Whitney stratifications and Plücker-type formulas

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Abstract

We present a geometrically explicit computation of the degree of the dual variety of a complex projective variety V in terms of Euler-Poincaré characteristics of the strata of the canonical Whitney stratification of V and their general plane sections and local vanishing Euler-Poincaré characteristics of the strata along their boundary strata. This result is essentially an application to the case of a conic singularity of a formula due to Lê Dũng Tráng and the author relating the multiplicities of the local polar varieties at a point x of a complex analytic space X of the closures of the strata of a Whitney stratification of X with the vanishing Euler characteristics of X along the strata.

Vanishing Euler characteristics 1

Let $X = \bigcup_\alpha X_\alpha$ be a Whitney stratified complex analytic set of dimension d. Given $x \in X_\alpha$, choose a local embedding $(X, x) \subset (\mathbf{C}^n, 0)$. Set $d_\alpha = \dim X_\alpha$. For each integer $i \in [d_\alpha + 1, d]$ there exists a Zariski open dense subset $W_{\alpha,i}$ in the Grassmannian $G(n-i, n)$ and for each $L_i \in W_{\alpha,i}$ a semi-analytic subset E_{L_i} of the first quadrant of $\mathbf{R}^2$, of the form $\{(\epsilon, \eta) | 0 < \epsilon < \epsilon_0, 0 < \eta < \phi(\epsilon)\}$ with $\phi(\epsilon)$ a certain Puiseux series in ϵ, such that the homotopy type of the intersection $X \cap (L_i + t) \cap \mathbf{B}(0, \epsilon)$ for $t \in \mathbf{C}^n$ is independent of $L_i \in W_{\alpha,i}$ and (ϵ, t) provided that $(\epsilon, |t|) \in E_{L_i}$.

Vanishing Euler characteristics 2

Moreover, this homotopy type depends only on the stratified set X and not on the choice of $x \in X_\alpha$ or the local embedding.

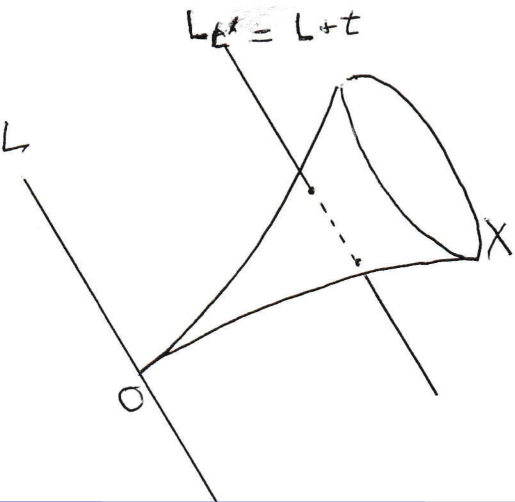
In particular the Euler-Poincaré characteristics $\chi_i(X, X_\alpha)$ of these homotopy types are invariants of the stratified analytic set X.

Vanishing Euler characteristics 3

The Euler-Poincaré characteristics $\chi_i(X, X_\alpha),\ i \in [d_\alpha + 1, d]$ are called the local vanishing Euler-Poincaré characteristics of X along X_α.

Example 1

Let d be the dimension of $X \subset \mathbf{C}^n$. Taking $X_\alpha = \{x\}$, which is permissible by Whitney's lemma, and $i = d$ gives $\chi_d(X, \{x\}) = m_x(X)$, the multiplicity of X at the point x.



Example 2

Assume that $(X, x) \subset (\mathbf{C}^n, 0)$ is a hypersurface with isolated singularity at the point x (taken as origin in $\mathbf{C}^n$), defined by $f(z_1, \ldots, z_n) = 0$. By Whitney's lemma , in a sufficiently small neighborhood of x, the minimal Whitney stratification is $(X \setminus \{x\}) \cup \{x\}$, and we have

$$\chi_i(X, \{x\}) = 1 + (-1)^{n-1-i}\mu^{(n-i)}(X, x), \qquad (*)$$

where $\mu^{(k)}(X, x)$ is the Milnor number of the restriction of the function f to a general linear space of dimension k through x.

Let us recall that the Milnor number $\mu^{(n)}(X, x)$ of an isolated singularity of hypersurface in $\mathbf{C}^n$ as above is defined algebraically as the multiplicity in $\mathbf{C}\{z_1, \ldots, z_n\}$ of the Jacobian ideal $j(f) = \left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle$, which is also the dimension of the $\mathbf{C}$-vector space $\frac{\mathbf{C}\{z_1,\ldots,z_n\}}{j(f)}$ since in this case the partial derivatives form a regular sequence.

