Protomathematics, perception and the meaning of mathematical objects

Paru dans ``Images and Reasoning'' Grialou, Longo, Okada, Eds., Keio University, Tokyo 2005

Bernard Teissier
Institut mathématique de Jussieu, UMR 7586 du CNRS

1. Introduction

Assuming that mathematical logic is, among other things, a mathematical model of reasoning, why is it that when a mathematician understands a theorem, his mental activities seem to have so little in common with the logical exposition of the proof ? The feeling of understanding is often, at least for part of the proof, of the nature of an illumination brought about by the feeling that the mechanisms proposed are ``right'', in the light of our intuition and our experience of ``how things work''.
It is more as if someone explained to you a climbing route, with the way to surmount each difficulty to get to the top, and at the end, you realize that yes, this will very probably take you to the top. This conviction relies very much on your intuition and your experience of the difficulties. In this metaphor, the logic of the proof would correspond to the programming of a robot to make the climb. The mathematician is pretty much convinced that this is possible only once HE has been there (in Mathematics, this is called *carefully checking the proof*), and then the problem of programming a robot to do the same has ceased to interest him.
There is a fundamental difference in that the robot does not know what is significance, and in Mathematics, the choice of statements, based on their significance, is crucial. Also, this metaphor quickly reaches a limit, because the mathematician to some extent creates his own mountain ranges, albeit certainly not in my opinion in the gratuitous way sometimes suggested.

Usually a model of anything, including reasoning, is supposed to correspond quite closely to the experiments,
at least in their structure, and the model of mathematical reasoning that we have does not seem to satisfy this requirement.
Why is it so ? I think we have to look for an explanation in the fact that part of the mathematical construction is of a preverbal nature, and that

we give meaning to the statement of a theorem and to its proof by calling upon some basic correspondance between mathematical objects and our experience of the world.
It is therefore very interesting to explore in animals, and in particular higher mammals, the activities which may be indications of preverbal mathematical activity, without extending unduly the use of the term ``mathematical''(see below).

I am going to propose, according to the credo of the ``Geoco'' (Géométrie et Cognition) group created by Longo, Petitot, and myself (see [LPT]), that one should try to look at cognitive bases, especially the way we acquire data on our environment through our perceptual system, as one possible place where to seek protomathematical reasoning and even foundations of Mathematics. I am convinced that there are others such places, more closely connected to the very structure of what we call reasoning ; I will come back to them later.
To be provocative : one should not try to explain the meaning of words with other words. Certainly the conscience of this contradiction is not new, Wittgenstein for one was well aware of it (see [H]).
But what is novel is that now we have another place where to seek the source of meaning.

I propose that at least in some instances, ``meaning'' is of a preverbal nature. This is compatible with the idea, which originates in Thom's work (see [Th], [P1], [P2], [T1]), that language is connected to ``meaning'' by a generalized catastrophy in the sense of Thom, which is irreversible by rational thought. Very roughly, a generalized catastrophy is the limit of bifurcations, in which the possible states ramify indefinitely. The important thing is that it does not satisfy Curie's principle, according to which in Physics the structure (for example, that given by symmetries, see also the discussion in [L]) of the causes reappears in the effects; it is therefore, as Thom says, ([Th]., Chap. 6, F), *not a formalizable process*. Thom has proposed to use this as a model for the way the ``pregnancy'' of meanings invests the ``saliencies'' of language. One cannot use a rational process to go back from the language to the pregnancies. One can speculate that this is the origin of feelings of transcendence (see [T2]). I refer to [P2] for much more on this subject.

Untile quite recently this kind of statement was doomed to be thought completely wrong, or completely trivial, or in fact meaningless. However, the old dilemma about the possibility of non verbal thought is rapidly becoming obsolete as we recognize that certain **unvolontary** and **unconscious** activities of our brain are strikingly similar to what we are used to calling thought. The limits of reason and the limits of language are of course great classics of philosophy, but the discussion is all but impossible, as was pointed out repeatedly, because we cannot express, or even think about, what is on the other side of the boundary we feel. Therefore one finds in philosophical texts a recourse to various non rational non verbal experiences to evoke what lies on the other side of that boundary. This goes from ``divine understanding'' to ``a priori data of consciousness'', to relation to a past golden age when language was not dissociated from experience to various forms of mystic experience as in the first Wittgenstein.

All these are, for me, essentially of great poetic value in transforming the feeling that we have of a direct perception of reality into….words. They are not really acceptable as foundation of Mathematics or explanation of their meaning.

