A BOUQUET OF BOUQUETS FOR A BIRTHDAY

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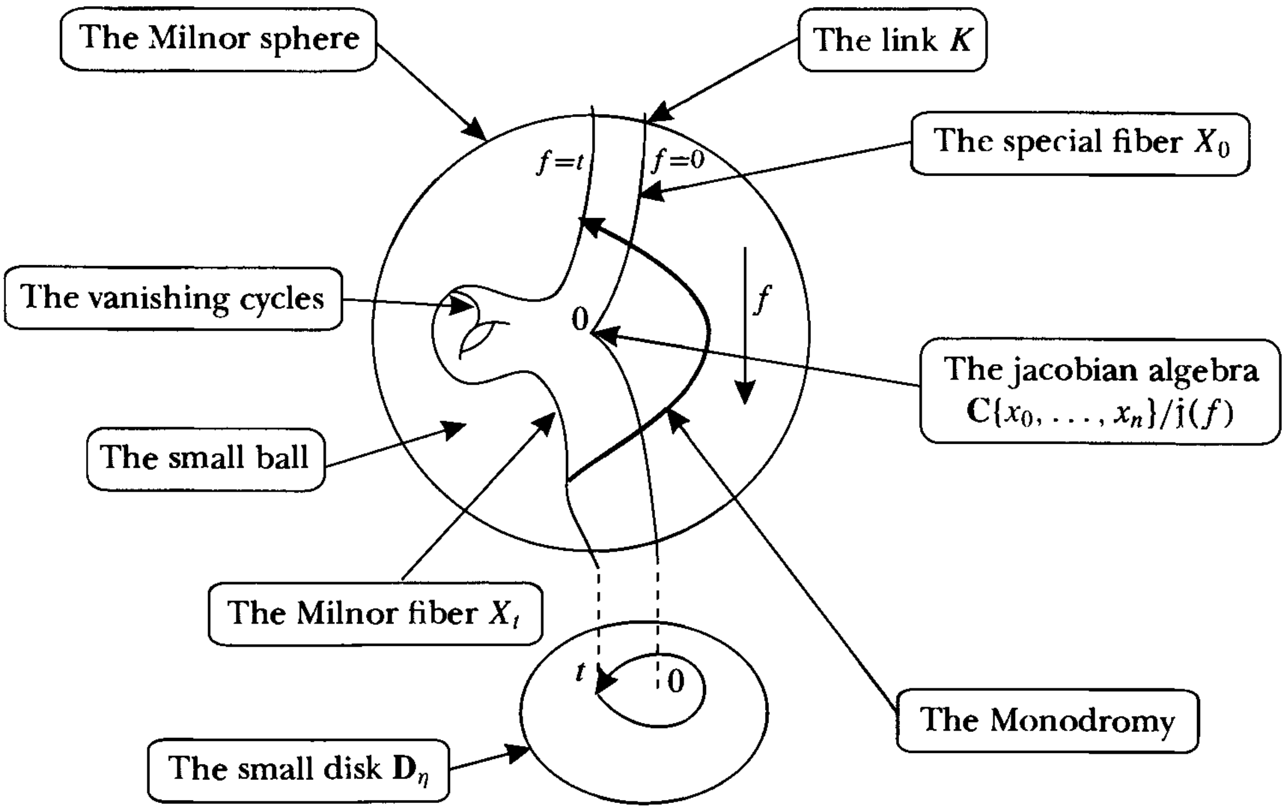
1. Introduction

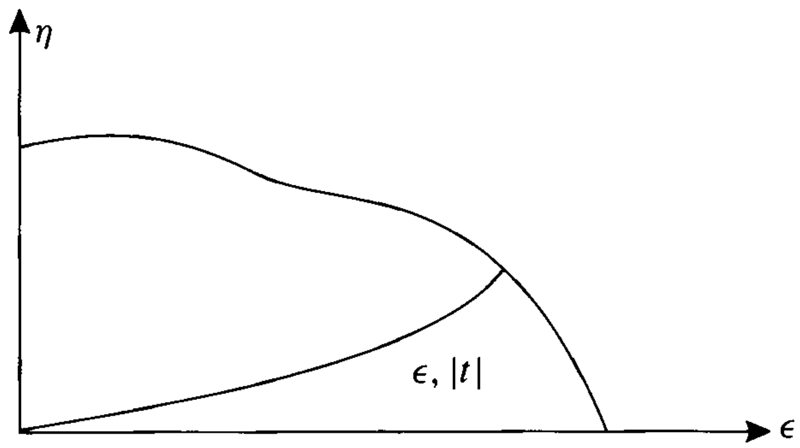
Milnor certainly gave to analytic singularity theory much more than he received; in exchange for a concrete description of exotic spheres, John Milnor gave to geometers working with singularities the following PICTURE for the understanding of the geometric behavior of an analytic function $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ in the neighborhood of a critical point.



This led to the development of the local topological study by differential-geometric methods of all singularities in analytic geometry, as opposed to a global algebro-geometric or cohomological study which had hitherto dominated, and to Thom's beautiful differential-geometric study of the generic singularities of mappings.

In fact, Milnor used this picture only for the case where $f^{-1}(0)$ has an isolated singularity at 0. In the general case he preferred to work with the restriction of the map f to the sphere $\mathbf{S}_\epsilon^{2n+1}$. There is a notable difference, since in the non-isolated singularity case the link $f^{-1}(0) \cap \mathbf{S}_\epsilon^{2n+1}$ is singular, and different from the boundary of the Milnor fiber $f^{-1}(t) \cap \mathbf{B}_\epsilon$. It was Lê who extended the picture's use to the general case. Let me now decorate it with results from [Mil], [Lê2–6]

In a neighborhood of 0 in the complex hypersurface $f^{-1}(0)$, as the point $x \in U$ tends to 0 the secant line $\overrightarrow{0x}$ tends to be contained in the tangent cone at 0, so there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$ the spheres $\mathbf{S}_\epsilon$ are transversal to $f^{-1}(0)$ in $\mathbf{C}^{n+1}$ in the sense of stratified spaces, so that all the intersections $\mathbf{S}_\epsilon \cap f^{-1}(0)$ are isotopic as stratified spaces. By Sard's Theorem and the openness of transversality, there exists $\eta_0 > 0$ such that for $0 < |t| < \eta_0$ the fiber $f^{-1}(t)$ is nonsingular and transversal to $\mathbf{S}_\epsilon$. In fact, there is an open region in a neighborhood of 0 in the first quadrant of the ϵ, η plane, limited by the positive ϵ axis and a semi-analytic curve, such that,



as long as $(\epsilon, |t|)$ lies in that region, these transversality results hold. Now we fix ϵ, η in that region and set $X_0 = f^{-1}(0) \cap \mathbf{B}_\epsilon$ and $X_t = f^{-1}(t) \cap \mathbf{B}_\epsilon$, where $\mathbf{B}_\epsilon$ is the open ball with center at 0 and radius ϵ and $|t| = \eta$. Then we have:

(1) The closure $\overline{X_0}$ is homeomorphic as a subspace of $\overline{\mathbf{B}_\epsilon}$ to the cone with vertex 0 over the *link* $K = f^{-1}(0) \cap \mathbf{S}_\epsilon$. In particular, it is contractible. If K is a topological sphere, X_0 is a topological manifold near 0. If

0 is a smooth point of $f^{-1}(0)$, i.e., a non-critical point of f, then K is an unknotted $(2n-1)$-sphere in $\mathbf{S}_\epsilon$.

(2) The space $f^{-1}(\mathbf{D}_\eta) \cap \mathbf{B}_\epsilon$ is contractible, since it retracts onto $f^{-1}(0) \cap \mathbf{B}_\epsilon$.

(3) Each fiber X_t is parallelizable and one proves by Morse theory (see [Lê5]) that it has the homotopy type of a CW-complex of dimension n.

(4) The (Milnor) link $K = \overline{X_0} \cap \mathbf{S}_\epsilon$ is $(n-2)$-connected.

(5) The map

$$f^{-1}(\mathbf{D}^*) \cap \mathbf{B}_\epsilon \to \mathbf{D}^*$$

induced by f is a fibration fiberwise diffeomorphic to the *Milnor fibration*:

$$\mathbf{S}_\epsilon \setminus K \to \mathbf{S}^1 \qquad z \mapsto \frac{f(z)}{|f(z)|}.$$

Consequently, by lifting the counterclockwise circulation around 0 of a point $t \in \mathbf{D}^*$ by $\tau \mapsto e^{2i\pi\tau}t$, we obtain a *Monodromy diffeomorphism*

$$h\colon X_t \to X_t$$

and we may choose

$$h|\partial X_t = \text{Identity}.$$

This diffeomorphism acts on the homology of X_t as an endomorphism h_* of the group $H_*(X_t, \mathbf{Z})$, which is still called the monodromy and is independent of the lifting of the circulation.

Note that by (5), which is proved by a transversality argument, all the fibers $\overline{X_t} = \overline{f^{-1}(t) \cap \mathbf{B}_\epsilon}$ are diffeomorphic as manifolds with boundaries as long as $(\epsilon, |t|)$ stay in the region symbolized in the figure above; X_t is called a *Milnor fiber* of f, ϵ_0 is called a *permissible radius* and $\mathbf{S}_\epsilon$ a *Milnor sphere*.