Topologically it is defined by the fact that for $0 < |\lambda| \ll \epsilon \ll 1$ the *Milnor fiber* $f^{-1}(\lambda) \cap \mathbf{B}(0, \epsilon)$, which is a smoothing of (X, x), has the homotopy type of a bouquet of $\mu^{(n)}(X, x)$ spheres of dimension $n - 1$.

The result in Example 2 is based on the fact that if $z_1 = 0$ is not a limit of tangent hyperplanes to X, then $f(0, z_2, \ldots, z_n) = \lambda$ for $\lambda \neq 0$ and $|z_i|, \ |\lambda|$ small enough and $f(t, z_2, \ldots, z_n) = 0$ for $t \neq 0$ and $|z_i|, \ |t|$ small enough, are both smoothings of the hypersurface with isolated singularity $f(0, z_2, \ldots, z_n) = 0$ and thus are diffeomorphic.

The conormal space 1

Let X^0 denote the non singular part of the (reduced) complex-analytic set $X \subset U \subset \mathbf{C}^n$, where U is an open set of $\mathbf{C}^n$.

The (projective) conormal space $C(X)$ of $X \subset U \subset \mathbf{C}^n$ is the closure in $X \times \check{\mathbf{P}}^{n-1}$ of the set of points $(x, H) \in X^0 \times \check{\mathbf{P}}^{n-1}$ such that H contains the direction of the tangent space to X at $x \in X^0$. The first projection induces an analytic map $\kappa\colon C(X) \to X$.

The conormal space 2

The (projective) conormal space $C(X)$ is a closed, reduced, complex analytic subspace of $X \times \check{\mathbf{P}}^{n-1}$ of dimension $n-1$. For any $x \in X$ the dimension of the fiber $\kappa_X^{-1}(x)$ is at most $n-2$. As a set, it is the set of limits in $\mathbf{P}^{n-1}$ of hyperplanes tangent to X at non singular points approaching x.

The conormal space 3

If $(X, 0)$ is the cone with vertex 0 over a projective variety $V \subset \mathbf{P}^{n-1}$, the fiber $\kappa^{-1}(0) \subset \check{\mathbf{P}}^{n-1}$ is the projectively dual variety $\check{V}$ of V.

This follows from the fact that the direction of the tangent space to X is constant along the homothety lines of X.

Polar varieties 1

Let us denote by $C(X)$ the conormal space of X, then we have the diagram:

$$
\begin{array}{ccc}
C(X) & \hookrightarrow & X \times \check{\mathbf{P}}^{n-1} \\
 & \overset{\lambda}{\searrow} & \downarrow{pr_2} \\
\downarrow{\kappa} & & \check{\mathbf{P}}^{n-1} \supset L^{d-k} \\
X \subset \mathbf{C}^n & &
\end{array}
$$

Polar varieties 2

Let $D_{d-k+1} \subset \mathbf{C}^n$ be a linear subspace of codimension $d-k+1$, for $0 \leq k \leq d-1$, and let $L^{d-k} \subset \check{\mathbf{P}}^{n-1}$ be the dual space of D_{d-k+1}, which is the linear subspace of $\check{\mathbf{P}}^{n-1}$ consisting of hyperplanes of $\mathbf{C}^n$ that contain D_{d-k+1}.

The next proposition provides the relation between the intuitive definition of local polar varieties as closures of sets of critical points on X^0 of linear projections and the conormal definition, which is useful for proofs.

Polar varieties 3

For a sufficiently general D_{d-k+1}, the image $\kappa(\lambda^{-1}(L^{d-k}))$ is the closure in X of the set of points of X^0 which are critical for the projection $\pi|_{X^0} : X^0 \to \mathbf{C}^{d-k+1}$ induced by the projection $\mathbf{C}^n \to \mathbf{C}^{d-k+1}$ with kernel $D_{d-k+1} = (L^{d-k})\check{}$.

Polar varieties 4

With the notation and hypotheses above, define for $0 \leq k \leq d-1$ the **local polar variety at** x as the germ at $x \in X$ of.

$$P_k(X; L^{d-k}) = \kappa(\lambda^{-1}(L^{d-k})),$$

or a suitably small representative, for general L^{d-k}.

A priori, we have just defined polar varieties set-theoretically, but since $\lambda^{-1}(L^{d-k})$ is empty or reduced and κ is a projective fibration over the smooth part of X we have:

The local polar variety $P_k(X; L^{d-k}) \subseteq X$ is a (germ of a) reduced closed analytic subspace of X, either of pure codimension k in X or empty.

