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NORMAL CONES AND SHEAVES OF RELATIVE JETS [1]

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Introduction

Given a subscheme, (or analytic subspace) Y of a relative scheme (or analytic space) X/S, one may wish to link the tangent cones of the fibers of X/S at points of Y with the normal cone of X along Y. (which generalises the normal bundle; see (1.8)). The main result of this work (see th. 2.3 and its avatar th. 3.6) is that if we imbed Y in $Y \times_S Y$ and $X \times_S Y$ diagonally, there exists a canonical sequence of morphisms of normal cones

$$(0) \to C_{Y \times_S Y, Y} \to C_{X \times_S Y, Y} \to C_{X, Y} \to (0)$$

which is 'exact' in the sense that $C_{X,Y}$ is a quotient of $C_{X \times_S Y, Y}$ by a natural action of $C_{Y \times_S Y, Y}$, if Y is smooth over the base S. Hence, in that case, $C_{X,Y}$ is flat over Y if and only if $C_{X \times_S Y, Y}$ is, and then the fiber of this last space above a point of Y is nothing but the tangent cone at this point to the fiber of X/S. The geometric meaning is that the various tangent cones to the fibers of X/S at points of Y glue up into a nice flat family parametered by Y if and only if $C_{X,Y}$ is flat over Y.

This exact sequence of normal cones gives rise locally to a split sequence of graded Algebras, looking only at the terms of degree 1, we must get an exact sequence of Modules and this is nothing but the well-known sequence

$$0 \to \mathscr{N}_{X,Y} \to i^*\Omega^1_{X/S} \to \Omega^1_{Y/S} \to 0.$$

The main result enables us to give, for a notion of relative normal flatness which we introduce in Section 3, a numerical criterion similar to Bennett's in [1], and to prove the existence of a relative Samuel stratification, i.e. a partition of X into subschemes (or subspaces) such that two points belong to the same subscheme (or subspace) if and only if the Samuel function of the fibers of X/S through these points are equal. (See Section 4).

Notations

Throughout this paper we work either in the category of schemes or in the category of analytic spaces defined over a valued, complete, non

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discrete, algebraically closed field. By scheme, we mean a non necessarily separated one.

If X is a scheme (resp. analytic space), we note $\mathcal{O}_X$ the structural sheaf of rings on X. If x is a point in X, $\mathcal{O}_{X,x}$ (resp. $\mathfrak{m}_{X,x}$) (resp. $\kappa(x)$) is the local ring of X at x (resp. its maximal ideal) (resp. its residue field).

Let $\mathscr{F}$ be an $\mathcal{O}_X$-Module, $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at x, $\mathscr{F}(x) = \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is the fiber of $\mathscr{F}$ at x.

0. Smooth morphisms and regular immersions

We recall here some basic properties we shall use in the later sections. For more details (maybe not strictly necessary at the first reading) one may refer to [4] exposé 2 or [5] Ch. IV, § 16 and 17, Ch. 0, § 15 and 19, and [2] exposé 13.

From now on the topological structure on a noetherian local ring $\boldsymbol{O}$ will always be that defined by its maximal ideal and completion will mean with respect to this topology.

0.1. Let us recall $\boldsymbol{A}$ being a topological ring, $\boldsymbol{B}$, $\boldsymbol{A}'$ topological $\boldsymbol{A}$-algebras, if $\boldsymbol{B}$ is formally smooth over $\boldsymbol{A}$, $\boldsymbol{B} \otimes_{\boldsymbol{A}} \boldsymbol{A}'$ is formally smooth over $\boldsymbol{A}'$. (In particular, if $\boldsymbol{P}$ is a prime ideal in $\boldsymbol{A}$, $\boldsymbol{A_P}$ the localization of $\boldsymbol{A}$ at $\boldsymbol{P}$, $\boldsymbol{B_P} = \boldsymbol{B} \otimes_{\boldsymbol{A}} \boldsymbol{A_P}$ is formally smooth over $\boldsymbol{A_P}$).

0.2. If $\boldsymbol{B}$ is formally smooth over $\boldsymbol{A}$, $\boldsymbol{C}$ formally smooth over $\boldsymbol{B}$, $\boldsymbol{C}$ viewed as an $\boldsymbol{A}$-topological algebra is formally smooth over $\boldsymbol{A}$.

0.3. $\boldsymbol{B}$ is formally smooth over $\boldsymbol{A}$, if and only if $\hat{\boldsymbol{B}}$ is formally smooth over $\hat{\boldsymbol{A}}$.

0.4. If $\boldsymbol{A} \to \boldsymbol{B}$ is a local morphism of noetherian local rings, $\boldsymbol{m}$ (resp. $\boldsymbol{K}$) the maximal ideal (resp. residue field) of $\boldsymbol{A}$, $\boldsymbol{B}$ is formally smooth over $\boldsymbol{A}$ if and only if $\boldsymbol{B}$ is $\boldsymbol{A}$-flat and $\boldsymbol{B}/\boldsymbol{mB} = \boldsymbol{B} \otimes_{\boldsymbol{A}} \boldsymbol{K}$ is formally smooth over $\boldsymbol{K}$ ([5], 0.19.7.1).

0.5. If $\boldsymbol{A}$ is a local noetherian $\boldsymbol{k}$-algebra where $\boldsymbol{k}$ is a field, $\boldsymbol{K}$ its residue field, if $\boldsymbol{K}$ is an extension of finite type of $\boldsymbol{k}$, $\boldsymbol{A}$ is formally smooth over $\boldsymbol{k}$ if and only if $\boldsymbol{A}$ is geometrically regular (i.e., for every finite extension $\boldsymbol{k}'$ of $\boldsymbol{k}$, the semi-local ring $\boldsymbol{A} \hat{\otimes}_{\boldsymbol{k}} \boldsymbol{k}'$ is regular).

0.6. Assumptions as in (0.5). Moreover assume $\boldsymbol{k} = \boldsymbol{K}$. Then $\boldsymbol{A}$ is formally smooth over $\boldsymbol{k}$ if and only if $\hat{\boldsymbol{A}}$ is $\boldsymbol{k}$-isomorphic with some ring of power series $\boldsymbol{k}[[T_1, \cdots, T_n]]$ ([5], 0.19.6.4).

0.7. From (0.4) and (0.6), one may easily deduce that if $\boldsymbol{A} \to \boldsymbol{B}$ is a local morphism of noetherian local rings residually trivial, $\boldsymbol{B}$ is formally smooth over $\boldsymbol{A}$ if and only if $\hat{\boldsymbol{B}}$ is $\hat{\boldsymbol{A}}$-isomorphic with some ring of power series $\hat{\boldsymbol{A}}[[T_1, \cdots, T_n]]$ ([5], 0.19.7.1.5).

0.8. Let $\boldsymbol{k}$ be a field, $\boldsymbol{K}$ an extension of $\boldsymbol{k}$. $\boldsymbol{K}$ is formally smooth over $\boldsymbol{k}$ if and only if $\boldsymbol{K}$ is a separable extension of $\boldsymbol{k}$ ([5], 0.19.6.1).

0.9. Let S be a locally noetherian scheme (resp. analytic space), $f: X \to S$ a morphism of schemes locally of finite type (resp. of analytic spaces). Let x be a point in X, $s = f(x)$.

The following conditions are equivalent:

(1) f is smooth at x.

(2) $\mathcal{O}_{X,x}$ is formally smooth over $\mathcal{O}_{S,s}$.

(3) f is flat at x and the canonical morphism f_s $X_s = f^{-1}(s) \to \operatorname{Spec} \boldsymbol{k}(s)$ is smooth at x.

(4) f is flat at x and $\mathcal{O}_{X_s,x}$ is geometrically regular. If $\kappa(x) = \kappa(s)$ (automatically satisfied in the analytic case), then the conditions (1) to (4) are also equivalent to the following one.

(5) $\hat{\mathcal{O}}_{X,x}$ is $\hat{\mathcal{O}}_{S,s}$ isomorphic to some ring of power series $\hat{\mathcal{O}}_{S,s}[[T_1, \cdots, T_n]]$.

0.10. A smooth morphism remains smooth after base extension.

0.11. The composition of smooth morphisms is a smooth morphism.

0.12. Assumptions as in (0.9). If f is smooth at x, $\mathcal{O}_{X,x}$ is a reduced ring if and only if $\mathcal{O}_{S,s}$ is reduced.

0.13. Assumptions as in (0.9). The set of points where f is smooth is open in X (perhaps empty).

0.14. Let $\boldsymbol{k}$ be a field, X an algebraic scheme over $\boldsymbol{k}$. Assume X is integral. X is smooth over $\boldsymbol{k}$ at its generic point if and only if its function field is a separable extension of $\boldsymbol{k}$.

Let us now come to regular immersions:

0.15. Let i: $Y \to X$ be an immersion of noetherian local schemes (resp. germs of analytic spaces). Let y be the closed point of Y (resp. the picked point on the germ Y), the following conditions are equivalent:

(1) i is a regular immersion.

(2) $\mathcal{I}_y = \operatorname{Ker} \mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}$ is generated by a regular sequence of elements for $\mathcal{O}_{X,y}$.

(3) The canonical morphism $\operatorname{Sym}_{\mathcal{O}_{Y,y}}[\mathcal{I}_y/\mathcal{I}_y^2] \to \operatorname{gr}_{\mathcal{I}_y} \mathcal{O}_{X,y}$ is an isomorphism and $\mathcal{I}_y/\mathcal{I}_y^2$ is a free $\mathcal{O}_{Y,y}$-Module.

(4) Let $\hat{X} = \operatorname{Spec} \hat{\mathcal{O}}_{X,y}$, $\hat{Y} = \operatorname{Spec} \hat{\mathcal{O}}_{Y,y}$, $\hat{\imath}$: $\hat{Y} \to \hat{X}$ is a regular immersion.

0.16. Assumptions as in (0.15). Let S be a noetherian local scheme (resp. germ of analytic space), s its closed point (resp. picked point). Assume that X and Y are S-schemes (resp. analytic spaces), i is an S-immersion and Y is flat over S. The condition (1) to (4) are equivalent to the following one:

(5) X is flat over S and i_s: $Y_s \to X_s$ where Y_s (resp. X_s) is the fiber of Y (resp. X)over s is a regular immersion.

0.17. Assumptions as in (0.16). Assume that Y and X are smooth over S. Then i is a regular immersion.

0.18. Assumptions as in (0.17). If the residual extension $\kappa(s) \to \kappa(x)$ is trivial then there exist $\hat{\mathcal{O}}_{S,s}$-isomorphisms $\varepsilon_1: \hat{\mathcal{O}}_{X,y} \to \hat{\mathcal{O}}_{S,s}[[Z_1, \cdots, Z_r, Y_1, \cdots, Y_s]]$, $\varepsilon_2: \hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{S,s}[[Y_1, \cdots, Y_s]]$, such that the diagram (of

$\hat{\mathcal{O}}_{S,s}$-algebras):

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,y} & \longrightarrow & \hat{\mathcal{O}}_{Y,y} \\ {\scriptstyle\varepsilon_1}\downarrow & & \downarrow{\scriptstyle\varepsilon_2} \\ \hat{\mathcal{O}}_{S,s}[[Z,Y]] & \xrightarrow{\omega} & \hat{\mathcal{O}}_{S,s}[[Y]] \end{array}$$

where ω is the canonical projection, is commutative.

0.19. Let $i: Y \to X$ be an immersion of schemes (resp. analytic spaces). i is a regular immersion at $y \in Y$ if the immersion i_y induced on the local schemes (resp. germs of analytic space) at y is a regular immersion. i is a regular immersion if it is at every point $y \in Y$.

0.20. If i is a regular immersion at y, there exists an open neighborhood U of y in X such that $i|U \cap Y: U \cap Y \to U$ is a regular immersion.

0.21. $i: Y \to X$ is a regular closed immersion if and only if: $\mathcal{I}$ being the Ideal defining Y in X, $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_Y$-Module and the canonical morphism $\mathrm{Sym}_{\mathcal{O}_Y}[\mathcal{I}/\mathcal{I}^2] \to \mathrm{gr}_{\mathcal{I}}\,\mathcal{O}_X$ is an isomorphism.

0.22. Let $f: X \to S$ be a morphism of schemes locally of finite type (resp. of analytic spaces). Assume the scheme S is locally noetherian. Let $i: Y \to X$ be an S-immersion. Let y be a point in Y. If X is smooth over S at $i(y)$ and Y smooth over S at y, i is a regular immersion at y.

0.23. Let $f: X \to S$ as in (0.22). Let $\Delta_f: X \to X \times_S X$ be the diagonal immersion. If f is smooth at x, Δ_f is a regular immersion at x. (Apply 0.22+0.10+0.11).