I find, however, that now that we understand better and better our perceptual system, in it much more natural to work with the hypothesis that what lies beyond the limit of language and of reason in our experience of the world is precisely constituted by those unvolontary and unconscious activities of our brain mentioned above, a part of which are involved in the perception and primary analysis of our environment, and which we share to a large extent with other primates.

Indeed, the perceptual approach to human mathematics presented here is highly compatible with the fascinating perspective of a non-human epistemology discussed in [H-L] after the description of the knot-tying abilities of the female Orang-Utan Wattana.

Coming back to the boundaries of language, I propose to give an example of what might be on the other side of that boundary, in a very restricted area of knowledge.

Since this is linked in an essential way to our understanding of the nature of language and one aspect of Mathematics is to be a kind of laboratory-made language, it is natural that the simplest examples should come from Mathematics. The reader will judge whether it satisfies the criterion of not explaining the meaning of words with other words.

To support the idea that the situation concerning non verbal thinking is changing thanks to the progress in understanding cognitive and perceptual mechanisms, I propose to study from this viewpoint the mathematical line. I claim that the concept of mathematical line which we use in understanding theorems comes by a protomathematical construction from our perceptual system. The proposition made here could not exist without the recent progress in the understanding of perception, for which influential sources are [B] and [N].

2. The example of the line

2.1 The visual line

Our perceptual system recognizes at least two different lines : the visual line and the vestibular line.
The visual line is a construction of our visual system, which detects curves and directions. It is, according to Jacques Ninio's definition (see [N])`` a curve which has the same direction at each of its points''.
It seems to be an established and non trivial fact that some events, probably in in our visual area V1, correspond to the detection of a contour or a curve, and that a subset of those events corresponds to the detection of a curve of curvature zero, that is, a line. I will not say more on this here, referring to [B] and [N].

2.2 The vestibular line

Our vestibular system detects all changes of orientation and accelerations.
When we are in movement and the vestibular system detects nothing, it is a well-defined state which corresponds to uniform translation. Here we have to note that time intervenes subjectively through our biological clocks but also because in view of the principle of Galilean relativity if our accelerometers detect nothing, we are in uniform translation for physical time.

So our vestibular system, with the help of our biological clocks, provides us with a line PARAMETRIZED BY TIME.
Here again we create a universal time which is comparable to the individual biological times, an important construction made possible by

the fact that our bodies perceive uniform translation.
I pass on the corrections effected by our perception which make us perceive uniform
translation even as we walk. One may add that steps of our walk provide the lattice of integers in the reals.
I refer again to [B]; its author is not, however, responsible for the somewhat extensive interpretation given here.

2.3 The Poincaré-Berthoz isomorphism

Poincaré said that our perception of a point in space is our perception of the gesture we have to make to seize an object placed there.
Alain Berthoz and his group at the Collège de France have made many experiments to understand this. I propose that the mathematical line is the result of the identification of the visual line with the vestibular line by what I call the Poincaré-Berthoz isomorphism. It says that we can represent our walk as a movement on a visual line, and conversely we can imagine ourselves walking along a visual line to go from one point to the other (Remember that this is the kind of thing which Einstein always declared to be essential tenets of his intuition) .
Thus by ``transfer of structure'' , since the vestibular line carries a parametrization by time, so does the visual line. Remark that there is no fixed origin on the visual line, while one may argue that the vestibular line originates where I am now. Perhaps even the idea that the vestibular line has in reality no determined origin is also due to the Poincaré-Berthoz isomorphism.

This ties in with what Giuseppe Longo views as the source of Platonism; the idea that the mathematician's conviction of the existence of mathematical objects is due to his perception of their invariance with respect to various codings. Here identification the Poincaré-Berthoz isomorphism creates an object which is by construction independant of the
perceptual coding. This is its mathematical existence, and explains something which has long puzzled me: the fact that the identification of time with the real line seems so obvious to everyone.

