Resolving singularities of plane analytic branches with one toric morphism

Rebecca Goldin and Bernard Teissier

To Heisuke Hironaka

Abstract

Let $(C,0)$ be an irreducible germ of complex plane curve. Let $\Gamma \subset \mathbf{N}$ be the semigroup associated to it and $C^\Gamma \subset \mathbf{C}^{g+1}$ the corresponding monomial curve, where g is the number of Puiseux exponents of $(C,0)$. We show, using the specialization of $(C,0)$ to $(C^\Gamma,0)$, that the same toric morphisms $Z(\Sigma) \to \mathbf{C}^{g+1}$ which induce an embedded resolution of singularities of $(C^\Gamma,0)$ also resolve the singularities of $(C,0) \subset (\mathbf{C}^{g+1},0)$, the embedding being defined by elements of the analytic algebra $\mathcal{O}_{C,0}$ whose valuations generate the semigroup Γ.

1 Introduction

Some corrections to the original text are made in red.

In the last few years Mark Spivakovsky has proposed a program to prove the resolution of singularities of excellent schemes. A part of this program is a new look at Zariski's local uniformisation theorem for arbitrary valuations. A fundamental object of study in this approach is the graded ring associated to the filtration of a local domain R naturally provided by a valuation of R.

In the special case where R is the local analytic algebra $\mathcal{O}$ of a plane branch C, with its unique valuation, this graded ring was studied in detail by Monique Lejeune-Jalabert and the second author (see [L-T], [T1]), as a special case of the $\overline{\mathrm{gr}}_I R$ appearing in their study of the "$\overline{\nu}_I$-filtration" of a ring (roughly, by the integral closures of the fractional powers of an ideal I of R).

Recall that one calls *branch* a germ of analytically irreducible excellent curve. We will in this paper deal only with complex analytic branches.

The following facts which appeared at that time in that special case may be relevant to the understanding of the role of the graded ring in the general case:

– The graded ring is the ring of the monomial curve C^Γ with the same semigroup Γ as the given plane branch (see definitions below).

– The generators of the semigroup are the intersection numbers with the given plane branch C of a transversal non-singular germ $x = 0$ and of plane branches $f_j(x, y) = 0$ with a smaller number j , $0 \leq j \leq g-1$, of Puiseux exponents and having with C maximal contact in the sense defined by M. Lejeune-Jalabert (see [Z], pp.16-17, [L-J]). In particular the initial forms in the graded ring of the images in the algebra $\mathcal{O}$ of the equations of these branches generate it as a $\mathbf{C}$-algebra.

– There exists a one parameter deformation of C^Γ having all its fibers except the special one isomorphic to C, and this deformation is equisingular in the sense that it has a simulateous resolution of singularities by normalization; the normalization of the total space of the family is non-singular and induces normalization for each fiber (see [T1]).

We may therefore hope that a process of resolution for C^Γ will induce resolution for C. In this paper we show that the curve C^Γ can be resolved by a single toric modification of its ambient space $\mathbf{C}^{g+1}$ (where g is the number of characteristic Puiseux exponents of the plane branch C) and that some of the toric modifications which resolve C^Γ also resolve the curve C if we view it as embedded in $\mathbf{C}^{g+1}$ by $g+1$ elements of its maximal ideal whose valuations generate the semigroup Γ. This is shown by a generalization of the usual non degeneracy argument, which can be found also in [O2], [O3]. We need more details than in these papers because we want to show that the toric modification resolves not only C^Γ but also C.

From this follows the rather interesting fact that *any plane branch with g Puiseux exponents can be embedded in $\mathbf{C}^{g+1}$ in such a way as to be resolved by one single toric modification, i.e., it is in some sense non-degenerate with respect to its Newton polyhedron*. Moreover, if we denote by $\pi\colon Z \to \mathbf{C}^{g+1}$ a toric map resolving C, and by $\pi|S'\colon S' \to S$ the strict transform by this toric map of a non-singular surface S containing our plane branch C, we can identify $\pi|S'$ to the composition of g toric modifications of non-singular surfaces, thus recovering the known fact that a plane branch can be resolved by a composition of g toric morphisms. The first non trivial example, the simplest branch with two characteristic exponents, is computed in an Appendix.

The description of this composition of toric maps of surfaces obtained in this way has the advantage over the "static" one of [O1] of showing explicitly its analytic dependence on the coefficients of the equation or the parametrization of the curve, in view of the results of [T1], and also of giving a geometric vision of the relationship with the resolution process of the "singular curves with maximal contact" of M. Lejeune-Jalabert, and probably also of the approximate roots à la Abhyankar. Indeed, the embedding of the plane branch C in $\mathbf{C}^{g+1}$ is obtained by adding to the coordinates x, y the images in $\mathcal{O}$ of the equations of plane branches with $< g$ characteristic exponents having maximal contact with C. The disadvantage is that the construction given here is for the time being restricted to irreducible germs.

It is tempting to ask whether in general, *given any germ of an algebraic or analytic space and a valuation of its local algebra, the germ can be embedded in an affine space in such a way as to be non-degenerate with respect to its Newton* *polyhedron and to the given valuation, in the sense that its strict transform is non-singular at the point specified by the valuation in a toric modification of the ambient space subordinate to the Newton polyhedron.*

This would be an effective local uniformization theorem and the results of this paper indicate that perhaps the specialization to the graded algebra is the key to such a result.

In the general case the graded algebra associated with a valuation is not even Nœtherian, and as the reader will see, the proofs in this paper rely on the extensive knowledge we have of the structure of the semigroup of a plane branch, a knowledge essentially lacking in dimension ≥ 3 (the case of dimension 2 is currently under study). In the paper "Valuations, Deformations, and toric Geometry " following this one the second author begins to prepare the way for a proof of the local uniformization theorem along these lines. It is worth noting that one of the results of [L-T] is that the graded algebra associated with the $\overline{\nu}_I$ filtration, i.e., $\overline{\mathrm{gr}}_I R$, is an R-algebra of finite type. Some very interesting finiteness results on the graded algebra associated with a valuation and its relation with Abhyankar's inequality have recently been obtained by O. Piltant ([P]).

This text is based on a lecture given at the singularities Seminar by the second author in January 1994, where examples largely stood in for proofs. The ideas are presented in the framework of complex geometry for simplicity, but the interested reader will see how to transform this into a characteristic-blind resolution for one-dimensional excellent henselian equicharacteristic local integral domains with an algebraically closed residue field.

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2 Puiseux expansion and the semigroup of a curve

Definition. Let $(C, 0) \subset (\mathbf{C}^2, 0)$ be a germ of an irreducible analytic curve (a plane branch) defined by the equation $f(X, Y) = 0, \quad f \in \mathbf{C}\{X, Y\}$. We will call $\mathcal{O} = \mathbf{C}\{X, Y\}/(f)$ the *local algebra of the curve* C, and denote by $\mathbf{m}_{\mathcal{O}}$ its maximal ideal. Note that $\mathcal{O} = \mathbf{C}\{x, y\}$ where x and y are the residue classes in $\mathbf{C}\{X, Y\}/(f)$ of X and Y respectively.

According to Newton and Puiseux, there is a parametric representation of C, called the *Newton-Puiseux expansion*, i.e an injection of the algebra $\mathcal{O}$ into $\mathbf{C}\{t\}$ described in suitable coordinates X, Y, t by:

$$\begin{aligned} X &= x(t) = t^n & (*) \\ Y &= y(t) = \textstyle\sum_{i \geq m} a_i t^i \quad \text{with m > n.} \end{aligned}$$

The Puiseux expansion of x and y; since the two series converge for small enough $\|t\|$, say $\|t\| < \epsilon$. Denoting by $\mathbf{D}(0, \epsilon)$ the disk with center 0 and radius ϵ in $\mathbf{C}$, the image of the map $\mathbf{D}(0, \epsilon) \to \mathbf{C}^2$ defined by $x(t), y(t)$ is a representative of the germ $(C, 0) \subset (\mathbf{C}^2, 0)$. From the integer n and the exponents appearing in the expansion of $y(t)$ one can extract the Puiseux characteristic exponents, which characterize the topological type of a small representative of our branch. (see [Z]). In particular, we may choose the coordinates X, Y in such a way that the exponent m appearing in the expansion of $y(t)$ is not divisible by the multiplicity n. It is then generally denoted by β_1. We shall do so from now on. More generally, any analytically irreducible germ of a curve has a normalization isomorphic to $\mathbf{C}\{t\}$, and its algebra is an analytic subalgebra of $\mathbf{C}\{t\}$.

Given this parametrization, certain properties of an analytic branch C, plane or not, can be described using the valuation on the algebra $\mathcal{O}$ induced via the injection $\mathcal{O} \hookrightarrow \mathbf{C}\{t\}$ by the t-adic valuation of $\mathbf{C}\{t\}$.

Definition. Given a branch $(C, 0)$ and its algebra $\mathcal{O} \hookrightarrow \mathbf{C}\{t\}$, let ν be the t-adic valuation on $\mathbf{C}\{t\}$. We define the *semigroup* Γ of $\mathcal{O}$ to be

$$\Gamma = \{\nu(\xi) : \ \xi \in \mathcal{O} \setminus \{0\}\}.$$

We follow here the common usage, which is to denote by the same letter Γ the semigroup deprived of its zero element, i.e., $\{\nu(\xi) : \ \xi \in \mathbf{m}_{\mathcal{O}}\}$ We say that the branch $(C, 0)$ *has the semigroup* Γ.

The semigroup Γ is finitely generated and has finite complement in $\mathbf{N}$ (see [Zariski, II.1] for the case of plane branches), and if $(C, 0)$ is a plane branch, its minimal set of generators $\{\overline{\beta}_0, \ldots, \overline{\beta}_g\}$ can be uniquely determined by:

(a) $\overline{\beta}_0 = n$

(b) $\overline{\beta}_i = \min\{z \in \Gamma \mid \ z \ \notin < \overline{\beta}_0, \ldots, \overline{\beta}_{i-1} >\}$

where $< \overline{\beta}_0, \ldots, \overline{\beta}_{i-1} > \subset \mathbf{N}$ is the semigroup generated by $\{\overline{\beta}_0, \ldots, \overline{\beta}_{i-1}\}$ (in particular, $\overline{\beta}_1 = \beta_1$). We then have that

1. $\{\overline{\beta}_0, \ldots, \overline{\beta}_g\}$ generates Γ,
2. $\overline{\beta}_0 < \cdots < \overline{\beta}_g$, and
3. $\gcd(\overline{\beta}_0, \ldots, \overline{\beta}_g) = 1$

We call the vector $(\overline{\beta}_0, \ldots, \overline{\beta}_g) \subset \mathbf{Z}^{g+1}$ the *weight vector*. Of course the datum of its semigroup Γ does *not* determine a branch up to analytic isomorphism. In fact, the datum of the semigroup Γ of a plane branch is equivalent to the datum of its Puiseux characteristic exponents (see [Z]). Among all the branches, plane or not, having a given semigroup, one may single out the affine curve $C^\Gamma \subset \mathbf{C}^{g+1}$ defined by the parametrization:

$$C^\Gamma : \ u_i = t^{\overline{\beta}_i} \quad 0 \leq i \leq g. \tag{1}$$

Consider $(C^\Gamma, 0)$ as the germ of a curve. Its algebra

$$\mathcal{O}_{C^\Gamma,0} = \mathbf{C}\{t^{\overline{\beta}_0}, \ldots, t^{\overline{\beta}_g}\} \subset \mathbf{C}\{t\}$$

clearly has semigroup Γ. It is shown in [T1] that all branches having a given semigroup Γ are complex analytic deformations of $(C^\Gamma, 0)$ and these deformations are in some sense *equisingular*.