0.24. Conversely if Δ_f is a regular immersion at x and f is flat at x, f is smooth at x.

1. Some functorial properties of $\mathcal{P}_{X/S}$, $\mathrm{gr}_Y X$ and a nice commutative diagram

We recall here some definitions and properties of the differential invariants used in the categories of schemes and analytic spaces. The reader is referred in the algebraic case to [5], Chap. IV, § 16, in the analytic case to [2], exposé n^0. 14.

1.1 Let $f: X \to S$ be a morphism of schemes (resp. analytic spaces), $\Delta_f: X \to X \times_S X$ the diagonal immersion, $i = 1,2$: $p_i: X \times_S X \to X$ the canonical projections. There exists a projective system of $\mathcal{O}_X$-Algebras $(\mathcal{P}^n_{X/S})_{n \geqq 0}$ called the system of relative jets of X/S. (One calls it also the system of relative principal parts.)

For simplicity, assume Δ_f is a closed immersion and let $\mathcal{D}$ be the associated $\mathcal{O}_{X \times_S X}$-Ideal. $\mathcal{P}^n_{X/S} = p_{1*}(\mathcal{O}_{X \times_S X}/\mathcal{D}^{n+1})$. Let

$$\mathcal{P}_{X/S} = \bigoplus_{n \geqq 0} \mathcal{P}^n_{X/S}.$$

1.2. If S is a locally noetherian scheme and f is locally of finite type or if f is a morphism of analytic spaces, $n \geqq 0$, $\mathscr{P}^n_{X/S}$ is a coherent $\mathscr{O}_X$-Module.

1.3 Let $i: Y \to X$ be an immersion. Let us define $\mathscr{P}_{X/S}(Y) = i^*(\mathscr{P}_{X/S})$.

1.4. Let $\mathrm{gr}^0 \mathscr{P}_{X/S}(Y) = \mathscr{O}_Y$, $n \geqq 1$,

$$\mathrm{gr}^n \mathscr{P}_{X/S}(Y) = \mathrm{Ker}\,(\mathscr{P}^n_{X/S}(Y) \to \mathscr{P}^{n-1}_{X/S}(Y));$$

$\bigoplus_{n \geqq 0} \mathrm{gr}^n \mathscr{P}_{X/S}(Y)$ has a canonical structure of $\mathscr{O}_Y$-Algebra called $\mathrm{gr}\,\mathscr{P}_{X/S}(Y)$. If $i = \mathrm{id}_X$, we write simply $\mathrm{gr}\,\mathscr{P}_{X/S}$ instead of $\mathrm{gr}\,\mathscr{P}_{X/S}(X)$.

Note that by definition $\mathrm{gr}^1 \mathscr{P}_{X/S} = \Omega^1_{X/S}$ and $\mathscr{O}_X$ being $\mathscr{O}_X$-flat, $\mathrm{gr}^1 \mathscr{P}_{X/S}(Y) = i^*(\Omega^1_{X/S})$.

1.5. Let us recall that the nth infinitesimal neighborhood of Y for i is (if i is a closed immersion and $\mathscr{I}$ denotes the associated $\mathscr{O}_X$-Ideal) the subscheme (resp. analytic subspace) of X defined by $\mathscr{I}^{n+1}$.

1.6. Finally, let us denote by $\mathrm{gr}_Y X$ the graded $\mathscr{O}_Y$-Algebra associated with i. Again if i is closed, $\mathrm{gr}_Y X$ is $\mathrm{gr}_{\mathscr{I}}\,\mathscr{O}_X = \bigoplus_{n \geqq 0} \mathscr{I}^n/\mathscr{I}^{n+1}$.

1.7. If S is a locally noetherian scheme and f is locally of finite type, or if f is a morphism of analytic spaces, $\mathrm{gr}_Y X$ is an $\mathscr{O}_Y$-Algebra of finite presentation.

1.8. If $C_{X,Y}$ is $\mathrm{Spec}\,\mathrm{gr}_Y X$ (resp. $\mathrm{Specan}\,\mathrm{gr}_Y X$) the canonical morphism $C_{X,Y} \to Y$ is the normal cone of X along Y.

Given a commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & X & \longrightarrow & S \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle h} & & \uparrow \\ Y' & \longrightarrow & X' & \longrightarrow & S' \end{array}$$

where i and i' are immersions, one gets canonical maps:

$$h^*(\mathscr{P}_{X/S}) \to \mathscr{P}_{X'/S'} \tag{1.9}$$

$$k^*(\mathrm{gr}\,\mathscr{P}_{X/S}(Y)) \to \mathrm{gr}\,\mathscr{P}_{X'/S'}(Y') \tag{1.10}$$

$$k^*(\mathrm{gr}_Y X) \to \mathrm{gr}_{Y'} X' \tag{1.11}$$

1.12. If the diagram on the right hand side is cartesian, then (1.9) is an isomorphism.

1.13. If h is flat and the diagram on the left hand side is cartesian, then (1.10) is an isomorphism.

1.14. Let y be a point in Y, s its image in S, X_x, Y_x, S_s the local scheme, (resp. germ of analytic space) of X at x, Y at x, S at s. The stalk of $\mathscr{P}_{X/S}(Y)$, $\mathrm{gr}\,\mathscr{P}_{X/S}(Y)$, $\mathrm{gr}_Y X$ at y is $\mathscr{P}_{X_y/S_s}(Y_y)$, $\mathrm{gr}\,\mathscr{P}_{X_y/S_s}(Y_y)$, $\mathrm{gr}_{Y_y} X_y$.

1.15. Given an immersion $i: Y \to X$ of S-schemes (resp. S-analytic spaces), we get canonically immersions of S-schemes (resp. S-analytic spaces)

$\delta(i): Y \to X \times_S Y$, $\delta'(i): Y \times_S Y \to X \times_S Y$ such that

$$\begin{tikzcd} Y \arrow[r, "\delta(\mathrm{Id}\,Y)"] \arrow[dr, "\delta(i)"'] & Y \underset{S}{\times} Y \arrow[d, "\delta'(i)"] \\ & X \underset{S}{\times} Y \end{tikzcd}$$

is commutative, hence graded $\mathcal{O}_Y$-Algebras $\mathrm{gr}_Y\, Y \times_S Y$ and $\mathrm{gr}_Y\, X \times_S Y$ and normal cones $C_{Y \times_S Y, Y}$ and $C_{X \times_S Y, Y}$.

1.16. Let us note that, if X and S are Spec of complete noetherian local rings and if the associated residual extension is trivial, $Y \times_S Y$, $X \times_S X$, $X \times_S Y$ are local schemes. Denote by $Y \hat{\times}_S Y$, $X \hat{\times}_S X$, $X \hat{\times}_S Y$ the Spec of the completion of their respective local rings. We obtain commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "\hat{\Delta}_f"] \arrow[dr, "\Delta_f"'] & X \underset{S}{\hat{\times}} X \arrow[r, "\hat{p}_1"] \arrow[d] & X \\ & X \underset{S}{\times} X \arrow[ur, "p_1"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d, "\mathrm{Id}\,Y"'] & Y \underset{S}{\hat{\times}} Y \arrow[r] \arrow[d] & Y \underset{S}{\times} Y \arrow[d] \\ Y \arrow[r] & X \underset{S}{\hat{\times}} Y \arrow[r] & X \underset{S}{\times} Y \end{tikzcd}$$

The canonical morphisms $\mathrm{gr}_Y X \times_S Y \to \mathrm{gr}_Y X \hat{\times}_S Y$ and $\mathrm{gr}_Y\, Y \times_S Y \to \mathrm{gr}_Y\, Y \hat{\times}_S Y$ are isomorphisms. If $\hat{\mathscr{D}}$ is the ideal defining X in $X \hat{\times}_S X$, $\mathscr{P}^n{}_{X/S}$ is canonically isomorphic to $\hat{p}_{1*}(\mathcal{O}_{X \hat{\times}_S X}/\hat{\mathscr{D}}^{n+1})$.

1.17. Let us consider the following commutative diagram of graded $\mathcal{O}_Y$-Algebras:

$$\begin{tikzcd} \mathrm{gr}_Y\, X \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}"] & \mathrm{gr}_Y\, X \underset{S}{\times} Y \arrow[r, "\beta"] \arrow[d, "\lambda"] & \mathrm{gr}_Y\, Y \underset{S}{\times} Y \arrow[d, "\mu"] \\ \mathrm{gr}_Y\, X \arrow[r, "\alpha'"] & \mathrm{gr}\, \mathscr{P}_{X/S}(Y) \arrow[r, "\beta'"] & \mathrm{gr}\, \mathscr{P}_{Y/S} \end{tikzcd}$$

where α and β arise from the functoriality property (1.11) and the commutative diagram:

$$\begin{tikzcd} Y \underset{S}{\times} Y \arrow[r, "\delta'(i)"] & X \underset{S}{\times} Y \arrow[r, "p_1"] & X \\ Y \arrow[r, "\mathrm{Id}"] \arrow[u, "\delta(\mathrm{Id})"] & Y \arrow[r, "\mathrm{Id}"] \arrow[u, "\delta(i)"] & Y \arrow[u, "i"'] \end{tikzcd}$$

where β' arises from the functoriality property (1.10) and the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i"] & X \\ Y \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id}"] & Y \arrow[u, "i"'] \end{tikzcd}$$

where α'_n, λ_n, μ_n, the homogeneous component of degree n of α', λ, μ are induced respectively by:

$$d^n_{X/S}\colon \mathcal{O}_X \to \mathcal{P}^n_{X/S}$$

the $\mathcal{O}_S$-linear map obtained from the action of $\mathcal{O}_X$ on $\mathcal{O}_{X\times_S X}/\mathcal{D}^{n+1}$ by p_2.

$$\Lambda^n(i)\colon \mathcal{O}_{X\times_S Y} = p_1^*(\mathcal{O}_X) \to p_1^*(\mathcal{P}^n_{X/S}) \to \delta(i)_*(\mathcal{P}^n_{X/S}(Y))$$

$$M^n\colon \mathcal{O}_{Y\times_S Y} \xrightarrow{\qquad\qquad\qquad} \delta(\mathrm{Id})_*(\mathcal{P}^n_{Y/S})$$

Note that $M^n = \Lambda^n(\mathrm{Id}\, Y)$.

LEMMA (1.18): *The $\mathcal{O}_Y$-Algebra $\mathcal{P}^n_{X/S}(Y)$ is canonically $\mathcal{O}_Y$-isomorphic to the structural sheaf of nth infinitesimal neighborhood of Y for the immersion $\delta(i)$, $\mathcal{O}_{Y_{\delta(i)}}[n]$ endowed with the structure of $\mathcal{O}_Y$-Algebra from the projection $X\times_S Y$ on Y.*

PROOF: From the commutative diagram:

$$\begin{tikzcd} X \arrow[r] & S \\ X\times_S Y \arrow[u, "p_1"] \arrow[r] & Y \arrow[u, "f\circ i"'] \end{tikzcd}$$

and property (1.12) we get an isomorphism of $\mathcal{O}_Y$-Algebras:

$$p_1^*(\mathcal{P}^n_{X/S}) \to \mathcal{P}^n_{X\times_S Y/Y}.$$

Applying now [5] Chap. IV.16.4.11, resp. [2], with the section $\delta(i)$, we have now an isomorphism of $\mathcal{O}_Y$-Algebra:

$$\delta(i)^*(\mathcal{P}^n_{X\times_S Y/Y}) \to \mathcal{O}_{Y_{\delta(i)}}[n].$$

But $i = p_1\circ\delta(i)$, so finally the isomorphism of $\mathcal{O}_Y$-Algebras:

$$i^*(\mathcal{P}^n_{X/S}) = \mathcal{P}^n_{X/S}(Y) \to \mathcal{O}_{Y_{\delta(i)}}[n].$$

LEMMA (1.19): *λ and μ of diagram* 1.17 *are isomorphisms.*

It is enough to prove that λ is an isomorphism. From Lemma (1.18), we get a canonical isomorphism: $\mathrm{gr}^n\mathcal{P}_{X/S}(Y) \to \mathrm{gr}^n_Y X\times_S Y$. It is easy to see that it is reciprocal of λ.

REMARK (1.20): If S is a locally noetherian scheme and f is locally of finite type, or f is a morphism of analytic space, $\mathrm{gr}\,\mathcal{P}_{X/S}(Y)$ is an $\mathcal{O}_Y$-Algebra of finite presentation. Immediate from (1.7) and (1.19).

REMARK (1.21): If $f\colon Y\to S$ is smooth over S at x, the stalk at x of $\Omega^1_{Y/S}$ is a locally free $\mathcal{O}_{Y,y}$-module and its symmetric algebra is canonically isomorphic to the stalk at x of $\operatorname{gr} \mathscr{P}_{Y/S}$. Immediate from (1.19) and (0.23), (0.21).

