\}[\{w_j\}_{j \in P_-}]$ in the same way as before so that $\tilde{\mathcal{G}}_i$ is the ideal generated by the $\{u_0^{\alpha_0} \cdots u_g^{\alpha_g} : \sum_{k=0}^{g} \alpha_k \overline{\beta}_k \geq i\}$, one sees that the definition of P_- and (2.8) imply:

$$in_{\tilde{\mathcal{G}}}\,F_i^- = f_i(u_0, \ldots, u_g)$$

and therefore, for every point $\mathbf{w} = (w_j) \in \mathbb{C}^{\tau_-}$, the quotient filtration $\mathcal{G}_{\overline{w}}$ of the filtration $\tilde{\mathcal{G}}$ in the algebra $\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}}$ of the fiber $G_\Gamma^{-1}(\mathbf{w})$ has, as its associated graded algebra (in which one always uses the fact that $f_1, \ldots, f_g$ is a regular sequence):

$$gr_{\mathcal{G}_{\mathbf{w}}}\,\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}} \cong \mathbb{C}[u_0, \ldots, u_g]/(f_1, \ldots, f_g) = \mathbb{C}[C^\Gamma].$$

As a result, the associated graded algebra of $\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}}$ with respect to the filtration $\mathcal{G}_{\mathbf{w}}$ is an integral domain. This implies that $\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}}$ is an integral domain and, in particular, that the function $\xi \to \nu_{\mathcal{G}_{\mathbf{w}}}(\xi)$ (cf. (1.1.1)) is a (discrete) valuation of $\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}}$. The valuation ring $V \subset Tot\,(\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}})$ (the fraction field of $\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}}$) can therefore only be the normalisation of $\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}}$, and thus, $\mathcal{G}_{\mathbf{w}}$ must coincide with the filtration by the $\overline{\mathcal{M}}^i$ of 1.1. It follows that $\overline{gr}_{\mathcal{M}}\,\mathcal{O}_{X_{\mathbf{w}},\mathbf{0}} = \mathbb{C}[C^\Gamma]$, which shows that $(X_{\mathbf{w}}, \mathbf{0})$ has Γ for its semigroup.

Finally, it is clear that the section $\tilde{\Sigma}$ of $\mathbb{C}^{g+1} \times \mathbb{C}^{\tau_-} \to \mathbb{C}^{\tau_-}$, given by $\tilde{\Sigma}(\mathbf{w}) = (\mathbf{0}, \mathbf{w})$, has its image inside X^- since if $j \in P_-$, then $s_{ij}(u_0, \ldots, u_g)$ of degree $> n_i\overline{\beta}_i$ cannot be constant; $\tilde{\Sigma}$ therefore induces a section $\Sigma : \mathbb{C}^{\tau_-} \to X^-$ of G_Γ, which, by 2.11.12, has the desired property : $\Sigma(\mathbf{w})$ is the unique singular point of $X_{\mathbf{w}}$ and has Γ as its semigroup. This completes the proof of 2.10. $\square$

3. First applications

3.1 Miniversal equisingular deformations.

3.1.1. COROLLARY 1 (of 2.10). *Any (plane or not) branch* $(C,\mathbf{0})$, *whose semigroup* Γ *is such that* C^Γ *is a complete intersection, possesses a miniversal constant semigroup deformation with nonsingular base.*

In particular,

3.1.2. COROLLARY 2 (WAHL [WA], also see [T-1] exp. II, [T-2] ch. III). *Any plane branch possesses a miniversal equisingular deformation* $G_E : X_E \overset{\Sigma_E}{\longrightarrow} S_E$ *whose base* S_E *is nonsingular, and which has a section* Σ_E *that picks out the unique singular point of each fiber.*

PROOFS OF 3.1.1 AND 3.1.2. It suffices to apply the openness of versality in the form of the product decomposition theorem (Addendum 2.1) in order to verify that the base of the miniversal deformation of a branch $(C,\mathbf{0})$ with constant semigroup Γ is the product of a nonsingular space of dimension $2\delta(\Gamma)-\tau(C,\mathbf{0})$ (where $\tau(C,\mathbf{0})$ is the dimension of the base of the miniversal deformation of $(C,\mathbf{0})$, and, by 2.7, $2\delta(\Gamma)=\tau(C^\Gamma,\mathbf{0})$) with the germ of $\mathbb{C}^{\tau-}$ at points $\mathbf{w}$ of $\mathbb{C}^{\tau-}$ such that $(G_\Gamma^{-1}(\mathbf{w}),\Sigma(\mathbf{w}))\cong(C,\mathbf{0})$. Such points $\mathbf{w}$ exist by Theorem 1. This shows 3.1.1; and 3.1.2 follows immediately from this and (II, §3), since two plane branches are equisingular if and only if they have the same semigroup Γ. □

3.1.3. Remark: The description of $G_\Gamma : X_\Gamma \to D_\Gamma$, given in 2.11.12, only exhibits linear terms in the w_j $(j\in P)$. This is in contrast to Theorem 8.2 of [Wa], which displays a certain intrinsic nonlinearity in the product decomposition theorem, relative to the point of view adopted here. □

3.1.4. Remark: Let $D\subset\mathbb{C}^\tau$ be the discriminant (cf. Addendum) of the miniversal deformation $X\to\mathbb{C}^\tau$ of C^Γ, where Γ is the semigroup of a plane branch. We first note that $\mathbb{C}^{\tau-}=D_\Gamma\subset D$. In fact, we can be more precise as follows. At a point $\mathbf{w}$ of $\mathbb{C}^{\tau-}$ such that $(X_\mathbf{w},\Sigma(\mathbf{w}))$ is a plane branch, the multiplicity of D (as a hypersurface) is the Milnor number of $(X_\mathbf{w},\Sigma(\mathbf{w}))$, which equals $2\delta(\Gamma)$ ([Mi, [Ri]). Like the embedding dimension of a fiber, the multiplicity is also an upper semicontinuous function. Since D_Γ is nonsingular, the fact (cf. [T-3], Th. 2) that constancy of the Milnor number in a deformation of plane branches insures their equisingularity then implies:

$$D_\Gamma=\overline{D_\mu}\,,$$

where $\mu=2\delta(\Gamma)$ is the Milnor number of $(X_\mathbf{w},\Sigma(\mathbf{w}))$ (cf. II, No. 2 and [T-2]), and $\overline{D_\mu}$ denotes the set of points $s\in D$ such that the multiplicity of D at s is $\geq\mu$ (compare with [T-2] chap. III). □

3.2 Characterization of semigroups of plane branches.

3.2.1. PROPOSITION. *Let* $\Gamma=<\overline{\beta}_0,\ldots,\overline{\beta}_g>$ *be a semigroup such that* $\mathbb{Z}_+-\Gamma$ *is finite and satisfying (cf. 2.2.2):*

(1) $n_i\cdot\overline{\beta}_i\in<\overline{\beta}_0,\ldots,\overline{\beta}_{i-1}>\quad(1\leq i\leq g)$

$(2)_\ell$ ℓ *of the indices* $i, 1\leq i\leq g-1$, *are such that* $n_i\overline{\beta}_i<\overline{\beta}_{i+1}$.

Then there exists a branch $(C,\mathbf{0})\subset(\mathbb{C}^{g+1-\ell},\mathbf{0})$ *such that* Γ *is the semigroup of* $(C,\mathbf{0})$. *In particular, if all the indices* $i(1\leq i\leq g)$ *satisfy* $n_i\overline{\beta}_i<\overline{\beta}_{i+1}$, *then* Γ *is the semigroup of a plane branch (the converse was proved in 2.2.2).*

(The last result[4] can also be easily verified by using (*) of 2.2.1 to define the β_q inductively ($\beta_0 = \overline{\beta_0}$) and by verifying that the branch

$$x = t^{\beta_0}, \qquad y = t^{\beta_1} + t^{\beta_2} + \cdots + t^{\beta_g}$$

has Γ as its semigroup. However, I prefer the proof given here.)