Polar varieties 5

The multiplicity $m_x(P_k(X; L^{d-k}))$ at $x \in X$ of $P_k(X; L^{d-k})$ is independent of the embedding in an affine space and of the general linear space L^{d-k} and depends only on the analytic type of X at x.

Polar varieties 6

If (X, x) is a hypersurface with isolated singularity the multiplicities at x of the polar varieties can be computed from the $\mu^{(k)}(X, x)$; we have the equalities

$$m_x(P_k(X, x)) = \mu^{(k+1)}(X, x) + \mu^{(k)}(X, x) \text{ for } k = 0, \ldots, n-1.$$

At this point it is important to note that the equality $m_x(P_{n-2}(X, x)) = \mu^{(n-1)}(X, x) + \mu^{(n-2)}(X, x)$ which, by what we have just seen, implies the equality

$$\chi_1(X, \{x\}) - \chi_2(X, \{x\}) = (-1)^{n-2} m_x(P_{n-2}(X, x)),$$

implies, setting $d = n - 1$ to go back to familiar notations, the general formula

$$\chi_{d-k}(X, \{x\}) - \chi_{d-k+1}(X, \{x\}) = (-1)^k m_x(P_k(X, x)),$$

simply because an affine space $L_{d-k} + t$ can be viewed as the intersection of an $L_1 + t$ for a general L_1 with a general vector subspace L_{d-k-1} of codimension $d - k - 1$ through the point x taken as origin of $\mathbf{C}^n$, and

The general formula is:

Theorem (Lê-Teissier) With the conventions just stated, and for any Whitney stratified complex analytic set $X = \bigcup_\alpha X_\alpha \subset \mathbf{C}^n$, we have for $x \in X_\alpha$ the equality

$$\chi_{d_\alpha+1}(X, X_\alpha) - \chi_{d_\alpha+2}(X, X_\alpha) = \sum_{d_\beta > d_\alpha} (-1)^{d_\beta - d_\alpha - 1} m_x(P_{d_\beta - d_\alpha - 1}(\overline{X_\beta}, x))(1 - \chi_{d_\beta+1}(X, X_\beta)),$$

where it is understood that $m_x(P_{d_\beta - d_\alpha - 1}(\overline{X_\beta}, x)) = 0$ if $x \notin P_{d_\beta - d_\alpha - 1}(\overline{X_\beta}, x)$.

Key ingredients in the proof of this formula are:

1. For a general projection $\pi : \mathbf{C}^n \to \mathbf{C}^{d-k+1}$ the kernel of π is transversal to the tangent cone of the corresponding polar variety $P_k(X; L^{d-k})$ so that the image of $P_k(X; L^{d-k})$ in $\mathbf{C}^{d-k+1}$ has the same multiplicity as $P_k(X; L^{d-k})$.
2. The map $\pi|X \colon X \to \mathbf{C}^{d-k+1}$ is *descriptible* in the sense that source and target can be stratified in such a way that each stratum in the source is a topological bundle over its image.
3. In complex analytic geometry the complement of a closed union of strata in its "tubular neighborhood" as provided by the Whitney conditions, has zero Euler-Poincaré characteristic.
4. In addition, the existence of fundamental systems of good neighborhoods of a point of $\mathbf{C}^n$ relative to a Whitney stratification also plays an important role.

Let $V \subset \mathbf{P}^{n-1}$ be a projective variety of dimension d. We assume that it is not contained in a hyperplane.

1) If $V = \bigcup_\alpha V_\alpha$ is a Whitney stratification of V, denoting by $X_\alpha \subset \mathbf{C}^n$ the cone over V_α, we have that $X = \{0\} \bigcup_\alpha X^*_\alpha$, where $X^*_\alpha = X_\alpha \setminus \{0\}$, is a Whitney stratification of X. It may be that (V_α) is the minimal Whitney stratification of V but $\{0\} \bigcup X^*_\alpha$ is not minimal, for example if V is itself a cone.

2) Again with X_α denoting the cone over V_α, if $L_i + t$ is an i-codimensional affine space in $\mathbf{C}^n$ it can be written as $L_{i-1} \cap (L_1 + t)$ with vector subspaces L_i and for general directions of L_i we have, denoting by $\mathbf{B}(0,\epsilon)$ the closed ball with center 0 and radius ϵ, for small ϵ and $0 < |t| \ll \epsilon$:

$$\chi_i(X,\{0\}) := \chi(X \cap (L_i + t) \cap \mathbf{B}(0,\epsilon)) = \chi(V \cap H_{i-1}) - \chi(V \cap H_{i-1} \cap H_1),$$

where $H_i = \mathbf{P}L_i \subset \mathbf{P}^{n-1}$.

