Foreword

The main ideas of the proof of resolution of singularities of complex-analytic spaces presented here were developed by Hironaka in the late 1960's and early 1970's. The difficulty of presenting with sufficient clarity these subtle and original ideas, as well as the other interests of the co-authors, are some of the causes of the unusually long delay before the final redaction. During this period, a number of proofs of resolution of singularities in all dimensions by compositions of permissible blowing-ups, all inspired by Hironaka's general approach, have appeared, the validity of some of them extending beyond the complex-analytic case. The proof has been so streamlined that while in the 1960's and 1970's it was considered to be one of the most difficult proofs produced by Mathematics it can now be the subject of an advanced University course. In the bibliography below one finds a small sample of proofs and expositions.

So, setting apart the epilogue on singularities of foliations, is this book of historical interest only? We think that it is not at all the case, and that reading and perusing it will be very rewarding for any mathematician interested in the subject of resolution of singularities, and more generally in singularity theory. Unlike most of the recent proofs, the redaction is not oriented towards a proof of a canonical or algorithmic resolution of singularities, or even towards a most "economical" proof.

What is presented in this book is in fact a masterly study of the infinitely near "worst" singular points of a complex-analytic spaces obtained in finite sequences of permissible blowing-ups and of the way to tame them using subspaces of the ambient space having "maximal contact" with the singular space at a given point. This taming proves by an induction on the dimension that there exist finite sequences of permissible blowing-ups at the end of which the worst infinitely near points have disappeared, and this is essentially enough to obtain resolution of singularities.

Here permissible means that the center of blowing-up is non singular and contained in a "Samuel stratum" of X, where the Hilbert–Samuel stratum of the local rings of the points of X is largest (in the lexicographic order). For infinitely near points "worst" is taken as meaning that the Hilbert–Samuel function is the same as that of the worst points on X. The totality of all such infinitely near worst points is of a nature similar to the Zariski–Riemann manifold or, more relevant in this case, the *voûte étoilée*, the analogue in complex-analytic geometry of the Zariski–Riemann manifold created by Hironaka. The fact that the structure of this extremely complicated object can be described, at least locally, by resolutions of the singularities of an *idealistic exponent*, which is slightly more general than a singular space, living in a non singular space of lower dimension with maximal contact with the given space, is the very foundation of Hironaka's method in his 1964 proof for algebraic varieties and still appears, under various guises since maximal contact does not exist as such, in most attempts to prove resolution in positive characteristic. It appears here in a very purified form which, we believe, makes it much easier to understand the fundamental mechanisms at work in all proofs of birational resolution in high dimensions.

In the complex-analytic case, in comparison to the algebraic case, there is an essential new difficulty due to the fact that permissible local centers of blowing-up do not globalize, even at the price of adding singularities of Zariski closures and complicating the induction. The recent proofs use some form of uniqueness in the local construction, for example precise conditions on the centers of blowing-ups using the exceptional divisors created by previous blowing-ups, which imply, according to a classical scheme, that locally defined centers will glue up. Hironaka's method is completely different, and may serve as a template for solving certain globalization problems where there is no natural local uniqueness, by creating new objects (in this case *groves* and *polygroves*) which contain more information and glue up naturally.

If we add that this book contains an elegant presentation of all the prerequisites of complex-analytic geometry, including basic definitions and theorems needed to follow the development of ideas and proofs, and that the epilogue presents the use of similar ideas in the resolution of singularities of complex-analytic foliations, we have given our reasons to find this book so useful and interesting. We think it will be a great help to the members of the younger generation who wish to understand one of the most fundamental results in algebraic and analytic geometry and invent possible extensions and applications of the methods created to prove it.

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