PROOF. One first verifies without difficulty that the vectors $(u_2, 0, \ldots, 0)$, $(0, u_3, 0, \ldots, 0), \ldots, (0, 0, \ldots, u_g, 0)$ are $\mathbb{C}-$independent modulo $\mathcal{N}$ in $\mathbb{C}[C^\Gamma]^g$ (cf. 2.5.2). Let $i_1, \ldots, i_\ell$ be the ℓ indices referred to in (2). Condition (1) implies that C^Γ is a complete intersection (2.2.3), and therefore that we can apply Theorem 3 (2.10). This shows us that the deformation, obtained by restricting the projection $\mathbb{C}^{g+1} \times \mathbb{C}^\ell \to \mathbb{C}^\ell$ to X', defined by the ideal $(F'_1, \ldots, F'_g)$, where

$$\begin{cases} F'_i = f_i(u_0, \ldots, u_i) & \text{if } i \in \{1, \ldots, g\} - \{i_1, \ldots, i_\ell\} \\ F'_i = f_i + \lambda_i u_{i+1} & \text{if } i \in \{i_1, \ldots, i_\ell\}\,, \end{cases}$$

(the λ_i are among the w_j of 2.5.2)
is a constant semigroup deformation since if $i \in \{i_1, \ldots, i_\ell\}$, then

$$deg\, f_i = n_i \overline{\beta}_i < \overline{\beta}_{i+1} = deg\, u_{i+1}.$$

Now, if $\boldsymbol{\lambda} \in \mathbb{C}^\ell$ is such that each $\lambda_i \neq 0$, $(i \in \{i_1, \ldots, i_\ell\})$, the fiber $(X'_{\boldsymbol{\lambda}}, \mathbf{0})$ is contained in the (transversal) intersection of ℓ nonsingular hypersurfaces $F'_i = 0$ $(i \in \{i_1, \ldots, i_\ell\})$. Up to isomorphism, therefore, $(X'_{\boldsymbol{\lambda}}, \mathbf{0}) \subset (\mathbb{C}^{g+1-\ell}, \mathbf{0})$, and the semigroup of $(X'_{\boldsymbol{\lambda}}, \mathbf{0})$ must be Γ since it is independent of $\boldsymbol{\lambda}$. This proves 3.2.1. □

3.2.2. Remark: By eliminating the u_i, $2 \leq i \leq g$, the above method allows one to construct quite easily the equation of a plane branch with given semigroup (or equivalently, a given set of Puiseux exponents). □

3.2.3. Remark: There exist semigroups Γ such that C^Γ is a complete intersection but Γ cannot be the semigroup of a plane branch, i.e. satisfying (1) but not $(2)_{g-1}$ of 3.2.1. For example, let $\Gamma =< 9, 21, 22 >$. One has $e_0 = 9, e_1 = 3, e_2 = 1, n_1 = n_2 = 3$, and

$$n_1\overline{\beta}_1 = 63 = 7 \cdot 9 \in < 9 >$$
$$n_2\overline{\beta}_2 = 66 = 5 \cdot 9 + 21 \in < 9, 21 >\,,$$

which imply that C^Γ is the complete intersection (cf. 2.2.3):

$$C^\Gamma : \begin{cases} u_1^3 - u_0^7 = 0 \\ u_2^3 - u_0^5 u_1 = 0\,, \end{cases}$$

but $n_1\overline{\beta}_1 > \overline{\beta}_2 = 22$, so that Γ is not the semigroup of a plane branch (cf. 2.2.2).□

3.3 On the branches whose module of differentials has maximum torsion (a second look at Zariski's article [Z]).

In this part I apply the "jump" property of τ (Addendum 2.9) to prove a theorem that generalizes the main result of Zariski's article [Z].

[4] MM. J. Bertin et Carbonne (Toulouse) have recently informed me that the characterization of semigroups of plane branches was proved in an article of Brezinsky, Proc. A.M.S. No. 2 (1972) pg. 381.

3.3.1. THEOREM 4. *Let $\mathcal{O}$ be the algebra of a branch $(C, \mathbf{0})$ whose semigroup Γ is such that C^Γ is a complete intersection. Let T be the $\mathcal{O}-$torsion submodule of $\Omega^1_{\mathcal{O}}$, the module of differentials of $\mathcal{O}$. Then*
(A) $dim_{\mathbb{C}}\, T = \tau(C, \mathbf{0}) \leq 2\delta(\Gamma)$
(B) $dim_{\mathbb{C}}\, T = 2\delta(\Gamma)$ *if and only if* $(C, \mathbf{0})$ *is isomorphic to* $(C^\Gamma, \mathbf{0})$,
where $\tau(C, \mathbf{0})$ denotes the dimension of the base of the miniversal deformation of $(C, \mathbf{0})$.

PROOF. Since C^Γ is a complete intersection by assumption, so too is $(C, \mathbf{0})$ (proof of 2.3). In particular, $(C, \mathbf{0})$ is therefore Gorenstein. One can then apply the theorem of local duality to show

$$dim_{\mathbb{C}}\, T = dim_{\mathbb{C}}\, H^0_{\mathcal{M}}(\Omega^1_{\mathcal{O}}) = dim_{\mathbb{C}}\, Ext^1_{\mathcal{O}}\,(\Omega^1_{\mathcal{O}}, \mathcal{O}) = \tau(C, \mathbf{0})$$

(cf. Pinkham [Pi], ch. III, 10.4). Since $(C, \mathbf{0})$ is a deformation of $(C^\Gamma, \mathbf{0})$, it now follows from the product decomposition theorem (Addendum 2.1) that

$$\tau(C, \mathbf{0}) \leq \tau(C^\Gamma, \mathbf{0}) = 2\delta(\Gamma) \quad \text{(cf. 2.7).}$$

This proves (A).

To prove (B), we note that in the one parameter deformation provided by Theorem 1 (1.3), each fiber, except possibly the special fiber, is isomorphic to $(C, \mathbf{0})$. If $(C, \mathbf{0})$ is not isomorphic to $(C^\Gamma, \mathbf{0})$, one can then apply the "jump property" of τ (Addendum 2.9) to conclude that $\tau(C, \mathbf{0}) < \tau(C^\Gamma, \mathbf{0}) = 2\delta(\Gamma)$. (In place of Theorem 1, one could use the fact that in the miniversal deformation $G : (X, C^\Gamma) \to (S, \mathbf{0})$ of C^Γ, the analytic type of the fibers X_s remains constant when s varies in an orbit of the $\mathbb{C}^*$ action on S (2.5). Of course, $X_0 = C^\Gamma$.)

This completes the proof of 3.3.1. □

3.3.2. Remark: In the article of Zariski [Z], the integer $2\delta(\Gamma)$ is replaced by the conductor c of Γ. The two quantities are equal since $\mathcal{O}$ is Gorenstein. Zariski shows by very different methods that if $(C, \mathbf{0})$ is a plane branch such that $dim_{\mathbb{C}}\, T = c$, then $(C, \mathbf{0})$ is isomorphic to a branch defined by the equation $Y^n - X^m = 0$. Of course, any monomial curve in the plane must be of this type. □

Chapter II: Application to the study of the moduli space

1. An example: the moduli space associated to $\Gamma =< 4, 6, 2s + 7 >$.

1.1. I want to begin this chapter with an example showing how one can apply the results of the preceding chapter in order to recover a result of the Course (IV, §3): the moduli space associated to the characteristic $(4, 6, 2s + 1)$ consists of a single point whenever the integer s is at least 3.

