THE MEANING OF THE FORMULA

A tribute to Gilles Châtelet

B. Teissier

Université Paris Cité and
Sorbonne Université, CNRS,
IMJ-PRG, F-75013 Paris,
France.

bernard.teissier@imj-prg.fr

An offensive philosophy cannot content itself with endlessly rambling on about the 'status' of scientific objects. It must stand resolutely in the vanguard of the obscure, not seeing the irrational as 'diabolical' and resistant to articulation, but as the means by which new dimensions can emerge.

Gilles Châtelet, *Les enjeux du mobile*.

Gilles Châtelet (1945-1999) was a mathematician, physicist and philosopher whose philosophy of science I greatly appreciate, as well as the passion with which he developed it, particularly in [C].

René Thom (1923-2002) was an extraordinary mathematician and philosopher who used mathematics to construct qualitative models of changes in the state of systems as a function of changes in the parameters on which these systems depend. These models are universal in the sense that they do not depend on particular systems but organise their evolution. Some of his ideas have become famous under the name of *Catastrophe Theory*. His fundamental work on the subject is [Th1].

Abstract: A reflection on the relation between rationality and the meaning of the tools it uses, centered on the approaches of Gilles Châtelet and René Thom.

1 INTRODUCTION

At last! more than two centuries after Kant, our culture is beginning to seriously question the dominance of what is commonly called *rationality* and its limits. One symptom of the relevance of this questioning is the resurgence of anti-scientific schools of thought, sometimes of religious origin. It seems to me that part of our society's disaffection with the sciences stems from a misunderstanding of the nature of rational thought, which is perceived as having a strong inhuman component. Along with René Thom, Gilles Châtelet is one of those who have had the courage, in modern times, to emphasise the extent to which the most rational sciences, and in particular physics and mathematics, are also human sciences, i.e., dependent on human nature.

As far as Thom was concerned, in fact, it was more a question of creating a rational language, of mathematical origin, sufficiently flexible and rich to be able to talk about phenomena, such as the creation of language or the stages of embryogenesis, which seemed to escape classical rational thought. And this led him to a vision of the meaning of forms and their dynamics that was so innovative that it is still poorly understood except by a few experts. Gilles Châtelet, seemingly in the opposite direction, was relentless in his search for the forks in the road of thought, those mysterious, misty places where new ideas and great questions are born. And he sought them in our humanity, in our gestures rather than in formalism, while recognising the heuristic value of the latter.

Both, apparently looking for different things, were in fact two of the great thinkers of an *adult* rationality, if we accept that rationality must be understood as a *dialogue following the rules of rational thought* between man and his environment. There are other modes of dialogue, such as art and poetry.

From this point of view, reductionism is the infantile disease of rationality that consists in asserting that the dialogue must end one day, and that man will have the last word. This is contrary to all plausibility, even if man has, in areas where his perception of the world is on the right scale and even in certain areas where it is not, said many things that he finds very interesting, and that nature does not contradict.

This desire to have the last word obviously greatly diminishes the quality of the dialogue, and it has another consequence: it encourages man to place himself in the role of God[1] , and thus to deny his humanity in order to be quite certain of always being right (this is also known as the quest for objectivity, in the name of which some people shoot at anything that seems to come under the heading of introspection or psychology). An infantile disease, I tell you!

Thom in [Th2] and Châtelet in [C] had clearly understood that the real problem of modern rationality, which has tools for founding truth, is the problem of the foundations of meaning. It is these foundations which give the dialogue its quality and encourage us to pursue it. The important thing is that they are useful for the dialogue, which implies on the one hand that we can use them for reliable (verifiable) intellectual constructs and on the other that they have meaning for us. The objects and concepts of science must be compatible with these two requirements.

In fact, this compatibility is rather analogous to that which exists between the administrative life of a citizen and his real life. In this case, the citizen, *alias* the mathematical or physical object or concept, *leads a double life* in our scientific imagination: one life as an axiomatically defined object, and another far more exciting life as an object endowed with meaning. It is above all in this real life that it is used in dialogue with the world.

[1] As René Char wrote*: It seems that God always has the last word, but he says it in such a low voice that no one ever hears him.*

And Gilles Châtelet tirelessly tracked down the occurences of extensions to the world of ideas of our experience of space and movement, which he calls "gestures" for short. Such extensions, for example by an 'allusive stratagem' or a concentration of ambiguity, provoke the simultaneous creation, often by a partly unconscious process, of a mathematical or physical object and its meaning. He understood that the objects and concepts we create are there first and foremost to carry a meaning, to perform a 'gesture' in the mathematical or physical domain or to resolve an ambiguity, and only very distantly afterwards to have a 'status' that enables them to be bearers of truth.

Once you have seen things in this way, you re-read the history of scientific ideas with fascination and, like Châtelet, you follow Oresme in his attempt to define speed rigorously in a framework where the division of magnitudes of different natures does not exist, and Grassmann in his capture of space in the nets of the algebra he creates for this purpose. One of the intuitive foundations of this algebra is that a rectangle *must* be a product, while Oresme is constrained by the idea that a product *must* be an area. Both refer to the Greek view that we 'see' a product of two lengths as the area of the rectangle that has them as sides. But for Grassmann, the product of vectors *generates* space; the Greek idea has been transfigured. We are really rediscovering rationality as a dialogue in which concepts and objects are created by the effect of human impulses very strongly linked to our perception of the world. To question the objectivity of these concepts and objects is quite simply irrelevant compared to what drives them. It is the infantile attitude mentioned above.

