Valuations on equicharacteristic complete noetherian local domains

Bernard Teissier

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Abstract

Given an equicharacteristic noetherian complete local domain R and a rational valuation ν on R we show that there exist an algebra $S = k[\widehat{(u_i)_{i\in I}}]$ equipped with a weight, or monomial valuation, and a surjection $\pi\colon S \to R$ such that the valuation ν is induced by the weight on S in the sense that $\nu(x)$ is the maximum weight of counterinages of x in S. Moreover, the kernel of π is generated by overweight deformations of binomials corresponding to generators of the relations between generators $(\gamma_i)_{i\in I}$ of the semigroup $\Gamma = \nu(R\setminus\{0\})$. In all this, the index set I is an ordinal $\leq \omega^{\dim R}$.

1 Extended abstract

Given an equicharacteristic complete noetherian local domain R with algebraically closed residue field k, we study the relation between a zero-dimensional valuation ν of R centered at the maximal ideal and its associated graded ring $\mathrm{gr}_\nu R$ with respect to the filtration defined by the valuation. We shall be interested mostly in *rational* valuations, in the sense that the extension $R/m \subset R_\nu/m_\nu$ is trivial, where R_ν is the valuation ring of ν. Then by general properties of valuations, each nonzero homogeneous component of the k-algebra $\mathrm{gr}_\nu R$ is a 1-dimensional k-vector space. In this case the graded algebra is essentially the semigroup algebra $k[t^\Gamma]$ where $\Gamma = \nu(R\setminus\{0\})$ is the semigroup of values of the valuation. The main idea is that in the event that $\mathrm{gr}_\nu R$ is a finitely generated k-algebra, some of the birational toric maps which provide embedded pseudo-resolutions for the affine toric variety corresponding to $\mathrm{gr}_\nu R$ (see [3]) also provide local uniformizations for ν on R, and when $\mathrm{gr}_\nu R$ is not finitely generated, then the same should be true for some of the birational toric maps which (pseudo) resolve the affine variety defined by a well chosen finite subset of the set of binomial equations describing the relations between the generators of Γ. If $\mathrm{gr}_\nu R$ is a finitely generated k-algebra, or equivalently Γ is a finitely generated semigroup, the valuation ν is necessarily Abhyankar (for zero-dimensional valuations this means that the value group is $\mathbf{Z}^r$ with $r = \dim R$). The converse is true up to a birational map on Spec$R$ followed by localization at the center of the valuation and completion (see [6]).

In general, the semigroup of a valuation on a noetherian local domain is well ordered, so that it has a minimal system of generators

$$\Gamma = \langle \gamma_1, \ldots, \gamma_i, \ldots \rangle = \langle (\gamma_i)_{i \in I} \rangle$$

indexed by an ordinal $I \leq \omega^h$, where ω is the ordinal of $\mathbf{N}$ and h is the rank, or (archimedian) height, of the valuation. If Γ is a numerical semigroup, it is finitely generated by a classical result of Dickson; this is the case when R is one dimensional.

As shown in [4], the fact that R is noetherian and the valuation ν is rational imply the existence of a presentation of $\mathrm{gr}_\nu R$ as a quotient of a polynomial ring, possibly in countably many variables, by a binomial ideal:

$$k[(U_i)_{i\in I}]/(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell \in L} \simeq \mathrm{gr}_\nu R, \quad \text{with } \lambda_\ell \in k^*.$$

Here the variables U_i are in bijection with the generators γ_i of Γ and the isomorphism sends each U_i to an element $\overline{\xi}_i$, of degree γ_i, of a minimal system of generators of the k-algebra $\mathrm{gr}_\nu R$.

In order to relate the the embedded pseudo-resolutions for the (generalized) affine toric variety corresponding to $\mathrm{gr}_\nu R$ with local uniformizations of the valuation ν on R, the main tools are the concept of *overweight deformation* of a prime binomial ideal, and the *valuative Cohen theorem*.

Weights and overweight deformations

For this summary, a weight on $k[(u_i)_{i\in I}]$ or $k[[(u_i)_{i\in I}]]$ will be a morphism of semigroups $w\colon M(I) \to \Phi_{\geq 0}$, where $M(I)$ denotes the semigroup of monomials in the u_i, and Φ a totally ordered abelian group, which attributes to each variable u_i a weight $w(u_i) = \gamma_i \in \Phi_{\geq 0}$. In our case, Φ will be the value group of the valuation and the image of the map w is the semigoup $\Gamma \subset \Phi_{\geq 0}$.

A weight is compatible with a binomial ideal (an ideal generated by binomials) if each generating binomial is homogeneous.

The weight of a polynomial, or a series, is the minimum weight of its terms.

Given a weight which is compatible with it, an *overweight deformation* of a binomial is an expression

$$F = u^m - \lambda_{mn} u^n + \sum_{w(u^p) > w(u^m)} c_p u^p \in k[[(u_i)_{i\in I}]].$$

If we have any number of binomials and a compatible weight we do the same and consider the deformations

$$F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(u^p) > w(u^{m^\ell})} c_p^{(\ell)} u^p \in k[[(u_i)_{i\in I}]].$$

One has to add the condition that *the initial binomials of the F_ℓ generate the ideal F_0 of initial forms of the elements of the ideal F generated by the F_ℓ.*