We therefore fix an integer $s \geq 3$. The formula (II, 3.11) shows us that the semigroup of a plane branch with characteristic $(4, 6, 2s+1)$ is generated by $\overline{\beta}_0 = 4, \overline{\beta}_1 = 6, \overline{\beta}_2 = 2s + 7$, i.e. $\Gamma =< 4, 6, 2s + 7 >$. Thus, the curve C^Γ lies in $\mathbb{C}^3$ and is parametrized by:

$$C^\Gamma : \begin{cases} u_0 = t^4 \\ u_1 = t^6 \\ u_2 = t^{2s+7} \end{cases}$$

and the calculations done in Ch. I, 2.2.3 give us the equations[5]

$$C^\Gamma : \begin{cases} u_1^2 - u_0^3 = 0 \\ u_2^2 - u_0^{s+2} u_1 = 0\,. \end{cases}$$

We will now determine the miniversal deformation of C^Γ. To do this, we are helped by the fact that we know (ch. 1, 2.7)

$$\tau(C^\Gamma, \mathbf{0}) = dim_{\mathbb{C}}\, \mathbb{C}[C^\Gamma]^2/\mathcal{N} = 2\delta(\Gamma),$$

where $\mathcal{N}$ is the submodule of $\mathbb{C}[C^\Gamma]^2$ generated by the images of (cf. ch. 1, 2.5.2)

$$\boldsymbol{\gamma}_0 = \begin{pmatrix} -3u_0^2 \\ -(s+2)u_0^{s+1}\, u_1 \end{pmatrix}, \quad \boldsymbol{\gamma}_1 = \begin{pmatrix} 2u_1 \\ -u_0^{s+2} \end{pmatrix}, \quad \boldsymbol{\gamma}_2 = \begin{pmatrix} 0 \\ 2u_2 \end{pmatrix}.$$

Now, since C^Γ is a complete intersection, and in particular, Gorenstein, $2\delta(\Gamma)$ equals the conductor c of Γ (II, §1), which equals $2s + 10$ (because Γ contains all even integers ≥ 4, and thus, all odd integers $\geq 2s + 11 = \overline{\beta}_0 + \overline{\beta}_2$). We therefore need $2s + 10$ $\mathbb{C}-$linearly independent vectors in $\mathbb{C}[C^\Gamma]^2$ modulo $\mathcal{N}$, and it is not difficult to verify that the equivariant miniversal deformation of C^Γ is described by

[5] For $0 \leq s \leq 2$, C^Γ is a complete intersection, but Γ is not the semigroup of a plane branch. This gives us a simpler example than Ch. I, 3.2.3.

the following pair of functions, using the notations of ch. 1, 2.5.2:

$$\begin{aligned}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} u_1^2 - u_0^3 \\ u_2^2 - u_0^{s+2}\, u_1 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + w_3 \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + w_4 \begin{pmatrix} 0 \\ u_0 \end{pmatrix} + w_5 \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \\ + w_6 \begin{pmatrix} 0 \\ u_1 \end{pmatrix} + w_7 \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \\ + w_8 \begin{pmatrix} u_0^2 \\ 0 \end{pmatrix} + w_9 \begin{pmatrix} u_0 u_1 \\ 0 \end{pmatrix} + w_{10} \begin{pmatrix} u_0 u_2 \\ 0 \end{pmatrix} + w_{11} \begin{pmatrix} 0 \\ u_0^2 \end{pmatrix} \\ + w_{12} \begin{pmatrix} 0 \\ u_0 u_1 \end{pmatrix} + w_{13} \begin{pmatrix} 0 \\ u_1^2 \end{pmatrix} \\ + \sum_{j=4}^{s+1} w_{10+j} \begin{pmatrix} 0 \\ u_0^j \end{pmatrix} + \sum_{k=2}^{s} w_{10+s+k} \begin{pmatrix} 0 \\ u_0^k u_1 \end{pmatrix}.\end{aligned}$$

(I take the opportunity here to include a remark of M. Merle, who has indicated to me that the calculation of the miniversal deformation done in ([T-1], exp. II, Remark 5.6) is false. This however does not affect the validity of that Remark.)

The only vectors $\mathbf{s}_j$ appearing on the right side of the above equation whose coefficient w_j has negative degree, i.e. such that $j \in P_-$, are easily seen to be $\begin{pmatrix} u_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} u_0 u_2 \\ 0 \end{pmatrix}$. Indeed, $j \in P_-$ means that each component s_{ij} of $\mathbf{s}_j$ satisfies $deg\, s_{ij} > n_i \overline{\beta}_i = deg\, f_j$, where $deg\, u_0 = 4$, $deg\, u_1 = 6$, and $deg\, u_2 = 2s + 7$. Here, $deg\,(u_1^2 - u_0^3) = 12$, $deg\,(u_2^2 - u_0^{s+2}\, u_1) = 4s + 14$, and $deg\, u_0^s = 4s$, $deg\, u_0^s u_1 = 4s + 6$, which are both $< 4s + 14$, while $deg\, u_2 = 2s + 7 \geq 13 > 12$ when $s \geq 3$.

We therefore know, thanks to Theorem 3 (ch. 1, 2.10), how to write the miniversal constant semigroup deformation of C^Γ:

where $X_\Gamma \subset \mathbb{C}^3 \times \mathbb{C}^2$ (with coordinates $(u_0, u_1, u_2, w_7, w_{10})$) is defined by:

$$(*) \qquad \begin{cases} u_1^2 - u_0^3 + w_7 u_2 + w_{10} u_0 u_2 = 0 \\ u_2^2 - u_0^{s+2}\, u_1 = 0\,. \end{cases}$$

Since each fiber has exactly one singular point, $u_0 = u_1 = u_2 = 0$, those that are plane branches are exactly those for which $w_7 \neq 0$, and by Theorem 1, up to isomorphism, any plane branch with characteristic $(4, 6, 2s+1)$ appears in this way. A simple calculation shows that whenever $w_7 \neq 0$, one can then change variables as follows:

$$\begin{aligned} u_0' &= (w_7 + w_{10} u_0)^{\frac{4}{2s-5}} \cdot u_0 \\ u_1' &= (w_7 + w_{10} u_0)^{\frac{6}{2s-5}} \cdot u_1 \\ u_2' &= (w_7 + w_{10} u_0)^{\frac{2s+7}{2s-5}} \cdot u_2 \end{aligned}$$

so that (*) becomes:

$$(*') \qquad \begin{cases} (u_1')^2 - (u_0')^3 + u_2' = 0 \\ (u_2')^2 - (u_0')^{s+2}\, u_1' = 0\,. \end{cases}$$

In this way, we have shown that any plane branch which appears in the miniversal constant semigroup deformation of C^Γ, thus, any plane branch with semigroup Γ, is isomorphic to the branch defined by (*′), or more simply, by the equation (in $\mathbb{C}[u_0, u_1]$)

$$(u_1^2 - u_0)^3 - u_0^{s+2} u_1 = 0,$$

by eliminating u_2, as suggested by ch. 1, 3.2.2. Evidently, this shows that the moduli space is indeed reduced to a single point in this case.

But we can also observe here, in addition, that when $w_7 = 0$ and $w_{10} \neq 0$, that the following change of variables:

$$u_0' = w_{10}^{\frac{4}{2s-1}} \cdot u_0$$
$$u_1' = w_{10}^{\frac{6}{2s-1}} \cdot u_1$$
$$u_2' = w_{10}^{\frac{2s+7}{2s-1}} \cdot u_2$$

transforms (*) into the pair of equations:

$$\begin{cases} (u_1')^2 - (u_0')^3 + u_0' u_2' = 0 \\ (u_2')^2 - (u_0')^{s+2}\, u_1' = 0\,, \end{cases}$$

which shows that any two fibers of G_Γ such that $w_7 = 0, w_{10} \neq 0$ are isomorphic branches.

By Addendum 2.7, we have therefore proved (since C^Γ is not isomorphic to any other fiber of G_Γ) the following.

1.2. Proposition. *The quotient space of* $\mathbb{C}^2$ *(with coordinates* $\mathbf{w} = (w_7, w_{10})$*), defined by the equivalence relation*

$$\mathbf{w} \sim \mathbf{w}' \quad \textit{iff} \quad (G_\Gamma^{-1}(\mathbf{w}), \mathbf{0}) \cong (G_\Gamma^{-1}(\mathbf{w}'), \mathbf{0}),$$

and given the quotient topology, consists of three points P_I, P_{II}*, and* P_{III}*, corresponding respectively to* $w_7 \neq 0$*,* $w_7 = 0, w_{10} \neq 0$*, and* $w_7 = w_{10} = 0$ *(i.e.* C^Γ*). The closure of* $\{P_I\}$ *is* $\{P_I, P_{II}, P_{III}\}$*, that of* $\{P_{II}\}$ *is* $\{P_{II}, P_{III}\}$*, and* P_{III} *is the only closed point. Moreover,* P_I *corresponds to the plane branches.*

We will study other moduli spaces from this point of view in the following sections.