Definition. Given a branch $(C, 0)$ and its semigroup Γ, we call the curve C^Γ described above by the parametrization (1) the *monomial curve* associated to C. If $(C, 0)$ is a plane germ, the monomial curve C^Γ is a complete intersection in $\mathbf{C}^{g+1}$, where g is the number of Puiseux characteristic exponents of $(C, 0)$.

3 Deforming Curves

The curve C^Γ mentioned above, as we shall see, is the center of a finite-dimensional flat family of branches which contains (up to complex analytic isomorphism of germs) every germ of analytically irreducible curve (or branch) with semigroup Γ.

We begin by showing (see [T1]) that given any branch $(C, 0)$, one can construct explicitly a one-parameter analytic deformation of $(C^\Gamma, 0)$, whose general fibres are isomorphic to the curve $(C, 0)$. Let us take for simplicity the case of a plane branch. Consider the parametric representation of the curve C as in (*). After a change of coordinates we may assume that it has the form :

$$\begin{aligned} x(t) &= t^{\overline{\beta}_0} \\ y(t) &= t^{\overline{\beta}_1} + \textstyle\sum_{j > \overline{\beta}_1} c_j^{(1)} t^j. \end{aligned}$$

Note that $\overline{\beta}_0$ is the multiplicity n of C. By the definition of the $\overline{\beta}_i' s$, there exist elements $\xi_i(t) \in \mathcal{O} = \mathbf{C}\{x, y\}$, $2 \le i \le g$ such that

$$\xi_i(t) = t^{\overline{\beta}_i} + \sum_{j > \overline{\beta}_i} c_j^{(i)} t^j.$$

where $\xi_0 = x(t)$. Now consider the family, parametrized by v, of curves parametrized by t, in $\mathbf{C}^{g+1} \times \mathbf{C}$:

$$\mathcal{X}(v) \left\{ \begin{aligned} x &= t^{\overline{\beta}_0} \\ y &= t^{\overline{\beta}_1} + \textstyle\sum_{j > \overline{\beta}_1} c_j^{(1)} v^{j - \overline{\beta}_1} t^j \\ u_2 &= t^{\overline{\beta}_2} + \textstyle\sum_{j > \overline{\beta}_2} c_j^{(2)} v^{j - \overline{\beta}_2} t^j \\ \cdot &= \cdot \\ \cdot &= \cdot \\ \cdot &= \cdot \\ u_g &= t^{\overline{\beta}_g} + \textstyle\sum_{j > \overline{\beta}_g} c_j^{(g)} v^{j - \overline{\beta}_g} t^j \ . \end{aligned} \right.$$

Proposition 3.1. *The branch $(\mathcal{X}(0), 0)$ is isomorphic to $(C^\Gamma, 0)$, and for any $v_0 \neq 0$, the branch $(\mathcal{X}(v_0), 0)$ is isomorphic to $(C, 0)$ by the isomorphism $u_i(t) \mapsto v_0^{-\overline{\beta}_i} u_i(v_0 t)$ (where we set $x = u_0,\ y = u_1$).*

Proof. For $v = 0$ it is obvious, and for $v \neq 0$, since $\mathcal{O}$ is generated as an analytic subalgebra of $\mathbf{C}\{t\}$ by $x(t), y(t)$, it follows from the fact that the $\xi_i(t)$ are in $\mathcal{O}$ (see [T1]). ◌

The semigroups of plane branches are characterized by the following arithmetical properties (see [T1]); let us set $e_i = \gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_i), e_{i-1} = n_i e_i$. Then

$$\begin{array}{ll} n_i\overline{\beta}_i \in & < \overline{\beta}_0, \ldots, \overline{\beta}_{i-1} > \\ n_i\overline{\beta}_i < & \overline{\beta}_{i+1} \end{array}$$

The first relation implies the existence of integers $\ell_j^{(i)}$ $, 1 \leq i \leq g$ $, 0 \leq j \leq i-1$, such that

$$n_i\overline{\beta}_i = \ell_0^{(i)}\overline{\beta}_0 + \cdots + \ell_{i-1}^{(i)}\overline{\beta}_{i-1}$$

So that on the curve C^Γ as parametrized, we have equations:

$$\begin{array}{lclclcl} f_1 & = & u_1^{n_1} & - & u_0^{\ell_0^{(1)}} & = & 0 \\ f_2 & = & u_2^{n_2} & - & u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} & = & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ f_g & = & u_g^{n_g} & - & u_0^{\ell_0^{(g)}} \cdots u_{g-1}^{\ell_{g-1}^{(g)}} & = & 0. \end{array} \qquad (**)$$

It is shown in [T1] that these equations define the curve C^Γ, which is therefore a complete intersection in $\mathbf{C}^{g+1}$. Now using the theory of miniversal deformations (see [T3]), we have:

Theorem 3.2. [Teissier] *There exists a germ of a flat morphism*

$$p\colon\ (\mathcal{X}_u, 0) \to (\mathbf{C}^{\tau_-}, 0)$$

endowed with a section σ such that $(p^{-1}(0), 0)$ is analytically isomorphic to $(C^\Gamma, 0)$, and for any representative of the germ of the morphism p and any branch $(C, 0)$ with semigroup Γ, there exists $v_C \in \mathbf{C}^{\tau_-}$ such that $(p^{-1}(v_C), \sigma(v_C))$ is analytically isomorphic to $(C, 0)$. Moreover, $(\mathcal{X}_u, 0)$ is embedded in $(\mathbf{C}^{g+1} \times \mathbf{C}^{\tau_-}, 0)$ in such a way that p is induced by the second projection.

Proof. See [T1]; this is the miniversal constant semigroup deformation of C^Γ, and it corresponds to the part of the base of the miniversal (equivariant) deformation of C^Γ which is spanned by the coordinates with negative weight. It is shown in [T1], 4.4 that $\tau_- \leq \#(\mathbf{N} \setminus \Gamma)$.

More precisely the miniversal constant semigroup deformation of C^Γ is a map $p\colon\ \mathcal{X}_u \to \mathbf{C}^{\tau_-}$, where $\mathcal{X}_u$ is embedded in $\mathbf{C}^{g+1} \times \mathbf{C}^{\tau_-}$ in such a way that p is induced by the second projection, and defined by the equations

$$\begin{array}{lclcl} F_1 & = & f_1 + \sum_{r=1}^{\tau_-} v_r \phi_{r,1}(u_0,\ldots,u_g) & = & 0 \\ \cdot & & \cdot & = & \cdot \\ \cdot & & \cdot & . & \cdot \\ \cdot & & \cdot & . & \cdot \\ F_g & = & f_g + \sum_{r=1}^{\tau_-} v_r \phi_{r,g}(u_0,\ldots,u_g) & = & 0 \end{array}$$

where the $\phi_{r,j}$ are polynomials and each monomial in $(u_0,\ldots,u_g)$ appearing in $\phi_{r,j}$ is of weight $> n_j\overline{\beta}_j$ when each u_k is given the weight $\overline{\beta}_k$; the equation f_j is then homogeneous of weight $n_j\overline{\beta}_j$. In fact one may choose the vectors ϕ_r to have only one nonzero coordinate which is a monomial $\phi_{r,j}$. Moreover, the images vectors $\phi_1,\ldots,\phi_{\tau_-}$ in $\mathbf{C}[C^\Gamma]^{g+1}/N$, where N is a certain jacobian submodule of $\mathbf{C}[C^\Gamma]^{g+1}$, are linearly independant over $\mathbf{C}$.

The basic property of the miniversal deformation is that for any deformation $d\colon(\mathcal{Y},y)\to(S,0)$ of the germ $(C^\Gamma,0)$ such that all the fibers have a singular point with semigroup Γ there exists an analytic map germ $h\colon(S,0)\to(\mathbf{C}^{\tau_-},0)$ such that d is isomorphic to the deformation obtained from p by pull-back by h. Moreover, since C^Γ is quasi-homogeneous, it is also the case for p in the sense that there are weights on the v_i such that the equations F_j are homogeneous of degree $n_j\overline{\beta}_j$. These weights are negative, which is the reason for the notation $\mathbf{C}^{\tau_-}$; in fact the weight w_r of v_r is $n_j\overline{\beta}_j$ minus the weight of $\phi_{r,j}$, this difference being independent of j.

The recipe to find vectors ϕ_r making the deformation miniversal is due to J. Mather; it is the following (see [T1]): Consider the ring of algebraic functions $\mathbf{C}[C^\Gamma]$ on the monomial curve, and the submodule N of $\mathbf{C}[C^\Gamma]^g$ generated by the vectors $\partial_{u_i}f$ where $f=(f_1,\ldots,f_g)$. The quotient $\mathbf{C}[C^\Gamma]^g/N$ is a finite dimensional vector space over $\mathbf{C}$, and so we may choose vectors $\psi_i\in\mathbf{C}[u_0,\ldots,u_g]^g$, each having a single nonzero coordinate which is a monomial, such that the natural images of the ψ_i in $\mathbf{C}[C^\Gamma]^g/N$ form a basis of this vector space. The ϕ_r are those among the $\psi_i=(0,\ldots,0,u^{q_{i,j}},0,\ldots,0)$ such that if $u^{q_{i,j}}$ is at the j-th line, the weight of $u^{q_{i,j}}$ is $> n_j\overline{\beta}_j$, so that in the corresponding deformation the only equation modified, $f_j+v_iu^{q_{i,j}}$ is modified by a term of weight greater than that of f_j.

In particular, the one-parameter specialization of a branch C to C^Γ which we have seen above can be obtained in this way, up to isomorphism, by a map $h_1\colon(\mathbf{C},0)\to(\mathbf{C}^{\tau_-},0)$. The map p is equivariant with respect to the action of the group $\mathbf{C}^*$ on $\mathcal{X}_u$ (resp $\mathbf{C}^{\tau_-}$) which is described by $u_k\mapsto\lambda^{\overline{\beta}_j}u_k,\ v_r\to\lambda^{w_r}v_r$ (resp. only the action on the v_r). If $v_C\in\mathbf{C}^{\tau_-}$ is a point corresponding to a branch isomorphic to $(C,0)$, the image of h_1 is contained in the orbit of v_C under this action of $\mathbf{C}^*$. The action of $\mathbf{C}^*$ on $\mathbf{C}^{\tau_-}$ ensures that any branch with semigroup Γ appears up to isomorphism as a fiber in any representative of the germ $p\colon(\mathcal{X}_u,0)\to(\mathbf{C}^{\tau_-},0)$. ↺

4 Resolution Using Toric Morphisms

We produce *toric morphisms* $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{C}^{g+1}$ to resolve the singularities of $C^\Gamma \subset \mathbf{C}^{g+1}$, and then show that the same morphisms which resolves C^Γ resolve any fibre of the miniversal constant semigroup deformation $p\colon\ \mathcal{X}_u \to \mathbf{C}^{\tau_-}$. In particular, since all curves with semigroup Γ are represented as fibers of this deformation, Theorem 3.2 implies that $\pi(\Sigma)$ resolves $(C, 0)$. The toric resolution of the curve C^Γ will be a consequence of a generalization of the work of Varchenko [V] (expounded also in [M]) for "convenient" or "commode" functions, whose Newton polyhedron intersects each axis and which are non-degenerate.