If, in addition, 0 is an isolated critical point, then:

(6*) The space X_t has the homotopy type of a bouquet of n-spheres; actually $\overline{X_t}$ is obtained from a ball $\mathbf{D}^{2n}$ by simultaneously attaching a number of handles of index n. (See [Mil], p. 58, and [L-P] for the case $n = 2$.)

(7*) Since there is no singularity of the map f on the sphere $\mathbf{S}_\epsilon$, the induced mapping $\bigcup_{t \in \mathbf{D}_\eta} \partial X_t \to \mathbf{D}_\eta$ is a differentiable fibration, so the Milnor link K is a differentiable manifold diffeomorphic to the boundary of the Milnor fiber.

(8*) The closures of the fibers of the fibration of (5) above are diffeomorphic to those of the fibers of the Milnor fibration.

Let me sketch the proof of (6) in the isolated singularity case. All the following considerations are local; let us agree that we consider only points in $\mathbf{B}_\epsilon \times \mathbf{D}_\eta$ but continue to denote the map by f. The idea is to analytically deform the given map f to a map which has only complex Morse critical points (locally expressible as a sum of squares of holomorphic coordinates). Consider the 1-parameter family of maps given by

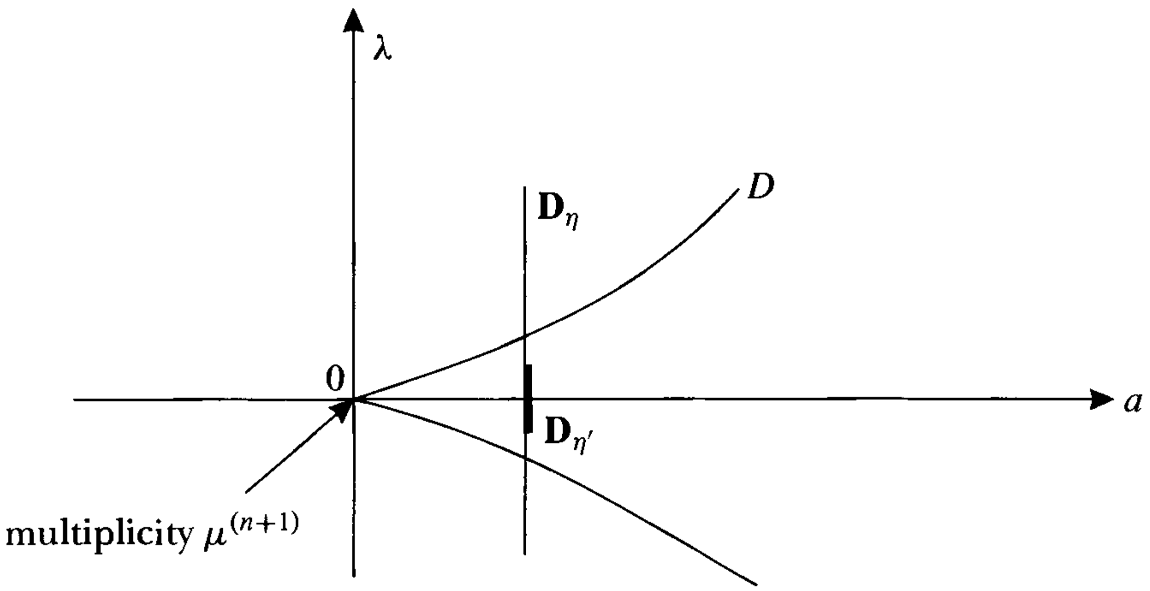
$$\lambda = f(x) + ax_0;$$

we wish to study the critical points and critical values of the associated map $F\colon \mathbf{C}^{n+1} \to \mathbf{C}^2$ described by $x \mapsto (f_a(x), a)$ where $f_a(x) = f(x) + ax_0$. The critical points of F constitute the curve in $\mathbf{C}^{n+1} \times \mathbf{C}$ with equations

$$\frac{\partial f}{\partial x_0} + a = 0,\ \frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0.$$

From here on we have to assume that the hyperplane $x_0 = 0$ is "sufficiently general" (this can be made precise). Then we have:

The image in $\mathbf{C}^2$ of this critical curve by the map F is a plane curve, say D (for discriminant); and the key points are the following:



(i) There is no critical point of F with $\lambda = 0$, except 0, and the map F induces a topological fibration $\mathbf{B}_\epsilon \setminus F^{-1}(D) \to \mathbf{D}_\eta \times \mathbf{D}_\tau \setminus D$ above the complement of the discriminant D in a small polycylinder $|\lambda| < \eta$, $|a| < \tau$.

(ii) If we fix a value of $a \neq 0$ (sufficiently close to 0) and a sufficiently small value $\eta' \ll \eta$, the map f_a has no critical value in $\mathbf{D}_{\eta'}$, so that $f_a^{-1}\mathbf{D}_{\eta'} \to \mathbf{D}_{\eta'}$ is a topological fibration and $f_a^{-1}\mathbf{D}_{\eta'}$ has the homotopy type of a general fiber of f_a (or of f—it is the same, since the complement of D is connected).

(iii) The space $f_a^{-1}(\mathbf{D}_\eta)$ is contractible, as we saw in (2) above.

(iv) The curve D, algebraically defined as the image of the critical locus, as in [T4], has no multiple component. This is equivalent to the fact that the map f_a has only complex Morse singularities in $\mathbf{B}_\epsilon$, so that the real analytic function $|f_a|$ is a Morse function on $\mathbf{B}_\epsilon$. By an application due to Lê ([Lê2], [Lê6]) of a general principle of Thom already used by Andreotti-Frankel for the distance function to an algebraic set (see [A-F]), its index is $n+1$ at each of its critical point.

Thus, the Milnor fiber X_t, which is the fiber of the fibration induced by F above the point $a = 0$, $\lambda = t$, becomes contractible after adding a finite number of $(n+1)$-cells; it has to be homotopic to a bouquet of n-spheres.

Finally we remark that the number of these cells, which is the number of critical points of the function f_a on $\mathbf{B}_\epsilon$, is equal to the number of intersection points with D of a line in the (λ, a)-plane, parallel to the λ axis $a = 0$; this number is the intersection multiplicity at 0 of the λ axis with the plane curve D. By the projection formula, it is equal to the intersection multiplicity of the critical curve $C \subset \mathbf{C}^{n+2}$ with the hyperplane $a = 0$. By the general theory of intersection multiplicity, this last number is equal, since the curve C is a complete intersection, to the dimension as a complex vector space of the quotient of the ring of convergent power series by the ideal generated by the partial derivatives of f,

$$\mu^{(n+1)}(f, 0) = \dim_{\mathbf{C}} \mathbf{C}\{x_0, \ldots, x_n\}/(\partial f/\partial x_0, \ldots, \partial f/\partial x_n),$$

which is called the *Milnor number* of the (isolated) critical point of f at 0 or of the (isolated) singularity at 0 of the fiber $X_0 = f^{-1}(0)$, and then written $\mu^{(n+1)}(X_0, 0)$.

In fact, the Milnor number is the multiplicity at the origin of the curve $D \subset \mathbf{C}^2$, since one can show that in this case the analytically irreducible components of the curve D are all tangent to the curve $\lambda = 0$, and therefore the intersection number at 0 of the λ axis with D is equal to the multiplicity of D at 0.

Milnor's Theorem shows that for a hypersurface with isolated singularity, only the middle dimensional homology group $H_n(X_t, \mathbf{C})$ is nonzero, and isomorphic to $\mathbf{C}^{\mu}$. Furthermore, the homotopical construction of the Milnor fiber sketched above shows that the integral homology has no torsion, so that $H_n(X_t, \mathbf{Z})$ is a lattice in $H_n(X_t, \mathbf{C})$, called the Milnor lattice of the singularity. It is then natural to study the behavior of this Milnor lattice, especially with respect to the bilinear form on $H_n(X_t, \mathbf{C})$ given by the intersection of cycles, much as one does in differential geometry, in the hope of finding subtle invariants of the singularity, subtle enough for example to give an obstruction to a given singularity appearing as a deformation of another. Some references for this topic are [Br4], [Br5], [E].

At this point, one should also mention the mixed Hodge structure on $H_n(X_t, \mathbf{C})$, first constructed by Steenbrink in [St1], and which has provided much information on the analytical behavior of a function with a critical point. Here a fundamental result is the "semicontinuity of the spectrum" of Arnol'd, Steenbrink, Varchenko (see [A-V-G], [St 2]). Another fundamental point is the analytic expression for the intersection form of cycles on the Milnor fiber through a sequence of "higher residues" found by K. Saito and studied by Varchenko, Kashiwara and M. Saito. In all these constructions, the interpretation of the complex homology $H_n(X_t, \mathbf{C})$ of the Milnor fibers X_t, viewed as a locally constant sheaf of vector spaces on the punctured disk $\mathbf{D}^*_\eta$, as the space of solutions of a differential equation in t (the Gauss-Manin system), and the integral homology $H_n(X_t, \mathbf{Z})$ as a lattice (the Brieskorn lattice) in that locally constant sheaf, due to Brieskorn (see [Br3]), plays a fundamental role; it describes the analytic structure of the variation of the homology classes of X_t with t and the results above must be understood in this context. This can now, after work of Malgrange, Deligne, be deemed to be a part the theory of $\mathcal{D}$-modules (see [Ph2]).