2. Compactification of the moduli space of plane branches

2.1. Let $\Gamma \subset \mathbb{Z}_+$ be the semigroup of a plane branch, and $D_\Gamma \simeq \mathbb{C}^{\tau_-}$ the base of a representative $G_\Gamma : X_\Gamma \overset{\Sigma_\Gamma}{\longrightarrow} D_\Gamma$ of the miniversal constant semigroup deformation of C^Γ (see ch. 1, 2.10), where Σ_Γ is the section that picks out the unique singular point of each fiber. One can define an equivalence relation on D_Γ as follows:

$\mathbf{w} \sim \mathbf{w}'$ $(\mathbf{w}, \mathbf{w}' \in D_\Gamma)$ *iff* the germs $(G_\Gamma^{-1}(\mathbf{w}), \Sigma(\mathbf{w}))$ and $(G_\Gamma^{-1}(\mathbf{w}'), \Sigma(\mathbf{w}'))$ are analytically isomorphic.

2.2. I will call the moduli space associated to the semigroup Γ the quotient space $D_\Gamma/\sim$ (with quotient topology). I will denote the space by $\tilde{M}_\Gamma$, or by $\tilde{M}$ for simplicity and set $m : D_\Gamma \to \tilde{M}_\Gamma$ to be the canonical mapping.

2.3. Theorem 5.
1. *$\tilde{M}_\Gamma$ is quasi-compact (i.e. a quotient of a compact space that is not necessarily separated) and connected.*
2. *The moduli space M_Γ of plane branches, corresponding to the equisingularity class with semigroup Γ, is an open, dense, and connected subset of $\tilde{M}_\Gamma$.*
3. *$M_\Gamma = \tilde{M}_\Gamma$ is and only if Γ is generated by two elements (i.e. $g = 1$).*

Proof. $\tilde{M}_\Gamma$ is a quotient of the quotient space of D_Γ determined by the natural action of $\mathbb{C}^*$ (cf. ch. 1, 2.5, 2.10), which is clearly compact, thanks to ch. 1, 2.8. An alternative argument one might prefer is to note that the mapping $m : D_\Gamma \to \tilde{M}_\Gamma$ assigns to each point $\mathbf{w} \in D_\Gamma$ the equivalence class modulo isomorphism of the fiber $(G_\Gamma^{-1}(\mathbf{w}), \Sigma(\mathbf{w}))$. By Theorem 1, m therefore satisfies the property that its restriction to any open neighborhood U of $\mathbf{0}$ in D_Γ is surjective. Since D_Γ is locally compact, $\tilde{M}_\Gamma$ must be quasi-compact.

The proof of (2) uses the semicontinuity of the embedding dimension of fibers of a morphism, which itself follows from two facts. The first is the upper semicontinuity of the dimension of fibers of a coherent sheaf of modules. The second is the existence of a coherent sheaf $\mathcal{P}^1_{X_\Gamma/D_\Gamma}$ on X_Γ (the sheaf of relative jets) such that

$$dim_{\mathbb{C}}\, \mathcal{P}^1_{X_\Gamma/D_\Gamma}(x) = im\ dim_x\, (G_\Gamma^{-1}(G_\Gamma(x)), x),$$

where $im\ dim_x$ denotes the embedding dimension at x[6], which equals the dimension of the Zariski tangent space $(\mathcal{M}/\mathcal{M}^2)^*$ at x. (For more details, see Seminar Cartan 60-61, fasc. 2, exposés of Grothendieck, and [L-T-1].)

In particular, $\overline{D}_\Gamma^{(3)} = \{\mathbf{w} \in D_\Gamma : im\ dim\, (G_\Gamma^{-1}(\mathbf{w}), \Sigma(\mathbf{w})) \geq 3\}$ is a closed analytic subset of D_Γ whose complement is nonempty. This is due to the fact that Γ is the semigroup of a plane branch. Thus Theorem 1 (ch. 1) insures the existence of $\mathbf{w} \in D_\Gamma$ such that $(G_\Gamma^{-1}(\mathbf{w}), \Sigma(\mathbf{w})) \subset (\mathbb{C}^2, \mathbf{0})$, i.e. $im\ dim\, G_\Gamma^{-1}(\mathbf{w}) = 2$. Moreover, each fiber of G_Γ is singular unless $\Gamma = \mathbb{Z}_+$, in which case $D_\Gamma = \{\mathbf{0}\}$, and therefore $im\ dim\, G_\Gamma^{-1}(\mathbf{w}) \geq 2\ \forall \mathbf{w} \in D_\Gamma$. As a result,

$$(2.3.1) \qquad D_\Gamma - D_\Gamma^{(3)} = D_\Gamma^{(2)} = \{\mathbf{w} \in D_\Gamma : (G_\Gamma^{-1}(\mathbf{w}), \Sigma_\Gamma(\mathbf{w})) \text{ is a } \textit{plane} \text{ branch}\}$$

is an open dense subset of D_Γ. I observe that the image set $m(D_\Gamma^{(2)}) \subset \tilde{M}_\Gamma$ is the moduli space of plane branches with semigroup Γ. Moreover, the fact that D_Γ is nonsingular implies $D_\Gamma^{(2)}$ is connected.

Thus, to complete the proof of (2), it suffices to verify that the quotient topology on $m(D_\Gamma^{(2)})$ agrees with the topology on the space M_Γ that comes from the Puiseux series expansions (III, 2.2). The first point is that it is certainly true that set theoretically

$$m(D_\Gamma^{(2)}) = M_\Gamma \subset \tilde{M}_\Gamma.$$

The fact that the topologies agree follows from the "continuity of zeroes of an ideal defining a flat deformation as a function of its coefficients", and the theorem cited in (ch. 1, 2.11) and used as in 2.11.4.

[6] minimal dimension of a smooth space into which the singular germ embeds.

2.3.2. Part (3) follows from these facts: i) the embedding dimension is an analytic invariant; ii) $D_\Gamma^{(2)}$ and its complement $D_\Gamma^{(3)}$ are unions of orbits of the $\mathbb{C}^*$ action on D_Γ; iii) $M_\Gamma = \tilde{M}_\Gamma$ if and only if $D_\Gamma^{(2)} = D_\Gamma$ (i.e. if and only if the embedding dimension of C^Γ equals 2). Now, to say that $\overline{\beta}_0, \ldots, \overline{\beta}_g$ form a minimal generating system for Γ (ch. 1, 1.1.2) is equivalent to saying that the embedding dimension of C^Γ, which is a priori at most $g+1$, in fact equals $g+1$ (arguing as in ch. 1, 1.10). Thus, $M_\Gamma = \tilde{M}_\Gamma \iff g = 1$, completing the proof of (3). □

2.4. Remark: $\tilde{M}_\Gamma$ is a "natural compactification" of the moduli space M_Γ of plane branches, but it is not a minimal compactification. For example, we have seen in the preceding section when $\Gamma =< 4, 6, 2s + 7 >$, that M_Γ is compact since $M_\Gamma = \{P_I\}$, but $M_\Gamma \neq \tilde{M}_\Gamma$. However, it was shown in the Course (Chap. IV) that this was the only example where M_Γ is compact and $g > 1$.

3. The generic component of the moduli space

3.1. In the preceding section, we have only used the semicontinuity of the embedding dimension of fibers of a morphism, applied to the miniversal constant semigroup deformation $G_\Gamma : X_\Gamma \overset{\Sigma_\Gamma}{\longrightarrow} D_\Gamma$ of C^Γ of a plane branch (or more generally, such that C^Γ is a complete intersection, in which case one should replace M_Γ by the moduli space of branches with semigroup Γ and of minimal embedding dimension).

In this section, we will use, in addition, the semicontinuity of the dimension $\tau(X_{\mathbf{w}}, \mathbf{0})$ ($\mathbf{0} = \Sigma_\Gamma(\mathbf{w})$) of the base of the miniversal deformation of the fibers $X_{\mathbf{w}}$. In fact, I will redo a part of the constructions given in Chapter VI §1 -3 of the Course by applying the ideas developed in the discussion above. The fact that I am not able to write explicitly a miniversal deformation of C^Γ ($g > 1$) forces me to use the language of exact sequences. In addition, since I was not able to preserve all the notations of (VI) for various reasons, I included a small summary of the differences (see 3.5) to help the reader compare the discussion below with that of the Course.