This also extends to more complicated curves,
and for example, (see [B]) one can give a vestibular verification of the

fact that the sum of the angles of a triangle is equal to π.
A blindfolded subject is asked to start from a point, walk for a fixed distance, turn by some angle, and again walk for some distance. then he is asked to stop and turn towards his/her starting point. It works !
One important consequence of the fact that when we think of a geometric line, or a curve, we easily imagine ourselves walking on it, is that it suggests concepts of speed, length of a curve, curvature, etc.
If we agree that we have a direct intuition in this way of the length of a curve, it is perhaps not surprising that the concept of area, which has no clear perceptual counterpart, has appeared so much later (in Homer, the size of cities is measured by the length of their perimeter...). Our perceptual system is I think embarrassed to compare the areas of two lakes of quite different shapes. Of course in pre-agricultural societies the survival value of area evaluation is much lower than that of length. The invention of area is a decisive moment of Mathematics, lost in the mists of time. The precise comparison of areas of simple shapes occupies a substantial part of Euclid's elements, and it is essentially reduced to comparison of lengths, which our perceptual system ``understands'' much better..
Our notion of space is similarly the result of the identification of the space constructed by our visual system and the space of our movements, as Poincaré emphasized.
It is in this sort of identification, I believe, that the Foundations of the meaning Mathematics lie, and also the source of the intuition which allows us to give ``meaning" to mathematical statements and proofs.
For example, by construction the vestibular line is equipped with an addition, namely the addition of distances or times, and the fact that an infinite sum which has to converge from the point of view of distance appears, in the absence of a theory of convergence, to be infinite from the viewpoint of time may be conjectured to have given rise to Zeno's paradox.
Of course, the mathematical constructions provide many degrees of sophistication above this basic one. The line is only the first stage of an amazingly elaborate construction. The perceptual fact that we feel the continuity of movement on the vestibular line endows the line with the structure of a *continuum,* a fact that has been the source of a number of

important mathematical questions. Nevertheless I hope that this interpretation would have pleased René Thom who often said that he wished that Mathematics were founded on geometry (in fact, on the continuum) rather than on the integers.

3. Other examples

3.1 Small ordinals

Let me illustrate how these basic intuitions of our surroundings may help to provide intuitions of infinite phenomena.
Let us consider the theory of ordinals, which is a basic tenet of the theory of (infinite) ordered sets. Recall that an ordered set is a set on which a binary operation ``x is less than or equal to y'' is given, which satisfies three axioms which need not be recalled here.
Remark that the concept of order itself is a result of the identification of the visual and vestibular lines: ``before'' in time is identified with ``on the near side of". Now infinity is the idea of walking forever, and the intuition that by walking you eventually get there.
The theory of ordinals is the theory of *well ordered* sets, which means ordered sets such that any subset has a smallest element. In particular, there is a smallest element in the set of elements larger that any subset, finite or infinite. Any well ordered set containing the set of integers as a strict subset contains an element ω which represents the result of the action of walking forever along the vestibular line.
In the viewpoint proposed here there is a need for us to represent the result of such an action, simply because we can conceive it. This need is nonverbal but definitely mathematical in some sense.
However, if we want to understand the ordinals ω, $\omega+1$,..., $\omega+k$, etc., we need more geometry, as follows :
Imagine the set of points with integral coordinates (i,j) in the plane, and decide that (i,j)<(i',j') if i<i', or i=i' and j<j'. Given any two distinct points, one is less than the other for this order. Note that the points lying on the vertical axis i=0 are less than any point whose first coordinate is >0. Now to imagine w+1, we pick any point with first coordinate i=1, say (1,j). The image is that you walk along the axis i=0 forever starting from some point, say (0,0), but now you can say that after ``walking

forever'', you reach a new type of point , namely (1,j). Then ω +2 is represented by the point (1,j+1), and so on. After walking forever on the line i=1which is in fact a column in my representation, you can decide that you reach a point (2,k), which will represent 2 ω.
Here there is a certain arbitrariness, because we can define many different well ordered subsets of the set of all pairs of integers (i,j), which is not itself well ordered; to determine a well ordered subset we may proceed as follows: choose a smallest element in each of the columns i= 0, 1,2,....and decide that all the elements above this element in the same column belong to our subset. You see that after walking forever on the column i=t, I **must** choose a point of arrival on the next column, because my intuition refuses to accept walking from minus infinity to somewhere, even if I have convinced it to accept the idea of walking forever from my present position.