Generalizations. The following generalization of the fibration theorem is due to Lê:

THEOREM 1 (Lê, [Lê3]). Let $\mathcal{U}$ be a neighborhood of 0 in $\mathbf{C}^N$, and $X \subset \mathcal{U}$ be a closed complex analytic subset of $\mathcal{U}$ containing 0. Let $f : \mathcal{U} \to \mathbf{C}$ be a holomorphic function such that $f(0) = 0$. Then there exists a semi-analytic subset P as above in the (ϵ, η) plane such that for $(\epsilon, \eta) \in P$, the map f induces a topological fibration

$$X \cap \mathbf{B}_\epsilon \cap f^{-1}(\mathbf{D}^*_\eta) \rightarrow \mathbf{D}^*_\eta.$$

In particular, one can define vanishing cycles for f on X as the elements of the homology of this fiber, and build from this a *sheaf of vanishing cycles.*

2. Singularities and spheres

In 1961 Mumford published the striking theorem that a normal point on a complex surface S near which S is a topological manifold (which is the case if the link is a topological sphere) is a nonsingular point (see [Mu]); Hirzebruch decided to study the higher dimensional case.

Around 1964 Pham, computing the variation of integrals representing the amplitude of diffusion of interacting particles as a function of their energy-impulsion, was led to study the monodromy of polynomials of the form

$$x_0^{a_0} + \cdots + x_n^{a_n}$$

and he gave an explicit representation of the Milnor fiber, in this case diffeomorphic to $x_0^{a_0} + \cdots + x_n^{a_n} = 1$, as a join. I will not go into this here, but only recall (see [Ph1]) that the Milnor number in this case is $\mu = \prod_0^n (a_i - 1)$ and the characteristic polynomial of the monodromy endomorphism h_* is

$$\Delta(t) = \prod_{\omega_i \in \mu_{a_i} - \{1\}} (t - \omega_0 \cdot \ldots \cdot \omega_n)$$

If we denote by $K(a_0, \ldots, a_n)$ the corresponding Milnor link, we find that for $K(2, \ldots, 2, 3)$ we have:

$$\Delta(t) = t^2 - t + 1 \qquad \text{for } n \text{ odd}$$
$$\Delta(t) = t^2 + t + 1 \qquad \text{for } n \text{ even}$$

So for n odd, $\Delta(1) = 1$ and $K(2, \ldots, 2, 3)$ is a topological sphere of dimension $2n - 1 = 1, 5, 9, 13, \ldots$. In dimensions 1 and 5 it has to be a standard sphere, but in dimension 9 one gets an exotic sphere, so that

$$(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^3 = 0) \cap \mathbf{S}^{11}$$

is an exotic 9-sphere. Indeed (see [Mi1]), whenever the differentiable manifold K is a topological sphere, since it is the boundary of the Milnor fiber which is a $(n-1)$-connected parallelizable manifold, it belongs to the subgroup $bP_{2n} \subset \Theta_{2n-1}$ of the group Θ_{2n-1} of C^∞ manifolds which are topological $(2n-1)$-spheres, consisting of those elements which bound a parallelizable $2n$-manifold.

It follows from the h-cobordism theorem that the diffeomorphism type of K is completely determined

- by the signature of the intersection pairing

$$H_n(X_t, \mathbf{Z}) \otimes H_n(X_t, \mathbf{Z}) \to \mathbf{Z} \qquad \text{if } n \text{ is even}$$

- by the Kervaire invariant $c(X_t) \in \mathbf{Z}/2\mathbf{Z}$ if n is odd.

By a result of Levine [Le] the characteristic polynomial $\Delta(t)$ of the monodromy endomorphism completely determines the Kervaire invariant, so that for n odd it determines the differentiable structure of the link K of an isolated singularity when K is a topological sphere. Since the characteristic polynomial of the monodromy endomorphism corresponding to a Pham-Brieskorn polynomial $x_0^{a_0}+\cdots+x_n^{a_n}$ has been computed as a function of $a_0, \ldots, a_n$ (see above), this determination of the diffeomorphism type of K is crucial for the proof of the results which follow.

It was known ([K-M]) that cardinality of $bP_{2k} \leq 2$ if k is odd, and that bP_{4m} is always cyclic, of order

$$\frac{\sigma_m}{8} = \epsilon_m 2^{2m-4}(2^{2m-1}-1) \times \text{numerator of } \frac{4B_m}{m}$$

where B_m is the m^{th} Bernoulli number, $\epsilon_m = \pm 1$, and $\epsilon_m = 1$ if m is odd.

For n even, say $n = 2m$, Brieskorn proved the following:

THEOREM 2 (Brieskorn, [H1], [H2], [Br1], [Br2]).

(a) The link $K(2, \ldots, 2, p, q) \subset S^{4m+1}$ with $2m-1$ exponents equal to 2, and p, q coprime odd numbers, is a topological sphere of dimension $4m-1$ belonging to the subgroup $bP_{4m} \subset \Theta_{4m-1}$.

(b) In particular

$$K(2, \ldots, 2, 3, 6k-1) = (-1)^m k g_m,$$

where g_m is an element of bP_{4m} of index 8 and for $k = 1, 2, \ldots, \sigma_m/8$ one obtains the $\sigma_m/8$ different spheres of bP_{4m}.

(c) There is a similar result for odd n, so that every odd-dimensional differentiable manifold which bounds a parallelizable manifold and is a topological sphere is diffeomorphic to a Milnor Link $K(a_0, \ldots, a_n)$.

So for $n = 4$, $k = 1, 2, \ldots, 28$, the links $K(2, 2, 2, 3, 6k-1)$ give us the 28 classes of 7-spheres in bP_8.

For $n = 6$, $k = 1, 2, \ldots, 992$, the links $K(2, 2, 2, 2, 2, 3, 6k-1)$ give us the 992 classes of 11-spheres of bP_{12}. For any odd n such that $bP_{2n} \neq 0$, the link

$$K(\underbrace{2, \ldots, 2}_{n \text{ times}}, 3)$$

is the exotic element.

The study of the topological, differentiable, or metric structure of the link K has also provided invariants of the singularity, as in the work of W. Neumann and J. Seade: W. Neumann showed in [N] that for normal complex surfaces (for example hypersurfaces in $\mathbf{C}^3$ with an isolated singularity), the oriented homeomorphism type of the link K determines the topology of a resolution of singularities. In many cases, $\pi_1(K)$ suffices. This is a beautiful chapter of the study of three-dimensional topology, closely connected to the work of Waldhausen. Note, however, that this is much weaker than determining the local topology in $\mathbf{C}^3$ of the surface near the singular point (we will see more about this below; see also [D1], [E-N-S], [Se], and [W]).

The problem of determining which $(n-2)$-connected differentiable manifolds of dimension $2n-1$ bounding parallelizable manifolds occur as links of isolated hypersurface singularities, is not solved except for plane curves ($n = 1$); see [D1]. Some lens spaces, for example, appear as links (= intersection with a small sphere centered at a singular point) of complex singularities, but not of hypersurfaces. Moreover, the links may well be diffeomorphic without the singularities being equivalent, and another question is how much information the link K by itself, and the embedding $K \subset \mathbf{S}^{2n+1}$, respectively, contains about the singularity $(X, 0)$.

The case of plane curves ($n = 2$) already displays this phenomenon; all irreducible plane curve germs have a link diffeomorphic to $\mathbf{S}^1$, but it is the embedding of this knot in $\mathbf{S}^3$ as a knot which determines the local topology of the germ. For general plane curve germs, the relationship between the link of the singularity, its local topology, the Milnor fiber and its monodromy, the Puiseux characteristic exponents of the parameterizations of the branches (= analytically irreducible components) of the curve, its resolution of singularities and algebraic features of the local ring, are rather well, but not completely, understood. One outstanding question was posed by Milnor:

QUESTION 3 (Milnor, [Mil]). Is it true that the gordian number of the link of a germ of a plane curve is equal to the "δ invariant" of the singularity?

The δ invariant of a plane curve defined near 0 by $f(x, y) = 0$ has many equivalent algebraic definitions, but I choose the following: it is the maximum number of singularities which may appear in the same fiber of an arbitrarily small deformation $f(x, y) + \epsilon g(x, y, \epsilon)$ of the map f, tending to 0 with ϵ. See [B-W] for more information.

This question has apparently been answered affirmatively recently. It has very interesting higher dimensional analogues.

3. The Monodromy

Milnor investigated the monodromy not only because its characteristic polynomial gives the discriminant of the intersection pairing on the Milnor fiber and therefore a way of determining whether the link K is a topological sphere, but also because the monodromy is a fundamental object in geometry. In the sixties, several authors had begun the study of the monodromy of a family of degenerating algebraic varieties, notably Landman, and Clemens and Grothendieck who used resolution of singularities of the singular fiber in a 1-parameter degenerating family $G: X \to \mathbf{D}$ to study the monodromy of the general fiber X_t under circulation of t around $0 \in \mathbf{D}$. The fact that Milnor put the local approach in a clear framework opened, in this direction also, a whole array of new problems on the monodromy, which had hitherto been studied almost exclusively in a global setting (in the fibers).