3.2. Let Γ be a semigroup of a plane branch, and let $G_\Gamma : X_\Gamma \overset{\Sigma_\Gamma}{\longrightarrow} D_\Gamma$ be the miniversal constant semigroup deformation of the monomial curve C^Γ. By the Addendum, §2, 2.5, the set of points $\mathbf{w} \in D_\Gamma$ such that $\tau(X_{\mathbf{w}}, \mathbf{0})$ assumes its minimal value is an open analytic subset V_0 of D_Γ. (In fact, this set is algebraic since it is also stable under the $\mathbb{C}^*$ action on D_Γ.) Furthermore, as we noted in the preceding section, the set $D_\Gamma^{(2)}$ (of $\mathbf{w} \in D_\Gamma$ such that $(X_{\mathbf{w}}, \mathbf{0})$ is a plane branch) is an open, dense, and analytic (in fact, it is, of course, algebraic) subset of D_Γ. Setting $V = V_0 \bigcap D_\Gamma^{(2)}$ (actually, it seems quite likely that $V_0 \subset D_\Gamma^{(2)}$), and $\tau_{min} = min\, \tau(X_{\mathbf{w}}, \mathbf{0})$, it follows that:

$$\mathbf{w} \in V \longleftrightarrow (X_{\mathbf{w}}, \mathbf{0}) \textit{is a plane branch and } \tau(X_{\mathbf{w}}, \mathbf{0}) = \tau_{min}.$$

3.3. By the product decomposition theorem (Addendum 2.1), the germ of D_Γ at any $\mathbf{w}$ is a product:

$$(D_\Gamma, \mathbf{w}) \simeq (\mathbb{C}^{\tau_0 - \tau_{\mathbf{w}}} \times D_{\mu,\mathbf{w}}, \mathbf{0})$$

where $\tau_{\mathbf{w}} = \tau(X_{\mathbf{w}}, \mathbf{0})$ and $D_{\mu,\mathbf{w}}$ is the base of the miniversal equisingular deformation of the branch $(X_{\mathbf{w}}, \mathbf{0})$ (equisingular = *either* constant semigroup *or* equisingular in Zariski's sense if $(X_{\mathbf{w}}, \mathbf{0})$ is a plane branch).

We will use the notation: $q(\mathbf{w}) = dim\, D_{\mu,\mathbf{w}}$ $(q(\mathbf{0}) = dim\, D_\Gamma = \tau_-$, see ch. 1 §2).

3.4. The isomorphism in 3.3 implies the following about the dimensions:

$$q(\mathbf{0}) = \tau_0 - \tau_{\mathbf{w}} + q(\mathbf{w}) \quad (\tau_0 = 2\delta(\Gamma),\ \text{ch. } 1,\ 2.7).$$

3.5. It is this equation that is cited in the Course (VI, 2.7) (with $g = 1$), where C^Γ (the curve $Y^{\overline{\beta}_0} - X^{\overline{\beta}_1} = 0$) is denoted C_0, D_Γ is identified as $Eqs\, Defr\, C_0$, $\tau = \tau_0$ is denoted N', $\tau_- = q(\mathbf{0})$ is denoted N, $\tau_{\mathbf{w}}$ is denoted q' when $\mathbf{w} \in V$, $q(\mathbf{w})$ is denoted q when $\mathbf{w} \in V$, $X_{\mathbf{w}}$ is denoted $C_{\mathbf{v}}$, and $D_{\mu,\mathbf{w}}$ is denoted $Eqs\, Defr\, C_{\mathbf{v}}$.

3.6. Since $\mathbf{w} \in V$ implies $\tau_{\mathbf{w}} = \tau_{min}$, we see that $q(\mathbf{w})$ is independent of $\mathbf{w} \in V$. Setting $q(\mathbf{w}) = q$ for $\mathbf{w} \in V$, it follows that 3.4 can be rewritten as follows:

$$q = q(\mathbf{0}) - (\tau_{\mathbf{0}} - \tau_{min}) = \tau_- - (2\delta(\Gamma) - \tau_{min}).$$

3.7. Definition. We will say that a branch $(X_{\mathbf{w}}, \mathbf{0})$ is general if $\mathbf{w} \in V = D_\Gamma^{(2)} \bigcap V_0$.

In the case $g = 1$, this definition coincides with that of (VI, §3). □

Theorem 6. *The dimension of the generic component M_1 of the moduli space $\tilde{M}_\Gamma$ of branches with semigroup Γ is:*

$$dim\, M_1 = q = q(\mathbf{w}) \quad (\mathbf{w} \in V),$$

and the intersection of the generic component with $M_\Gamma \subset \tilde{M}_\Gamma$ is Zariski open and dense.

Theorem 7. *The set of general points of M_Γ (corresponding to the set of general branches) is contained in the generic component of M_Γ and contains a Zariski open subset of M_Γ.*

I only want to summarize the procedure of the proof in the language of this Appendix (see VI, §3). With help from the theorem of Rosenlicht (VI, §1), one shows that the moduli mapping $m : D_\Gamma \to \tilde{M}_\Gamma$ induces a rational dominant mapping

$$\varphi^* : V \longrightarrow M_1$$

where M_1 is the generic component of $\tilde{M}_\Gamma$.

The product decomposition theorem, applied to the morphism G_Γ, now implies the following. In a neighborhood of a point $\mathbf{w} \in V$ where φ^* is defined and coincides with m, we have an induced mapping

$$m_{\mathbf{w}} : D_{\mu,\mathbf{w}} \longrightarrow M_1$$

where $D_{\mu,\mathbf{w}}$ is the base of the miniversal equisingular deformation of $(X_{\mathbf{w}}, \mathbf{0})$. By 3.3 and 3.4, since $\mathbf{w} \to \tau_{\mathbf{w}} = \tau_{min}$ is constant along $D_{\mu,\mathbf{w}}$, we are allowed to apply (Addendum 2.9.2) with $S^1 = S_{\tau_{\mathbf{w}}} \bigcap D_{\mu,\mathbf{w}}$ in order to conclude that $\mathbf{w}$ is an isolated point in its fiber $(\varphi^*)^{-1}\big(\varphi^*(\mathbf{w})\big)$. Thus, $\varphi^* : D_{\mu,\mathbf{w}} \to M_1$ is (locally) a finite morphism, from which Theorem 6 follows. Theorem 7 is verified similarly, using the same argument as in (VI, §3).

3.8. As a result, to calculate the dimension of the generic component of the moduli space, 3.6 tells us that it suffices to calculate τ_- and τ_{min} (given that $2\delta(\Gamma)$ is easy to compute).

In (VI, §2, calculation of N when $g = 1$), Zariski showed without too much difficulty that $\tau_- = \frac{(\overline{\beta}_0-3)(\overline{\beta}_1-3)}{2} + \frac{\overline{\beta}_1}{\overline{\beta}_0} - 1$. However, it turned out to be quite difficult to compute q, therefore τ_{min}, even when $\Gamma =< \overline{\beta}_0, \overline{\beta}_0 + 1 >$[7]

One can, however, perhaps view the computation in 3.1, which showed $q = 0$ when $\Gamma =< 4, 6, 2s+7 >$ (i.e. we were able to show: $\tau_- = 2$ and $\tau_0 - \tau_{min} = 2$), as a procedure that might be capable of extending these results to $g > 1$.

3.9. The "τ constant strata" of the moduli space.

According to (Addendum 2.5), there exists a finite partition of D_Γ, $D_\Gamma = \bigcup D_{\Gamma,t}$, where each $D_{\Gamma,t}$ is a locally closed subspace of D_Γ , defined by the condition $\mathbf{w} \in D_{\Gamma,t} \leftrightarrow \tau(X_{\mathbf{w}}, \mathbf{0}) = t$. Their images $M_t = m(D_{\Gamma,t}) \subset \tilde{M}_\Gamma$ will be called "τ constant strata" of $\tilde{M}_\Gamma$.

Mumford asked if the M_t are separated (in the quotient topology). I will show here a considerably weaker property that follows from the jumping behavior of τ.