2. The key exact sequence of cones

Given Y a scheme (resp. analytic space), we say that C is a Y-cone if C is Y-isomorphic to the Spec (resp. Specan) of an $\mathcal{O}_Y$-positively graded, augmented Algebra of finite presentation generated by its degree 1 elements. It is equivalent to saying that there exists on C an action of the multiplicative group $\boldsymbol{G}_{m,Y}$ induced by a Y-immersion of C in a locally trivial Y-vector bundle. We note $(0) = \operatorname{Spec} \mathcal{O}_Y$ (resp. $\operatorname{Specan} \mathcal{O}_Y$).

Remark that we have canonical Y-morphisms, $p: C \to (0)$, $v: (0) \to C$. p is the structural morphism, v is the vertex.

Definition (2.1): *Let* $(0) \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to (0)$ *be a sequence of Y-morphisms of Y-cones. We say that it is exact if there exists a covering* $(Y_i)_{i \in I}$ *of Y by open sets such that if* $C_i = C \times_Y Y_i$, $C'_i = C \times_Y Y_i$, $C''_i = C \times_Y Y_i$ *there exists a commutative diagram of Y_i-cones*

$$\begin{tikzcd}
(0) \arrow[r] & C'_i \arrow[r] & V_i \arrow[r] & V''_i \arrow[r] & (0) \\
(0) \arrow[r] & C'_i \arrow[r, "\alpha_i"'] \arrow[u, "\mathrm{Id}"'] & C_i \arrow[r, "\beta_i"'] \arrow[u] & C''_i \arrow[r] \arrow[u] & (0)
\end{tikzcd}$$

where:

(i) *the vertical arrows are closed immersions*

(ii) *the upper horizontal sequence is an exact and split sequence of trivial vector bundles* (*i.e., Spec* (*resp. Specan*) *of the symmetric Algebra of a free* $\mathcal{O}_{Y_i}$*-Module*)

(iii) *the action of* C'_i *by translation on* V_i *induces an action on* C_i *and* ${}^{\beta}{}_{i}$ *induces a* Y_i*-isomorphism of* $C_i/\alpha_i(C'_i)$ *on* C''_i.

We leave to the reader as an exercise to check that (iii) may be changed to:

(iii′) $C_i \simeq V_i \times_{V''_i} C''_i$

or

(iii″) *A splitting of the upper exact sequence induces a left inverse of* α_i, *so that the induced morphism* $C_i \to C'_i \times_{Y_i} C''_i$ *is a* Y_i*-isomorphism.*

Remark (2.2): Note that if $(0) \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to (0)$ is exact, α is an immersion and β is surjective.

Theorem (2.3): *Let X be an S-scheme, $i: Y \to X$ an S-immersion. Assume*:

(1) *X and S are local schemes, Spec of noetherian complete local rings*

(2) *x (resp. s) being the closed points of X (resp. S), the residual extension $\kappa(s) \to \kappa(x)$ is trivial*

(3) *Y is formally smooth over S at x.*

Then the canonical sequence

$$(0) \to C_{Y \times_S Y, Y} \to C_{X \times_S Y, Y} \to C_{X, Y} \to (0)$$

is an exact sequence of Y-cones.

PROOF: Let $\boldsymbol{O}$ (resp. $\boldsymbol{A}$) (resp. $\boldsymbol{S}$) be the local ring of X (resp. Y) at x, (resp. S) at s. Let $\boldsymbol{M}$ be the maximal ideal of $\boldsymbol{O}$ and choose $\tau = (\tau_1, \cdots, \tau_n)$ to be a system of generators of $\boldsymbol{M}$. Let $\psi : \boldsymbol{S}[[t]] \to \boldsymbol{O}$ be the $\boldsymbol{S}$-morphism, such that $1 \leqq i \leqq n$, $\psi(t_i) = \tau_i$. ψ is surjective. From (2) it follows that $\boldsymbol{O} = \boldsymbol{S} + \boldsymbol{M}$.

Take $f \in \boldsymbol{O}$. Thus there exists $f_0 \in \boldsymbol{S}$, $h_i \in \boldsymbol{O}$, such that:

$$f = f_0 + \sum_{i=1}^{n} h_i \tau_i.$$

Now, applying to each h_i the same process, and so on, we get for $\alpha = (\alpha_1, \cdots, \alpha_n)$, $f_\alpha \in \boldsymbol{S}$, such that: $f = \sum_{\alpha : |\alpha| \leqq n} f_\alpha \tau^\alpha \operatorname{mod} \boldsymbol{M}^{n+1}$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\tau^\alpha = \tau_1^{\alpha_1} \cdots \tau_n^{\alpha_n}$. $\boldsymbol{O}$ being complete $f = \sum_\alpha f_\alpha \tau^\alpha$ and ψ being continuous $f = \psi(\sum_\alpha f_\alpha t^\alpha)$. Let $\boldsymbol{R} = \boldsymbol{S}[[t]]$, $Z = \operatorname{Spec} \boldsymbol{S}[[t]]$; then, we have an S-immersion of X in Z formally smooth over S at x.

From [5] (0.19.6.4 and 0.19.7.1), the $\boldsymbol{S}$-algebra $\boldsymbol{A}$ is $\boldsymbol{S}$-isomorphic to a ring of formal power series over $\boldsymbol{S}$, $\boldsymbol{S}[[y]]$. Hence, we have a surjective $\boldsymbol{S}$-morphism $\bar{\psi} : \boldsymbol{S}[[t]] \to \boldsymbol{S}[[y]]$. We can find (cf. 0.18) an $\boldsymbol{S}$-isomorphism $\lambda : \boldsymbol{S}[[y, z]] \to \boldsymbol{S}[[t]]$ such that

$$\bar{\psi} \circ \lambda(y) = y, \qquad \bar{\psi} \circ \lambda(z) = 0.$$

So, up to S-isomorphisms, the S-immersion of Y in Z corresponds to the canonical projection $\boldsymbol{S}[[y, z]] \to \boldsymbol{S}[[y]]$.

By functoriality, we get a commutative diagram (of Y-cones)

$$\begin{tikzcd}
(0) \arrow[r] & C_{Y \times_S Y, Y} \arrow[r] & C_{Z \times_S Y, Y} \arrow[r] & C_{Z, Y} \arrow[r] & (0) \\
(0) \arrow[r] & C_{Y \times_S Y, Y} \arrow[r] \arrow[u, "\mathrm{Id}"'] & C_{X \times_S Y, Y} \arrow[r] \arrow[u, "(\mathrm{I})"'] & C_{X, Y} \arrow[r] \arrow[u] & (0)
\end{tikzcd}$$

corresponding to the following one (of $\boldsymbol{A}$-algebras)

$$\begin{tikzcd}
\operatorname{gr}_Y Z \arrow[r] \arrow[d] & \operatorname{gr}_Y Z \times_S Y \arrow[r] \arrow[d] & \operatorname{gr}_Y Y \times_S Y \arrow[d] \\
\operatorname{gr}_Y X \arrow[r] & \operatorname{gr}_Y X \times_S Y \arrow[r] & \operatorname{gr}_Y Y \times_S Y
\end{tikzcd}$$

Condition (i) appears to be trivially satisfied.

Look now at condition (ii). As already observed in Section 1, one does not change the normal cones by replacing the usual tensor product by the completed one. For simplicity, call x the closed point of Z, $Y\hat{\times}_S Y$, $X\hat{\times}_S Y$, $Z\hat{\times}_S Y$; we have a commutative diagram of S-algebras:

$$\begin{tikzcd}
\mathcal{O}_{Z,x} \arrow[r] \arrow[d] & \mathcal{O}_{Z\hat{\times}_S Y,x} \arrow[r] \arrow[d] & \mathcal{O}_{Y\hat{\times}_S Y,x} \arrow[d] \\
S[[y,z]] \arrow[r, "\varphi_1"] & S[[y,z,y']] \arrow[r, "\varphi_2"] & S[[y,y']]
\end{tikzcd}$$

where vertical arrows are S-isomorphisms, $\varphi_1(y)=y$, $\varphi_1(z)=z$, $\varphi_2(y)=y$, $\varphi_2(z)=0$, $\varphi_2(y')=y'$. So the ideal of $\mathcal{O}_{Z\hat{\times}_S Y,x}$ (resp. $\mathcal{O}_{Y\hat{\times}_S Y,x}$) defining Y in $Z\hat{\times}_S Y$ (resp. $Y\hat{\times}_S Y$) appears to be generated by $(z,y-y')$ (resp. $(y-y')$). Consider now the commutative diagram (of S-algebras)

$$\begin{tikzcd}
S[[y,z,y']] \arrow[r, "\varphi_2"] \arrow[d, "\sigma"] & S[[y,y']] \arrow[d, "\sigma_0"] \\
S[[y,z,y'']] \arrow[r, "\varphi_2^\sigma"] & S[[y,y'']]
\end{tikzcd}$$

where $\sigma(y)=y$, $\sigma(z)=z$, $\sigma(y')=y-y''$, $\sigma_0(y)=y$, $\sigma_0(y')=y-y''$

$$\varphi_2^\sigma(y)=y,\quad \varphi_2^\sigma(z)=0,\quad \varphi_2^\sigma(y'')=y''$$

σ and σ_0 are S-isomorphisms and $\sigma(y-y')=y''$.

Now we can identify $\mathrm{gr}_Y Z$ (resp. $\mathrm{gr}_Y Z\hat{\times}_S Y$) (resp. $\mathrm{gr}_Y Y\hat{\times}_S Y$) with $\mathrm{gr}_{(z)} S[[y,z]]$, (resp. $\mathrm{gr}_{(z,y'')} S[[y,z,y'']]$), (resp. $\mathrm{gr}_{(y'')} S[[y,y'']]$), $\tilde{\alpha}$ with $\mathrm{gr}(\sigma\cdot\varphi_1)$, $\tilde{\beta}$ with $\mathrm{gr}(\varphi_2^\sigma)$. Finally, by letting: $Z=\mathrm{cl}\, z \bmod (z)^2$ or $\bmod (z,y'')^2$, $Y''=\mathrm{cl}\, y'' \bmod (y'')^2$ or $(z,y'')^2$ (as no confusion may possibly happen), it turns out that: $\tilde{\alpha}: S[[y]][z]\to S[[y]][Z,Y'']$ is simply the canonical injection, $\tilde{\beta}: S[[y]][Z,Y'']\to S[[y]][Y'']$ is simply the canonical projection, so that (ii) holds clearly.

We shall prove now that (iii′) is satisfied, i.e., that:

$$\begin{tikzcd}
\mathrm{gr}_Y Z \arrow[r] \arrow[d] & \mathrm{gr}_Y Z\hat{\times}_S Y \arrow[d] \\
\mathrm{gr}_Y X \arrow[r] & \mathrm{gr}_Y X\hat{\times}_S Y
\end{tikzcd}$$

is a cocartesian diagram (in A-algebras). The vertical arrows being surjective, this amounts to saying that the ideal generated in $\mathrm{gr}_Y Z\hat{\times}_S Y$ by the image of the kernel of the left hand side arrow is the kernel of the right hand side arrow.

Recall now ([7] II.2. Lemma 5) that, $\boldsymbol{B}$ being a noetherian ring, $\boldsymbol{I}$, $\boldsymbol{J}$ ideals in $\boldsymbol{B}$, by definition, the sequence $0 \to \operatorname{in}_{\boldsymbol{J}}(\boldsymbol{B}, \boldsymbol{I}) \to \operatorname{gr}_{\boldsymbol{J}} \boldsymbol{B} \to \operatorname{gr}_{\boldsymbol{J}} \boldsymbol{B}/\boldsymbol{I} \to 0$ is exact. If, for $f \in \boldsymbol{B}$, $f \neq 0$, we note $v_{\boldsymbol{J}}(f) = \{\sup n : f \in \boldsymbol{J}^n\}$ and $\operatorname{in}_{\boldsymbol{J}} f = \operatorname{cl} f \bmod \boldsymbol{J}^{v_{\boldsymbol{J}}(f)+1}$, it is easily seen that $\operatorname{in}_{\boldsymbol{J}}(\boldsymbol{B}, \boldsymbol{I})$ is generated by all $\operatorname{in}_{\boldsymbol{J}} f$ with $f \in \boldsymbol{I}$, $f \neq 0$.

If $\boldsymbol{I}$ is the ideal in $\boldsymbol{S}[[y, z]]$ defining X in Z, the ideal $\boldsymbol{K}$ generated by $\varphi_1(\boldsymbol{I})$ in $\boldsymbol{S}[[y, z, y']]$ is that defining $X \hat{\times}_S Y$ in $Z \hat{\times}_S Y$.