For example, if we denote by F_j the set of points of X_t fixed by the iterated diffeomorphism h^j, the zeta function of the monodromy

$$\zeta(s) = \exp\left(\sum_{j=1}^{\infty} \chi(F_j)\frac{s^j}{j}\right)$$

is well defined for a sufficiently small ball $\mathbf{B}_\epsilon$ and $t \ll \epsilon$.

Brieskorn in [Br3] interpreted the monodromy as the monodromy of a local system on $\mathbf{C}$, the (locally constant) sheaf of solutions of a differential equation, the Gauss-Manin connection associated with the Milnor fibration. He also showed that the eigenvalues of the local monodromy are roots of unity. This is equivalent to the fact that the monodromy is quasi-unipotent: for some integer k the monodromy endomorphism h_* of $H_n(X, \mathbf{C})$ satisfies

$$(h_*^k - \mathrm{Id})^{n+1} = 0.$$

Lê also proved, in [Lê7], the generalization of this monodromy theorem to the monodromy deduced from his fibration theorem quoted above; the integer n is then the dimension of the fiber $f^{-1}(0) \subset X$.

He also proved that the monodromy associated to an analytically irreducible analytic function of two variables $f(x, y)$ is unipotent. In the opposite direction, Malgrange in [M] gave an example showing that the exponent $n + 1$ actually occurs as minimal exponent of nilpotency.

Moreover, one can find a representative (up to isotopy, that is) of the monodromy which has no fixed point, as was shown by Lê in [Lê4].

A'Campo showed how to compute the zeta function of the monodromy from the numerical data associated to a resolution of singularities of the hypersurface $f = 0$. Later Thom and Sebastiani, probably motivated by Pham's construction of the Milnor fiber of his polynomials, proved that if we consider two isolated singularities of hypersurfaces, written in different variables, say $f(x_0, \dots, x_n)$ and $g(w_0, \dots, w_m)$, then the Milnor fiber of $f(x_0, \dots, x_n) + g(w_0, \dots, w_m)$ is topologically a join of the Milnor fibers of f and g, the homology of this Milnor fiber is the tensor product of the homologies of the Milnor fibers of f and of g, and the Monodromy is the tensor product of their Monodromies. This turned into a very useful tool for the computation of monodromies and intersection matrices. The Thom-Sebastiani result was extended to non-isolated singularities by Oka-Sakamoto, and further extended by Deligne.

4. The Milnor number as a topological invariant

Now let us go back to the homology of X_t in the isolated singularity case.

Saying that the singularity is isolated is equivalent, by Hilbert's Nullstellensatz, to saying that some power of the maximal ideal $\mathfrak{m} = (x_0, \dots, x_n)$ of the local ring $\mathcal{O}_{n+1} = \mathbf{C}\{x_0, \dots, x_n\}$ is contained in the *jacobian ideal* $\mathfrak{j}(f) = (\partial f/\partial x_0, \dots, \partial f/\partial x_n)\mathcal{O}_{n+1}$ which defines the critical locus of the map f. Therefore the Milnor number, which we defined as the dimension (as a complex vector space) of the *Jacobian algebra*

$$\mathcal{O}_{n+1}/\mathfrak{j}(f) = \mathbf{C}\{x_0, \dots, x_n\}\big/(\partial f/\partial x_0, \dots, \partial f/\partial x_n)$$

of f at 0 is finite, and clearly grows with the complexity of the singularity.

It is natural to ask how sensitive this number μ is to the local topology of the map f, or of the fibers X_0 or X_t. To make things more precise, let us define an equivalence relation between germs of hypersurfaces:

DEFINITION 4. Two germs of hypersurfaces (X_1, x_1) and (X_2, x_2) in $\mathbf{C}^{n+1}$ have the same topological type if they have representatives $(X_1, x_1) \subset (U_1, 0) \subset (\mathbf{C}^{n+1}, 0)$ and $(X_2, x_2) \subset (U_2, 0) \subset (\mathbf{C}^{n+1}, 0)$, where X_i is closed in U_i, such that there exists an homeomorphism $\varphi\colon (U, 0) \to (V, 0)$ such that $\varphi(X_1) = X_2$.

Similarly, two germs of maps f_1, f_2: $(\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^p, 0)$ have the same topological type if they have representatives $f_i: (U_i, 0) \to (V_i, 0)$ which are conjugate by homeomorphisms of the $(U_i, 0)$, $(V_i, 0)$.

It is not difficult to check that if f_1 and f_2 have the same topological type at 0, then for small enough representatives (i.e., in Milnor balls) so do their fibers $X_{1,0}$ and $X_{2,0}$ (which are both contractible anyway), as well as their "general fibers" $X_{1,t}$ and $X_{2,t}$. In particular, if two germs of maps f_1, f_2: $(\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ with isolated singularities have the same topological type, they have the same Milnor number. Lê has also shown (see [Lê3]) that in this case, the local monodromies on the homology groups $H^n(X_{i,t}, \mathbf{Z})$ are conjugate endomorphisms. Easy examples show that two isolated singularities of hypersurfaces may well have the same Milnor number and be topologically inequivalent. We shall see below, however, that in an analytic family of isolated singularities, the constancy of the Milnor number has strong consequences.

5. The Milnor number as a topological and algebraic invariant

An important point is that the Milnor number is a topological invariant defined algebraically.

In contrast, consider the multiplicity of a hypersurface at a point, which we may assume to be the origin; in this case, the multiplicity of the origin as a singular point of a hypersurface $f = 0$ is the order m of the power series f at 0, i.e., the degree of the homogeneous polynomial of lowest degree appearing in the Taylor expansion at the origin: $f(x_0, \ldots, x_n) = f_m(x_0, \ldots, x_n) + f_{m+1}(x_0, \ldots, x_n) + \cdots$. The multiplicity at the origin of the hypersurface may also be defined as follows: for a sufficiently small ϵ, a direction of line $\ell \in \mathbf{P}^n$, viewed as a linear map ℓ: $(\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^n, 0)$, of sufficiently general direction, and a sufficiently general point $t \in \mathbf{C}^n$ with $0 < |t| \ll \epsilon$, the line $\ell^{-1}(t)$ of direction ℓ intersects $X_0 \cap \mathbf{B}_\epsilon$ in m points. Since this number of points is equal to the topological Euler characteristic of the intersection, we have:

$$m = \chi(X_0 \cap \ell^{-1}(t) \cap \mathbf{B}_\epsilon).$$

The multiplicity is also defined in a purely algebraic way: Let $\mathcal{O}_{X_0,0} = \mathbf{C}\{x_0, \ldots, x_n\}/(f)$ denote the local algebra of the fiber X_0 at the origin, and $\mathfrak{m}$ its maximal ideal; then for large ν the dimensions of the complex vector spaces $\mathcal{O}_{X_0,0}/\mathfrak{m}^\nu$ coincides with the values of a rational polynomial of degree n, the leading term of which is $m \cdot (\nu^n/n!)$. This construction extends naturally to define the (Krull) dimension $\operatorname{Dim} \mathcal{O}$ of a noetherian local ring $\mathcal{O}$ (e.g., an analytic algebra), as the degree of the polynomial and its multiplicity as the coefficient of $\nu^n/n!$ in the polynomial giving $\operatorname{length}_{\mathcal{O}} \mathcal{O}/\mathfrak{m}^\nu$ for large ν. It also extends naturally to associate an integer $e_{\mathcal{O}}(\mathfrak{q})$ to any ideal $\mathfrak{q}$ of a noetherian local ring $\mathcal{O}$ such that the length $\mathcal{O}/\mathfrak{q}$ is finite; this is the multiplicity in the sense of commutative algebra of the ideal $\mathfrak{q}$ in $\mathcal{O}$. If the ring $\mathcal{O}$ is Cohen-Macaulay, which is the case for a ring of convergent power series, and if the primary ideal $\mathfrak{q}$ can be generated by exactly $\operatorname{Dim} \mathcal{O}$ elements, which is true for the jacobian ideal, then we have $e_{\mathcal{O}}(\mathfrak{q}) = \operatorname{length}_{\mathcal{O}}(\mathcal{O}/\mathfrak{q})$, where the length is the length as a $\mathcal{O}$-module, which in our case, since $\mathcal{O}$ is a **C**-algebra, is just the dimension as a vector space of $\mathcal{O}/\mathfrak{q}$. Finally, setting $\mathcal{O}_{n+1} = \mathbf{C}\{x_0, \ldots, x_n\}$, we have the equality

$$e_{\mathcal{O}_{n+1}}(\mathfrak{j}(f)) = \mu^{(n+1)}(X, 0).$$

This is useful since, although the Milnor number is defined algebraically as the dimension as a complex vector space of the "jacobian algebra" associated with the map f at the origin, from the algebraic viewpoint, even to check that it depends only on the germ at 0 of the zero-set $f = 0$ is not obvious; one has to show that if we replace $f \in \mathbf{C}\{x_0, \ldots, x_n\}$ by uf, where $u \in \mathbf{C}\{x_0, \ldots, x_n\}$ is an invertible element, i.e., $u(0) \neq 0$, the dimension as a complex vector space of the jacobian algebra does not change. A natural way to do this begins with the remarks that this dimension is also the multiplicity, in the sense of commutative algebra, of the jacobian ideal $\mathfrak{j}(f)$ in the local ring $\mathbf{C}\{x_0, \ldots, x_n\}$, and that f is "almost" in the ideal $\mathfrak{j}(f)$ in the sense that for p large enough, we have $f\mathfrak{j}(f)^p \subset \mathfrak{j}(f)^{p+1}$ (this is one way of expressing the fact that Euler's relation for homogeneous polynomials (which is the case $p = 0$) is true asymptotically for arbitrary power series $f \in \mathfrak{m}$). This is similar to the fact used near the beginning of section 1 that the limits of secant lines from 0 are in the tangent cone. The result then follows from the stability properties of the multiplicity, which imply that, since u is invertible, the multiplicity of the ideal $\left(u\frac{\partial f}{\partial x_i} + f\frac{\partial u}{\partial x_i}\right)$ is equal to that of the ideal $\mathfrak{j}(f)$.