3.9.1. PROPOSITION. *Each M_t satisfies the property (T_0), that is, given any two points $m_1 \neq m_2 \in M_t$, there exists an open set containing m_1 but not m_2.*

PROOF. Assume that any open set containing m_1 must also contain m_2. This means that there exist fibers in X_Γ with analytic type corresponding to m_2 arbitrarily close to a fiber with analytic type corresponding to m_1. Thus, there exists a one parameter family of branches whose generic fiber (i.e. all but the special fiber, cf. ch. 1, §1, thm. 1) has the analytic type of m_2 while the special fiber has the analytic type of m_1. Since $m_1 \neq m_2$, the "jump property" of τ (Addendum 2.9.1) implies $\tau_{m_1} > \tau_{m_2}$, which contradicts the assumption that $\tau_{m_1} = \tau_{m_2} = t$. □

4. A connection between the moduli space of projective curves and Weierstrass points with semigroup Γ

4.1. Here I only want to remark that $D_\Gamma (= \mathbb{C}^{\tau_-})$ is precisely that part of the miniversal deformation $G : (X, C^\Gamma) \to (\mathbb{C}^\tau, \mathbf{0})$ of C^Γ that Pinkham ([Pi], ch. IV, No. 13) was forced to avoid in order to apply the theory of "negatively homogeneous" deformations. His result is that there exist:

i) $\tilde{G}^+ : \tilde{X}^+ \to \{\mathbf{0}\} \times \mathbb{C}^{\tau_+} \simeq \mathbb{C}^{\tau_+}$, $\tilde{X}^+$ containing the Zariski open set $X^+ = G^{-1}(\{\mathbf{0}\} \times \mathbb{C}^{\tau_+})$, and such that $\tilde{G}^+\big|_{X^+} = G\big|_{X^+}$,

ii) a section $W_\Gamma : \mathbb{C}^{\tau_+} \to \tilde{X}^+$ whose image is $\tilde{X}^+ - X^+$,

iii) a Zariski open dense subset $U \subset \mathbb{C}^{\tau_+}$,

such that for each $\mathbf{w} \in U$, the fiber $\tilde{X}^+_{\mathbf{w}}$ is a projective nonsingular curve with genus $\gamma = \delta(\Gamma)$ and $W_\Gamma(\mathbf{w}) \in \tilde{X}^+_{\mathbf{w}}$ is a Weierstrass point with semigroup Γ.

[**4.1.1.** The semigroup Γ of a point p on a nonsingular irreducible projective curve C is the set of orders of poles at p of rational functions on C that are regular on $C - \{p\}$. The point is *ordinary* if the smallest nonzero element of Γ is $\gamma + 1$ where γ is the genus of C. It is a Weierstrass point if it is not ordinary.]

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[7] This computation has been done in general, when $g = 1$, by Ch. Delorme : *Sur la dimension d'un espace de singularité*, C.R.A.S. t. 280, mai 1975, pgs. 1287-89.

From Pinkham's result, one deduces the existence of a morphism $U \to \mathcal{M}_{\gamma,1}$, the moduli space of *pointed* nonsingular projective curves of genus $\gamma(=\delta(\Gamma))$, where the point in question is, evidently, a Weierstrass point with semigroup Γ. Moreover, Pinkham shows that the fibers of the morphism $U \to \mathcal{M}_{\gamma,1}$ are exactly the orbits of the $\mathbb{C}^*$ action on U described in ch. 1, §2.

4.2. This contrasts with the fact, observed in the example calculated in §1 of this chapter, that the fibers of $m : V \to M_\Gamma$ (in the notations of the preceding subsection) can be of dimension 2.

4.3. In any case, one can detect a geometric relation between the singularities of branches with semigroup Γ and Weierstrass points with the same semigroup inside the base of the versal deformation $G : X \to \mathbb{C}^\tau$ of C^Γ, whenever Γ is the semigroup of a plane curve and $\mathbb{C}^\tau = \mathbb{C}^{\tau_+} \times \mathbb{C}^{\tau_-}$, where:

1. $\mathbb{C}^{\tau_-}$ contains a Zariski open set V such that $X_V \overset{\Sigma_\Gamma}{\curvearrowleft\!\!\longrightarrow} V$ has a section, and $\Sigma_\Gamma(\mathbf{w})$ is a singularity of a plane branch with semigroup Γ. (In fact, Σ_Γ is defined on all of $\mathbb{C}^{\tau_-}$ and $\Sigma_\Gamma : \mathbb{C}^{\tau_-} \to X^-$ (cf. ch. 1, 2.11.12.)
   One uses this to define the morphism $m : V \to M_\Gamma$ (cf. §2 of this chapter).
2. $\mathbb{C}^{\tau_+}$ contains a Zariski open set U such that $X_U \to U$ can be extended to $\tilde{X}_U \overset{W_\Gamma}{\curvearrowleft\!\!\longrightarrow} U$, where $W_\Gamma(\mathbf{w})$ is a Weierstrass point with semigroup Γ on the nonsingular $\tilde{X}_{\mathbf{w}}$.

4.4. Remark: Since $\mathbf{w} \to \delta(X_{\mathbf{w}}, \mathbf{0})$ is constant along $\mathbb{C}^{\tau_-} \times \{\mathbf{0}\}$ (ch. 1, 2.10, 2.11), an argument of the type "geometry of the discriminant", like that given in ([T-4], §3), shows that $\tau_- < \delta(\Gamma)$, and therefore, that $\tau_+ > \delta(\Gamma)$ (assuming always that C^Γ is a complete intersection).

4.5. In fact, Pinkham shows that every smooth projective curve, having a Weierstrass point with semigroup Γ, appears, up to isomorphism, as the fiber of $\tilde{G}^+$ over each point of an orbit of the natural $\mathbb{C}^*$ action on $\mathbb{C}^{\tau+}$ in such a way that the Weierstrass point is picked out by W_Γ (cf. [Pi], §13, 13.11).

4.6. This geometric relation provided by C^Γ between Weierstrass points with semigroup Γ and singular points with semigroup Γ is not really so surprising because on $\tilde{C}^\Gamma = C^\Gamma \cup \{\infty\}$, the origin is a singular point with semigroup Γ, and ∞ is a "Weierstrass point with semigroup Γ", due to the fact that each function $t^{\overline{\beta}_i}$ has a pole at ∞ of order $\overline{\beta}_i$, and these $g+1$ functions generate the rational function field on $\tilde{C}^\Gamma$. The only surprising point is the precise complementary nature of these two points of view, a feature brought out by our focus upon the miniversal deformation. A certain part of all of this continues to make sense even when C^Γ is not a complete intersection.

Addendum

This addendum contains no new result, but does collect together, for the reader's convenience, some well known facts about miniversal deformations from ([G], [S], [Tj], [T-1], [T-2], ch. III) that were used in Chapter VI of the Course, as well as the Appendix.

0. Notations and an existence result.

0.1. Let $(X_0, \mathbf{0}) \subset \mathbb{C}^{n+k}, \mathbf{0})$ be a germ of a complex analytic space of dimension n with isolated singularity. A *miniversal deformation* of $(X_0, \mathbf{0})$ then exists ([Tj], [G]), that is, a commutative diagram of germs of analytic spaces

$$\begin{array}{ccc} (X_0, \mathbf{0}) & \xrightarrow{\text{inclusion}} & (X, \mathbf{0}) \\ \downarrow & & \downarrow{\scriptstyle G} \\ \{\mathbf{0}\} & \xrightarrow{\text{inclusion}} & (S, \mathbf{0}) \end{array}$$

where G is a flat morphism, which is *versal* as a deformation and satisfies a certain minimality property. Versality means that for any other deformation of $(X_0, \mathbf{0})$,

$$\begin{array}{ccc} (X_0, \mathbf{0}) & \xrightarrow{\text{inclusion}} & (Z, \mathbf{0}) \\ \downarrow & & \downarrow{\scriptstyle H} \\ \{\mathbf{0}\} & \xrightarrow{\text{inclusion}} & (Y, \mathbf{0}) \end{array}$$

such that H is flat, there must be a base change morphism $h : (Y, \mathbf{0}) \to (S, \mathbf{0})$ such that $Z \cong X \times_S Y$. The minimality property is expressed by imposing the property that the tangent map of h is uniquely determined by H, which implies that the dimension of S is minimal among all bases of deformations G that satisfy the versality condition (whence the terminology). We will often be sloppy and refer to the miniversal deformation G. In fact, such a deformation is only unique up to a (nonunique) isomorphism.