4.1 The Toric Morphism

For the notions and basic results of toric geometry used in this section, we refer to [C]. A toric morphism is locally described by monomial maps $\pi(a)\colon \mathbf{C}^{g+1} \to \mathbf{C}^{g+1}$ where $a = (a^0, \ldots, a^g)$ with $a^j \in \mathbf{N}_0^{g+1}$. for all j and $\mathrm{span}\{a^0, \ldots, a^g\} = \mathbf{R}^{g+1}$:

$$\begin{array}{rcl} u_0 & = & {y_0}^{a^0_0} \cdot \cdots \cdot {y_g}^{a^g_0} \\ u_1 & = & {y_0}^{a^0_1} \cdot \cdots \cdot {y_g}^{a^g_1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u_g & = & {y_0}^{a^0_g} \cdot \cdots \cdot {y_g}^{a^g_g} \end{array}$$

More precisely, a fan (see [Oda], Chap. 1) Σ with support $\mathbf{R}_+^{g+1}$ is a decomposition of the positive quadrant $\mathbf{R}_+^{g+1}$ into rational simplicial cones σ_α with the properties that any face of such a cone is also a part of the fan and that the intersection of two of them is a face of each. A fan is *non-singular* if the primitive integral vectors of the 1-skeleton of each cone σ of dimension $g + 1$ form a basis of the integral lattice $\mathbf{N}_0^{g+1}$. We have set $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

To a fan Σ are associated a toric variety $Z(\Sigma)$ and a toric (equivariant) map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{C}^{g+1}$. The variety $Z(\Sigma)$ is obtained by glueing up affine varieties $Z(\sigma)$ corresponding to each cone of maximum dimension, and if the fan is non-singular, so is $Z(\Sigma)$; in this case, for each cone σ of maximal dimension $Z(\Sigma)$ is isomorphic to $\mathbf{C}^{g+1}$ and the map $\pi(\sigma)$ induced by $\pi(\Sigma)$ is equal to $\pi(a^0, \ldots, a^g)$ where the a^j are the primitive integral vectors of the 1-skeleton of the cone σ. An upper convex map (see [C], [Oda] A.3 p. 182-185) $m\colon \mathbf{R}_+^{g+1} \to \mathbf{R}_+$ taking integral values on the integral points and linear in each cone $\sigma \in \Sigma$ determines a divisor on $Z(\Sigma)$, which in the chart $\pi(a^0, \ldots, a^g)\colon Z(\Sigma) \to \mathbf{C}^{g+1}$ has the equation

$$y_0^{m(a^0)} \ldots y_g^{m(a^g)} = 0.$$

We will call it the exceptional divisor of the toric map $\pi(\Sigma)$ corresponding to the upper convex function m. Just like in the absolute case, the datum of such a map is equivalent, in the case where the function m is *strictly convex,* to the datum of an embedding $Z(\Sigma) \subset \mathbf{C}^{g+1} \times \mathbf{P}^N$ such that the exceptional divisor is the pull back of a hyperplane section of $\mathbf{P}^N$. Such an upper convex map is called a *support function* .

The *Newton polyhedron* of a function $f\colon \mathbf{C}^{g+1} \to \mathbf{C}$ can be used to define fans and a support function.

Definition. For a function $f = \sum_{p\in\mathbf{N}^{g+1}} f_p u^p$, let supp $f = \{p \in \mathbf{N}^{g+1} : f_p \neq 0\}$ and $\mathcal{N}_+(f) =$ *boundary of the convex hull of* $(\{\text{supp } f\} + \mathbf{R}_+^{g+1})$ in $\mathbf{R}_+^{g+1}$. We call $\mathcal{N}_+(f)$ the *Newton polyhedron* of the function f. Note that $\mathcal{N}_+(f)$ has finitely many compact faces and that its non-compact faces of dimension $\leq g$ are parallel to coordinate hyperplanes.

Define the function m by:

$$m(q) = \inf_{p\in\mathcal{N}_+(f)} < q, p >$$

and define an equivalence relation: two vectors are equivalent if and only if the same elements of $\mathcal{N}_+(f)$ minimize the inner product with each vector. In other words,

$$\begin{array}{ccc} q & \sim & q' \\ & \Updownarrow & \\ \{p \in \mathcal{N}_+(f) \mid < q, p >= m(q)\} & = & \{p \in \mathcal{N}_+(f) \mid < q', p >= m(q')\} \end{array} \qquad (***)$$

The function m is homogenous and piecewise linear, thus the equivalence classes (***) form a fan; to each class (and corresponding set of $p \in \mathcal{N}_+(f)$), we associate a cone σ which consists of those vectors whose inner product is minimized by (and only by) these p. We obtain a convex rational fan Σ_0 in $\mathbf{R}_+^{g+1}$ with vertex 0. By construction, the function m is linear on each cone $\sigma \in \Sigma$, and it is easy to verify that it is strictly upper convex. The non-compact faces of the boundary $\mathcal{N}(f)$ of $\mathcal{N}_+(f)$ correspond to cones σ in the fan which are contained in a coordinate hyperplane of $\mathbf{R}^{g+1}$. Moreover, given a subset $J \subset \{0, \ldots, g\}$, we can consider the projection of the Newton polyhedron to the subspace $\mathbf{R}^J = \{p \in \mathbf{R}^{g+1} \mid p_k = 0 \text{ for } k \notin J\}$ of $\mathbf{R}^{g+1}$. If that projection contains the origin, then for all $a \in \check{\mathbf{R}}^{g+1} \mid a_k = 0$ for $k \notin J$, we have $m(a) = 0$, and therefore we may assume that the cone $\mathbf{R}_+^J$ is in the fan, so that the basis vectors of $\mathbf{R}_+^J$ are in the 1-skeleton of the fan.

By a theorem of Kempf, Mumford, et al. (see [TE], pp. 32-35, [Oda], p.23), any fan can be refined into a non-singular fan, still containing as faces the spaces $\mathbf{R}_+^J$ such that $m(a) = 0$ for $a \in \mathbf{R}_+^J$. The function m is of course linear in each cone of the finer fan and upper convex (but not strictly so in general), and we are in the situation described above.

If we now consider several functions $f_j; 1 \leq j \leq k$, to each of them corresponds a Newton polyhedron, and therefore a fan $\Sigma_0^{(j)}$ and a support function m_j.

Definition. Let Σ_0 be the fan consisting of the intersections of the cones of the fans $\Sigma_0^{(j)}$. The fan Σ_0 is the least fine common refinement of all the $\Sigma_0^{(j)}$.

It is also the fan associated in the manner we have just seen to a single Newton polyhedron, which is the *Minkowski sum* of the Newton polyhedra of the f_j and is also the Newton polyhedron of the product $f_1 \dots f_k$.

Recall that the Minkowski sum $\mathcal{N}_1 + \mathcal{N}_2$ of two Newton polyhedra $\mathcal{N}_1,\ \mathcal{N}_2$ is the convex domain spanned by vector sums $\{p_1 + p_2\ \mid p_1 \in \mathcal{N}_1,\ p_2 \in \mathcal{N}_2\}$. It is a commutative and associative operation. If $\mathcal{N}_k$ is the Newton polyhedron of f_k, $\mathcal{N}_1 + \mathcal{N}_2$ is the Newton polyhedron of $f_1 f_2$.

All the functions m_j are linear in each cone of Σ_0. Taking a non-singular refinement Σ of Σ_0 as described above, we finally have a non-singular fan with support functions m_j and thus, since this fan is also a refinement of the "trivial" fan of $\mathbf{R}_+^{g+1}$, we get a proper toric map $\pi(\Sigma) \colon Z(\Sigma) \to \mathbf{C}^{g+1}$ where $Z(\Sigma)$ is a non singular toric variety, and each function $f_j \circ \pi(\Sigma)$ defines a map from $Z(\Sigma)$ to $\mathbf{C}$.

To study the effect of the modification $\pi(\Sigma)$ on the functions f_j, we restrict ourselves to a chart $Z(\sigma)$ of $Z(\Sigma)$ corresponding to a cone σ whose primitive integral vectors are denoted by $a = (a^0, \dots, a^g)$. We assume that they form a basis of the integral lattice, and we shall write $\sigma =< a^0, \dots, a^g >$ for the convex cone spanned by the vectors $(a^0, \dots, a^g)$. If we write the function

$$f_j = \sum_{p \in \mathbf{N}^{g+1}} f_p^{(j)} u^p \ ,$$

where $u^p = u_0^{p_0} u_1^{p_1} \cdots u_g^{p_g}$, then the composition

$$f_j \circ \pi(a^0, \dots, a^g) = \sum_{p \in \mathbf{N}^{g+1}} f_p^{(j)} y_0^{<a^0,p>} \cdots y_g^{<a^g,p>}, \tag{2}$$

where $<,>$ is the standard inner product, can be written

$$\begin{aligned} f_j \circ \pi(a) &= y_0^{m_j(a^0)} \dots y_g^{m_j(a^g)} \textstyle\sum_{p \in \mathbf{N}^{g+1}} f_p^{(j)} y_0^{<a^0,p>-m_j(a^0)} \cdots y_g^{<a^g,p>-m_j(a^g)} \\ &= y_0^{m_j(a^0)} \dots y_g^{m_j(a^g)} \tilde{f}_j(y_0, \dots, y_g) \ . \end{aligned}$$

The function $\tilde{f}_j$ is called the *strict transform* of f_j; it satisfies $\tilde{f}_j(0) \neq 0$.

We use the following fact, which is obvious from the definitions:
For each $j,\ 1 \leq j \leq k$, there is a unique $p^{(j)} \in \mathcal{N}_+(f_j)$ such that $< a^i, p^{(j)} >= m_j(a^i),\ 0 \leq i \leq g$, and the strict transforms take the form

$$\tilde{f}_j = f_{p^{(j)}}^{(j)} + \sum_{p \in \mathcal{N}_+(f_j) \setminus \{p^{(j)}\}} f_p^{(j)} y_0^{<a^0,p>-m_j(a^0)} \cdots y_g^{<a^g,p>-m_j(a^g)}.$$

Moreover we can compute the critical locus of $\mathrm{crit}\pi(\Sigma)$ of $\pi(\Sigma)$, which locally is the critical locus of $\pi(\sigma)$ for each $\sigma =< a^0, \dots, a^g >\in \Sigma$.

By direct inspection we find the following relation for the jacobian matrix jac$\pi(\sigma)$ of $\pi(\sigma)$:

$$y_0 \cdots y_g \text{ jac}\pi(\sigma) = u_0 \dots u_g \det(a^0, \dots, a^g),$$

setting $\alpha_j = \left(\sum_{i=0}^{g} a_i^j\right) - 1$ this means since $\det(a^0, \dots, a^g) = \pm 1$,

$$\text{jac}\pi(\sigma) = \pm y_0^{\alpha_0} \dots y_g^{\alpha_g}.$$

So the divisor $y_j = 0$ is contained in the critical locus of $\pi(\sigma)$ if and only if a^j is not a coordinate vector.

Note that the divisor $y_j = 0$ is contained in $\pi(\sigma)^{-1}(0)$ if and only if the vector a^j has all its coordinates different from zero.