This requires a proof, which is based on the Thom-Mather theorem.

3) For every stratum X^*_α of X, we have the equalities $\chi_i(X, X^*_\alpha) = \chi_i(V, V_\alpha)$.

4) The polar variety $P_k(X, 0)$ is the cone over the polar variety $P_k(V)$ defined *à la Todd* as the closure in V of the set of points in V^0 where the tangent space is contained in a hyperplane containing $\mathbf{P}D_{d-k+1}$.

5) If the dual $\check{V} \subset \check{\mathbf{P}}^{n-1}$ is a hypersurface, its degree is equal to $m_0(P_d(X,0)) = \deg P_d(V)$, which is the number of non singular critical points of the restriction to V of a general linear projection $\mathbf{P}^{n-1} \setminus L_2 \to \mathbf{P}^1$.

Note that we will apply statements 2), 3) and 4) not only to the cone X over V but also to the cones $\overline{X_\beta}$ over the closed strata $\overline{V_\beta}$.

To prove statement 2), we first remark that it suffices to prove the result for $i = 1$ since we can then apply it to $X \cap L_{i-1}$. By an appropriate choice of coordinates assume that the hyperplane L_1 defined by $z_1 = 0$ in $\mathbf{P}^{n-1}$ with homogeneous coordinates $(z_1, \ldots, z_n)$ and is transversal to the strata of V.

The affine chart $\mathbf{A}^{n-1} \subset \mathbf{P}^{n-1}$ defined by $z_1 \neq 0$ admits coordinates $\frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}$. Consider the restrictions of the distance function to $0 \in \mathbf{A}^{n-1}$ to the strata of V which are algebraic and by Bertini-Sard's theorem and Thom-Mather's topological triviality theorem, we obtain that there exists a radius R_0 such that, denoting by $\mathbf{D}(0, R)$ the ball centered at 0 and with radius R in $\mathbf{A}^{n-1}$, the spheres of radius R are transversal to all the strata of V for $R \geq R_0$ and we have that the homotopy type of $V \cap \mathbf{D}(0, R)$ is constant for $R \geq R_0$ and equal to that of $V \setminus V \cap H$ where H is the hyperplane at infinity $z_1 = 0$.

Given $t \neq 0$ and $\epsilon > 0$, the application $(z_1 : \ldots : z_n) \mapsto (t, t\frac{z_2}{z_1}, \ldots, t\frac{z_n}{z_1})$ maps isomorphically $V \cap \mathbf{D}(0, R)$ onto $X \cap (L_1 + t) \cap \mathbf{B}_\epsilon$ if $R = \frac{\epsilon}{|t|}$. It suffices to take $|t|$ small enough with respect to ϵ to ensure that $\frac{\epsilon}{|t|} > R_0$ to obtain statement 2).

Statement 3), $\chi_i(X, X_\alpha^*) = \chi_i(V, V_\alpha)$, follows from the fact that locally at any point of X_α^*, the cone X, together with its stratification, is the product of V, together with its stratification, by the generating line through x of the cone, and product by a disk does not change the Euler characteristic.

Statements 4) and 5) follow from definitions. Indeed, remembering that the fiber $\kappa^{-1}(0)$ of the conormal map $\kappa\colon C(X) \to X$ is the dual variety $\check{V}$, the last statement follows from the very definition of polar varieties. Given a general line L^1 in $\check{\mathbf{P}}^{n-1}$, the corresponding polar curve in X is the cone over the points of V where a tangent hyperplane belongs to the pencil L^1; it is a finite union of lines and its multiplicity is the number of these lines, which is the number of corresponding points of V.

Plücker formula in the case the dual is a hypersurface 1

Given the projective variety $V \subset \mathbf{P}^{n-1}$ equipped with a Whitney stratification $V = \bigcup_{\alpha \in A} V_\alpha$, denote by d_α the dimension of V_α. The local formula and the previous results yield, if the projective dual $\check{V}$ is a hypersurface in $\check{\mathbf{P}}^{n-1}$, the formula:

$$(-1)^d \deg \check{V} = \tag{1}$$

$$\chi(V) - 2\chi(V \cap H_1) + \chi(V \cap H_2) - \sum_{d_\alpha < d} (-1)^{d_\alpha} \deg P_{d_\alpha}(\overline{V_\alpha})(1 - \chi_{d_\alpha+1}(V, V_\alpha)) \tag{2}$$

.

Here H_1, H_2 denote general linear spaces of codimension 1 and 2 respectively, $\deg P_{d_\alpha}(\overline{V_\alpha})$ is the number of critical points of a general linear projection $\overline{V_\alpha} \to \mathbf{P}^1$, which is the degree of $\check{\overline{V_\alpha}}$ when it is a hypersurface and 0 otherwise, and is equal to 1 if $d_\alpha = 0$.