0.2. Proposition ([Tj], [S]). *If $(X_0, \mathbf{0}) \subset (\mathbb{C}^{n+k}, \mathbf{0})$ is the germ of a complete intersection isolated singularity, the base $(S, \mathbf{0})$ of the miniversal deformation is nonsingular, and any versal deformation of $(X_0, \mathbf{0})$ with nonsingular base is isomorphic to $G \times id\,(\mathbb{C}^t) : (X \times \mathbb{C}^t, \mathbf{0}) \to (S \times \mathbb{C}^t, \mathbf{0})$. Moreover, to any deformation $H : (Z, \mathbf{0}) \to (Y, \mathbf{0})$ of $(X_0, \mathbf{0})$ there is a coherent $\mathcal{O}_Z$ module $T^1_{Z/Y}$ and morphism of coherent $\mathcal{O}_Y$ modules $\Theta_H : (\Omega^1_Y)^\wedge \to H_*(T^1_{Z/Y})$ that satisfy the following properties:*

1. *the construction of $T^1_{Z/Y}$ is compatible with base changes $h : Y' \to Y$, i.e. if $Z' = Z \times_Y Y' \xrightarrow{p} Z$, then $T^1_{Z'/Y'} \simeq p^* T^1_{Z/Y}$;*

2. $T^1_{Z/Y}(\mathbf{0}) = T^1_{Z_0,\mathbf{0}} = T^1_{X_0,\mathbf{0}}$ *is a finite dimensional* $\mathbb{C}-$*vector space* (*"vector space of infinitesimal deformations of* $(X_0,\mathbf{0})$"); *one sets* $\tau(X_0,\mathbf{0}) = dim_{\mathbb{C}}\, T^1_{X_0,\mathbf{0}}$. *A description of* $T^1_{X_0,\mathbf{0}}$ *as* $\mathcal{O}^k_{X_0,\mathbf{0}}/\mathcal{N}$ *is used in* (*ch. 1, 2.5.2*) (*see [Tj]*). *One also has* $T^1_{X_0,\mathbf{0}} \simeq Ext^1_{\mathcal{O}_{X_0,\mathbf{0}}}\left(\Omega^1_{\mathcal{O}_{X_0,\mathbf{0}}}, \mathcal{O}_{X_0,\mathbf{0}}\right)$.
3. *Since* $(X_0,\mathbf{0})$ *is a complete intersection, a deformation* $G:(X,\mathbf{0})\to(S,\mathbf{0})$ *with nonsingular base is versal (resp. miniversal) if and only if* $\Theta_G(\mathbf{0}) : E_{S,\mathbf{0}} \to T^1_{X_0,\mathbf{0}}$ *is surjective (resp. an isomorphism), where* $E_{S,\mathbf{0}}$ *is the (Zariski) tangent space of* S *at* $\mathbf{0}$.

0.2.1. Remark: In the particular case where $k=1$, i.e. $(X_0,\mathbf{0})\subset(\mathbb{C}^{n+1},\mathbf{0})$ is a hypersurface with isolated singularity, $T^1_{X_0,\mathbf{0}}$ is nothing other than the underlying vector space of the analytic algebra for the singular subspace of $(X_0,\mathbf{0})$, that is, $T^1_{X_0,\mathbf{0}} \simeq \mathcal{O}_{X_0,\mathbf{0}}/j'\cdot\mathcal{O}_{X_0,\mathbf{0}}$, where j' is the n^{th} Fitting ideal of the module of differentials $\Omega^1_{X_0,\mathbf{0}}$ (cf. [T-2], ch. III). This follows from the classical presentation of $\Omega^1_{X_0,\mathbf{0}}$ as the cokernel of

$$(f)/(f^2) \xrightarrow{(d)} \Omega^1_{\mathbb{C}^{n+1}} \bigotimes_{\mathcal{O}_{\mathbb{C}^{n+1}}} \mathcal{O}_{X_0}$$

(where $f\in\mathbb{C}\{z_0,\ldots,z_n\}$ is such that $f=0$ defines $(X_0,\mathbf{0})$ and $(d)(gf) =$ class of $d(gf)$ in $\Omega^1_{\mathbb{C}^{n+1}} \bigotimes_{\mathcal{O}_{\mathbb{C}^{n+1}}} \mathcal{O}_{X_0}$), and from the definition of the n^{th} Fitting ideal as

$$j' = (\partial f/\partial z_0,\ldots,\partial f/\partial z_n)\mathcal{O}_{X_0,\mathbf{0}}.$$

0.2.2. We recall that by parts (2), (3) of the Proposition, one can associate to any complete intersection isolated singularity $(X_0,\mathbf{0})$, a miniversal deformation with nonsingular base $G:(X,\mathbf{0})\to(S,\mathbf{0})$ such that $dim\,(S,\mathbf{0}) = \tau(X_0,\mathbf{0})$.

If $(X_0,\mathbf{0})$ is a hypersurface $\{f(z_0,\ldots,z_n)=0\}$, then

$$\tau(X_0,\mathbf{0}) = dim_{\mathbb{C}}\,\mathbb{C}\{z_0,\ldots,z_n\}\big/\left(f,\partial f/\partial z_0,\ldots,\partial f/\partial z_n\right),$$

which implies that $\tau(X_0,\mathbf{0}) \le \mu(X_0,\mathbf{0}) = dim_{\mathbb{C}}\,\mathbb{C}\{z_0,\ldots,z_n\}\big/\left(\partial f/\partial z_0,\ldots,\partial f/\partial z_n\right)$ is the Milnor number (cf. [Mi]) (compare with ch. 1, 3.3).

1. The discriminant (cf. [T-2], ch. III) and the $\mu-$constant stratum.

Let $G:(X,\mathbf{0})\to(S,\mathbf{0})$ be the miniversal deformation of a complete intersection isolated singularity $(X_0,\mathbf{0})$.

There exists a germ of a reduced hypersurface $(D,\mathbf{0})\subset(S,\mathbf{0})$, satisfying the following property:

For any sufficiently small representative of G, $s\in D$ *if and only if the fiber* X_s *has at least one singular point.*

In other words, D is the image by G of the critical subspace $C\subset(X,\mathbf{0})$ (a subspace defined by a Fitting ideal, with underlying point set the set of points that are singular in their fiber). The Preparation theorem then tells us that $G\big|_C : C\to S$ is a *finite* morphism (for a sufficiently small representative). In the case where $(X_0,\mathbf{0})$ is a hypersurface, the multiplicity of $(D,\mathbf{0})$ at $\mathbf{0}$ equals the Milnor number $\mu(X_0,\mathbf{0})$ (0.2.2), which is a topological invariant of $(X_0,\mathbf{0})$. This fact led me to introduce in [T-1, exp. 2] a closed analytic subspace $(D_\mu,\mathbf{0})\subset(D,\mathbf{0})$,

$D_\mu = \{s \in D : m_s(D) = \mu\}$ (for a sufficiently small representative of D), where $m_s(D)$ denotes the multiplicity of the hypersurface D at s, and to conjecture that this space is nonsingular.

This conjecture is proved in ch. 1, 3.1 of the Appendix, when $(X_0, \mathbf{0})$ is a plane branch, and in general for germs of reduced plane curves by using the theorem μ *constant* $\Leftrightarrow$ *equisingularity* (cf. [T-3], Th. 2) to identify the germ $(D_\mu, \mathbf{0})$ with an analytic representative of the formal miniversal equisingular deformation, which is known to be nonsingular by Wahl [Wa].

2. Summary of results about the miniversal deformation.

2.1. Product Decomposition Theorem ([T-2], ch. III, §1, 2.1). *Let $(X_0, \mathbf{0})$ be the germ of a complete intersection isolated singularity. Any sufficiently small representative of the miniversal deformation $G : (X, \mathbf{0}) \to (S, \mathbf{0})$ of $(X_0, \mathbf{0})$ satisfies the following property:*

For any $s \in S$, let $x_i(s)$ $(1 \leq i \leq \ell)$ be the singular points of the fiber $X_s = G^{-1}(s)$. Then there exists a product decomposition (non canonical) of S in a neighborhood of s

$$S \simeq S_1 \times \cdots \times S_\ell \times \mathbb{C}^t\,, \qquad \text{where } t = \tau(X_0, \mathbf{0}) - \sum_{i=1}^{\ell} \tau(X_s, x_i(s)),$$

wuch that in some neighborhood of each $x_i(s)$, G is isomorphic to

$$id\,(S_1 \times \cdots \times S_{i-1}) \times G_i \times id\,\left(S_{i+1} \times \cdots \times S_\ell \times \mathbb{C}^t\right) :$$
$$(S_1 \times \cdots \times S_{i-1}) \times X_i \times \left(S_{i+1} \times \cdots \times S_\ell \times \mathbb{C}^t\right) \longrightarrow (S_1 \times \cdots \times S_{i-1}) \times S_i \times \left(S_{i+1} \times \cdots \times S_\ell \times \mathbb{C}^t\right),$$

where $G_i : X_i \to S_i$ is the miniversal deformation of the germ of the complete intersection isolated singularity $(X_s, x_i(s))$.