So, after transformation by σ as above, what we have to check is that: $\operatorname{in}_{(z, y'')}(\boldsymbol{S}[[y, z, y'']], \sigma(\boldsymbol{K}))$ is generated in $\operatorname{gr}_{(z, y'')} \boldsymbol{S}[[y, z, y'']]$ identified with $\boldsymbol{S}[[y]][Z, Y'']$ by $\tilde{\alpha}(\operatorname{in}_{(z)}(\boldsymbol{S}[[y, z]], \boldsymbol{I}))$.

Take an element g in $\sigma(\boldsymbol{K}) \neq 0$. There exists $f_i \in \boldsymbol{I}$, $Q_i \in \boldsymbol{S}[[y, z, y'']]$ such that:

$$g = \sum_{i=1}^{m} Q_i(y, z, y'') f_i(y, z).$$

Let $v = v_{(z, y'')}(g)$. Write $Q_i = \sum_{\alpha \in N_0{}^s, |\alpha| \leqq v} Y''^{\alpha} Q_{i\alpha}(y, z) + R_i$ with $R_i \in (y'')^{v+1}$.

$$\operatorname{in}_{(z, y'')}(g) = \operatorname{in}_{(z, y'')}\Big(\sum_{\alpha \in N_0{}^s, |\alpha| \leqq v} Y''^{\alpha} \big(\sum_{i=1}^{m} Q_{i\alpha} f_i \big)\Big).$$

Let $g_\alpha = \sum_{i=1}^{m} Q_{i\alpha} f_i$; $g_\alpha \in \boldsymbol{I}$ and $v_{(z)}(g_\alpha) \geqq v - |\alpha|$; if not, let

$$\mu = \inf_{\alpha, |\alpha| \leqq v} v_{(z)}(g_\alpha) + |\alpha|; \qquad \mu < v,$$

and

$$0 = \sum_{\alpha: |\alpha| \leqq v} Y''^{\alpha} \operatorname{cl} g_\alpha \bmod (z, y'')^{\mu - |\alpha| + 1}$$

But, $\operatorname{cl} g_\alpha \bmod (z, y'')^{\mu - |\alpha| + 1}$ is in $\boldsymbol{S}[[y]][Z]$, so it vanishes and for $\alpha: |\alpha| \leqq v$, $g_\alpha \in (z)^{\mu+1}$. So, finally,

$$\operatorname{in}_{(z, y'')}(g) = \sum_{\alpha: |\alpha| \leqq v} Y''^{\alpha} (\operatorname{cl} g_\alpha \bmod (z, y'')^{v - |\alpha| + 1}).$$

So there exists some α such that $g_\alpha \notin (z)^{v - |\alpha| + 1}$, and restricting $\sum$ only to those α:

$$\operatorname{in}_{(z, y'')}(g) = \sum_{\alpha}^{*} Y''^{\alpha} \cdot \tilde{\alpha}[\operatorname{in}_{(z)}(g_\alpha)].$$

Remark (2.4): In fact, we proved that for every S-immersion of X in Z, a local scheme, Spec of a noetherian complete local ring, formally smooth over S

$$(0) \to C_{Y \underset{S}{\times} Y, Y} \to C_{Z \underset{S}{\times} Y, Y} \to C_{Z, Y} \to (0)$$

is an exact sequence of trivial vector bundles and diagram (I) is cartesian.

COROLLARY (2.5): *Let $f: X \to S$ be a morphism of analytic spaces (over a complete, valued, non discrete, algebraically closed field* $\boldsymbol{k}$*), $i: Y \to X$ an S-immersion, x a point in Y, s its image in S.*

(i) *If Y is smooth over S at x, there exists an open neighborhood U (resp. V) of x (resp. s) in X (resp. S) such that $f(U) = V$ and that, letting X_0 (resp. Y_0) (resp. S_0) to be the restriction of X (resp. Y) (resp. S) on U (resp. $U \cap Y$) (resp. V), the canonical sequence of Y_0-cones:*

$$(0) \to C_{Y_0 \underset{S_0}{\times} Y_0, Y_0} \to C_{X_0 \underset{S_0}{\times} Y_0, Y_0} \to C_{X_0, Y_0} \to (0)$$

is exact.

(ii) *If Y is smooth over S, then*

$$(0) \to C_{Y \underset{S}{\times} Y} \to C_{X \underset{S}{\times} Y, Y} \to C_{X, Y} \to (0)$$

is exact.

PROOF: The exactness of a sequence of cones being of local nature, (ii) is an immediate consequence of (i).

Now, things being local around x, we may assume that i is a closed immersion and f is a separated morphism. Let us choose an immersion of an open neighborhood U_1 of x in X in an open set Ω of some affine $\boldsymbol{k}$-space E^n. Since Y is smooth over S at x, there exists an open neighborhood W (resp. V) of x (resp. s) on Y (resp. S) such that $f(W) = V$, $W \subset U_1$ *and* $f|W: W \to V$ is smooth.

Let U be any open set contained in $f^{-1}(V) \cap U_1$, whose intersection with Y is W. Clearly, $f(U) = V$. Now, by shrinking Ω if necessary, we may also assume that the immersion of U in Ω is closed, so that combining with $f|U$, we get a closed V-immersion $j_0: U \to V \times \Omega$. Let $Z_0 = V \times \Omega$; finally, Y_0 is smooth over S_0, $i_0: Y_0 \to X_0$ is a closed S_0-immersion and $j_0: X_0 \to Z_0$ is a closed S_0-immersion in an analytic space Z_0 smooth over S_0.

By functoriality, we get a commutative diagram of Y_0-cones:

$$\begin{tikzcd}
(0) \arrow[r] & C_{Y_0 \underset{S_0}{\times} Y_0, Y_0} \arrow[r] & C_{Z_0 \underset{S_0}{\times} Y_0, Y_0} \arrow[r, "{[\mathrm{I}]}"'] & C_{Z_0, Y_0} \arrow[r] & (0) \\
(0) \arrow[r] & C_{Y_0 \underset{S_0}{\times} Y_0, Y_0} \arrow[r] \arrow[u] & C_{X_0 \underset{S_0}{\times} Y_0, Y_0} \arrow[r] \arrow[u] & C_{X_0, Y_0} \arrow[r] \arrow[u] & (0).
\end{tikzcd}$$

Condition (i) holds clearly.

Y_0 (resp. Z_0), (resp. $Y_0 \times_{S_0} Y_0$)(resp. $Z_0 \times_{S_0} Y_0$) being smooth over S_0, $j_0 \circ i_0$ (resp. $\delta(\mathrm{Id}\, Y_0)$) (resp. $\delta(j_0 \circ i_0)$) *is a regular immersion, so that* $\mathrm{gr}^1_{Y_0} Z_0$ (resp. $\mathrm{gr}^1_{Y_0} Y_0 \times_{S_0} Y_0$) (resp. $\mathrm{gr}^1_{Y_0} Z_0 \times_{S_0} Y_0$) is a locally free $\mathcal{O}_{Y_0}$-Module and its symmetric Algebra is canonically isomorphic to $\mathrm{gr}_{Y_0} Z_0$ (resp.

$\mathrm{gr}_{Y_0} Y_0 \times_{S_0} Y_0)$ (resp. $\mathrm{gr}_{Y_0} Z_0 \times_{S_0} Y_0)$ (see Section 0). As an exact sequence of free module splits, it is enough to show that the upper horizontal sequence is exact and diagram I is cartesian, or equivalently that:

$$0 \to \mathrm{gr}^1_{Y_0} Z_0 \to \mathrm{gr}^1_{Y_0} Z_0 \underset{S_0}{\times} Y_0 \to \mathrm{gr}^1_{Y_0} Y_0 \underset{S_0}{\times} Y_0 \to 0$$

is exact and diagram

$$\begin{array}{ccc} \mathrm{gr}_{Y_0} Z_0 & \longrightarrow & \mathrm{gr}_{Y_0} Z_0 \underset{S_0}{\times} Y_0 \\ \downarrow & & \downarrow \\ \mathrm{gr}_{Y_0} X_0 & \longrightarrow & \mathrm{gr}_{Y_0} X_0 \underset{S_0}{\times} Y_0 \end{array}$$

is cocartesian. To do so, we have to prove the same with the induced sequence and diagram of stalks at every point y of Y_0, and $\hat{\mathcal{O}}_{Y_0, y}$, completion of $\mathcal{O}_{Y_0, y}$ with respect to the $\mathfrak{m}_{Y_0, y}$-adic topology, being a faithfully flat $\mathcal{O}_{Y_0, y}$-module, it is actually enough to do it after this base extension.

But, generally $\mathscr{Y}$ being an analytic subspace of some analytic space $\mathscr{X}$ and letting $\mathscr{Y}_y = \operatorname{Spec} \mathcal{O}_{\mathscr{Y}, y}$, $\hat{\mathscr{Y}}_y = \operatorname{Spec} \hat{\mathcal{O}}_{\mathscr{Y}, y}$, $\mathscr{X}_y = \operatorname{Spec} \mathcal{O}_{\mathscr{X}, y}$, $\hat{\mathscr{X}}_y = \operatorname{Spec} \hat{\mathcal{O}}_{\mathscr{X}, y}$,

$$(\mathrm{gr}_{\mathscr{Y}} \mathscr{X})_y \underset{\mathcal{O}_{\mathscr{Y}, y}}{\otimes} \hat{\mathcal{O}}_{\mathscr{Y}, y} = \mathrm{gr}_{\hat{\mathscr{Y}}_y} \hat{\mathscr{X}}_y$$

and $\mathscr{X}_1$, $\mathscr{X}_2$ being $\mathscr{S}$-analytic spaces, x_1, x_2 points of X_1, X_2 whose image in $\mathscr{S}$ is s

$$(\mathscr{X}_1 \underset{\mathscr{S}}{\times} \hat{\mathscr{X}}_{2, x_1 \times x_2}) = \hat{\mathscr{X}}_{1, x_1} \underset{\mathscr{S}_s}{\hat{\times}} \hat{\mathscr{X}}_{2, x_2}$$

(where $\hat{\times}$ means that the usual tensor product has been replaced by the completed one). So, it follows immediately from the proof of Corollary (2.3) after noticing that all residual extensions being trivial in analytic geometry, and Y_0 (resp. Z_0) being smooth over S_0, $\hat{Y}_{0, y}$ (resp. $\hat{Z}_{0, y}$) is formally smooth over $\hat{S}_{0, f_0(y)}$ and there is no residual extension.

COROLLARY (2.6): *Let $f: X \to S$ be a morphism of schemes, locally of finite type, $i: Y \to X$ an S-immersion, x a point of X, s its image in S. Assume S is locally noetherian.*

(i) *If Y is smooth over S at x, there exists an open neighborhood U (resp. V) of x (resp. s) in X (resp. S) such that $f(U) = V$ and that, letting X_0 (resp. Y_0) (resp. S_0) to be the restriction of X (resp. Y) (resp. S) on U (resp. $U \cap Y$) (resp. V), the canonical sequence of Y_0-cones:*

$$(0) \to C_{Y_0 \underset{S_0}{\times} Y_0, Y_0} \to C_{X_0 \underset{S_0}{\times} Y_0, Y_0} \to C_{X_0, Y_0} \to (0)$$

is exact.

(ii) *If* Y *is smooth ver* S, *then*

$$(0) \to C_{Y \underset{S}{\times} Y, Y} \to C_{X \underset{S}{\times} Y, Y} \to C_{X, Y} \to (0)$$

is exact.

PROOF: As above, (ii) follows from (i).