5 Existence of a toric resolution

In this section, we study a germ at 0 of a complete intersection $X \subset \mathbf{C}^{g+1}$ defined by a set of equations $\{f_1 = f_2 = \cdots = f_k = 0\}$ with $f_j \in \mathbf{C}\{u_0, \dots, u_g\}$, $j = 1, \dots k$. We assume that the series f_j have no constant term. We use the notations introduced above and consider a toric map of non-singular spaces $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{C}^{g+1}$. We make our computations in a chart corresponding to a regular simplicial cone $\sigma =< a^0, \dots, a^g >$ of Σ.

Definition. Given a family of functions $\{f_j\}_{1 \le j \le k}$ as above, and a toric morphism associated to a regular fan Σ such that all the support functions $(m_j)_{1 \le j \le k}$ are linear in each cone σ of Σ, for each map $\pi(\sigma)\colon \mathbf{C}^{g+1}(\sigma) \to \mathbf{C}^{g+1}$, we will call the divisor $y_0^{m(a^0)} \dots y_g^{m(a^g)}$, where $m(a) = \sum_{j=1}^{k} m_j(a)$, the *toric exceptional divisor of* $\pi(\sigma)$ *associated to the support functions* m_j. This definition globalizes to $\pi(\Sigma)$.

From what we have seen one deduces for each $\pi(\sigma)$ with $\sigma =< a^0, \dots, a^g >$ the following statements:

Proposition 5.1. *The fiber $\pi(\sigma)^{-1}(0) \subset \mathbf{C}^{g+1}$ is the union of the intersections $y_{i_1} = \cdots = y_{i_t} = 0$ over all minimal families $J = (i_1, \dots, i_t)$ of indices such that for each ℓ , $0 \le \ell \le g$, there is an $i \in J$ such that the ℓ-th coordinate of a^i is $\neq 0$. In particular the divisorial part of $\pi(\sigma)^{-1}(0)$ is the union of the divisors $y_i = 0$ for those i such that the vector a^i has no zero component.*
The critical locus crit$(\pi(\sigma))$ *is the union of the divisors $y_s = 0$ for those s such that a^s is not a coordinate vector of $\mathbf{Z}^{g+1}$.*

For any set of functions $\{f_j\}$, we label the compact faces of $\mathcal{N}_+(f_j)$ by γ_j. Any compact face of the Newton polyhedron $\mathcal{N}_+ = \sum_{j=1}^{k} \mathcal{N}_+(f_j)$ which is the Minkowski sum of the Newton polyhedra of the $\{f_j\}$ will be of the form $\gamma = \gamma_1 + \cdots + \gamma_k$. Each face γ_j in turn is of the form

$$\gamma_j = \gamma_j(I) = \{p \in \mathcal{N}_+(f_j) \mid < a^h, p >= m_j(a^h) \text{ for } h \in I\} \ ,$$

where I is a subset of $\{0,\ldots,g\}$. If the face $\gamma_j(I)$ is not compact, it contains a line parallel to a coordinate axis, say the ℓ^{th}, so that there is an ℓ, $0\le\ell\le g$, such that for all $h\in I$ we have $a_\ell^h=0$.

Definition. We call the set of functions $\{f_j\}_{1\le j\le k}$ *non-degenerate* if for all compact faces $\gamma=\gamma_1+\cdots+\gamma_k$ of $\mathcal{N}_+=\sum_{j=1}^k\mathcal{N}_+(f_j)$, denoting by $f_j|_{\gamma_j}$, the sum

$$f_j|_{\gamma_j}=\sum_{p\in\gamma_j}f_p^{(j)}u^p,$$

the $k\times(g+1)$ matrix

$$\begin{pmatrix} \partial_{u_0}f_1|_{\gamma_1} & \cdot & \cdot & \cdot & \partial_{u_g}f_1|_{\gamma_1}\\ \partial_{u_0}f_2|_{\gamma_2} & \cdot & \cdot & \cdot & \partial_{u_g}f_2|_{\gamma_2}\\ \cdot & & & & \cdot\\ \cdot & & & & \cdot\\ \cdot & & & & \cdot\\ \partial_{u_0}f_k|_{\gamma_k} & \cdot & \cdot & \cdot & \partial_{u_g}f_k|_{\gamma_k} \end{pmatrix}$$

has maximal rank k on $(\mathbf{C}^*)^{g+1}$. This means that the equations

$$f_1|_{\gamma_1}=\cdots=f_k|_{\gamma_k}=0$$

define a non singular complete intersection in $(\mathbf{C}^*)^{g+1}$. For example a single function f is non-degenerate if for each compact face γ of $\mathcal{N}_+(f)$, the $(g+1)$-vector $(\partial_{u_i}f|_\gamma)_i$ does not vanish in $(\mathbf{C}^*)^{g+1}$; this is the original definition of Varchenko [V].

Definition. The morphism $\pi:\tilde{X}\longrightarrow X$ is a *resolution* if the following conditions hold:

- π is a proper morphism,
- $\tilde{X}$ is non-singular, and
- $\tilde{X}\setminus\pi^{-1}(\mathrm{Sing}X)\longrightarrow X\setminus\mathrm{Sing}X$ is an isomorphism.

If the morphism π is a toric morphism $\pi(\Sigma)$, the last condition is a consequence of the inclusion

$$\tilde{X}\cap\mathrm{crit}(\pi(\Sigma))\subset\pi(\Sigma)^{-1}(\mathrm{Sing}X).$$

However, this condition is difficult to check if X is not itself a toric subvariety, which motivates the following definition of a toric pseudo-resolution, after recalling that the strict transform $\tilde{X}$ of $X\subset\mathbf{C}^{g+1}$ by $\pi(\Sigma)$ is the closure in $Z(\Sigma)$ of $\pi(\Sigma)^{-1}(X)\setminus\mathrm{crit}(\pi(\Sigma))$.

Definition. A toric morphism $\pi(\Sigma)\colon Z(\Sigma)\to\mathbf{C}^{g+1}$ is a *toric (embedded) pseudo-resolution* of a subvariety $X\subset\mathbf{C}^{g+1}$ if the strict transform $\tilde{X}$ of X by $\pi(\Sigma)$ is smooth and transversal to a stratification of the critical locus of $\pi(\Sigma)$.

Note that a toric pseudo-resolution is not necessarily a resolution of singularities in the usual sense since it may not induce an isomorphism

$$\tilde{X} \setminus \pi(\Sigma)^{-1}(\mathrm{Sing}X) \to X \setminus \mathrm{Sing}X.$$

A toric pseudo-resolution only induces an isomorphism

$$\tilde{X} \setminus \mathrm{crit}(\pi(\Sigma)) \to X \setminus \mathrm{disc}(\pi(\Sigma)),$$

where $\mathrm{disc}(\pi(\Sigma))$ is the image by $\pi(\Sigma)$ of $\mathrm{crit}(\pi(\Sigma))$, and since $\tilde{X}$ is non singular, the inclusion $\mathrm{Sing}X \subset \mathrm{disc}(\pi(\Sigma))$ holds. In the case of a single function f, Varchenko introduced in [V] the condition of being *commode* which is equivalent to asking that if a primitive vector a^i of a cone σ (of a fan compatible with the Newton polyhedron of f) is contained in a hyperplane (which means that if in the chart $\mathbf{C}^{g+1}(\sigma)$ the divisor $y_i = 0$ is not contained in $\pi(\sigma)^{-1}(0)$), then we have $m(a^i) = 0$; it follows then from the conditions satisfied by our fans that a^i must in fact be a basis vector, so that $y_i = 0$ is not in the critical locus. For commode functions, in fact, the toric exceptional divisor and the critical locus of $\pi(\Sigma)$ both coincide set theoretically with $\pi(\Sigma)^{-1}(0)$ so that a toric pseudo-resolution of $f(u_0, \ldots, u_g) = 0$ is also a resolution in the usual sense.

For complete intersections, Oka introduced in [O3] a notion of *convenient* generalizing Varchenko's definition. It is much too strong for our purposes. Our monomial curve is only 1-convenient in the sense of [O3], and the product of its equations is far from being commode in the sense of [V].

However, in the case of a curve, a toric embedded pseudo-resolution is an embedded resolution in the usual sense unless the curve is contained in a coordinate hyperplane.

5.1 The Inverse Image of X by the Morphism $\pi(\Sigma)$.

Theorem 5.2. *If the set of functions $\{f_j\}_{1\leq j\leq k}$ defining the complete intersection $X \subset \mathbf{C}^{g+1}$ is non-degenerate at the origin, there exists a neighborhood U of 0 in $\mathbf{C}^{g+1}$ such that in $\pi(\Sigma)^{-1}(U)$ the strict transform of X by $\pi(\Sigma)$ is non-singular and transversal in $Z(\Sigma)$ to the strata of a stratification of the divisor $\pi(\Sigma)^{-1}(0)$.*

Proof. We consider for each $\pi(\sigma)$ a natural stratification of $\mathbf{C}^{g+1}(\sigma)$ such that $\pi(\sigma)^{-1}(0)$ is a union of strata, as follows: For each $I \subset \{0,\ldots,g\}$, define S_I to be the constructible subset of $\mathbf{C}^{g+1}(\sigma)$ defined by $y_i = 0$ for $i \in I$, $y_i \neq 0$ for $i \notin I$. The sets S_I form a partition of $\mathbf{C}^{g+1}(\sigma)$ by non-singular varieties.

If the subset I is such that for each ℓ , $0 \leq \ell \leq g$, there is a $j \in I$ such that the ℓ-th coordinate of a^j is $\neq 0$ (see 5.1), the stratum S_I is contained in $\pi(\sigma)^{-1}(0)$, and conversely, so that $\pi(\sigma)^{-1}(0)$ is a union of strata. This stratification of each $\mathbf{C}^{g+1}(\sigma)$ is compatible with the chart decomposition of $Z(\Sigma)$ and so gives a stratification of $Z(\Sigma)$. Moreover, this stratification satisfies the Whitney conditions, so that to check the transversality of a non-singular subspace to every stratum in a neighborhood it is sufficient to check transversality to the strata contained in $\pi(\sigma)^{-1}(0)$; as we have seen, these strata correspond to compact faces of the Newton polyhedra. Now we can compute in each chart $\mathbf{C}^{g+1}(\sigma)$ the restriction of each $\tilde{f}_j$ to S_I:

$$\tilde{f}_j|_{S_I} = \sum_{p|<a^i,p>=m_j(a^i) \text{ for } i\in I} f_p^{(j)} y_{j_1}^{<a^{j_1},p>-m_j(a^{j_1})} \ldots y_{j_\ell}^{<a^{j_\ell},p>-m_j(a^{j_\ell})},$$

where $\{j_1,\ldots,j_\ell\} = \{0,\ldots,g\} \setminus I$.
The set $\{p \,|< a^i, p >= m_j(a^i) \text{ for } i \in I\}$ is by definition a face $\gamma_{j,I}$ of the Newton polyhedron $\mathcal{N}_+(f_j)$, and this face is compact by the assumption on I.