Let us consider an overweight deformation (F_ℓ) of a prime binomial ideal F_0 in the power series ring $k[[u_1,\ldots,u_N]]$, and the map

$$\pi\colon k[[u_1,\ldots,u_N]]\to R=k[[u_1,\ldots,u_N]]/(F_1,\ldots,F_s).$$

Proposition 1.1. *1) The map which associates to $x\in R$ the maximum of the weights of the elements of $\pi^{-1}(x)$ is well defined and is a valuation ν on R.*
2) The associated graded ring of $k[[u_1,\ldots,u_N]]$ with respect to the weight filtration is $k[U_1,\ldots,U_N]$ and the map

$$\Pi\colon k[U_1,\ldots,U_N]\to k[U_1,\ldots,U_N]/F_0=\mathrm{gr}_\nu R$$

is the associated graded map of π with respect to the weight and valuation filtrations.
3) Given $\tilde{x}\in\pi^{-1}(x)$, we have $w(\tilde{x})=\nu(x)$ if and only if $\mathrm{in}_w(\tilde{x})\notin F_0$.

The power series ring adapted to Γ
Let $(u_i)_{i\in I}$ be variables indexed by the elements of the minimal system of generators $(\gamma_i)_{i\in I}$ of the semigroup Γ of the rational valuation ν on R. Give each u_i the weight $w(u_i)=\gamma_i$ and let us consider the set of power series $S=\sum_{e\in E}d_eu^e$ where $(u^e)_{e\in E}$ is any set of monomials in the variables u_i and $d_e\in k$.

By a theorem of Campillo-Galindo (see [1]), the semigroup Γ being well ordered is combinatorially finite, which means that for any $\phi\in\Gamma$ the number of different ways of writing ϕ as a sum of elements of Γ is finite. This is equivalent to the fact that the set of exponents e such that $w(u^e)=\phi$ is finite: for any given series the map $w\colon E\to\Gamma,\ e\mapsto w(u^e)$ has finite fibers. Each of these fibers is a finite set of monomials in variables indexed by a totally ordered set, and so can be given the lexicographical order and order-embedded into an interval $1\leq i\leq n$ of $\mathbf{N}$. and thus produce an embedding $E\subset(\Gamma\times\mathbf{N})_{lex}$ which induces a total order on E, for which it is well ordered. When E is the set of all monomials, this gives a total monomial order.

The combinatorial finiteness implies that this set of series $S=\sum_{e\in E}d_eu^e$ is a k-algebra, which we denote by $k[\widehat{(u_i)_{i\in I}}]$. It is endowed with a weight $w(S)$, which is the minimum weight of the terms of S, and a topology defined by the weight filtration. It is shown in [6] that the algebra is spherically complete with respect to the monomial valuation given by the weight. Since the weights of the elements of a series form a well ordered set and only a finite number of terms of the series have minimum weight, the associated graded ring of $k[\widehat{(u_i)_{i\in I}}]$ with respect to the filtration by weights is the polynomial ring $k[(U_i)_{i\in I}]$.

Proposition 1.2. (The valuative Cohen theorem, see [4]) *Assuming that the local noetherian equicharacteristic domain R is complete, with a rational valuation ν, and fixing a field of representatives $k\subset R$, there exist choices of representatives $\xi_i\in R$ of the $\overline{\xi_i}$ generating the k-algebra $\mathrm{gr}_\nu R$ such that the surjective map of k-algebras $k[(U_i)_{i\in I}]\to\mathrm{gr}_\nu R,\ U_i\mapsto\overline{\xi_i}$, is the associated graded map of a continuous surjective map*

$$k[\widehat{(u_i)_{i\in I}}]\to R,\ u_i\mapsto\xi_i,$$

of topological k-algebras, with respect to the weight and valuation filtrations respectively. The kernel of this map can be generated up to closure by overweight deformations of binomials generating the kernel of $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$, $U_i \mapsto \overline{\xi}_i$. *If ν is of rank one or if Γ is finitely generated, any choice of representatives ξ_i is permitted.*

Note that here we have generalized the concept of overweight deformation to the case of countably many binomials.

Corollary 1.1. *Let ν be a rational valuation on a complete equicharacteristic noetherian local domain R. If the semigroup Γ of the valuation on R is finitely generated, the ring R is obtained by an overweight deformation from the quotient of a power series ring by the binomial ideal encoding relations between the generators of Γ.*

As a consequence of this, it is a combinatorial problem, relatively easy in the rank one case, to show that if in addition the residue fied k is algebraically closed some of the embedded toric resolutions of $\mathrm{gr}_\nu R$ give local uniformization for ν on Spec R, once it is re-embedded in the same ambient space as Spec $\mathrm{gr}_\nu R$. Associated with the fact recalled above that the semigroups of values of rational Abhyankar valuations on excellent equicharacteristic local domains are finitely generated up to ν-modification and completion, this gives a proof of local uniformization for Abhyankar valuations of equicharacteristic excellent local domains with an algebraically closed residue field.

In the general case of a rational valuation ν of a complete equicharacteristic noetherian local domain the valuative Cohen theorem allows us to produce a sequence of Abhyankar *semivaluations* ν_B of R of, that is, Abhyankar valuations on r-dimensional quotients R/K_B of R, where r is the rational rank of ν, which are indexed by finite subsets B ordered by inclusion of the index set I and have the property that for any element $x \in R$, the valuation $\nu(x)$ is equal to $\nu_B(x)$ for large enough B. This is an extension of the abyssal phenomenon which was described for the plane in [4] and an asymptotic approximation property of rational valuations.

All the references below except the first one and the last one are available, with corrections, at http://webusers.imj-prg.fr/ bernard.teissier/articles-Teissier.html

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