It is enough to prove (i) under the following additional hypothesis: X is an affine scheme, S is a noetherian affine scheme, f is of finite type, i is a closed immersion, $f|Y: Y \to S$ is surjective and smooth. X being affine and f of finite type, there exists Z smooth over S (one may choose some $S[T_1, \cdots, T_n]$) and a closed S-immersion $j: X \to Z$. As in Corollary (2.5), $j \circ i$, $\delta(\mathrm{Id}\, Y)$, $\delta(j \circ i)$ are regular immersions and it is enough to show that, at every point y of Y, the sequence (resp. diagram) of stalks induced by

$$0 \to \mathrm{gr}_Y^1 Z \to \mathrm{gr}_Y^1 Z \underset{S}{\times} Y \to \mathrm{gr}_Y^1 Y \underset{S}{\times} Y \to 0$$

$$\text{(resp.}\quad \begin{array}{ccc} \mathrm{gr}_Y^1 Z & \longrightarrow & \mathrm{gr}_Y^1 Z \underset{S}{\times} Y \\ \Big| & & \Big| \\ \mathrm{gr}_Y^1 X & \longrightarrow & \mathrm{gr}_Y^1 X \underset{S}{\times} Y \end{array} \quad)$$

is exact (resp. cocartesian). Here, we use a trick to kill the nasty residual extension $\kappa(f(y)) \to \kappa(y)$ which possibly may occur. $f|Y: Y \to S$ is smooth, so it is flat and thus $\mathcal{O}_{S, f(y)} \to \mathcal{O}_{Y, y}$ and $\mathcal{O}_{Y, y} \to \mathcal{O}_{Y \times_S Y, \delta(\mathrm{Id}\, Y)(y)}$ are faithfully flat. (We obtain the second arrow from the first one by base change followed by localization at a suitable prime ideal and one knows that every flat morphism of noetherian local rings is faithfully flat). Apply this base extension. After easy functorial computation (Section 1) we reduce ourselves to prove the same replacing S (resp. Y) (resp. X) (resp. Z) by the local scheme of Y (resp. $Y \times_S Y$) (resp. $X \times_S Y$) (resp. $Z \times_S Y$) at y (resp. $\delta(\mathrm{Id})(y)$) (resp. $\delta(i)(y)$) (resp. $\delta(j \circ i)(y)$). But now, all those local schemes have the same residue field $\kappa(y)$ at their closed point and smoothness is preserved. So that, it is, in fact, enough to prove our statement with S, Y, X, Z local schemes and trivial residual extension $\kappa(f(y)) \to \kappa(y)$. But (as in the analytic case) if Y is the Spec of the completion of the local ring of Y at its closed point, the canonical projection $\hat{Y} \to Y$ is faithfully flat, so that after this base change, and usual functorial computation (see again Section 1) our proposition becomes an immediate consequence of Theorem (2.3) and Remark (2.4).

REMARK (2.7): Under the hypothesis of Theorem (2.3) (resp. i) Corollary (2.5)) (resp. (i) of Corollary (2.6)) $\mathrm{gr}_Y X \times_S Y$ being lo around x on Y a tensor product of $\mathrm{gr}_Y X$ and $\mathrm{gr}_Y Y \times_S Y$, from (1.19), we deduce that $\mathrm{gr}\,\mathscr{P}_{X/S}(Y)$ is locally around x on Y a tensor product of $\mathrm{gr}_Y X$ and $\mathrm{gr}\,\mathscr{P}_{Y/S}$.

Remark (2.8): Assumptions as in Remark (2.7). $\mathrm{gr}^1\,\mathscr{P}_{X/S}(Y)$ is locally around x on Y a direct sum of $\mathrm{gr}^1_Y X$ and $\mathrm{gr}^1\,\mathscr{P}_{Y/S}$. But by definition $\mathrm{gr}^1_Y X = \mathscr{N}_{X,Y}$ is the conormal sheaf of the immersion $i\colon Y \to X$, $\mathrm{gr}^1\,\mathscr{P}_{X/S} = \Omega^1_{X/S}$, $\mathrm{gr}^1\,\mathscr{P}_{Y/S} = \Omega^1_{Y/S}$, and $\mathcal{O}_X$ being $\mathcal{O}_X$-flat, $\mathrm{gr}^1\,\mathscr{P}_{X/S}(Y) = i^*(\mathrm{gr}^1\,\mathscr{P}_{X/S}) = i^*(\Omega^1_{X/S})$. Thus we recover the well-known exact sequence of Jacobi-Zariski (see [5])

$$0 \to \mathscr{N}_{X,Y} \to i^*(\Omega^1_{X/S}) \to \Omega^1_{Y/S} \to 0.$$

Corollary (2.9): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space) and Y a subscheme (resp. analytic subspace) of x. At any point $y \in Y$ such that Y is smooth over S at y, the following conditions are equivalent:*
(i) $(\mathscr{P}_{X/S}(Y))_y$ *is* $\mathcal{O}_{Y,y}$*-flat*
(ii) $(\mathrm{gr}\,\mathscr{P}_{X/S}(Y))_y$ is $\mathcal{O}_{Y,y}$*-flat*
(iii) $(\mathrm{gr}_Y X)_y$ is $\mathcal{O}_{Y,y}$*-flat.*

Proof: The equivalence of (i) and (ii) does not depend upon the smoothness of Y over S at y. Let us remark also that since we are dealing with sums of $\mathcal{O}_{Y,y}$-modules of finite type, we may replace flat by free in the assertion. Assume $(\mathscr{P}_{X/S}(Y))_y$ $\mathcal{O}_{Y,y}$-free i.e. each $(\mathscr{P}^n_{X/S}(Y))_y$ $\mathcal{O}_{Y,y}$-free, $n \geqq 0$. Then the exact sequences

$$0 \to (\mathrm{gr}^n\,\mathscr{P}_{X/S}(Y))_y \to (\mathscr{P}^n_{X/S}(Y))_y \to (\mathscr{P}^{n-1}_{X/S}(Y))_y \to 0$$

show that $(\mathrm{gr}^n\,\mathscr{P}_{X/S}(Y))_y$ is $\mathcal{O}_{Y,y}$-free for all $n \geqq 0$, hence also $(\mathrm{gr}\,\mathscr{P}_{X/S}(Y))_y$. Conversely, assume that each $\mathrm{gr}^n\,\mathscr{P}_{X/S}(Y))_y$ is $\mathcal{O}_{Y,y}$-free. Since $(\mathrm{gr}^0\,\mathscr{P}_{X/S}(Y))_y = (\mathscr{P}^0_{X/S}(Y))_y = \mathcal{O}_{Y,y}$, the same exact sequences show by induction on n that each $(\mathscr{P}^n_{X/S}(Y))_y$ is $\mathcal{O}_{Y,y}$-free, hence also $(\mathscr{P}_{X/S}(Y))_y$.

The equivalence of (ii) and (iii) comes from Corollary (2.6) (resp. (2.5)) which tells us that we have a (non-canonical) isomorphism (see Remark (2.7))

$$(\mathrm{gr}\,\mathscr{P}_{X/S}(Y))_y \simeq (\mathrm{gr}_Y X)_y \mathop{\hat{\otimes}}_{\mathcal{O}_{Y,y}} (\mathrm{gr}\,\mathscr{P}_{Y/S})_y$$

and since Y is smooth over S, in view of Remark 1.21, $(\mathrm{gr}\,\mathscr{P}_{Y/S})_y$ is a direct sum of free $\mathcal{O}_{Y,y}$-modules. But $\mathrm{Tor}_{\mathcal{O}_{Y,y}}$ commutes with direct sums.

Corollary (2.10): *Assumptions as in Theorem* (2.3). *The following conditions are equivalent:*
(i) $\mathscr{P}_{X/S}(Y)$ *is* $\mathcal{O}_Y$*-flat*
(ii) $\mathrm{gr}\,\mathscr{P}_{X/S}(Y)$ *is* $\mathcal{O}_Y$*-flat*
(iii) $\mathrm{gr}_Y X$ *is* $\mathcal{O}_Y$*-flat.*

It follows from Theorem (2.3) by the same arguments as those used in (2.9).

3. Relative normal flatness and W-normal flatness

DEFINITION (3.1): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space) and Y a subscheme (resp. analytic subspace) of X. We say that X/S is normally flat along Y/S (or that X is normally flat along Y over S) at a point $y \in Y$ if the following conditions are satisfied:*

(a) *Y is smooth over S at y*

(b) $(\mathrm{gr}_Y X)_y$ *is $\mathcal{O}_{Y,y}$-flat, i.e., the equivalent conditions of Corollary* (2.9) *hold at y.*

We say that X/S is normally flat along Y/S if it is so at every point $y \in Y$.

PROPOSITION (3.2): *If X/S is normally flat along Y/S, for any base extension $S' \to S$, setting $X' = X \times_S S'$, $Y' = Y \times_S S'$, the canonical map of Y'-cones* (1.11)

$$C_{X',Y'} \to C_{X,Y} \times_Y Y'$$

is an isomorphism.

PROOF: We may localize ourselves at $y' \in Y'$. Let y be the image of y' by the canonical projection $p: Y' \to Y$. Since Y is smooth over S at y, and $\mathrm{gr}_Y Y = \mathcal{O}_Y$ is $\mathcal{O}_Y$-flat, $\mathscr{P}^n_{Y/S}$ is $\mathcal{O}_Y$-flat, for all $n \geqq 0$. Hence the following sequences:

$$0 \to (p^* \operatorname{gr}^n \mathscr{P}_{X/S}(Y))_{y'} \to (p^* \mathscr{P}^n_{X/S}(Y))_{y'} \to (p^* \mathscr{P}^{n-1}_{X/S}(Y))_{y'} \to 0$$

$$0 \to (p^* \operatorname{gr}^n \mathscr{P}_{Y/S})_{y'} \to (p^* \mathscr{P}^n_{Y/S})_{y'} \to (p^* \mathscr{P}^{n-1}_{Y/S})_{y'} \to 0$$

are exact for all $n \geqq 0$. But by the nice behaviour of $\mathscr{P}^n_X$ under base extension [Section 1], they coincide with the following:

$$0 \to (\operatorname{gr}^n \mathscr{P}_{X'/S'}(Y'))_{y'} \to (\mathscr{P}^n_{X'/S'}(Y'))_{y'} \to (\mathscr{P}^{n-1}_{X'/S'}(Y'))_{y'} \to 0$$

$$0 \to (\operatorname{gr}^n \mathscr{P}_{Y'/S'})_{y'} \to (\mathscr{P}^n_{Y'/S'})_{y'} \to (\mathscr{P}^{n-1}_{Y'/S'})_{y'} \to 0$$

respectively. So that we get canonical isomorphisms of graded $\mathcal{O}_{Y',y'}$-algebras:

(3.2.1) $$(p^* \operatorname{gr} \mathscr{P}_{X/S}(Y))_{y'} \simeq (\operatorname{gr} \mathscr{P}_{X'/S'}(Y'))_{y'}$$

(3.2.2) $$(p^* \operatorname{gr} \mathscr{P}_{Y/S})_{y'} \simeq (\operatorname{gr} \mathscr{P}_{Y'/S'})_{y'}.$$

But on the other hand, from Corollary (2.6) (resp. (2.5)) we have the (non canonical) isomorphism (see remark (2.7))

$$(\operatorname{gr} \mathscr{P}_{X/S}(Y))_y \simeq (\operatorname{gr}_Y X)_y \underset{\mathcal{O}_{Y,y}}{\otimes} (\operatorname{gr} \mathscr{P}_{Y/S})_y$$

hence

$$(p^* \operatorname{gr} \mathscr{P}_{X/S}(Y))_{y'} \simeq (p^* \operatorname{gr}_Y X)_{y'} \underset{\mathcal{O}_{Y',y'}}{\otimes} (p^* \operatorname{gr} \mathscr{P}_{Y/S})_{y'}$$

i.e., by (3.2.1) and (3.2.2).

$$(\operatorname{gr} \mathscr{P}_{X'/S'}(Y'))_{y'} \simeq (p^* \operatorname{gr}_Y X)_{y'} \underset{\mathcal{O}_{Y',y'}}{\otimes} (\operatorname{gr} \mathscr{P}_{Y'/S'})_{y'} \tag{3.2.3}$$

hence in the commutative diagram of graded $\mathcal{O}_{Y',y'}$-algebras

$$\begin{tikzcd}
(p^* \operatorname{gr}_Y X)_{y'} \arrow[r] \arrow[d] & (\operatorname{gr} \mathscr{P}_{X'/S'}(Y'))_{y'} \arrow[r] \arrow[d, equal] & (\operatorname{gr} \mathscr{P}_{Y'/S'})_{y'} \arrow[d, equal] \\
(\operatorname{gr}_{Y'} X')_{y'} \arrow[r] & (\operatorname{gr} \mathscr{P}_{X'/S'}(Y'))_{y'} \arrow[r] & (\operatorname{gr} \mathscr{P}_{Y'/S'})_{y'}
\end{tikzcd}$$

both lines represent exact sequences of cones. The upper one by (3.2.3), and the lower one because by base extension Y' remains smooth over S' at y', (0.10) and Corollary (2.6) (resp. (2.5)). It follows that the vertical arrow must be an isomorphism.

COROLLARY (3.3): *If X is normally flat along Y over S, then after any base extension $S' \to S$. $X' = X \times_S S'$ is normally flat along $Y' = Y \times_S S'$ over S'.*

PROOF: Y' remains smooth over S', and $\operatorname{gr}_{Y'} X'$ is a flat $\mathcal{O}_{Y'}$-Module as the inverse image of $\operatorname{gr}_Y X$ by the canonical projection $p: Y' \to Y$, in view of the above proposition. We remark that this Corollary is also a direct consequence of Corollary (2.9) and the nice behaviour of $\mathscr{P}_{X/S}(Y)$ under base extension.