The function $\tilde{f}_j|_{S_I}$ is the restriction to S_I of a function on $\mathbf{C}^{g+1}(\sigma)$ which is independent of the y_i ; $i \in I$. Moreover, this last function differs from the composition with $\pi(\sigma)$ of the function

$$f_{j,\gamma_{j,I}} = \sum_{p\in\gamma_{j,I}} f_p^{(j)} u^p$$

by multiplication by a monomial in $(y_0,\ldots,y_g)$. Since on S_I the coordinates y_k; $k \notin I$ are $\neq 0$, the behavior of the functions $\tilde{f}_j|_{S_I}$ is determined by the behavior of the $f_{j,\gamma_{j,I}} \circ \pi(\sigma)$ at points where all the y_i are $\neq 0$. It follows that if the jacobian determinant of the functions $f_{1,\gamma_{1,I}}, \ldots f_{k,\gamma_{k,I}}$ has a $k \times k$ minor which does not vanish on $(\mathbf{C}^*)^{g+1}$ then the functions $\tilde{f}_1|_{S_I}, \ldots, \tilde{f}_k|_{S_I}$ define a non-singular subspace of codimension k in S_I (in particular it is empty if $k > \dim S_I$). This implies that the strict transform of X by $\pi(\Sigma)$ is non-singular and transversal to all the strata S_I in a neighborhood of $\pi(\sigma)^{-1}(0)$; in particular it is transversal to the critical locus of $\pi(\sigma)$. ◯

Applying this to each chart of $\pi(\Sigma)$ gives:

Corollary. *If the set of functions* $\{f_j\}_{1\leq j\leq k}$ *defining the complete intersection* $X\subset \mathbf{C}^{g+1}$ *is non-degenerate at* 0*, there exists a neighborhood* U *of* 0 *in* $\mathbf{C}^{g+1}$ *such that the strict transform* $\tilde{X}\to X$ *of* X *by* $\pi(\Sigma)$ *induces in* $\pi(\Sigma)^{-1}(U)$ *an embedded toric pseudo-resolution of singularities of* $X\cap U$*.*

5.2 The Toric Resolution of the Monomial Curve

Let C^Γ be the monomial curve derived from C. In this section, we study the embedded resolution of the curve C^Γ by a toric morphism from two different viewpoints, which correspond to its parametric and equational presentations respectively.

First, the monomial curve is the closure in $\mathbf{C}^{g+1}$ of an orbit of $\mathbf{C}^*$ described by

$$t\mapsto (t^{\overline{\beta}_0}, t^{\overline{\beta}_1},\ldots, t^{\overline{\beta}_g}).$$

By the theory of toric varieties (see [O], Chap 1) this orbit is described combinatorially by the linear map $\mathbf{Z}\to\mathbf{Z}^{g+1}$ such that the image of 1 is the *weight vector*

$$w=(\overline{\beta}_0,\overline{\beta}_1,\ldots,\overline{\beta}_g)\in\mathbf{Z}^{g+1}.$$

This means that the corresponding morphism of semigroup algebras

$$\mathbf{C}[u_0,\ldots,u_g]\to\mathbf{C}[t]$$

obtained from the dual map $\check{\mathbf{Z}}^{g+1}\to\check{\mathbf{Z}}$ by restriction to $\mathbf{N}^{g+1}$ is the morphism of algebras which corresponds to our parametrization of the monomial curve.

We use the notations of §4. Now if we take a regular fan Σ in $\mathbf{R}_+^{g+1}$ which is compatible with the weight vector w, in the sense that w is contained in an edge of any simplicial cone it meets outside 0, we obtain a toric map

$$\pi(\Sigma)\colon Z(\Sigma)\to\mathbf{C}^{g+1}.$$

To compute the strict transform of C^Γ by $\pi(\Sigma)$, we need to find out first which charts $Z(\sigma)$ of $Z(\Sigma)$, corresponding to $(g+1)$-dimensional simplicial cones σ of Σ, have an image which contains C^Γ. Seeking solutions of the form $y_i(t)=c_it^{\alpha_i}+\cdots$ to the equations $y_0^{a_i^0}.\ldots.y_g^{a_i^g}=t^{\overline{\beta}_i}$, we see that the vector $(\alpha_0,\ldots,\alpha_g)$ must satisfy $M.\alpha=w$ where $M=(a_i^j)$ is the matrix of the column vectors $(a^0,\ldots,a^g)$ which are the generators of the simplicial cone σ. Since we want the α's to be positive, this implies that σ contains w, so that by our assumption that the fan is compatible with w, the vector w must be equal to one of the a^j. We see that we need to consider only those simplicial cones $\sigma=\ <a^0,\ldots,a^g>$, where up to reordering we have $w=a^0$. Then, w is the image of the coordinate vector $(1,0,\ldots,0)$ by the linear map $p(\sigma)\colon\mathbf{Z}^{g+1}\to\mathbf{Z}^{g+1}$ described by the matrix M, i.e., sending the i-th basis vector to a^i. The strict transform of C^Γ by the map $\pi(\sigma)$ is again a monomial curve, but its weight vector is the pull back of w by $p(\sigma)$, which is the basis vector $(1, 0, \ldots, 0)$; in fact, the simplicial cone σ is the image by $p(\sigma)$ of the cone generated by the basis vectors. So the strict transform of C^Γ has the parametric representation

$$y_0 = t, y_1 = \cdots = y_g = 1,$$

and it is indeed non singular and transversal to the exceptional divisor.

Remark. 1) The regular simplicial fan compatible with w is by no means unique; however, there are, at least in the case of plane branches, algorithms which produce such fans. Their description is beyond the intent of this text, but the reader can get some idea of the problem by looking at the example which concludes it.
2) This resolution process works for any monomial curve: the embedded toric resolution reduces entirely to the combinatorial problem of finding a regular simplicial fan Σ in $\mathbf{R}_+^{g+1}$ compatible with the weight vector. We can summarize this in:

Theorem 5.3. *Let $C^\Gamma \subset \mathbf{C}^{g+1}$ be a monomial curve with weight vector $w \in \mathbf{N}^{g+1}$. For any regular simplicial fan Σ in $\mathbf{R}_+^{g+1}$ compatible with w, the toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{C}^{g+1}$ has the property that the strict transform of C^Γ is non singular and transverse to the exceptional divisor. In fact, it appears only in the charts corresponding to simplicial cones containing w and there it is defined parametrically, up to reordering of the coordinates, by $y_0 = t, y_1 = \cdots = y_g = 1$.* ↺

This construction of the resolution is simpler than the following one, but not as easily adaptable to the study of the effect of the toric map $\pi(\Sigma)$ on deformations of C^Γ such as our curve $C \subset \mathbf{C}^{g+1}$. The other viewpoint is to explore the implications of the previous section on C^Γ. We reprove the result above from this viewpoint. As specified in Equations (**), C^Γ is defined by equations $\{f_j = 0\}_{1 \le j \le g}$, where each $f_j = u_j^\alpha - M_j$ and M_j is a monomial in $u_0, \ldots, u_{j-1}$. Note that $\mathrm{supp} f_j$ has exactly two elements, corresponding to the exponents of the two monomials in f_j, and each Newton polyhedron $\mathcal{N}_+(f_j)$ has only one compact face, which is a segment. For each function f_j, let $\Sigma_0^{(j)}$ be the fan representing the equivalence classes derived from the function m_j. Let Σ be a non-singular fan obtained by refining $\Sigma_0 = \bigcap_j \Sigma_0^{(j)}$, and $\mathcal{N}_+$ be the corresponding Newton polyhedron. We refer to the compact face of the Newton polyhedron $\mathcal{N}_+(f_j)$ as γ_j.

Proposition 5.4. *The curve defined by $\{\tilde{f}_1 = \tilde{f}_2 = \cdots = \tilde{f}_g = 0\}$ intersects the divisor $\pi(\sigma)^{-1}(0)$ only in the charts $\mathbf{C}^{g+1}(\sigma)$, $\sigma =< a^0, \ldots, a^g >$, in which for some b the vector a^b is the weight vector $(\overline{\beta}_0, \ldots, \overline{\beta}_g)$. If a^b is the weight vector, then $(\Pi_{i=0}^g y_i = 0) \bigcap \{\tilde{f}_1 = \tilde{f}_2 = \cdots = \tilde{f}_k = 0\}$ is contained in $\{y_b = 0\}$ but not in any other hyperplane $y_i = 0$ for $i \neq b$.*

Proof. Each strict transform $\tilde{f}_j$ has only two terms

$$\tilde{f}_j = f^{(j)}_{p_0^{(j)}} + f^{(j)}_{p_1^{(j)}} y_0^{<a^0, p_1^{(j)}> - m_j(a^0)} \cdots y_g^{<a^g, p_1^{(j)}> - m_j(a^g)}$$

since each equation f_j has only two terms. The only possibility for the g equations $\tilde{f}_j = 0$ to have a common root with $y_i = 0$ for some i is that $< a^i, p_1^{(j)} >= m_j(a^i) =< a^i, p_0^{(j)} >$. But this implies that a^i is constant on the Minkowski sum of the g segments constituting the Newton polyhedra of the f_j. By the structure of the equations, this sum is of dimension g; a^i is uniquely determined as the primitive normal vector of this face, and it has to be the weight vector a^b.
Indeed, the monomial map $\pi(\sigma)\colon\ Z(\sigma) \to \mathbf{C}^{g+1}$ maps the strict transform $\tilde{C^\Gamma}$ to C^Γ; if the coordinate y_b corresponds to the weight vector, we see that y_b must be a coordinate on the strict transform of C^Γ, and all other coordinates equal to 1. In other words, the equations in $Z(\sigma)$ of the desingularization of C^Γ are $(y_i = 1,\ \text{for } i \neq b)$; they are the strict transforms of the equations of C^Γ, which have only ± 1's as coefficients. ⟲

Theorem 5.5. *Let C^Γ be a monomial curve defined by $\{f_j\}_{1\leq j\leq g}$ as above. The strict transform of $\{f_1 = \cdots = f_g = 0\} \cap (\mathbf{C}^*)^{g+1}$ by the morphism $\pi(\Sigma)$ is non-singular and transversal in $Z(\Sigma)$ to the strata of $\pi(\Sigma)^{-1}(0)$.*

Proof. This follows directly from Theorem 5.2 if we can prove the fact that the family of functions $\{f_j\}_{1\leq j\leq g}$ is non-degenerate.

But there is only one compact face for the Newton polyhedron $\sum_{j=1}^{g} \mathcal{N}_+(f_j)$; it is the (Minkowski) sum $\sum_{j=1}^{g} \gamma_j$. By Proposition 5.4, this compact face corresponds to the subset $I = \{j\} \subset \{0, \ldots, g\}$ such that a^j is the weight vector. Moreover, for each j, the function f_j is equal to f_{j,γ_j}, so that we only have to check that the equations of the monomial curve define a non-singular complete intersection in $(\mathbf{C}^*)^{g+1}$ and we have

$$df_1 \wedge \cdots \wedge df_g = n_1 \ldots n_g u_1^{n_1-1} \ldots u_g^{n_g-1} du_1 \wedge \cdots \wedge du_g + \cdots$$

so the differential form does not vanish outside of the coordinate hyperplanes: the set of equations $f_1, \ldots, f_g$ is non-degenerate.
Note that unlike the first proof, this one encounters a (minor) difficulty if we work over a field of characteristic dividing one of the n_i. ⟲

6 Simultaneous Resolution

We will now show that some toric morphisms not only resolves C^Γ, but simultaneously resolves all curves in the miniversal deformation with constant semigroup $\mathcal{X}_u$ of C^Γ which we saw in section 3.
Definition. Let $f : (X, 0) \to (Y, 0)$ be a flat map with reduced fibres and Y reduced. We say (see [T2]) that f admits a *very weak simultaneous resolution* if, for all sufficiently small representatives, there exists a proper morphism $\pi :\ \tilde{X} \to X$ such that:

1. The composition $q = f \circ \pi : \tilde{X} \to Y$ is an analytic submersion, i.e., q is flat, and for all $y \in Y$, the fiber $\tilde{X}(y) = q^{-1}(y)$ is non-singular.