Note that some authors use the unfortunate terminology "multiplicity" of f at 0 for the Milnor number; if one wants to speak of multiplicity, then "Jacobian multiplicity" would be acceptable for the Milnor number.

In any case, Zariski asked the following question:

QUESTION 5 (Zariski, [Z1]). Is the multiplicity of a germ of hypersurface at a point an invariant of the local topological type?

Except in the case $n = 1$, or very special cases (e.g., if one of the hypersurfaces has multiplicity 2), the answer is not known. A positive answer would make the multiplicity a topological invariant defined algebraically, just like the Milnor number.

Instead of considering two hypersurfaces and trying to understand what it means for them to have the same Milnor number or the same multiplicity, one may consider an easier question: Let $F(v; x_0, \ldots, x_n) \in \mathbf{C}\{t; x_0, \ldots, x_n\}$ be a convergent power series satisfying $F(v; 0, \ldots, 0) = 0$ and thus defining a germ at 0 of an analytic family of germs $(X_v, 0) \subset (\mathbf{C}^{n+1}, 0)$. Assume that for a small representative, the Milnor number $\mu(X_v, 0)$ is independent of v; does it imply that the topological type is also independent of v? This was proved by Lê and Ramanujam under a dimension restriction:

THEOREM 6 (Lê-Ramanujam, [L-R]). In an analytic family of isolated singularities of analytic hypersurfaces of dimension $n \neq 2$, the constancy of the Milnor numbers of the fibers implies the constancy of their topological type.

The assumption $n \neq 2$ is needed because the proof uses the h-cobordism theorem. To remove this assumption is an open problem. The main difficulty of the proof is the fact that in an analytic family $F(v; x_0, \ldots, x_n) = 0$, although a Milnor sphere $\mathbf{S}_{\epsilon_0}$ for the special fiber$F(0; x_0, \ldots, x_n) = 0$ remains transversal to $F(v; x_0, \ldots, x_n) = 0$ for sufficiently small $|v|$ by the openness of transversality, it may no longer be a Milnor sphere there; there may be non-transversal spheres of smaller radius. So, for small v, the fiber $F(v; x_0, \ldots, x_n) = t \cap \mathbf{S}_{\epsilon_0}$ for small t is diffeomorphic to the Milnor fiber of $F(0; x_0, \ldots, x_n)$ but is not necessarily a Milnor fiber of $F(v; x_0, \ldots, x_n)$. A key step of the proof is to use the assumption "μ constant" to see that for small v and t the part of the fiber $F(v; x_0, \ldots, x_n) = t$ contained between a Milnor sphere $\mathbf{S}_{\epsilon_v}$ and the sphere $\mathbf{S}_{\epsilon_0}$ is an h-cobordism. Then one applies the h-cobordism theorem.

Thus, the following question is weaker than Zariski's question on the topological invariance of the multiplicity:

QUESTION 7. Is the multiplicity constant in an analytic family of germs of hypersurfaces with isolated singularities and constant Milnor number?

Except for families of curves (see [Lê1]), this question is also unsolved.

Now it happens that there is a way of putting both the Milnor number and the multiplicity in a sequence of numbers attached to the analytic isomorphism type of a germ of hypersurface.

The basic remark is geometric: by the general finiteness theorems of Thom and Whitney, given any analytic family of germs of complex (or real) analytic spaces, there exists a strict analytic subset S of the parameter space T such that for $t \notin S$, the germs $X(t)$ are all topologically equivalent.

Given a germ $(X, 0) \subset (\mathbf{C}^N, 0)$, we can consider the family of the intersections $(X \cap H, 0)$ of X with k-codimensional vector subspaces of $H \subset \mathbf{C}^N$, parameterized by the appropriate Grassmanian $G(n, k)$. There is a strict algebraic subspace $T \subset G$ such that for $H \notin T$, all the reduced (i.e., set theoretic) intersections $([X \cap H], 0)$ have the same topological type. In particular, if $(X, 0)$ is a reduced hypersurface germ, or a reduced complete intersection, for $H \notin T$ the intersection $(X \cap H, 0)$ is reduced, and all these germs have the same topological type, so the topological type of a general linear section of a germ is well defined.

Therefore, the Milnor number $\mu^{(n)}(X, 0)$ of a general hyperplane section of a hypersurface $X \subset \mathbf{C}^{n+1}$ is well defined; it is finite if and only of the singular locus of the hypersurface is of dimension ≤ 1.

So if we have a germ of a hypersurface $(X, 0) \subset (\mathbf{C}^{n+1}, 0)$ with isolated singularity, we can associate to it the sequence of the Milnor numbers of its general plane sections of all dimensions:

$$\mu^{(*)}(X, 0) = \big(\mu^{(n+1)}(X, 0), \mu^{(n)}(X, 0), \dots, \mu^{(1)}(X, 0), \mu^{(0)}(X, 0)\big)$$

The number $\mu^{(i)}(X, 0)$ is the Milnor number of the restriction of f to a general plane of dimension i through the origin. Note that $\mu^{(1)}(X, 0) = m - 1$ where m is the order of the series f at 0, or the multiplicity at 0 of the hypersurface $f = 0$, and it is convenient to set $\mu^{(0)}(X, 0) = 1$. We shall see below that this whole sequence is an analytic invariant of the germ $(X, 0)$. It is not, however, an invariant of the topological type, but if we slightly strengthen the notion of topological type to mean the equality of the topological types of general plane sections of all dimensions, then $\mu^{(*)}(X, 0)$ is an invariant of this. This being seen, the main point is that the constancy of $\mu^{(*)}(X_t, 0)$ is a necessary and sufficient condition for the total space of a family $\mathcal{X} = \bigcup X_v$ of germs of hypersurfaces with isolated singularities to satisfy the Whitney conditions along its singular locus. Let $F(z_0, z_1, \dots, z_n; v_1, \dots, v_k) = 0$ define an analytic family $\mathcal{X} \subset \mathbf{C}^{n+1} \times \mathbf{C}^k$ of germs at 0 of hypersurfaces ($F \in \mathbf{C}\{z_0, z_1, \dots, z_n; v_1, \dots, v_k\}$, with $F(0, \dots, 0, v_1, \dots, v_k) = 0$) which means that $Y = \{0\} \times \mathbf{C}^k$ is contained in $\mathcal{X}$. We have:

THEOREM 8 (Teissier, Briançon-Speder, [T2], [B-S], [T8]). In an analytic family of germs of hypersurfaces with isolated singularities, the constancy of the sequence $\mu^*(X_t, 0)$ is equivalent to the fact that the pair of strata $\mathcal{X}\setminus Y$, Y satisfy the Whitney conditions in a neighborhood of the origin.

Let me recall that a pair of locally closed nonsingular disjoint analytic subspaces Z, Y of some complex space X, such that Y is contained in the closure of Z, is said to satisfy the Whitney conditions at a point $y \in Y$ if for some local embedding $(X, y) \subset (\mathbf{C}^N, 0)$ we have the following property: For all sequences (z_i, y_i) of points of $Z \times Y$, tending to y, if we consider the secant lines $\overline{z_i y_i}$ in $\mathbf{C}^N$ and the tangent spaces T_{Z,z_i}, any limits $(\ell, T) \in \mathbf{P}^{N-1} \times G(N, \dim Z)$ of their directions satisfy $\ell \subset T$.

An important concept for the sequel is that of *Whitney stratification* of a complex analytic space X; it is a locally finite partition

$$X = \bigcup_\alpha X_\alpha$$

of X into nonsingular analytic subspaces X_α such that the closure $\overline{X_\alpha}$ of each of them is a closed complex subspace of X, as well as $\overline{X_\alpha} \setminus X_\alpha$, and whenever $X_\alpha \cap \overline{X_\beta} \neq \emptyset$, then X_α is included in $\overline{X_\beta}$ and at each point of X_α the pair of strata (X_β, X_α) satisfies Whitney's conditions. By a fundamental theorem of Thom-Mather ([Mat]) the Whitney conditions imply that each $\overline{X_\beta}$, and in particular X itself, which is a locally finite union of $\overline{X_\beta}$'s, is locally topologically trivial along each X_α.