In particular, after any base extension $S' \to S$ such that S' is regular, X' is normally flat along Y' (in the classical sense of ([7] Chap. II), i.e., Y' is now regular, and $\operatorname{gr}_{Y'} X'$ $\mathcal{O}_{Y'}$-flat). One may ask whether the converse is true, and we have:

PROPOSITION (3.4): *Let X be a scheme locally of finite type over a locally noetherian reduced scheme (resp. a relative complex analytic space with S reduced) and Y a subscheme (resp. analytic subspace) of X, smooth over S. X is normally flat along Y over S if and only if, for any base extension $S' \to S$ where S' is the spectrum of a discrete valuation ring (resp. the unit disk in $\mathbb{C}$), $X' = X \times_S S'$ is normally flat along $Y' = Y \times_S S'$.*

PROOF: It suffices to see the 'if' part. To check that $\operatorname{gr}_Y X$ is $\mathcal{O}_Y$-flat, we apply the valuative criterion for flatness [EGA IV 11.8 resp. [8] Chap. 0]: Let $h: S' \to Y$ be any morphism such that S' is as in the Proposition. By composition we obtain a base extension $S' \to S$, and Y'/S' is endowed with a section $s: S' \to Y'$:

$$\begin{tikzcd} Y' \arrow[r, "p"] \arrow[d, shift right] & Y \arrow[d] \\ S' \arrow[u, shift right, "s"'] \arrow[r] \arrow[ur, "h"] & S \end{tikzcd}$$

Now again Y' is smooth over S', and our assumption that $\operatorname{gr}_{Y'} X'$ is $\mathcal{O}_{Y'}$-flat gives us by Corollary (2.9) that $\mathscr{P}_{X'/S'}(Y')$ is also $\mathcal{O}_{Y'}$-flat. Using once more the fact that $\mathscr{P}_{X'/S'}(Y') = p^*\mathscr{P}_{X/S}(Y)$, we see that this implies that $h^*\mathscr{P}_{X/S}(Y) = s^*\mathscr{P}_{X'/S'}(Y')$ is $\mathcal{O}_{S'}$-flat, and since Y, being smooth over S which is reduced, is itself reduced, and h is arbitrary, the valuative criterion tells us that $\mathscr{P}_{X/S}(Y)$ is $\mathcal{O}_Y$-flat, Q.E.D. by Corollary (2.9).

REMARK: If S is not reduced, the proposition is not true as is shown by the following example:

$$X = \operatorname{Spec} \boldsymbol{k}[\varepsilon][T]/\varepsilon \cdot T \qquad S = Y = \operatorname{Spec} \boldsymbol{k}[\varepsilon] \quad \text{where } \varepsilon^2 = 0.$$

One can generalize the concept of normal flatness in another direction:

DEFINITION (3.5) (*Hironaka*): *Let X be a scheme (resp. analytic space), Y a subscheme (resp. analytic subspace) of X, and W a subscheme (resp. analytic subspace) of Y. We say that X is W-normally flat along Y at a point $x \in W$ if the following conditions hold:*

(a) *W is regularly imbedded in Y at x.*

(b) *The canonical sequence of W-cones*

$$(0) \to C_{Y,W} \to C_{X,W} \to C_{X,Y} \underset{Y}{\times} W \to (0)$$

is exact in a neighborhood of x in W.

We say that X is W-normally flat along Y if it is so at every point of W.

With this definition, we find that we can translate our main result as:

THEOREM (3.6): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space), Y a subscheme (resp. analytic subspace) of X flat over S.*

Consider the immersions (as in Section 1)

$$Y \xrightarrow{\delta(\mathrm{Id}_Y)} Y \underset{S}{\times} Y \xrightarrow{\delta'(i)} X \underset{S}{\times} Y$$

$X \times_S Y$ is Y-normally flat along $Y \times_S Y$ at a point $y \in Y$ if and only if Y is smooth over S at y.

Indeed, if Y is smooth over S at y, the diagonal immersion $Y \to Y \times_S Y$ is regular at y (0.23) and condition (b) is exactly Corollary (2.3) (resp. (2.5)) since $C_{X \times_S Y, Y \times_S Y} \times_{Y \times_S^x Y} Y = C_{X,Y}$.

The converse follows only from (a) in view of (0.4).

4. Computation of $\dim_{\kappa(x)} \mathscr{P}^n_{X/S}(x)$ and applications

DEFINITION (4.1): *Let $\mathcal{O}$ be a noetherian local ring, $\boldsymbol{m}$ its maximal ideal. The Samuel function of $\mathcal{O}$, $H^1_{\mathcal{O}}$: $\mathbb{N} \to \mathbb{N}$ is defined by*

$$H^1_{\mathcal{O}}(v) = \mathit{length}_{\mathcal{O}}\, \mathcal{O}/\boldsymbol{m}^{v+1}.$$

For the purpose of comparing the Samuel functions of two noetherian local rings of different dimensions one is led to define by induction functions $H^i_{\mathcal{O}}$ by

$$H^i_{\mathcal{O}}(v) = \sum_{\mu=0}^{v} H^{i-1}_{\mathcal{O}}(\mu)$$

and it is easily checked that

$$H^{1+d}_{\mathcal{O}} = H^1_{\mathcal{O}[T_1, \cdots, T_d]_{\boldsymbol{M}}} \qquad \text{where } \boldsymbol{M} = \boldsymbol{m}[T_1, \cdots, T_d].$$

So if we want to compare $\mathcal{O}$, $\mathcal{O}'$, say $\dim \mathcal{O}' - \dim \mathcal{O} = d$, the natural comparison is that of $H^1_{\mathcal{O}'}$ and $H^{1+d}_{\mathcal{O}}$.

DEFINITION (4.2): *Let $\boldsymbol{k}$ be a field, and $\mathcal{O}$ a $\boldsymbol{k}$-algebra which is local and noetherian, and such that the residual extension $\boldsymbol{k} \to \kappa = \mathcal{O}/\boldsymbol{m}$ is of finite type. The arithmetic Samuel function of the $\boldsymbol{k}$-algebra $\mathcal{O}$ is by definition:*

$${}^{a}H_{\mathcal{O}} = H^{1+d}_{\mathcal{O}} \qquad \textit{where } d = d_{\boldsymbol{k}}(\mathcal{O}) = \mathit{tr.deg}_{\boldsymbol{k}}\, \kappa.$$

LEMMA (4.3): *Let $\boldsymbol{k}$ be a field and $\mathcal{O}$ a local noetherian $\boldsymbol{k}$-algebra whose residue field is of finite type over $\boldsymbol{k}$. Let $\boldsymbol{k}'$ be a finite radicial extension of $\boldsymbol{k}$, so that $\mathcal{O}' = \mathcal{O} \otimes_{\boldsymbol{k}} \boldsymbol{k}'$ is a local $\boldsymbol{k}'$-algebra whose residue field is of finite type over $\boldsymbol{k}'$. Then*

$${}^{a}H_{\mathcal{O}}(v) \leqq {}^{a}H_{\mathcal{O}'}(v) \qquad \textit{for all } v \geqq 0$$

and if $\mathcal{O}$ contains a prime ideal $\boldsymbol{P}$ such that
(a) *$\mathcal{O}/\boldsymbol{P}$ is a formally smooth $\boldsymbol{k}$-algebra*
(b) *$gr_{\boldsymbol{P}}\mathcal{O}$ is a flat $\mathcal{O}/\boldsymbol{P}$-algebra*
we have equality.

PROOF: Clearly $d_{\boldsymbol{k}'}(\mathcal{O}') = d_{\boldsymbol{k}}(\mathcal{O})$ so we are reduced to proving that $H^1_{\mathcal{O}}(v) \leqq H^1_{\mathcal{O}'}(v)$ for all $v \geqq 0$. By induction on $[\boldsymbol{k}' : \boldsymbol{k}]$ it is enough to prove the lemma in the case where $\boldsymbol{k}' = \boldsymbol{k}[X]/X^p - u$ with $u \in \boldsymbol{k} - \boldsymbol{k}^p$. Now consider $\kappa \otimes_{\boldsymbol{k}} \boldsymbol{k}'$. If it is not a field, u has a pth root in κ. We choose an element U in $\mathcal{O}$ such that $U^p \bmod \boldsymbol{m}$ is u, make the change of variables $Y = X - U$ and apply Proposition 10 of [6]. If it is a field, in particular if κ is a separable extension of $\boldsymbol{k}$, we have by the flatness of $\mathcal{O} \to \mathcal{O}'$ that $\boldsymbol{m}\mathcal{O}' = \boldsymbol{m}'$ *and* $\mathrm{gr}_{\boldsymbol{m}'} \mathcal{O}' = \mathrm{gr}_{\boldsymbol{m}} \mathcal{O} \otimes_{\mathcal{O}/\boldsymbol{m}} \mathcal{O}'/\boldsymbol{m}'$, hence equality. Now the general equality case reduces to the preceding one in the following way: By the flatness of $\mathcal{O}'$ over $\mathcal{O}$, conditions a) and b) hold for $\boldsymbol{P}\mathcal{O}'$ in $\mathcal{O}'$ with respect to $\boldsymbol{k}'$. By [7, Remark to Cor. 3] we have that $H^1_{\mathcal{O}} = H^{1+d}_{\mathcal{O}_{\boldsymbol{P}}}$, $H^1_{\mathcal{O}'} = H^{1+d}_{\mathcal{O}'_{\boldsymbol{P}\mathcal{O}'}}$ where $d = \dim \mathcal{O}/\boldsymbol{P} = \mathcal{O}'/\boldsymbol{P}\mathcal{O}'$. Since $\mathcal{O}/\boldsymbol{P}$ is formally smooth over $\boldsymbol{k}$, the residual extension of the $\boldsymbol{k}$-algebra $\mathcal{O}_{\boldsymbol{P}}$ is now separable (0.14), and we apply the preceding equality case to $\mathcal{O}_{\boldsymbol{P}}$, which gives us $H^1_{\mathcal{O}_{\boldsymbol{P}}} = H^1_{\mathcal{O}'_{\boldsymbol{P}\mathcal{O}'}}$, hence $H^1_{\mathcal{O}} = H^1_{\mathcal{O}'}$.

DEFINITION (4.4): *Let $\boldsymbol{k}$ be a field and $\mathcal{O}$ a local noetherian $\boldsymbol{k}$-algebra whose residue field is of finite type over $\boldsymbol{k}$. The Samuel function $H_{\mathcal{O}/\boldsymbol{k}}$ of $\mathcal{O}$ over $\boldsymbol{k}$ is defined by:*

$$H_{\mathcal{O}/\boldsymbol{k}}(\nu) = \operatorname*{Sup}_{\boldsymbol{k}' \in \mathcal{R}} {}^{\mathrm{a}}H_{\mathcal{O}'}(\nu)$$

where $\boldsymbol{k}'$ runs through the set $\mathcal{R}$ of all finite radicial extensions of $\boldsymbol{k}$, and $\mathcal{O}'$ is the $\boldsymbol{k}'$-algebra $\mathcal{O} \otimes_{\boldsymbol{k}} \boldsymbol{k}'$.

PROPOSITION (4.5): *$\mathcal{O}$ being as above, there exists a finite radicial extension $\boldsymbol{k}_0$ of $\boldsymbol{k}$ such that, if $\mathcal{O}_0 = \mathcal{O} \otimes_{\boldsymbol{k}} \boldsymbol{k}_0$,*

$$H_{\mathcal{O}/\boldsymbol{k}} = {}^{\mathrm{a}}H_{\mathcal{O}_0}.$$

If $\mathcal{O}$ contains a prime ideal $\boldsymbol{P}$ such that
(a) *$\mathcal{O}/\boldsymbol{P}$ is a formally smooth $\boldsymbol{k}$-algebra*
(b) *$gr_{\boldsymbol{P}}\mathcal{O}$ is a flat $\mathcal{O}/\boldsymbol{P}$-module*
we may choose $\boldsymbol{k}_0 = \boldsymbol{k}$.

(This happens in particular if the residue field of $\mathcal{O}$ is a separable extension of $\boldsymbol{k}$, since then the maximal ideal $\boldsymbol{m}$ of $\mathcal{O}$ satisfies conditions (a) and (b).)