2. For all $y \in Y$, the induced morphism $\tilde{X}(y) \to X(y)$ is a resolution of the singularities of $X(y)$. Let Y_1 be the image of a section $\sigma : Y \to X$.

We say that f admits a *weak simultaneous resolution along* Y_1 if it also satisfies the condition:

1. The morphism $q_{Y_1} : (\pi^{-1}(Y_1))_{\text{red}} \to Y$ induced by q is locally topologically a fibration, in the sense that every point $\tilde{x} \in \pi^{-1}(Y_1)$, has an open neighborhood $U \in (\pi^{-1}(Y_1))_{\text{red}}$ such that $U_{Y_1} \simeq V \times \tilde{Y}_0$, where V is an open set in Y, $\tilde{Y}_0$ is an open neighbourhood of $\tilde{x}$ in the fibre of q_{Y_1} passing through $\tilde{x}$, and $\simeq$ is a Y-homeomorphism.

We say that f admits a *strong simultaneous resolution along* Y_1 if it also satisfies the condition:

The morphism $q_{Y_1} : (\pi^{-1}(Y_1)) \to Y$ induced by q is locally analytically trivial (before reduction), in the sense that every point $\tilde{x} \in \pi^{-1}(Y_1)$, has an open neighborhood $U \in (\pi^{-1}(Y_1))$ such that $U_{Y_1} \simeq V \times \tilde{Y}_0$, where V is an open set in Y, $\tilde{Y}_0$ is an open neighbourhood of $\tilde{x}$ in the fibre of q_{Y_1} passing through $\tilde{x}$, and $\simeq$ is a Y-isomorphism.

An *embedded resolution* for $\mathcal{X} \subset \mathbf{C}^N$ is a bimeromorphic map $\pi\colon Z \to \mathbf{C}^N$ where Z is smooth, the exceptional locus E of π is a divisor with normal crossings in Z and the strict transform of $\mathcal{X}$ is smooth and transversal to the canonical stratification of E. As usual, $\mathbf{C}^N$ stands for an open subset of $\mathbf{C}^N$.

Recall from Section 3 that the miniversal deformation of the curve C^Γ yields a family of curves, $p\colon \mathcal{X}_u \to \mathbf{C}^{\tau_-}$, embedded in $\mathbf{C}^{g+1} \times \mathbf{C}^{\tau_-}$ in such a way that p is induced by the second projection, and where $\mathcal{X}_u$ is defined by the equations

$$\begin{array}{lcll} F_1 & = & f_1 + \sum_{r=1}^{\tau_-} v_r \phi_{r,1}(u_0, \ldots, u_g) & = 0 \\ \cdot & & \cdot & \quad \cdot \\ \cdot & & \cdot & \quad \cdot \\ \cdot & & \cdot & \quad \cdot \\ F_g & = & f_g + \sum_{r=1}^{\tau_-} v_r \phi_{r,g}(u_0, \ldots, u_g) & = 0 \end{array}$$

where the $\phi_{r,j}$ are polynomials and each monomial in $(u_0, \ldots, u_g)$ appearing in $\phi_{r,j}$ is of weight $> n_j\overline{\beta}_j$ when each u_k is given the weight $\overline{\beta}_k$; the equation f_j is then homogeneous of weight $n_j\overline{\beta}_j$. We have $\mathcal{X}_u(0) \simeq C^\Gamma$. Let $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{C}^{g+1}$ be the toric modification associated to a regular fan in $\mathbf{R}_+^{g+1}$ compatible with the sum of the Newton polyhedra of the equations f_j of C^Γ. By construction, C^Γ is resolved by the morphism $\pi(\Sigma)$.

Theorem 6.1. *If Σ is a regular fan compatible with the Newton polyhedra of the equations $(**)$ of C^Γ, the morphism*

$$\Pi(\Sigma) = \pi(\Sigma) \times \mathrm{Id}_{\mathbf{C}^{\tau_-}} : Z(\Sigma) \times \mathbf{C}^{\tau_-} \to \mathbf{C}^{g+1} \times \mathbf{C}^{\tau_-}$$

induces by restriction to the strict transform $\tilde{\mathcal{X}}_u$ a resolution of singularities of $\mathcal{X}_u$, which is a strong simultaneous resolution for $p\colon \mathcal{X}_u \to \mathbf{C}^{\tau_-}$ with respect to the subspace $Y_1 = \{0\} \times \mathbf{C}^{\tau_-}$. In addition, it is an embedded resolution for $\mathcal{X} \subset \mathbf{C}^{g+1} \times \mathbf{C}^{\tau_-}$.

Proof. Consider in a chart $\mathbf{C}^{g+1}(\sigma) \times \mathbf{C}^{\tau_-}$ the composition

$$F_j \circ \Pi(\sigma) = f_j \circ \Pi(\sigma) + \sum_{r=1}^{\tau_-} v_r(\phi_{r,j} \circ \pi(\sigma)).$$

Since for each support function $m(a)$ and fixed q the function $< a, q > /m(a)$ is a continuous function of a, and since the $\phi_{j,r}$ are polynomials, after possibly refining the fan Σ (this is where we have to impose conditions on the toric morphisms which resolve C^Γ), we may assume that for each cone $\sigma =< a^b, a^1, \ldots, a^g >$ containing the weight vector a^b, for each j, $1 \le j \le g$, and for each monomial appearing in one of the polynomials $\phi_{j,r}$, the inequality $< a^b, q > > m_j(a^b)$, which is equivalent to the weight inequalities mentioned above, implies the inequalities $< a^s, q > \ge m_j(a^s)$ for $1 \le s \le g$. We can therefore write the composition $F_j \circ \Pi(\sigma)$ as follows:

$$F_j \circ \Pi(\sigma) = y_0^{m_j(a^0)} \ldots y_g^{m_j(a^g)}(\tilde{f}_j + \sum_{r=1}^{\tau_-} v_r \tilde{\phi}_{r,j}),$$

where $\tilde{\phi}_{r,j} = \sum_q \tilde{\phi}_{r,j,q} y_0^{<a^b,q>-m_j(a^b)} \ldots y_g^{<a^g,q>-m_j(a^g)}$. From this follows, since it is true for the $\tilde{f}_j$, that at least for small $\|v\|$ the $\tilde{F}_j = \tilde{f}_j + \sum_{r=1}^{\tau_-} v_r \tilde{\phi}_{r,j}$ define a non-singular complete intersection $\tilde{\mathcal{X}}_u$ in $Z(\sigma) \times \mathbf{C}^{\tau_-}$: the strict transform of $\mathcal{X}_u$; moreover, each fiber over a point $v \in \mathbf{C}^{\tau_-}$ for sufficiently small $\|v\|$ is the resolution of the corresponding fiber of $\mathcal{X}_u(v)$. Let us now consider $\tilde{Y}_1 = (\pi(\sigma) \times \mathrm{Id}_{\mathbf{C}^{\tau_-}}|_{\tilde{\mathcal{X}}_u})^{-1}(Y_1)$. Its equations are $y_b = 0, \tilde{F}_j = 0$; by the Implicit Function theorem, it is non-singular and admits at least for small $\|v\|$ the coordinates $v_1, \ldots, v_{\tau_-}$; the morphism $\tilde{Y}_1 \to Y_1$ is not only a homeomorphism, but a local analytic isomorphism. Since there is an equivariant action of $\mathbf{C}^*$, all this is in fact true for all v.

Let us now consider a cone σ not adjacent to the weight vector. Let u^q be a monomial appearing as a $\phi_{r,j}$, set $G_j = f_j + v_r u^q$, and let $K_j \subset \{0, \ldots g\}$ be the set of those indices s such that $< a^s, q > < m_j(a^s)$. Set $n_j(a^s) =< a^s, q >$ for $s \in K_j$. Then $G_j \circ \Pi(\sigma)$ equals

$$\Pi_{s \notin K_j} y_s^{m_j(a^s)} \Pi_{s \in K_j} y_s^{n_j(a^s)} \big(\tilde{f}_j \Pi_{s \in K_j} y_s^{m_j(a^s)-n_j(a^s)} + v_r \Pi_{s \notin K_j} y_s^{<a^s,q>-m_j(a^s)}\big).$$

By our choice of σ we know that the $\tilde{f}_j$ do not vanish together on the exceptional divisor. The common zeroes of the strict transforms

$$\tilde{G}_j = \tilde{f}_j \Pi_{s \in K_j} y_s^{m_j(a^s)-n_j(a^s)} + v_r \Pi_{s \notin K_j} y_s^{<a^s,q>-m_j(a^s)}$$

constitute in the chart $\mathbf{C}^{g+1}(\sigma)\times\mathbf{C}^{\tau_-}$ the strict transform $\tilde{\mathcal{X}}_u$ of the space $\mathcal{X}_u$, therefore this strict transform meets the exceptional divisor only for $v_r=0$ and then it coincides with the union of some components in $\mathbf{C}^{g+1}(\sigma)\times\{0\}$ of the exceptional divisor of $\pi(\sigma)$. However, these components do not meet the strict transform (see 5.4) of the curve C^Γ, and therefore the inverse image of $\{0\}$ in $\tilde{\mathcal{X}}_u$ is not connected. Since $\mathcal{X}_u$ is the total space of an equisingular deformation of C^Γ it is analytically irreducible at the origin and by Zariski's main theorem the inverse image of $\{0\}$ must be connected. We therefore obtain a contradiction, which shows that for each cone $\sigma=< a^0,\dots,a^g >\in\Sigma$ all the monomials u^q appearing in the miniversal equisingular deformation of C^Γ satisfy the inequalities $< a^s,q > \geq m_j(a^s)$ for $0\leq s\leq g$. We are in the situation described at the beginning of the proof, and this shows that the map $\tilde{\mathcal{X}}_u\to\mathcal{X}_u$ induced by the toric modification $Z(\Sigma)\times\mathbf{C}^{\tau_-}\to\mathbf{C}^{g+1}\times\mathbf{C}^{\tau_-}$ is a weak embedded simultaneous resolution along Y_1.

Finally we remark that a refinement Σ' of our original regular fan corresponds to a birational toric map $Z(\Sigma')\to Z(\Sigma)$ which is an isomorphism outside the exceptional divisor. Moreover, in a chart corresponding to a cone σ adjacent to the weight vector, it is an isomorphism outside $\Pi_{s=1}^g y_s=0$; this open set contains the strict transform of C^Γ since we saw that it meets only $y_0=0$ (see 5.4). It follows from this that if the strict transform of $\mathcal{X}_u$ in $Z(\Sigma')\times\mathbf{C}^{\tau_-}$ is a simultaneous resolution of $\mathcal{X}_u$, such was already the case for the strict transform in $Z(\Sigma)\times\mathbf{C}^{\tau_-}$. This concludes the proof. ○

Corollary. *For a plane branch $(C,0)$ with g characteristic exponents, with semi group $\Gamma=<\overline{\beta}_0,\dots,\overline{\beta}_g>$ and given parametrically by $x=x(t),\ y=y(t)$, with $\nu(x)=\overline{\beta}_0$, $\nu(y)=\overline{\beta}_1$, let $\xi_i(t)\in\mathcal{O}_{C,0}=\mathbf{C}\{x(t),y(t)\}\subset\mathbf{C}\{t\}$ for $2\leq i\leq g$ be series such that*

$$(\nu(x),\nu(y),\nu(\xi_2),\dots,\nu(\xi_g))=(\overline{\beta}_0,\dots,\overline{\beta}_g),$$

where ν is the t-adic valuation. Then the embedding $(C,0)\to(\mathbf{C}^{g+1},0)$ given by

$$u_0=x(t),u_1=y(t),u_2=\xi_2(t),\dots,u_g=\xi_g(t)$$

has the property that its image is resolved by one toric modification.