Coming back to hypersurfaces with isolated singularities, it is a natural question to ask, for an analytic family $(X_v)_{v\in\mathbf{D}}$ of hypersurfaces with isolated singularity at 0, what the difference is between constancy of the topological type, which according to the Lê-Ramanujam theorem is equivalent, at least for $n \neq 2$, to "$\mu^{(n+1)}(X_v)$ constant", and the Whitney conditions, which according to the theorem just quoted are equivalent to "$\mu^*(X_v)$ constant". I had optimistically conjectured in 1972 that constancy of the topology implies Whitney conditions which gives, numerically, the slogan "μ constant implies μ^* constant"; a counterexample was found in 1975 by Briançon and Speder ([B-S1]).

These results show that the sequence of Milnor numbers is a rather precise topological invariant.

To show that it is also a good algebraic invariant, one may quote the results on the "simultaneous resolution" of families of curves and surfaces $f : X \to \mathbf{D}$. A (very weak) simultaneous resolution is a resolution $\pi : X' \to X$ of the singularities of X (i.e., the map π is proper, bi-meromorphic, isomorphic over the nonsingular part of X, and X' has no singularities), such that for any $t \in \mathbf{D}$, the map $X'_v \to X_v$ induced on the fibers is a resolution of singularities of X_v (in general, even if X' is nonsingular, X'_v may be singular, and also not bi-meromorphic to X_v). It follows from [Lê1], [T5] and the theory of equisingularity of Zariski that an analytic family of germs of plane curves with constant Milnor number has a simultaneous resolution (it is even "strong" in the sense of [T5]). Beautiful results in varying degrees of generalities due to Laufer ([La1]), Vaquié ([V]) and Kollár and Shepherd-Barron ([K-S]) have extended this to families of surfaces with isolated singularities: "μ constant implies simultaneous resolution". Laufer has even proved (in [La2]) a result which implies that "$\mu^{(*)}$ constant" implies strong simultaneous resolution for a family of surfaces in $\mathbf{C}^3$ with isolated singularity. The case of higher dimensions is completely open.

In any case, it was the necessity of giving an algebraic definition to the sequence $\mu^{(*)}(X, 0)$ which led me to introduce in [T2] the sequence of *mixed multiplicities* of two (or several) primary ideals in a noetherian local ring, which in a way "interpolates" between the multiplicities of these ideals. The $\mu^{(i)}(X, 0)$ appear as mixed multiplicities of the jacobian ideal $\mathfrak{j}(f)$ associated to an equation $f = 0$ of $(X, 0)$ and the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{n+1}$. The theory of mixed multiplicities has now become a chapter of commutative algebra (see [R]).

Coming back to a more topological definition, of $\mu^{(n)}(X, 0)$ for example, one finds the following interpretation:

Consider the *polar curve* $P_n(f, \ell) \subset \mathbf{C}^{n+1}$ of a map $f(z_0, z_1, \dots, z_n)$: $\mathbf{C}^{n+1} \to \mathbf{C}$ with isolated singularity, with respect to a linear form ℓ; it may be defined in this special case (see [T7]) as the critical locus of the map (f, ℓ): $\mathbf{C}^{n+1} \to \mathbf{C}^2$, hence is defined, in coordinates such that $\ell = z_0$, by the equations

$$\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_n} = 0.$$

Then we have the following

THEOREM 9 (Teissier, [T2], [T7]). For a sufficiently general linear form ℓ,

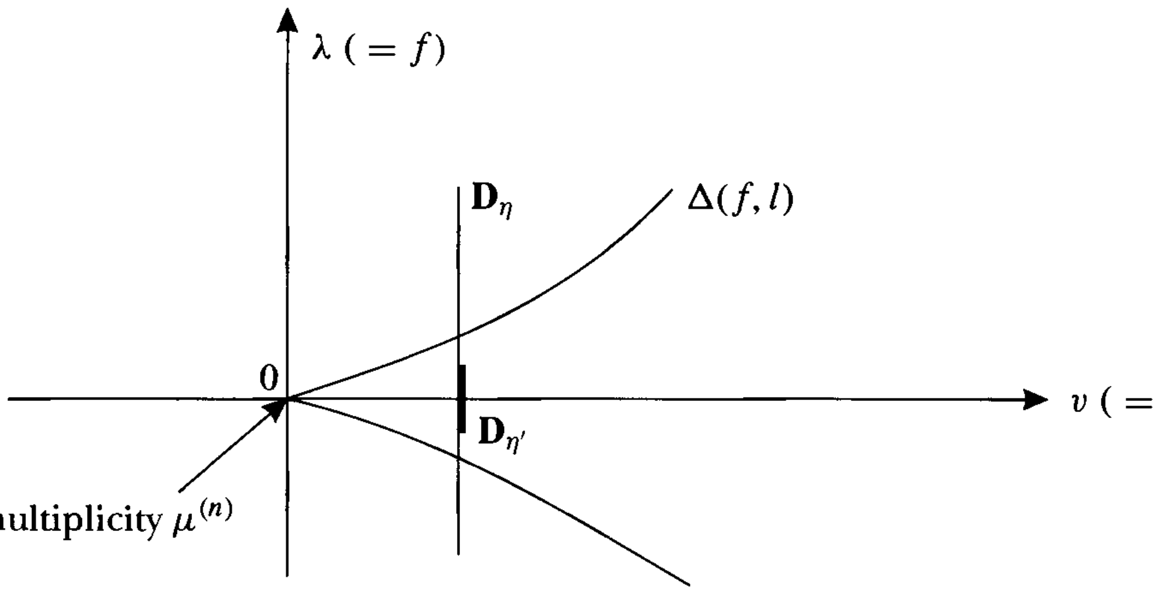
(a) the multiplicity at the origin of the corresponding polar curve is equal to the Milnor number of a general hyperplane section,

$$\mathrm{m}_0 P_n(f, \ell) = \mu^{(n)}(X, 0).$$

(b) The hyperplane $\ell = 0$ is transversal at 0 in $\mathbf{C}^{n+1}$ to the curve $P_n(f, \ell)$.

The statements of this theorem can be generalized to the wider context of the theory of relative polar varieties of maps $f: X \to \mathbf{C}$, without the assumption that f has an isolated critical point in a nonsingular space. Even statement (b), a special case of the *theorem of transversality of polar varieties* ([T7], [T8]), has, despite its apparent inevitability, many important consequences ([Lo], [T8]).

As a corollary in our special case, we have that the image in the plane $\mathbf{C}^2$ of the polar curve by the map $\varphi: \mathbf{C}^{n+1} \to \mathbf{C}^2$ given by $(\lambda = f, v = \ell)$, i.e., the *polar image* $\Delta(f; \ell)$ of the map f relative to the linear form ℓ (called the Cerf diagram by Thom and by Lê), is of multiplicity $\mu^{(n)}(X, 0)$ at the origin, and transversal to the line $\ell = 0$, which means that the tangent cone at the origin of the plane curve $\Delta(f; \ell)$, which is (as a set) a union of lines, does not contain $v = 0$, or equivalently, intersects $v = 0$ only at 0. In fact, in this case, the curve $\Delta(f; \ell)$ is even tangent to the line $\lambda = 0$.



Now we can argue exactly as in the proof of the fact that the Milnor fiber is a bouquet of $\mu^{(n+1)}(X, 0)$ spheres of dimension $n + 1$ to show:

THEOREM 10 (Lê, [Lê6]). The intersection

$$(f(z_0, \ldots, z_n) = 0) \cap (\ell(z_0, \ldots, z_n) = t) \cap \mathbf{B}_\epsilon$$

has, for $0 < \epsilon < \epsilon_0, 0 < |t| < \eta \ll \epsilon$ and ℓ sufficiently general, the homotopy type of a bouquet of $\mu^{(n)}(X, 0)$ spheres of dimension n.

Other results indicating that the sequence of Milnor numbers contains much information are the following:

Recall that the tangent cone at 0 of the hypersurface $f(x_0, \dots, x_n) = 0$ is by definition the cone $f_m(x_0, \dots, x_n) = 0$ where f_m is the homogeneous polynomial made from the terms of lowest degree m (the multiplicity) appearing in the Taylor expansion of f. Then we have:

PROPOSITION 11 (Teissier, [T2] + [T6]).

(i) The Milnor numbers of the general hyperplane sections of a hypersurface satisfy the inequalities

$$\frac{\mu^{(n+1)}}{\mu^{(n)}} \geq \frac{\mu^{(n)}}{\mu^{(n-1)}} \geq \cdots \geq \frac{\mu^{(1)}}{\mu^{(0)}}.$$

(ii) A hypersurface with isolated singularity X has the same topological type as its tangent cone (which is then reduced, with an isolated singularity at the vertex) if and only if all these inequalities are equalities, i.e., the sequence $\mu^{(*)}(X, 0)$ is such that $\mu^{(i)}(X, 0) = (\mu^{(1)}(X, 0))^i$; its singularity is then resolved by blowing up the origin.

PROPOSITION 12 (Teissier, [T2]). Given a hypersurface with isolated singularity $(X, 0) \subset (\mathbf{C}^{n+1}, 0)$, a hyperplane $H \subset \mathbf{C}^{n+1}$ through the origin is a limit direction of tangent hyperplanes to X at nonsingular points near 0 if and only if the Milnor number of the intersection $X \cap H$ is larger than the Milnor number of the intersection of X with a general hyperplane, which is $\mu^{(n)}(X, 0)$.