Conceiving of space as a product is one of the strongest ideas in geometry, one that drives the Cartesian vision as well as Riemann's local definition and, in its infinitesimal form, the definition of differential forms, which also contains the orientation of space necessary for the theory of integration. And nowhere have I found more forcefully expressed than in Châtelet's work the idea that it is really in this type of conception that the real driving forces of scientific construction lie.

2 WHERE ARE THE FOUNDATIONS?

In my opinion, the only criticism that can be made of Châtelet's presentation is that he does not specify the origin of the mental operations or gestures that he evokes so well to explain the 'true' meaning of the constructions of physics. He does place the origin in our 'being in the world' but stops there.

I have suggested elsewhere ([T1], [T2]) that it is useful to look for the source of meaning in the complex structure of our perceptual system, in the links between our different perceptions, particularly visual, motor and vestibular, which integrate them into a single perception of the world around us. To this structure must be added unconscious judgements and impulses such as those to determine causes or origins, to make analogies, to complete what is incomplete.

Certain proto-mathematical objects are very probably accessible to primates, and I like to say that a mathematician understands a demonstration when he or she has succeeded in explaining the situation to the primate inside him or her. For me, the 'primate' represents the *unconscious* sedimentation of our ancestors' experiences of the world into the structure of our thinking. For example, rational mechanics associates with any dynamic system (for example a collection of spinning tops spinning on each other or a sun/planets system), a space in which the temporal evolution of the system is represented by a curve, a trajectory, and this has a strong meaning for our primate, who has been throwing stones for hundreds of thousands of years. During their apprenticeship, and indeed throughout their lives, mathematicians learn to make sense of increasingly elaborate objects using such examples. And understanding a proof literally means that it makes sense to the person looking at it. This is very different from logical verification, and much more complicated to explain!

3 AFTERWORDS: MATHEMATICIANS NEED A SENSE OF THE FORMULA

Along with René Thom, Gilles Châtelet embodied the slogan I like to repeat:
"The mathematician must have a sense of the formula."[1]
This sentence, which is intended to be creatively ambiguous in the sense of William Empson in [E], suggests that the mathematician must not only know how to concentrate a lot of information in a few concepts and signs or formulas, but also know the origin of the meaning of each concept and each term in a formula. Those who reflect on their discipline must first try to determine, as mathematicians, the meaning of each concept or formula, before trying to make it the subject of a philosophical discourse.

The formulas relate to quantities that condense a very large amount of mathematical or physical information, such as volume, curvature, the number and nature of solutions to an equation, or energy, mass, temperature and many other concepts.

A good example (which remains to be treated from the point of view proposed here) is the theorem of Hopf (see the excellent book [M])

$$\chi(X) = \sum_{xi \in \mathrm{Zer}(v)} \mathrm{Ind}_{xi}\, v\,,$$

which asserts, for a compact differential variety without boundary X, the equality of its Euler-Poincaré characteristic and of the sum of the indices at its singular points (the points where it vanishes) of a differentiable and sufficiently general field v of vectors tangent to the variety.

The term on the left is topological, the one on the right is differential. In particular, the result implies that there is no vector field without zero on even-dimensional spheres. So, it is quite an elaborate result, which has had a prodigious mathematical legacy.

But how is this meaning to be determined? It is to this question that Châtelet provides some very original and valuable answers, of which I have tried to give an idea above: a concept may be the abstraction of a 'gesture', i.e., an atavistic experience of our primate, such as throwing a stone, or it may resolve an apparent contradiction. In all cases, it has a genesis. It seems a long time ago (in the 1930s) that Bourbaki could

---

[1] "Avoir le sens de la formule" translates in English as "To have a way with words" so that the sentence "Le mathématicien doit avoir le sens de la formule" presents the same ambiguity in a more amusing way.

write "Le traité prend les Mathématiques à leur début..." (The treatise takes Mathematics at their beginning) when writing Chapter 1 of the first Book of "Elements de Mathématique", which deals with the theory of sets, then barely forty years old.

Mathematics has no more beginnings than the human species; it has a history and a dynamic that are studied by specialists. But if, like Châtelet, we are also interested in their 'metaphysics', in what they reveal about our being in the world - in short, in what a true philosophy of mathematics should be concerned with - then we must not only revisit history, but also realise that he is describing the coastline of a continent where almost everything remains to be discovered.

Bibliography

[C] Châtelet, G., 1993, *Les enjeux du mobile, Mathématique, Physique, Philosophie*, Coll. Des Travaux, Le Seuil. English translation: *Science and Philosophy*, *Figuring Space, Mathematics, Physics and Philosophy*, Kluwer Academic Publishers, 2000.

[E] Empson, W., 1947, *Seven types of ambiguity*, Chatto and Windus, London.

[M] Milnor, J. *(1997) [1965]. Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics. Princeton, NJ: Princeton University Press.

[T1] Teissier, B., 2005, *Protomathematics, perception and the meaning of mathematical objects*, in "Images and Reasoning", edited by P. Grialou, G. Longo, M. Okada, CIRM, Keio University, Tokyo 2005.

[T2] Teissier, B., 2007, *Géométrie et Cognition : l'exemple du continu*, proceedings of the CNRS-LIGC Ecole thématique "Logique et interaction ; vers une géométrie du cognitif", Cerisy September 2006, directed by J.-B. Joinet and S. Tronçon, Hermann, "Visions des sciences", Paris 2009. Available from:
https://webusers.imj-prg.fr/~bernard.teissier/documents/Cerisy06final5.pdf

[Th1] Thom, R., 1977, *Stabilité structurelle et morphogénèse*, InterÉditions, Paris.

[Th2] Thom, R., 1988, *Esquisse d'une Sémiophysique*, InterEditions, Paris.