PROOF: By [5, Chap IV, 4.7.4] the $\boldsymbol{k}$-algebra $\mathcal{O}$ is of finite radicial multiplicity, i.e., there exists a finite radicial extension $\boldsymbol{k}_0$ of $\boldsymbol{k}$ such that the residue field κ_0 of $\mathcal{O}_0 = \mathcal{O} \otimes_{\boldsymbol{k}} \boldsymbol{k}_0$ is a separable extension of $\boldsymbol{k}_0$. By Lemma (4.3) we have $H_{\mathcal{O}/\boldsymbol{k}} = H_{\mathcal{O}_0/\boldsymbol{k}_0}$ and $H_{\mathcal{O}_0/\boldsymbol{k}_0} = {}^{\mathrm{a}}H_{\mathcal{O}_0}$. The second part of the assertion follows immediately from the equality case in Lemma 4.3.

DEFINITION (4.6): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space) and x a point of X. Let s be the image of x in S, X_s the fiber of X/S through x and $\kappa(s)$ the residue field of S at s. We define the relative Samuel function of X over S at x:*

$$H_{X/S,x} = H_{\mathcal{O}_{X_s,x/\kappa(s)}}.$$

We also define the arithmetic relative Samuel function

$$^{a}H_{X/S,x} = {}^{a}H_{\mathcal{O}_{X_s,x}}$$

and remark that one always has $^{a}H_{X/S,x}(v) \leqq H_{X/S,x}(v)$ *for all* $v \geqq 0$ *and equality holds in particular if the residual extensions are separable (e.g., analytic case).*

PROPOSITION (4.7): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space). One has*

$$\dim_{\kappa(x)} \mathscr{P}^n_{X/S}(x) = H_{X/S,x}(n)$$

for all $n \geqq 0$ *and any point* $x \in X$.

PROOF: Let s be the image of x in S, X_s the fiber of X/S through x. We have $\dim_{\kappa(x)} \mathscr{P}^n_{X_s/\kappa(s)}(x) = \dim_{\kappa(x)} \mathscr{P}^n_{X/S}(x)$ (1.14) and moreover, if $\boldsymbol{k}'$ is a finite radicial extension of $\kappa(s)$ and if x' is the point of $X'_s = X_s \times_{\kappa(s)} \boldsymbol{k}'$ mapped to x by the first projection, we have

$$\dim_{\kappa(x')} \mathscr{P}^n_{X'_s/\boldsymbol{k}'}(x') = \dim_{\kappa(x)} \mathscr{P}^n_{X_s/\kappa(s)}(x)$$

(1.12). Thanks to our definition of $H_{X/S,x}$ we also have $H_{X'_s/\boldsymbol{k}',x'} = H_{X_s/\boldsymbol{k},x}$. Since $\mathcal{O}_{X_s,x}$ satisfies the conditions of Proposition (4.5), we may thus, to prove our equality, reduce to the case where $\kappa(x)$ is a separable extension of $\kappa(s)$. In this case, the closure $\{\bar{x}\}$ of x in X_s is smooth over $\kappa(s)$ at x (0.8; 0.14). Applying now Lemma 1.19 and Corollary 2.6 (resp. 2.5) via remark 2.7, we find that we have isomorphisms:

$$\operatorname{gr} \mathscr{P}_{X_s/\kappa(s)}(x) \simeq (\operatorname{gr} \mathscr{P}_{X_s/\kappa(s)}(\bar{x}))_x \simeq (\operatorname{gr}_{\{\bar{x}\}} X_s)_x \underset{\kappa(x)}{\otimes} (\operatorname{gr} \mathscr{P}_{\{\bar{x}\}/\kappa(s)})$$

But on the one hand we have by (1.21) that

$$(\operatorname{gr} \mathscr{P}_{\{\bar{x}\}/\kappa(s)})_x \simeq \operatorname{Sym}_{\kappa(x)} \Omega^1_{\kappa(x)/\kappa(s)} \simeq \kappa(x)[T_1, \cdots, T_d]$$

where $d = d_{\kappa(s)}(\mathcal{O}_{X_s,x})$ and on the other hand we have

$$\dim_{\kappa(x)}(\operatorname{gr}^n_{\{\bar{x}\}} X_s)_x = \dim_{\kappa(x)} \mathfrak{m}^n_{X_s,x}/\mathfrak{m}^{n+1}_{X_s,s} = H^0_{\mathcal{O}_{X_s,x}}(n)$$

so that we have $\dim_{\kappa(x)} \operatorname{gr}^n \mathscr{P}_{X/S}(x) = H^d_{\mathcal{O}_{X_s,x}}(n)$ and by the exact sequences

$$0 \to (\operatorname{gr}^n \mathscr{P}_{X_s/\kappa(s)}(\bar{x}))_x \to (\mathscr{P}^n_{X_s/\kappa(s)}(\bar{x}))_x \to (\mathscr{P}^{n-1}_{X_s/\kappa(s)}(\bar{x}))_x \to 0$$

we get $\dim_{\kappa(x)} \mathscr{P}^n_{X/S}(x) = H^{1+d}_{\mathcal{O}_{X_s,x}}(n) = H_{\mathcal{O}_{X_s/\kappa(s)}}(n) = H_{X/S,x}(n)$. *QED.*

COROLLARY (4.8): *Let s_v be an integer. Then $X_{sv} = \{x \in X / H_{X/S,x}(v) \geqq s_v\}$ is a closed subscheme (resp. closed analytic subspace) of X.*

This follows immediately from the above proposition and the semicontinuity of the dimension of the fibers of coherent sheaves.

COROLLARY (4.9): (*A special case of Bennett's Theorem* 3 *in* [1]). *Let* $\mathcal{O}$

be a noetherian equicharacteristic local ring and $\boldsymbol{P}$ an ideal in $\mathcal{O}$ such that $\mathcal{O}/\boldsymbol{P}$ is regular. Then $gr_{\boldsymbol{P}}\mathcal{O}$ is a flat $\mathcal{O}/\boldsymbol{P}$-module if and only if

$$H^{1+d}_{\mathcal{O}_{\boldsymbol{P}}} = H^1_{\mathcal{O}}$$

where $d = dim\,\mathcal{O}/\boldsymbol{P}$.

PROOF: First one can reduce to the complete case. Let $\mathcal{O} \to \hat{\mathcal{O}}$ be the completion of $\mathcal{O}$. Then $\widehat{\mathcal{O}/\boldsymbol{P}} = \hat{\mathcal{O}}/\boldsymbol{P}\cdot\hat{\mathcal{O}}$ is regular, hence $\boldsymbol{P}\cdot\hat{\mathcal{O}}$ is a prime ideal of $\hat{\mathcal{O}}$. Using the flatness of $\hat{\mathcal{O}}$ over $\mathcal{O}$, it is easily seen (1.13) that $H^{1+d}_{\mathcal{O}_{\boldsymbol{P}}} = H^{1+d}_{\hat{\mathcal{O}}_{\boldsymbol{P}\hat{\mathcal{O}}}}$, $H^1_{\mathcal{O}} = H^1_{\hat{\mathcal{O}}}$ and $\mathrm{gr}_{\boldsymbol{P}\hat{\mathcal{O}}}\hat{\mathcal{O}} = \mathrm{gr}_{\boldsymbol{P}}\mathcal{O} \otimes_{\mathcal{O}/\boldsymbol{P}} \widehat{\mathcal{O}/\boldsymbol{P}}$. Since $\widehat{\mathcal{O}/\boldsymbol{P}}$ is a faithgully flat $\mathcal{O}/\boldsymbol{P}$-module, $\mathrm{gr}_{\boldsymbol{P}}\mathcal{O}$ is $\mathcal{O}/\boldsymbol{P}$-flat if and only if $\mathrm{gr}_{\boldsymbol{P}\cdot\hat{\mathcal{O}}}\hat{\mathcal{O}}$ is $\widehat{\mathcal{O}/\boldsymbol{P}}$-flat. Now we can use Cohen's structure theorem ([9], Theorem 27) which tells us that $\hat{\mathcal{O}}$ has a field of representatives, say κ. κ is also a field of representatives for $\hat{\mathcal{O}}/\boldsymbol{P}\cdot\hat{\mathcal{O}}$ which, being regular, is then isomorphic to

$$\kappa[[T_1, \cdots, T_d]] \qquad \text{where } d = \dim \mathcal{O}/\boldsymbol{P} = \dim \hat{\mathcal{O}}/\boldsymbol{P}\cdot\hat{\mathcal{O}}.$$

Hence $\hat{\mathcal{O}}/\boldsymbol{P}\hat{\mathcal{O}}$ is a formally smooth κ-algebra with trivial residual extension. By Corollary (2.10), all we have to prove is that, if $X = \operatorname{Spec}\hat{\mathcal{O}}$, $Y = \operatorname{Spec}\widehat{\mathcal{O}/P}$, $\mathscr{P}^n_{X/\kappa}(Y)$ is $\mathcal{O}_Y$-flat, for all $n \geqq 0$.

In view of the numerical criterion for flatness it is enough to check that if x is the closed point of Y and η its generic point, $\dim_{\kappa(x)}\mathscr{P}^n_{X/\kappa}(x) = \dim_{\kappa(\eta)}\mathscr{P}^n_{X/\kappa}(\eta)$. A straightforward computation shows that $\dim_{\kappa(x)}\mathscr{P}^n_{X/\kappa}(x) = H^1_{\hat{\mathcal{O}}}$; if we let $\boldsymbol{K}$ be the field of fractions of $\kappa[[T_1, \cdots, T_d]]$, we have by Theorem 2.3

$$(\mathrm{gr}\,\mathscr{P}_{X/\kappa}(Y))_\eta \approx (\mathrm{gr}_Y X)_\eta \underset{\boldsymbol{K}}{\otimes} (\mathrm{gr}_Y Y \underset{\kappa}{\times} Y)_\eta \simeq \mathrm{gr}_\eta X_\eta \underset{\boldsymbol{K}}{\otimes} \boldsymbol{K}[T_1, \cdots, T_d]$$

hence $\dim_{\kappa(\eta)}(\mathrm{gr}^n\mathscr{P}_{X/\kappa}(Y))_\eta = H^d_{\mathcal{O}_{\boldsymbol{P}}}$, and since $\mathscr{P}^n_{X/\kappa}(\eta) = (\mathscr{P}^n_{X/\kappa}(Y))_\eta$ by the exact sequences $0 \to (\mathrm{gr}^n\mathscr{P}_{X/\kappa}(Y))_\eta \to (\mathscr{P}^n_{X/\kappa}(Y))_\eta \to \mathscr{P}^{n-1}_{X/\kappa}(Y))_\eta \to 0$ *we find* that $\dim_{\kappa(\eta)}\mathscr{P}^n_{X/\kappa}(\eta) = H^{1+d}_{\mathcal{O}_{\boldsymbol{P}}}$. This also proves the converse.

REMARK (4.10): Since we have in fact used only the arithmetic Samuel function, Corollary (4.9) provides an independent proof of the equality of [7, remark to Cor. 3] which we used in the proof of the equality case in Proposition 4.5.

We are now ready to prove the numerical criterion for relative normal flatness:

THEOREM (4.11): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space) with S reduced, and Y a subscheme (resp. analytic subspace) of X. The following conditions are equivalent for a point $x \in Y$*

(i) *X/S is normally flat along Y/S at x*

(ii) *Y is smooth over S at x and the application $y \longrightarrow H_{X/S,y}$ is constant in a neighborhood of x in Y.*
(iii) *Y is smooth over S at x and the application $y \longrightarrow {}^{\mathrm{a}}H_{X/S,y}$ is constant in a neighborhood of x in Y.*

Furthermore, if these conditions are satisfied, there exists a neighborhood of x in Y at every point y of which X/S is normally flat along Y/S, and $H_{X/S,y} = {}^{\mathrm{a}}H_{X/S,y}$.

PROOF: By the openness of flatness (see [4] resp. [3]) and smoothness (see [4] resp. [2]) (i) implies that X/S remains normally flat along Y/S in some neighborhood of x in Y. By Corollary (2.9) and the numerical criterion for freeness of coherent sheaves on a reduced space ([4]) this in turn implies (ii), thanks to Proposition (4.7). Conversely, using again the openness property of smoothness, the same numerical criterion shows that (ii) implies that all the $\mathscr{P}^n_{X/S}(Y)$ are free in a neighborhood of x in Y, hence (i). Now Proposition (4.5) tells us that at any point y such that (i) holds, we have $H_{X/S,y} = {}^{\mathrm{a}}H_{X/S,y}$, hence we have shown that (ii) $\Rightarrow$ (iii).