2.2. Remark: The assertion in 2.1 is only interesting, of course, when $s \in D$, the discriminant locus of G. Otherwise, $t = \tau(X_0, \mathbf{0})$. However, one should observe that the conclusion is still meaningful even when $\ell = 1$ since it asserts that G is a versal (though not necessarily miniversal) deformation in the neighborhood of each point $x_i(s)$. □

2.3. Corollary. *Let $\Sigma \subset X$ be a closed analytic subspace of the critical locus C of G that is defined by conditions concentrated at each singular point of a single fiber $G^{-1}(s)$ of G. Defining $\Delta = G_*(\Sigma)$ $(\Delta \subset D)$, there is a neighborhood of s in which a decomposition of Δ exists*

$$\Delta = \bigcup_{i=1}^{\ell} \tilde{\Delta}_i\,, \qquad \text{where } \tilde{\Delta}_i = (S_1 \times \cdots \times S_{i-1}) \times \Delta_i \times \left(S_{i+1} \cdots \times S_\ell \times \mathbb{C}^t\right),$$

and $\Delta_i \subset S_i$ is the image of a subspace $\Sigma_i \subset X_i$, defined by the same conditions that define Σ. (One version of this occurs with the condition "has Γ as semigroup". This was used in ch. 1, 3.1.)

2.4. The preceding result motivates us to decompose the base S (of a sufficiently small representative) of the miniversal deformation into τ constant strata as follows:

2.5. Proposition. *For any sufficiently small representative of the miniversal deformation of a complete intersection isolated singularity, there exists a finite partition $S = \bigcup S_\tau$ of S into (locally closed) analytic subspaces such that:*

$$s \in S_\tau \Longleftrightarrow \sum_{x \in Sing\, X_s} \tau(X_s, x) = \tau.$$

(*Recall that since $C \to S$ is finite, $Sing\, X_s$ is a finite set.*)

Proof. In fact, this decomposition is just the stratification of S by the dimension of the fibers of the $\mathcal{O}_S$ coherent sheaf $G_* T^1_{X/S}$ (cf. Séminaire Cartan 60-61).□

2.6. In the rest of the discussion, I will use the notation $\tau_0 = \tau(X_0, \mathbf{0})$, and denote by S_{τ_0} the stratum of the partition from 2.5 that contains $\mathbf{0}$. Defining τ_s to denote $\sum_{x \in Sing\, X_s} \tau(X_s, x)$, one concludes that $\tau_0 \geq \tau_s$ for each $s \in S$ (applying, as usual, the upper semicontinuity of dimension principle of the fibers of a coherent sheaf), and that S_{τ_0} is a closed subspace of S.

2.6.1. One observes therefore that by 2.1, if $s \in S_{\tau_0}$ is such that the fiber X_s has exactly one singular point $x(s)$, then the germ of G at $x(s)$ is the miniversal deformation for $(X_s, x(s))$. □

2.7. Theorem ("*efficiency of the miniversal deformation*") ([T-1], exp. 1 §1). *Let $(X_0, \mathbf{0})$ be the germ of a complete intersection isolated singularity. Then any sufficiently small representative of the miniversal deformation $G : (X, \mathbf{0}) \to (S, \mathbf{0})$ satisfies the following property:*

the set of points $x \in X$ such that the germ (X_s, x) of the fiber of G containing x (i.e. $s = G(x)$) is analytically isomorphic to $(X_0, \mathbf{0})$ is precisely $\{\mathbf{0}\}$.

2.8. Corollary 1. (*first proved by Seidenberg [Sg] when $(X_0, \mathbf{0})$ is a plane curve*) *The following properties are equivalent for any deformation $H : (Z, \mathbf{0}) \to (Y, \mathbf{0})$ of a complete intersection isolated singularity $(X_0, \mathbf{0})$:*

1. *there exists a representative of H so that each fiber Z_y contains a point $z(y)$ such that the germ $(Z_y, z(y))$ is analytically isomorphic to $(X_0, \mathbf{0})$;*
2. *H is isomorphic to the trivial deformation $(X_0 \times Y, \mathbf{0}) \xrightarrow{pr_2} (Y, \mathbf{0})$, and there exists a germ of a section $\sigma : (Y, \mathbf{0}) \to (Z, \mathbf{0})$ of H such that σ is isomorphic to the section $y \to (0, y)$.*

In other words, any deformation of $(X_0, \mathbf{0})$ whose fibers are all isomorphic is locally analytically trivial.

2.9. Corollary 2. *(the "jump theorem for τ")*

2.9.1. First version:. *Let $H : (Z, \mathbf{0}) \to (Y, \mathbf{0})$ be (a representative of) a deformation of the complete intersection isolated singularity $(X_0, \mathbf{0})$ satisfying the following condition:*

there exists a dense open analytic set $V \subset Y$ such that the fibers $(Z_y, z(y))$ are all analytically isomorphic whenever $y \in V$.

Then, if $(Z_y, z(y))$ is NOT isomorphic to $(X_0, \mathbf{0})$ one must have:

$$\tau(Z_y, z(y)) < \tau_0 = \tau(X_0, \mathbf{0}).$$

Proof. It clearly suffices to prove the assertion when $Y = \mathbb{D}(= \{v \in \mathbb{C} : |v| < 1\})$ and $V = \mathbb{D} - \{0\}$ since $Y - V$ is then a closed nowhere dense analytic subspace of Y. Our deformation H can then be understood to be the pullback of the miniversal deformation $(X, \mathbf{0}) \to (S, \mathbf{0})$ of $(X_0, \mathbf{0})$, induced by a base change map $h : (\mathbb{D}, 0) \to (S, \mathbf{0})$.

We will assume $\tau(Z_y, z(y)) = \tau_0$ and deduce a contradiction.

The first point is to observe that $h(\mathbb{D}) \neq \{\mathbf{0}\}$ since if this were the case, then $(Z_y, z(y)) \simeq (X_0, \mathbf{0})$ would follow, which cannot be according to our hypothesis. Thus, $h(\mathbb{D})$ is indeed an arc contained entirely inside the τ constant stratum S_{τ_0} (2.5, 2.6), and 2.6 then implies that $z(y)$ is the unique singular point of the fiber Z_y. We can now apply the Product Decomposition Theorem 2.1 to conclude that G is the miniversal deformation of the germ $(Z_y, z(y))$.

As a result, we can therefore apply 2.7. This gives an open neighborhood $U_y \subset \mathbb{D} - \{0\}$ of any point $y \in \mathbb{D} - \{0\}$ such that for each $u \in U$, $h(u) = h(y)$ (since $(Z_u, z(u))$ is isomorphic to $(Z_y, z(y))$ by our hypothesis). Thus, $h(U_y) = \{y\}$. However, this clearly contradicts the fact that $h : \mathbb{D} \to S$ is a nontrivial arc. □

2.9.2. Second version:. *Let $S^1_{\tau_0} \subset S_{\tau_0}$ be an analytic subspace such that for each $s \in S^1_{\tau_0}$, the fiber X_s of the miniversal deformation has a unique singular point $x(s)$ (so that $\tau(X_s, x(s)) = \tau_0$ necessarily follows). Then:*

"the analytic type of the fibers $(X_s, x(s))$ varies continuously as a function of s, $s \in S^1_{\tau_0}$"
in the sense that for each $s \in S^1_{\tau_0}$ there exists a neighborhood V_s such that for any $v \in V_s - \{s\}$, $(X_v, x(v)) \not\cong (X_s, x(s))$.

Proof. This follows immediately from 2.1 and 2.7. □

2.9.3. Remark: 2.9.1 has been generalized recently in the context of formal geometry by Washburn [Ws], without assuming that the fibers have an isolated singularity. □

Question: Does 2.8 remain true when one no longer assumes that the fibers have an isolated singularity? nor complete intersection?

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