The case of a hypersurface with isolated singularities

Let $V \subset \mathbf{P}^{n-1}$ be a hypersurface of degree m with only isolated singularities $x_i \in V$. The formula in this case reduces to:

$$\deg \check{V} = m(m-1)^{n-2} - \sum_i (\mu^{(n-1)}(V, x_i) + \mu^{(n-2)}(V, x_i)),$$

where $\mu^{(n-1)}(V, x_i)$ is the Milnor number we have seen and $\mu^{(n-2)}(V, x_i)$ is the Milnor number of a general hyperplane section through x_i.

The formula gives a proof which is somewhat different from the original proof which was an algebraic computation of the number of tangent hyperplanes to the Milnor fiber at each singular point which were "absorbed" by the singularity as the Milnor fiber locally degenerated to V.

Using the properties of polar varieties and the previous theorem one can prove a similar formula in the case where the dual $\check{V}$ is not a hypersurface, and thus extend the theorem to all projective varieties. The degree of $\check{V}$ is then the multiplicity at the origin of the smallest polar variety of the cone X over V which is not empty. Now one uses again the equalities

$$m_x(P_k(X,x)) = m_x(P_k(X,x) \cap L_{d-k-1}) = m_x(P_k(X \cap L_{d-k-1}), x).$$

We are going to see that they tell us that:

The degree of $\check{V}$ is the degree of the dual of the intersection of V with a linear space of the appropriate dimension for this dual to be a hypersurface.

More precisely, when H is general hyperplane in $\mathbf{P}^{n-1}$, the following facts are consequences of the elementary properties of projective duality, and the property that tangent spaces are constant along the generating lines of a cone:

The dual of a general hyperplane section 1

- If $\check{V}$ is a hypersurface, the dual of $V \cap H$ is the cone with vertex $\check{H}$ over the polar variety $P_1(\check{V}, \check{H})$, the closure in $\check{V}$ of the critical locus of the restriction to $\check{V}^0$ of the projection $\pi\colon \check{\mathbf{P}}^{n-1} \to \check{\mathbf{P}}^{n-2}$ from the point $\check{H} \in \check{\mathbf{P}}^{n-1}$. Since we assume that V is not contained in a hyperplane, the degree of the hypersurface $\check{V}$ is ≥ 2, hence this critical locus is of dimension $n-3$ and the dual of $V \cap H$ is a hypersurface. In appropriate coordinates its equation is a factor of the discriminant of the equation of $\check{V}$.
- Otherwise, the dual of $V \cap H$ is the cone with vertex $\check{H}$ over $\check{V}$, i.e., the join $\check{V} * \check{H}$ in $\check{\mathbf{P}}^{n-1}$ of $\check{V}$ and the point $\check{H}$.

The dual of a general hyperplane section 2

In the case where $\check{V}$ is not a hypersurface, assuming that its codimension is $1+\delta(V)$ where $\delta(V)\geq 1$ is called the *dual defect* of V, we can compute its degree as follows: Let $L\subset \mathbf{P}^{n-1}$ be a general linear space of codimension $\delta(V)$. Then the intersections with L of the strata of the canonical Whitney stratification of V are strata of a Whitney stratification of $V\cap L$, the dual of $V\cap L$ is a hypersurface which is the join of $\check{V}$ and a $(\delta(V)-1)$-dimensional general linear space in $\check{\mathbf{P}}^{n-1}$ and **therefore has the same degree as** $\check{V}$.

The general Plücker-type formula

In the formula we gave in the case where $\check{V}$ is a hypersurface, we can use this to compute the degrees of the $\overline{\check{V}_\alpha}$ (including V itself) using their general linear sections to reduce to the case where all the duals are hypersurfaces.

So by induction we have a formula expressing the degree of $\check{V}$ in terms of the Euler characteristics of the strata of the minimal Whitney stratification of V and their general linear sections, and the local vanishing Euler characteristics of the Whitney strata of V along the strata of their boundaries since those of $V \cap L$ are some of those for the strata of V.

Conclusion

- One does not need characteristic classes or indeed any homology or cohomology theory to express the degree of the dual variety: the only things that ”move” are linear subspaces of $\mathbf{P}^{n-1}$ which have to be general.
- Since the degree of the dual variety is an integer, if one believes that it can be expressed in terms of the topological characters of the stratified space V, it is natural that it should be expressed in terms of Euler characteristics, and indeed that is what we did.

(minimal) bibliography

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