Let us now assume that (iii) is fulfilled. Let s be the image of x in S and X_s (resp. Y_s) the fiber of X/S (resp. Y/S) through x. (iii) tells us that ${}^{\mathrm{a}}H_{X_s/\kappa(s)}$ is constant in a neighborhood of x in Y_s, and that Y_s is smooth over $\kappa(s)$ at x. Setting $\mathcal{O} = \mathcal{O}_{X_s,x}$ which is equicharacteristic since it is a $\kappa(s)$-algebra, and $\boldsymbol{P}$ to be the ideal of Y_s in X_s at x, we find by Corollary (4.10) that $\mathrm{gr}_{\boldsymbol{P}}\mathcal{O}$ is a flat $\mathcal{O}/\boldsymbol{P}$-algebra, hence by Proposition 4.5 that we have ${}^{\mathrm{a}}H_{X_s/\kappa(s),x} = H_{X_s/\kappa(s),x}$, i.e., ${}^{\mathrm{a}}H_{X/S,x} = H_{X/S,x}$. Using again the openness of smoothness, this shows that (iii) $\Rightarrow$ (ii).

We now prepare ourselves to prove the existence of a 'relative Samuel stratification' which will enable us to state the above theorem in a more geometric way. From now on, analytic space will mean complex analytic space.

LEMMA (4.12): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space) and Y a reduced subscheme (resp. analytic subspace) of X. The set of points $y \in Y$ at which $\mathscr{P}_{X/S}(Y)$ is $\mathcal{O}_Y$-flat is the complement of a closed subscheme (resp. analytic subspace) Y_1 of Y and* $\dim Y_1 < \dim Y$.

PROOF: The result follows from the generic flatness theorem [4] (resp. [3]) in the following way: first, by Corollary (2.9), the flatness of $\mathscr{P}_{X/S}(Y)$ is equivalent to that of $\mathrm{gr}\,\mathscr{P}_{X/S}(Y)$, but by Remark (1.20) $\mathrm{gr}\,\mathscr{P}_{X/S}(Y)$ is a graded $\mathcal{O}_Y$-Algebra of finite presentation, corresponding to the Y-cone $C_{X\times_S Y,Y}$; By the generic flatness theorem, this cone is flat over Y outside of a closed subscheme (resp. analytic subset) C_1 and $\dim C_1 < \dim C_{X\times_S Y,Y}$. We are interested in flatness at points of the vertex (0). If no irreducible component of (0) is contained in C_1, we have our lemma. But this fact is also guaranteed by the generic flatness theorem, since otherwise the image of the non-flat locus of $C_{X\times_S Y,Y}$ would contain an irreducible component of Y.

LEMMA (4.13): *Let X be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space). Every point $x \in X$ has a neighborhood in which the application $x \longrightarrow H_{X/S,x}$ takes only a finite number of values (i.e., only a finite number of distinct Samuel functions appear).*

PROOF: We use the fact that for any Y subscheme (resp. analytic subspace) containing x, $\dim_{\kappa(x)} \mathscr{P}_{X/S}(x) = \dim_{\kappa(x)} \mathscr{P}_{X/S}(Y)(x)$. Let us first choose $Y^0 = X_{\text{red}}$. By Lemma 4.12, we find that there is a closed reduced subscheme (resp. analytic subspace) Y^1 of Y^0 such that $\dim Y^1 < \dim Y^0$ and $\operatorname{gr} \mathscr{P}_{X/S}(Y^0)$, hence also $\mathscr{P}_{X/S}(Y^0)$ is $\mathscr{O}_{Y^0}$-flat on $Y^0 - Y^1$, i.e., in view of Proposition 4.7, the application $x \longrightarrow H_{X/S,x}$ is constant on each connected component of $Y^0 - Y^1$. These are locally in finite number. We now apply Lemma 4.12 to Y^1 and in the same way find a reduced closed subscheme Y^2 of Y^1 such that: $x \longrightarrow H_{X/S,x}$ is constant on each connected component of $Y^1 - Y^2$, and $\dim Y^2 < \dim Y^1$. Since the dimension strictly decreases at each step, this has to stop after a finite number of steps, and that is clearly enough to prove the lemma.

LEMMA (4.14): *Let $\mathscr{S}$ be the set of sequences of integers with the product order. The application $X \to S$ (notations of Lemma 4.12, and same assumptions) given by $x \longrightarrow H_{X/S,x}$ is upper semi-continuous, i.e. if $s = (s_\nu) \in \mathscr{S}$, $\{x \in X : H_{X/S,x}(\nu) \geqq s_\nu \text{ for all } \nu \geqq 0\}$ is a closed subscheme (resp. analytic subspace) of X.*

PROOF: First we remark that this is a closed subset of X: if x does not belong to it, there exists a smallest integer ν_0 such that $H_{X/S,x}(\nu_0) < s_{\nu_0}$, But by Corollary 4.8, there exists then a neighborhood U of x such that for any $x' \in U$ we have $H_{X/S,x'}(\nu_0) \leqq H_{X/S,x}(\nu_0) < s_{\nu_0}$, i.e., no element of U belongs to the subset either. Let us now show that it is a locally closed subscheme (resp. analytic subspace): using Lemma 4.13, let us take a neighborhood V of x in which only a finite number of different Samuel functions appear, say $\mathrm{H}_1, \cdots, \mathrm{H}_r$. Assume that $\mathrm{H}_1, \cdots, \mathrm{H}_s$ $(0 \leqq s \leqq r)$ are those which satisfy: $\mathrm{H}_i(\nu) < s_\nu$ for some $\nu \geqq 0$. Let ν_i $(1 \leqq i \leqq s)$ be the smallest integer such that $\mathrm{H}_i(\nu_i) < s_{\nu_i}$ $(1 \leqq i \leqq s)$ and consider the closed subscheme (resp. analytic subspace) (in view of Corollary 4.8)

$$F = \{x' \in V : H_{X/S,x'}(\nu_1) \geqq s_{\nu_1} \text{ and } H_{X/S,x'}(\nu_2) \geqq s_{\nu_2} \text{ and } \cdots \text{ and } H_{X/S,x'}(\nu_s) \geqq s_{\nu_s}\}.$$

It is clear that for $x' \in V$, if $H_{X/S,x'}(\nu) \geqq s_\nu$ for all $\nu \geqq 0$, $x' \in F$ but the converse is also true since $H_{X/S,x'}(\nu_i) \geqq s_{\nu_i}$ implies that $H_{X/S,x'}$ is not one of the H_i for $0 \leqq i \leqq s$, hence we must have $H_{X/S,x'}(\nu) \geqq s_\nu$ for all $\nu \geqq 0$.

THEOREM (4.15): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space). There exists a locally finite partition of X into subschemes (resp. analytic subspaces) $X = \bigcup_{\alpha\in A} X_\alpha$ having the following properties:*
(i) *given $\alpha \in A$ there exists an application $H_\alpha: \mathbb{N} \to \mathbb{N}$ such that $x \in X_\alpha \Leftrightarrow H_{X/S,x} = H_\alpha$*
(ii) *$\bar{X}_\alpha$ and $\bar{X}_\alpha - X_\alpha$ are closed subschemes (resp. analytic subspaces) of X and* $\dim(\bar{X}_\alpha - X_\alpha) < \dim X_\alpha$.

PROOF: Let $\{\mathrm{H}_\alpha\}_{\alpha\in A}$ be the set of those applications $\mathbb{N} \to \mathbb{N}$ which actually occur as Samuel functions of some point $x \in X$. Clearly what we have to prove is that the sets

$$X_\alpha = \{x \in X : \mathrm{H}_{X/S,x} = \mathrm{H}_\alpha\}$$

are subschemes of X(resp. analytic subspaces).

First we observe that if $x \in \bar{X}_\alpha$, we must have $H_{X/S,x}(\nu) \geqq \mathrm{H}_\alpha(\nu)$ for all $\nu \geqq 0$. Otherwise there exists a smallest integer ν_0 such that $H_{X/S,x}(\nu_0) < s_{\nu_0}$ and by Corollary 4.8 the strict inequality must subsist in a neighborhood of x. But this neighborhood meets X_α, and we have our contradiction.

Let us now consider the closed subschemes (resp. analytic subspace) (see Lemma 4.14)

$$X_\alpha^* = \{x \in X : H_{X/S,x}(\nu) \geqq H_\alpha(\nu) \quad \text{for all } \nu \geqq 0\}.$$

In view of the above observation and Lemma 4.14 any $x \in \bar{X}_\alpha$ has an open neighborhood V such that

$$X_\alpha \cap V = (X_\alpha^* - \bigcup_{\beta\in \mathrm{B}(x)} X_\beta^*) \cap V$$

where $\mathrm{B}(x)$ is the finite set of those Samuel functions appearing in V and such that $\mathrm{H}_\beta(\nu) \geqq \mathrm{H}_\alpha(\nu)$ for all $\nu \geqq 0$ with $\mathrm{H}_\beta \neq \mathrm{H}_\alpha$. This shows that X_α is a subscheme (resp. analytic subspace) of X and the assertion (ii) is easy to obtain since $(\bigcup_{\beta\in\mathrm{B}(x)} X_\beta^*) \cap X_\alpha^* \cap V$ is a strict closed subscheme (resp. analytic subspace) of $X_\alpha^* \cap V$.

DEFINITION (4.16): *The subscheme (resp. analytic subspaces) X_α of X are called the (relative) Samuel strata of X/S.*

PROPOSITION (4.17): *Let X/S be a scheme locally of finite type over a locally noetherian scheme S (resp. a relative analytic space with S reduced)*

and Y a subscheme (resp. analytic subspace). The following conditions are equivalent at a point $y \in Y$.
(i) *X/S is normally flat along Y/S at y.*
(ii) *Y is smooth over S at y and locally around y contained in a Samuel stratum of X/S.*

PROOF: This is nothing but Theorem 4.11.

REMARK (4.18): The fibers of the Samuel strata of X/S at $s \in S$ are nothing but the Samuel strata of $X_s/\kappa(s)$.

5. An example of application

DEFINITION (5.1): *Let (X_0, x_0) be a germ of complex analytic space with isolated singularity. Let $F:(X,x) \to (S,s)$ be a deformation of (X_0, x_0) in the sense of [10] i.e. F is flat and we have a cartesian diagram:*

$$\begin{array}{ccc} (X_0, x_0) & \longrightarrow & (X, x) \\ \downarrow & & \downarrow{\scriptstyle F} \\ (s) & \longrightarrow & (S, s) \end{array}$$

We say that the deformation F is σ-normally flat if there exists a section σ of F such that X is relatively normally flat along $\sigma(S)$ at x (with respect to F).

REMARK (5.2): The section σ is not necessarily unique as shown by taking for X the family of plane curves given by the equation:

$$Y^3 + (X-T)^2(X+T)^2 Y + (X-T)^3(X+T)^3$$

considered as a deformation with parameter T of the germ of plane curve $Y^3 + X^4 Y + X^6 = 0$. The two sections σ_1 and σ_2 given by $(X-T, Y)$ and $(X+T, Y)$ make X σ-normally flat over the T-axis, since the Samuel function of a plane curve is determined by its multiplicity.

THEOREM (5.3): *Given a germ (X_0, x_0) of complex analytic space with isolated singularity, there exists a semi-universal σ-normally flat deformation of (X_0, x_0). (i.e. there exists a σ-normally flat deformation*

$$F_M: (X_M, x_M) \overset{\Sigma_M}{\leftrightarrows} (S_M, s_M)$$

of (X_0, x_0) such that any other is obtained from it by a base change uniquely determined at the first order.)

PROOF: (X_0, x_0) has a semi-universal deformation (See [10]). $F_U:(X_U, x_U) \to (S_U, s_U)$. Let S_M be the unique relative Samuel stratum of X_U/S_U containing x_U (theorem 4.15). Let us consider the mapping of S_M in S_U obtained by composing with F_U the inclusion i_M of S_M in X_U, and the corresponding base change:

$$\begin{array}{ccc} (X_M, x_M) & \longrightarrow & (X_U, x_U) \\ {\scriptstyle F_M}\downarrow & \overset{i_M}{\nearrow} & \downarrow{\scriptstyle F_U} \\ (S_M, x_U) & \longrightarrow & (S_U, s_U) \end{array}$$

F_M now has a section Σ_M corresponding to i_M and it is immediate to check that the σ-normally flat deformation $F_M\colon (X_M, x_M) \overset{\Sigma_M}{\leftrightarrows} (S_M, x_U)$ is semi-universal for σ-normally flat deformations, using § 4 and the fact that F_U is semi-universal.

REMARK (5.4): One can show by various methods (see [11] or [12]) that in the special case where (X_0, x_0) is a germ of hyper surface of dimension d_0, with isolated singularity of multiplicity m_0, writing τ_0 for the dimension of the base of the semi-universal deformation of (X_0, x_0), S_M is smooth at x_U, of dimension:

$$\dim_{x_U} S_M = \tau_0 + d_0 + 1 - \begin{bmatrix} d_0 + m_0 \\ m_0 - 1 \end{bmatrix}$$

REMARK (5.5): Example 5.2 shows that in general the image of S_M in S_U has self-intersection.

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