5.1 The invariant $\mu^{(n+1)}(X, 0) + \mu^{(n)}(X, 0)$. It has turned out that the sum of the Milnor number associated to an isolated singularity and of the Milnor number associated to a generic hyperplane section of this hypersurface appear in a number of apparently unrelated problems.

Consider a linear map $\ell\colon \mathbf{C}^{n+1} \to \mathbf{C}$ and its restriction to a hypersurface with isolated singularity $(X, 0) \subset (\mathbf{C}^{n+1}, 0)$. Let $H = \ker \ell$. If the intersection $(X \cap H, 0)$ has an isolated singularity, such a map has a *discriminant,* which is a complex subspace of $(\mathbf{C}, 0)$, therefore in our case is just the origin counted a certain number of times Δ_ℓ. This number is precisely $\mu^{(n+1)}(X, 0) + \mu^{(n)}(X \cap H, 0)$. Therefore, if the hyperplane H is not a limit of tangent hyperplanes to X at nonsingular points we have

$$\Delta_\ell = \mu^{(n+1)}(X, 0) + \mu^{(n)}(X, 0).$$

This is equivalent (see [T2]) to the fact that the multiplicity in the local ring $\mathcal{O}_{X,0}$ of the ideal $\mathfrak{j}'(f) = \mathfrak{j}(f)\mathcal{O}_{X,0}$ is given by

$$e_{\mathcal{O}_{X,0}}(\mathfrak{j}'(f)) = \mu^{(n+1)}(X,0) + \mu^{(n)}(X,0).$$

It also implies that if we consider a projective hypersurface $V \subset \mathbf{P}^{n+1}$ with isolated singularities, the *diminution of class* imposed by a singular point $x \in V$ is $\mu^{(n+1)}(V,x) + \mu^{(n)}(V,x)$; in other words, if V is of degree d, the class $\check{d}$ of V, which is the degree of the projectively dual hypersurface $\check{V} \subset \check{\mathbf{P}}^{n+1}$ is given by the formula

$$\check{d} = d(d-1)^n - \sum_{x \in \operatorname{sing} V} \left(\mu^{(n+1)}(V,x) + \mu^{(n)}(V,x)\right).$$

See [T5], [Lau]. This has recently been extended to a large class of projective varieties with isolated singularities by Kleiman (see [K]).

By a result of Langevin ([L]), if $f = 0$ is an equation for $(X,0) \subset (\mathbf{C}^{n+1},0)$, the limit of the integral of the curvature of the Milnor fibers $f^{-1}(t)$ inside balls of radius ϵ, as both t and ϵ tend to zero, with $|t|$ sufficiently smaller than ϵ, is given by

$$\lim_{\substack{\epsilon,t\to 0\\ |t|\ll\epsilon}} \int_{\mathbf{B}_\epsilon \cap f^{-1}(t)} |K| = \frac{\operatorname{Vol}(\mathrm{S}^{2n+1})}{2}\left(\mu^{(n+1)}(X,0) + \mu^{(n)}(X,0)\right).$$

Moreover, many of these results have been generalized; the algebraic definition of the Milnor number and the connection of the constancy of the invariant $\mu^{(*)}(X_t,0)$ with the Whitney conditions have been generalized recently by Gaffney (see [G]) to isolated singularities of complete intersections, using the concept of multiplicity of a submodule of a finite free module due to Buchsbaum-Rim. The theory of mixed multiplicities has also been extended to these modules by work of Gaffney, Rees (unpublished) and Kleiman-Thorup (to appear). The definition of the Milnor number has also been extended by Parusiński to non-isolated singularities (see [P], [P-P]). The expression of $\mu^{(n+1)}(X,0) + \mu^{(n)}(X \cap H,0)$ as the multiplicity of a discriminant has been generalized by Lê (see [Lê2]) to isolated singularities of complete intersections and more recently extended to the language of Lagrangian cycles by Sabbah ([S]). The formulas connecting integrals of curvature and polar multiplicities were generalized by Loeser in [Lo].

6. Further generalizations

We have seen the Milnor number given as the rank of a homology group, or an Euler characteristic (up to sign and ± 1), as the local degree of the gradient map grad $f: \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$, or as the intersection number at the origin of all the hypersurfaces defined by the vanishing of the partial derivatives of f. We also saw it appear as the multiplicity at the origin of a discriminant curve and as the multiplicity of the ideal generated by the partial derivatives of f in the ring of power series.

Now there is yet another avatar of the Milnor number: We may view the differential df of our function f as a section of the cotangent bundle of $\mathbf{C}^{n+1}$ defined near the origin,

$$T^*\mathbf{C}^{n+1} \to \mathbf{C}^{n+1}.$$

Let us denote by Λ_f the image of this section; it is a lagrangian subvariety of $T^*\mathbf{C}^{n+1}$. It is clear from the definition of intersection numbers that:

REMARK 13. The Milnor number $\mu^{(n+1)}(f, 0)$ is the intersection number at the point $(0, 0)$ of Λ_f and the zero section $T^*_{\mathbf{C}^{n+1}}\mathbf{C}^{n+1}$ of $T^*\mathbf{C}^{n+1}$, in the space $T^*\mathbf{C}^{n+1}$.

Deligne saw that this remark opened a far-reaching pathway (see [Lê 7]).

To say that 0 is not an isolated singularity is to say that Λ_f and the zero section of $T^*\mathbf{C}^{n+1}$ have *excess intersection.* In this case one knows how to associate an *intersection cycle* $\sum m_\alpha[C_\alpha]$ where the C_α are subvarieties of the intersection $\Lambda_f \cap T^*_{\mathbf{C}^{n+1}}\mathbf{C}^{n+1}$, which can be described rather explicitly by a method due to Vogel ([Vo]). The image of this cycle in $\mathbf{C}^{n+1}$ is called by D. Massey (see [Ma1], [Ma2]) the Lê cycle of the singularity at the origin; it is contained in the critical locus of the map f, and its numerical characters generalize the Milnor number: they can be used to describe the Milnor fiber by a handle decomposition specified by these characters, and Massey has shown that numerical conditions on the intersection multiplicities of the components of the Lê cycle with linear subspaces of $\mathbf{C}^{n+1}$ generalize the μ constant condition of a family f_v of isolated critical points and imply the constancy of the local Monodromy of the germs $(f_v, 0)$. These cycles are closely connected with the polar varieties.

The concept of a function $f: \mathbf{C}^{n+1} \to \mathbf{C}$ with an isolated singularity was generalized to the case of an arbitrary complex space by Lê as follows (see [Lê8]):

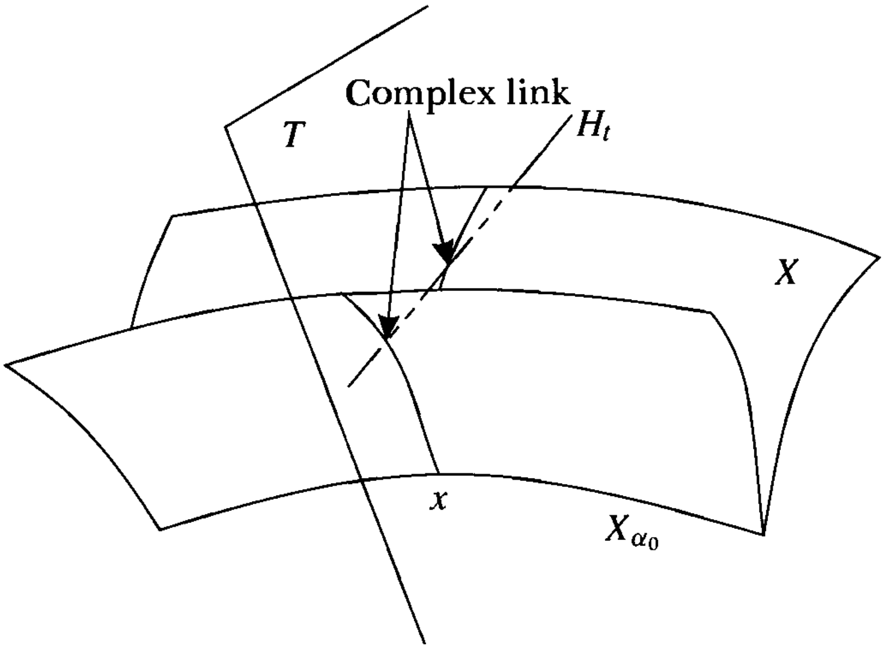
Let $X = \bigcup_\alpha X_\alpha$ be a Whitney-stratified complex analytic space and $f : X \to \mathbf{C}$ a complex analytic function; it has at a point $x \in X$ an isolated singularity if x has a neighborhood U in X such that the restriction $f|X_\alpha$ to each stratum has no singularity on $X_\alpha \cap U \setminus \{x\}$. Note that this depends on the stratification: we may add a nonsingular stratum T contained in X_α such that $f|T$ does not have an isolated singularity at a point $x \in T$. However, since there exists a unique minimal Whitney stratification, (see [T8]) we can give a meaning to the concept of a function with isolated singularity on any (reduced) complex analytic space.