7 The transforms of plane curves

Let us return to the construction of the miniversal constant semigroup deformation in §3. It is easy to check that the vectors $\phi_j\in\mathbf{C}[C^\Gamma]^g$, $1\leq j\leq g-1$, containing u_{j+1} at the j-th line and zero elsewhere, are independent modulo N with the notations of §3. In the miniversal constant semigroup deformation, the plane curves appear as those for which the coefficients of all these vectors are $\neq 0$ (see [T1]). For each point v_0 in the corresponding open set of $\mathbf{C}^{\tau_-}$, the equation of the corresponding plane branch is obtained as follows, in a small neighborhood of 0 :

One uses the first equation F_1 to express u_2 as a series in u_0, u_1, using the implicit function theorem. Then one substitutes this value in F_2 and uses this equation to express u_3 as a function of u_0, u_1, and so on. Finally the equation $F_{g-1} = 0$ allows us to express u_g as a series in u_0, u_1, and at this point we have the equations of a non-singular surface S_{v_0} containing our branch, thus explicitly shown to be planar.

The next step is to consider the strict transform $\tilde{S}_{v_0}$ of S_{v_0} in $\mathbf{C}^{g+1} \times \{v_0\}$ and the induced map $\tilde{S}_{v_0} \to S_{v_0}$ induced by $\Pi(\Sigma)$. One shows that, in a neighborhood of the point of the exceptional divisor picked by the strict transform of C, it factors through the toric map $S^1_{v_0} \to S_{v_0}$ which on S_{v_0} with the coordinates u_0, u_1 resolves the singularities of the branch $u_1^{n_1} - u_0^{\ell_1^{(0)}} = 0$. One then takes the strict transform of this equation as a new coordinate on $S^1_{v_0}$ and then the strict transform of C appears as a branch on $S^1_{v_0}$ with one less characteristic exponent; its equations are the strict transforms of $F_2, \ldots, F_g$ on $S^1_{v_0}$, in the new coordinates. Again one must look at the toric map (in the new coordinates) which resolves the first characteristic pair and show that it is dominated by $\tilde{S}_{v_0}$, and so on; finally we have factored the map $\tilde{S}_{v_0} \to S_{v_0}$ into the composition of g toric maps. This is similar to the process described by Spivakovsky in [S] of approximating a given valuation by a sequence of monomial valuations. However we see that if we allow a change in ambient space, the uniformization of a single monomial valuation, namely the t-adic valuation on the monomial curve, gives the uniformization of the t-adic valuation of C.

On the surface S, each of the coordinates $u_i,\ 2 \leq i \leq g$ is expressed as a series in u_0, u_1. For each $i,\ 2 \leq i \leq g$, the equation $u_i = 0$ on S defines a curve with i characteristic exponents having at the origin maximal contact with our original plane branch; this is a consequence of what we have seen in Corollary 6, and gives a geometric construction of curves (singular or not) having maximal contact in the sense of [L-J] with our plane branch; they are the curves on S_{v_0} defined by the equations $u_i = 0,\ i \geq 1$. Remark that for $i \geq 2$ they can also be written $f_{i-1}(u_0, \ldots, u_g) +$ higher degree, and with a little more work we find the structure of the approximate roots.

The construction just outlined needs to be detailed but we will not do it here since it is not part of the main purpose of this text. It is performed in the first non trivial special case (two characteristic exponents $(3/2, 7/4)$, i.e., semigroup $< 4, 6, 13 >$) at the end of the example to which we now turn.

8 Appendix : An example

Example. Suppose we have a plane branch C with semigroup $< 4, 6, 13 >$. The corresponding monomial curve C^Γ is described (see [T]) by the polynomials

$$\begin{array}{rcrcl} f_1 & = & {u_1}^2 - {u_0}^3 & = & 0 \\ f_2 & = & {u_2}^2 - {u_0}^5 u_1 & = & 0. \end{array}$$

As specified above, we construct $\mathcal{N}_+(f_1)$ and $\mathcal{N}_+(f_2)$, the Newton polygons, by the convex hull of the union of positive quadrants beginning at points corresponding to the exponents of the $\{u_i\}$'s . Thus,

$$\begin{array}{rcl} \mathcal{N}_+(f_1) & = & \text{convex hull of } \{\{(3,0,0),(0,2,0)\} + \mathbf{R}^3_+\} \\ \mathcal{N}_+(f_2) & = & \text{convex hull of } \{\{(5,1,0),(0,0,2)\} + \mathbf{R}^3_+\} \end{array}$$

and the corresponding fans are defined, respectively, by

$$\begin{array}{rcl} \Sigma_0^{(1)} & = & \text{the first quadrant of } \mathbf{R}^3 \text{ cut by the plane } 3x = 2y, \text{and} \\ \Sigma_0^{(2)} & = & \text{the first quadrant of } \mathbf{R}^3 \text{ cut by the plane } 5x + y = 2z. \end{array}$$

Then the equivalence class fan of f_1 and f_2 together is the intersection

$$\Sigma_0 = \Sigma_0^{(1)} \cap \Sigma_0^{(2)}$$

which has four maximal dimension cones:

1. $\langle (2,3,0),\ (0,1,0),\ (4,6,13),\ (0,2,1) \rangle$
2. $\langle (0,2,1),\ (0,0,1)\ (4,6,13) \rangle$
3. $\langle (2,0,5),\ (1,0,0),\ (2,3,0),\ (4,6,13) \rangle$
4. $\langle (0,0,1),\ (2,0,5),\ (4,6,13) \rangle$

We build here by an inductive method a refinement which resolves the fan $\Sigma_0 = \Sigma_0^{(1)} \cap \Sigma_0^{(2)}$. Notice that each maximal dimension cone spans $\mathbf{Z}^3$ (since the matrix of the spanning skeleton is unimodular). For each cone, we have listed the corresponding morphism $\pi(\sigma)$ in terms of the resulting $\{u_i\}$, the composition $f_i \circ \pi(\sigma)(y_0, y_1, y_2)$ for $i = 1, 2$, and the exceptional divisor $\pi(\sigma)^{-1}(0)$. We show that the two surfaces are transverse to each other and to the exceptional divisor. We start from a refinement for the fan in two dimensions associated to the first equation $u_1^2 - u_0^3 = 0$. The weight vector is $(2, 3)$ and from the geometric interpretation of the continued fraction expansion we find that a regular fan subdividing it is composed of the following four 2-dimensional cones and their faces:

1. $\sigma_1^{(2)} = \langle (1,0),\ (1,1) \rangle$
2. $\sigma_2^{(2)} = \langle (1,1),\ (2,3) \rangle$

3. $\sigma_3^{(2)} = \langle (2,3),\ (1,2)\rangle$

4. $\sigma_4^{(2)} = \langle (1,2),\ (0,1)\rangle$

Now we have to lift the vectors of the 1-skeleton to $\mathbf{R}^3$ in such a way that they form the 1-skeleton of a regular fan subdividing Σ_0. It suffices to show that in each cone of this fan both functions m_1 and m_2 are linear, i.e., that all linear forms take their minimum at a vertex of the sum $\mathcal{N}$ of the Newton polyhedra of f_1 and f_2. This Newton polyhedron has four vertices: $(3,0,2),(8,1,0),(0,2,2),(5,3,0)$, so we are especially interested in finding four cones having the weight vector as one of their faces.

Let us begin by lifting the weight vector $(2,3)$:
We seek integral vectors of the form $(2k,3k,z)$ with $k>0,\ z>0$; we know that they take their minimum value on the compact segment of the Newton polyhedron of f_1. On $\mathcal{N}_+(f_2)$ they take their minimum at $((5,1,0)$ if $13k\le 2z$, at $(0,0,2)$ if $13k\ge 2z$, at both if $13k=2z$. Remembering that we seek primitive vectors, we find three natural possibilities: $k=1, z=6$, giving the vector $a^1=(2,3,6)$, which takes its minimum at $(0,0,2)$, $k=1, z=7$ giving $a^2=(2,3,7)$, which takes its minimum at $(5,1,0)$, and of course the weight vector $a^0=(4,6,13)$ itself. We remark that $a^0=a^1+a^2$.
Let us now try to lift $(1,1)$; we seek vectors of the form (k,k,z); on $\mathcal{N}_+(f_1)$ they take their minimum value at $(0,2,0)$, and on $\mathcal{N}_+(f_2)$ at $(5,1,0)$ if $3k\le z$, at $(0,0,2)$ if $3k\ge z$. In this case we can take $k=1, z=3$, which gives the vector $a^3=(1,1,3)$ which takes its minimum on the compact segment of $\mathcal{N}_+(f_2)$.
Finally take $(1,2)$; we seek vectors $(k,2k,z)$ and there is one taking its minimum on the compact face of $\mathcal{N}_+(f_2)$: it corresponds to $k=1, z=3$, giving the vector $a'^4=(1,2,3)$. We remark that $a^0=a^2+a^3+a'^4$, and it is therefore tempting to take a'^4 as our fourth vector. However, it is not on the hyperplane $5x+y=2z$ so that the support function m_2 will *not* be linear on a cone spanned by $\{a^0,a^2,a'^4\}$. A better choice is to take $a^4=a^0-a^3=(3,5,10)$.

Now we check that the four cones (which we have decorated with the points of the Newton polyhedra of f_1 and f_2 where the support functions $m_1(a),\ m_2(a)$ take their minimum for $a\in\sigma_i$):

1. $\sigma_1^{(3)} = \langle a^0,a^1,a^3\rangle;\quad p_0^{(1)}=(0,2,0);\ p_0^{(2)}=(0,0,2)$

2. $\sigma_2^{(3)} = \langle a^0,a^2,a^3\rangle;\quad p_0^{(1)}=(0,2,0);\ p_0^{(2)}=(5,1,0)$

3. $\sigma_3^{(3)} = \langle a^0,a^2,a^4\rangle;\quad p_0^{(1)}=(3,0,0);\ p_0^{(2)}=(5,1,0)$

4. $\sigma_4^{(3)} = \langle a^0,a^1,a^4\rangle;\quad p_0^{(1)}=(3,0,0);\ p_0^{(2)}=(0,0,2)$

are unimodular and such that their union is a neighborhood of $\mathbf{R}_+a^0$ in $\mathbf{R}^3$; the second fact is obvious since the weight vector is in the interior of the convex hull of the σ_i $,1\le i\le 4$, and to check the first it suffices to check that one of them is unimodular. Now we can complete this subfan into a regular fan of $\mathbf{R}^3_+$, either by the same method or by invoking a general theorem, but we do not much care about it, since we have seen that the only charts where something interesting happens are those which correspond to a cone containing the weight vector in its 1-skeleton, and we have those above. Moreover, the projections in $\mathbf{R}^2$ of the σ_i form part of a regular subfan of the $\sigma_i^{(2)}$.

Let us now study the behavior of f_1, f_2 under the monomial maps corresponding to the $\sigma_i^{(3)}$, $1 \leq i \leq 4$. For economy we write σ_i for $\sigma_i^{(3)}$.

1. The cone σ_1 spanned by $\{(4,6,13),\ (2,3,6),\ (1,1,3)\}$

$$\pi(\sigma_1): \begin{array}{rcl} u_0 &=& {y_0}^4{y_1}^2y_2 \\ u_1 &=& {y_0}^6{y_1}^3y_2 \\ u_2 &=& {y_0}^{13}{y_1}^6{y_2}^3 \end{array}$$

Then,

$$\begin{array}{rcl} f_1 \circ \pi(\sigma_1)(y_0,y_1,y_2) &=& {y_0}^{12}{y_1}^6{y_2}^2(1-y_2) \\ f_2 \circ \pi(\sigma_1)(y_0,y_1,y_2) &=& {y_0}^{26}{y_1}^{12}{y_2}^6(1-y_1) \end{array}$$

and

$$\pi(\sigma_1)^{-1}(0) = \{y_0 = 0\} \cup \{y_1 = 0\} \cup \{y_2 = 0\}.$$

Here the strict transforms are

$$\begin{array}{rcl} \tilde{f}_1 &=& 1 - y_2 \\ \tilde{f}_2 &=& 1 - y_1. \end{array}$$

It is clear that the equations $\tilde{f}_1 = \tilde{f}_2 = 0$ define a non-singular complete intersection meeting the exceptional divisor transversally at the point $y_0 = 0, y_1 = y_2 = 1$.

To make the same computation for the other charts $\pi(\sigma_i)$ is in fact superfluous, since by construction of the toric modification, we will only observe the same phenomenon in a different chart.

For verification's sake let us compute for $\pi(\sigma_3)$:

2. The cone σ_3 spanned by $\{(4,6,13),\ (2,3,7),\ (3,5,10)\}$

$$\pi(\sigma_3): \begin{array}{rcl} u_0 &=& {y_0}^4{y_1}^2{y_2}^3 \\ u_1 &=& {y_0}^6{y_1}^3{y_2}^5 \\ u_2 &=& {y_0}^{13}{y_1}^7{y_2}^{10} \end{array}$$

Then,

$$\begin{array}{rcl} f_1 \circ \pi(\sigma_3)(y_0,y_1,y_2) &=& {y_0}^{12}{y_1}^6{y_2}^9(y_2-1) \\ f_2 \circ \pi(\sigma_3)(y_0,y_1,y_2) &=& {y_0}^{26}{y_1}^{13}{y_2}^{20}(y_1-1) \end{array}$$

And indeed it is the same situation viewed in another chart.

The miniversal deformation with constant semigroup of C^Γ is computed in [T1]: here $\tau_- = 2$ and $\mathcal{X}_u$ is defined in $\mathbf{C}^3 \times \mathbf{C}^2$ by the equations

$$\begin{aligned} F_1 &= u_1^2 - u_0^3 + v_1 u_2 + v_2 u_0 u_2 = 0 \\ F_2 &= u_2^2 - u_0^5 u_1 = 0 \end{aligned}$$

If we compose F_1 and F_2 with $\Pi(\sigma_1) = \pi(\sigma_1) \times \mathrm{Id}_{\mathbf{C}^2} : \ \mathbf{C}^{g+1}(\sigma_1) \times \mathbf{C}^2 \to \mathbf{C}^{g+1} \times \mathbf{C}^2$ we get

$$\begin{aligned} F_1 \circ \Pi(\sigma_1)(y_0, y_1, y_2) &= {y_0}^{12} {y_1}^{6} {y_2}^{2} (1 - y_2 + v_1 y_0 y_2 + v_2 y_0^5 y_1^2 y_2^2) \\ F_2 \circ \Pi(\sigma_1)(y_0, y_1, y_2) &= {y_0}^{26} {y_1}^{13} {y_2}^{20} (1 - y_1) \end{aligned}$$

The strict transform of $\mathcal{X}_u$ is defined in this chart by

$$\begin{aligned} \tilde{F}_1(y_0, y_1, y_2) &= 1 - y_2 + v_1 y_0 y_2 + v_2 y_0^5 y_1^2 y_2^2 = 0 \\ \tilde{F}_2(y_0, y_1, y_2) &= 1 - y_1 = 0 \end{aligned}$$

It is indeed a simultaneous resolution for $\mathcal{X}_u$. There remains to check that the strict transform of $\mathcal{X}_u$ does not meet the charts corresponding to cones that are not adjacent to the weight vector. This depends on our construction of a regular fan and will not be done here.

Let us now consider the plane branch with equations in $\mathbf{C}^3$ (for $v = v_0 \neq 0$):

$$\begin{aligned} F_1(v_0; u_0, u_1, u_2) &= u_1^2 - u_0^3 - v_0 u_2 = 0 \\ F_2(u_0, u_1, u_2) &= u_2^2 - u_0^5 u_1 = 0 \end{aligned}$$

It lies on the non-singular surface S_{v_0} with equation $F_1(v_0; u_0, u_1, u_2) = 0$. Let $\tilde{S}_{v_0}$ be the strict transform of S_{v_0} by the toric map $Z(\Sigma) \to \mathbf{C}^{g+1}$. As before we need to examine the situation only in a chart $Z(\sigma)$ where σ is adjacent to the weight vector. We take $a^0 = (4, 6, 13), \ a^1 = (2, 3, 7), \ a^2 = (3, 5, 10)$. So $\pi(\sigma)$ is described as:

$$\pi(\sigma): \quad \begin{aligned} u_0 &= {y_0}^4 {y_1}^2 {y_2}^3 \\ u_1 &= {y_0}^6 {y_1}^3 {y_2}^5 \\ u_2 &= {y_0}^{13} {y_1}^7 {y_2}^{10} \end{aligned}$$

and we have

$$(u_1^2 - u_0^3 + v_0 u_2) \circ \pi(\sigma) = y_0^{12} y_1^6 y_2^9 (1 - y_2 + v_0 y_0 y_1 y_2)$$

so that the equation of $\tilde{S}_{v_0}$ in this chart is

$$y_2(1 - v_0 y_0 y_1) = 1,$$

and the mapping $\tilde{S}_{v_0} \to S$ is described by

$$\begin{array}{rcl} u_0 & = & {y_0}^4{y_1}^2(\frac{1}{1-v_0y_0y_1})^3 \\ u_1 & = & {y_0}^6{y_1}^3(\frac{1}{1-v_0y_0y_1})^5 \end{array}$$

All computations are made in a neighborhood of the exceptional divisor, i.e., for $|y_0|$ small.

Now the toric map $S^1_{v_0} \to S_{v_0}$ which resolves the plane branch with one characteristic pair $u_1^2 - u_0^3 = 0$ has a chart:

$$\begin{array}{rcl} u_0 & = & x_0x_1^2 \\ u_1 & = & x_0x_1^3 \end{array}$$

and we can check that there is a factorization $\tilde{S}_{v_0} \to S^1_{v_0}$ given as follows in the coordinates y_0, y_1 on $\tilde{S}_{v_0}$:

$$\begin{array}{rcl} x_0 & = & 1 - v_0y_0y_1 \\ x_1 & = & y_0^2y_1(\frac{1}{1-v_0y_0y_1})^2 \end{array}$$

and we can view this factorization itself as composed of the maps

$$\begin{array}{rcl} x_0 & = & w_0 \\ x_1 & = & w_0^2w_1 \end{array}$$

and

$$\begin{array}{rcl} w_0 & = & 1 - v_0y_0y_1 \\ w_1 & = & y_0^2y_1 \end{array}$$

Now this last map is monomial after a change of the coordinates w_0, w_1 and moreover the substitution of the w's in the x's gives a monomial map $S^2_{v_0} \to S_{v_0}$ described as follows

$$\begin{array}{rcl} u_0 & = & w_0^5w_1^2 \\ u_1 & = & w_0^7w_1^3 \end{array}$$

which is again a monomial map, so that we have factorized our map $\tilde{S}_{v_0} \to S_{v_0}$ as a composition of two monomial maps, up to a change of variables (essentially a translation $w_0 \mapsto w_0 - 1$). Note the fact that it was convenient to refine part of the fan of the plane branch $u_1^2 - u_0^3 = 0$ from $(1,1),(2,3)$ to $(5,7),(2,3)$ before writing the translation, but that we could also have made a change in the variables y_0, y_1 (for $|y_0|$ small) to bring directly the map $\tilde{S}_{v_0} \to S^1_{v_0}$ into monomial form.

Finally, note that the strict transform on $\tilde{S}_v$ of our plane curve $C_{v_0} \subset S_{v_0}$ has equation $y_1 - 1 = 0$ In the general case, since we do not deform the last equation f_g of the monomial curve, the last equation of the strict transform of C_{v_0}, in the proper chart will still be of the form $y_k - 1 = 0$.

References

[C] D. Cox, *Toric varieties and toric resolutions.* This volume.

[F] W. Fulton, *Introduction to Toric Varieties.* Princeton University Press, 1993.

[TE] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal embeddings.* Lecture Notes in Math. 339, Springer-Verlag 1973.

[L-J] M. Lejeune-Jalabert, Thèse d'Etat, Université Paris 7, 1973.

[L-T] M. Lejeune-Jalabert, B. Teissier, *Cloture intégrale des idéaux et équisingularité.* In: Séminaire sur les singularités, Ecole Polytechnique, 1974-75. Institut Fourier, Université de Grenoble.

[M] M. Merle, *Polyèdre de Newton, Eventail et Désingularisation, d'après Varchenko.* In Séminaire sur les Singularités des Surfaces, Palaiseau, France, 1976-1977, Edited by M. Demazure, H. Pinkham, B. Teissier, Lecture Notes in Math. 777, Springer-Verlag 1980.

[Oda] T. Oda, *Convex bodies and algebraic geometry*, Ergeb. der Math., 3. Folge, Band 15, Springer-Verlag 1988.

[O1] M. Oka, *Geometry of plane curves via toroidal resolution.* In: Algebraic Geometry and Singularities, A. Campillo et al., editors, Progress in Math., No. 134, Birkhaüser, 95-121.

[O2] M. Oka, *Canonical stratification of non degenerate complete intersection varieties.* Jour. Math. Soc. Japan, 1990, **42**, 397-422.

[O3] M. Oka, *Non degenerate complete intersection singularities.* Actualités Mathématiques, Hermann, Paris, 1997.

[S] M. Spivakovsky, *Resolution of singularities.* Preprint, University of Toronto, 1994.

[T1] B. Teissier, Appendix in [Z].

[T2] B. Teissier, *Résolution simultanée II.* In: Séminaire sur les Singularités des Surfaces, Palaiseau, France, 1976-1977, Edited by M. Demazure, H. Pinkham, B. Teissier, Lecture Notes in Mathematics 777, Springer-Verlag, 1980.

[T3] B. Teissier, *The hunting of invariants in the geometry of discriminants.* In: Real and Complex Singularities, Nordic Summer School, Oslo 1976, Edited by Per Holm, Noordhoff and Sijthoff, 1977.

[V] A.N. Varchenko, *Zeta function of the monodromy and Newton's diagram.* Invent. Math. **37** (1976), 253-262.

[Z] Oscar Zariski. *Le Problème des Modules pour les Branches Planes*. Hermann, Paris, 1986.

Dept. of Mathematics
M.I.T.
Cambridge, MA 02139, USA
goldin@math.mit.edu

Laboratoire de Mathématiques de l'E.N.S.,
URA 762 du C.N.R.S.,
Ecole Normale Supérieure de Paris
45 Rue d'Ulm
75005 Paris, France
teissier@dmi.ens.fr