If we imbed locally (X, x) in $(\mathbf{C}^N, 0)$, this notion is equivalent to a conormal condition: Let V be a neighborhood of 0 in $\mathbf{C}^N$ in which X is closed, and let us consider an analytic function $\hat{f} : V \to \mathbf{C}$ extending the restriction $f|V \cap X$. By Whitney's condition, the union of the (nonsingular) conormal spaces $\bigcup_\alpha T^*_{X_\alpha \cap V} V$ is closed in the cotangent space T^*V, and the isolated singularity condition is equivalent to the fact that the image $\Lambda_f \subset T^*V$ of the differential $d\hat{f}$ and $\bigcup_\alpha T^*_{X_\alpha \cap V} V$ intersect only at the point $(x, d\hat{f}(x)) \in T^*V$.

A fundamental example of an isolated critical point of a function on a stratified space is that of a Morse singularity: the point x is a Morse singularity if the intersection of Λ_f and $\bigcup_\alpha T^*_{X_\alpha \cap V} V$ is a transversal intersection of nonsingular spaces at the point $(x, d\hat{f}(x)))$; this implies in particular that the point $(x, d\hat{f}(x))$ lies in only one of the Lagrangian subvarieties $T^*_{X_\alpha \cap V} V$, corresponding to the stratum X_{α_0} which contains x and does not lie in the closures of any of the other $T^*_{X_\alpha \cap V} V$. If $d\hat{f}(x) \neq 0$, i.e., if $d\hat{f}$ has rank 1 at x, then its kernel is tangent to X_{α_0} and is not a limit at x of tangent hyperplanes to the other strata. If $d\hat{f}(x) = 0$, the point we consider is $(x, 0)$ and is in the closure of all the conic Lagrangian varieties $T^*_{X_\alpha \cap V} V$. So in this case x is a Morse point only if X has only one stratum, i.e., is nonsingular at x, and f has at x a Morse singularity in the ordinary sense. Goresky and MacPherson have developed (see [G-M]) a Morse theory for stratified spaces, which uses the fact that on a stratified space, Morse functions in this sense are dense. An important object in the study of the topology of stratified spaces and in their Morse theory is the following, introduced in [L-T] and [G-M], and called in [G-M] the *complex link* of a stratum.

Let $X = \bigcup_\alpha X_\alpha$ be a Whitney-stratified complex analytic space, and x a point of X. Let us fix a local embedding $(X, x) \subset (\mathbf{C}^N, 0)$ of X near x, and an affine space $T \subset \mathbf{C}^N$ through 0, of dimension equal to the codimension of the stratum X_{α_0} containing x and transversal to that stratum.

Let $\mathbf{B}_\epsilon(x)$ denote the ball with center x and radius ϵ in $\mathbf{C}^N$. It can be shown (c.f. [L-T], [G-M]) that for a sufficiently general linear form $\ell\colon (\mathbf{C}^N, 0) \to (\mathbf{C}^{\dim X_{\alpha_0}+1}, 0)$, the homotopy type $\mathcal{L}(X, X_{\alpha_0})$ of the intersection $\mathbf{B}_\epsilon(x) \cap H_t \cap X$, where $H_t = \ell^{-1}(\ell(x) + t)$ is, for sufficiently small ϵ and $0 < |t| \ll \epsilon$, independent of all the choices made, and an analytic invariant of the germ (X, x), which moreover is constant as x varies on a Whitney stratum. It is called the *complex link* of the stratum X_{α_0} in X.



Grothendieck in SGA 2 defined the *rectified homotopical depth* rhd(X, x) (resp., *rectified homological depth* rHd(X, x)) of a complex space X at a point $x \in X$ as follows:

DEFINITION 14. One has rhd$(X, x) \geq n$ (resp., rHd$(X, x) \geq n$) if for any locally closed irreducible complex analytic subspace Y of X containing x, there exists an open neighborhood U of x in X such that any point $y \in Y \cap U$ has a fundamental system of neighborhoods U_α such that the topological pairs $(U_\alpha, U_\alpha \backslash Y)$ are $(n - \dim_x Y - 1)$-connected (resp., the integral homology $H_k(U_\alpha,\ U_\alpha \setminus Y)$ vanishes for $0 \leq k \leq n - \dim_x Y - 1$). The rectified depth is the largest n for which such an inequality holds. The rectified depth of X is the infimum of the depths at the points of X.

By a theorem of Hamm-Lê ([H-L]), to compute the relative homotopical or homological depth of X, one needs to test the vanishing of relative homotopy or homology groups at a point of each (connected) stratum of a Whitney stratification of X.

It is easy to see that one has the inequalities

$$\dim_x(X) \geq \mathrm{rHd}(X, x) \geq \mathrm{rhd}(X, x).$$

Lê proved the following:

THEOREM 15 (Lê, [Lê8]). Let $X = \bigcup_\alpha X_\alpha$ be a Whitney-stratified complex analytic space and x a point of X. The following conditions are equivalent:

(i) The equality $\mathrm{rhd}(X, x) = \dim_x(X)$ (resp., $\mathrm{rHd}(X, x) = \dim_x(X)$) holds.
(ii) The space X is equidimensional in a neighborhood of x and for any stratum X_α containing x in its closure, a complex Link $\mathcal{L}(X, X_\alpha)$ has the homotopy type (resp., the homology) of a bouquet of spheres of real dimension $\dim_x(X) - \dim X_\alpha - 1$.

Moreover, the equality $\mathrm{rhd}(X, x) = \dim_x(X)$ is equivalent to the following assertion: The space X is equidimensional in a neighborhood of x and, given a function $f\colon (X, x) \to (\mathbf{C}, 0)$ which has an isolated singularity with respect to the Whitney stratification, its Milnor fiber (see Theorem 1) has the homotopy type of a bouquet of spheres of dimension $\dim_x X - 1$.

These statements provide a rather complete generalization to the case of non-isolated singularities of the statements of the first paragraphs, and a rich supply of Bouquets in analytic geometry.

The spaces which satisfy the last conditions are called *Spaces with Milnor's property* in [Lê8] which I shall shorten to *Milnor spaces*, and they are topological analogues of the Cohen-Macaulay spaces (they satisfy vanishing conditions for the constant sheaf instead of the sheaf or holomorphic functions). In fact, one can prove that spaces which are local complete intersections are Milnor spaces. For a d-dimensional projective Milnor space $V \subset \mathbf{P}^N$, the Lefschetz theorem holds with the same bounds as in the nonsingular case: for a hyperplane H one has (see [H-L]):

$$\pi_k(V, V \cap H, x) = 0 \qquad \text{for any } k < d.$$

We have the following characterization of Milnor spaces:

PROPOSITION 16 (Lê, [Lê8]). A reduced complex analytic space X is a Milnor space if and only if for some Whitney stratification the complex link $\mathcal{L}(X, X_\alpha)$ has the homotopy type of a Bouquet for all strata X_α of dimension 0, and the homology type of a Bouquet for all other strata. This property then holds for all Whitney stratifications.

I think that it would be worthwhile to develop the geometric theory of *Milnor maps* $f : X \to Y$ between complex spaces, which are those for which the local rectified homotopical depth of X at $x \in X$ is the sum of the rectified homotopical depth of Y at $f(x)$ and of the rectified homotopical depth of $f^{-1}(f(x))$ at x; they would be the analogues of the flat maps of algebraic geometers.

Let me note also that the problem of describing the minimal stratification of a reduced complex space which satisfies the local topological triviality of the closure of each stratum condition along strata in its boundary is still open.

Conclusion

I have tried to present some of the developments to which Milnor's work in singularities has significantly contributed. I focused on the introduction of a geometric framework for the local study of singularities, the introduction of the Milnor number which became a fundamental invariant in part because of its triple topological, geometrical (as multiplicity of a discriminant), and algebraic nature, and the Bouquet theorem which is the first apparition of a fundamental local property of certain singular spaces, analogous in the sense of Grothendieck's SGA 2 to the Cohen-Macaulay property of the structure sheaf. The original condition of isolated singularity can now be forgotten, by and large, thanks to the use of Whitney and Thom stratifications.

This presentation is very far from being complete; any reasonably complete presentation would be of book length. One should add much more material, for example on the theory of equisingularity of Zariski ([Z2]), on quasi-homogeneous and other special singularities (especially the numerous works on the description of the local monodromy and the Brieskorn lattice), on the modularity of isolated singular points, the geometry of the discriminant of versal deformations and unfoldings, the μ-constant stratum of [T1] and their descriptions for special singularities and for those which are "sufficiently general" among those having a given Newton polyhedron, especially by Arnol'd and his school. One should also add to the last paragraph of this text the description of Whitney conditions in general by $\mu^{(*)}$-constant type conditions and the connection of the Milnor number with characteristic classes of singular spaces. In a more computational vein one should mention the role of the Milnor number as a measure of codimension of a singularity in Thom and Mather's theory of unfoldings, and in the theory of sufficiency of jets.

One could also describe many fundamental developments originating in the local viewpoint pioneered by Milnor, such as sheaves of vanishing cycles, mixed Hodge structures on their cohomology, the $\mathcal{D}$-module description of vanishing cycles by duality, the generalizations of the local Gauss-Manin connection of Brieskorn, the study of local systems and the relationship between the rectified homological depth and the perversity (in the sense of intersection homology) of the constant sheaf. Milnor's book started a stream, which is still running along merrily after mixing with many others.

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