3.2 Function spaces

Another basic example is that of function spaces, or functional analysis. The idea, due to V. Volterra, is to think of a function as a point in some space endowed with a ``topology'' which means essentially a way of defining the ``closeness'' of points. Think for example of all the differentiable functions on the line which decrease rapidly (in a precise technical sense) when the variable tends to infinity.
Then it has a meaning to say that two functions are ``close'', and we can apply to these spaces of functions the intuitions of our ambient space and what it means to be ``close'' there.
The study of spaces of functions is called ``functional analysis''. Many of the basic theorems of functional analysis are extensions to these infinite-dimensional spaces of functions of ``facts of life'' in our usual space. For example if a subset of the space is such that the segment joining any two of its points is entirely within the subset, it is called convex. Important theorems of functional analysis generalize the intuitive facts that disjoint closed convex sets can be separated by a hyperplane, or the fact that a convex set is the convex hull of its extreme points. Quite a few problems in functional analysis are connected with finding a continuous linear function on some function space which has certain properties with respect to some subset. Our intuition of the behaviour of hyperplanes (defined by the vanishing of linear functions) with respect to subsets *in our usual space* plays an absolutely fundamental role in functional

analysis for this reason.

4 Conclusion

In imitation of René Thom's famous aphorism ``The limit of truth is not error, it is meaninglessness'', I am tempted to propose the following : ``The limit of language is not mysticism, it is the perceptual system''.

Here by perceptual system I mean both the way we perceive relations in our environment, such as ``being on a line" but also the unvoluntary facts of perception such as the comparison of the sizes of two similar objects, the juxtaposition, automatic iteration, and many others which it would bery interesting to enumerate.

I believe that the construction of the mathematical line by identification of the visual and vestibular lines is a perceptual and non verbal fact which is part of our basic ``knowledge'' of what a line is. It is that knowledge and others of the same nature which ultimately enable us to understand the truth of mathematical theorems.

More generally, the perceptive system as described above consists of **preverbal organisations of our perception of the world** which, I insist again, are unvoluntary. Nevertheless, they provide the **ultimate reference for the meaning of our mathematical objects and reasonings**. The mathematician understands a proof when he has, with the help of his experience of course, decomposed it into steps which he can ``capture" with these preverbal organizations. Typically he reduces the truth of some assertion to what he feels would happen to him if he was moving on a line. Only then does he say ``I understand".

It must also be emphasized that the education of a mathematician teaches him to capture by his preverbal organizations quite abstract objects, an aptitude often called ``intuition".

Of course in today's mathematics, a process of *formal* reduction to these elementary components would be very long and complicated, probably impossible in practice in most cases. In fact the very identification of these primary components remains to be done. In addition to those mentioned at the beginning of this paragraph, the oppositions local/global, discrete/continuous, invariant/variant, comparable/incomparable, A implies B/B implies A, are just the

beginning of a very long list, which also contains the *desire* to find a cause for any phenomenon, and the brain's ability to detect analogies. The concept of limit/boundary itself, which I have used a lot above, is certainly a prime example of a perceptual reference for a mathematical object, which we share with many animals. The fact (let us call it Zeno's discovery) that a boundary can be reached either in finite or in infinite time may be a truly mathematical distinction, still important today to understand proofs precisely in the sense alluded to in the Introduction, but it is still quite close to our nature as primates.
On the other hand, if one wishes to compare this still extremely rudimentary proposal of foundation of Mathematics on the perceptive system to the classical foundations on logic, remember that it is well known that the redaction of a very simple ``real'' theorem in the language of set-theory is also an impossibly complicated task, and as I said in the beginning it does not even provide meaning!.

The use of the word mysticism in the beginning of this section is due to the close relationship between this kind of speculation and Wittgenstein's early views on the limits of language, for which I refer to Wittgenstein's texts but also to the extremely lucid book [H].
The Poincaré-Berthoz isomorphism does not refer to true relations between objects, so according to Wittgenstein it has no meaning and is only a source of philosophical *verbiage*. I respectfully beg to differ and to assert that such a relation is part of the meaning of the mathematical line.

It is perhaps not too daring to suggest that in order to understand what it means to understand a theorem, we have to accept that Mathematics is a fundamentally human science, whose sources of energy and paths of development are ultimately governed by human perception and human desires, which we share to some extent with primates.
This seems quite contrary to what the icon of Wittgenstein stands for. It is perhaps, however, a thought that he would have appreciated, had he been aware of the intricacies of the human perception and spontaneous mental activites, and therefore able to accept the use of the perceptual system as a cure for the ``disease of language'' which consists in explaining meaning with words when it does not refer to facts.

References

[B] Berthoz, A., 1997, *Le sens du Mouvement*, Editions Odile Jacob.
[H] Hadot, P., 2004, *Wittgenstein et les limites du langage*, Vrin.
[H-L] Herzfeld, C. & Lestel, D., 2005, *Topological ape: Knot tying and untying and the origins of mathematics*, This volume.
[N] Ninio, J., 1989, *L'empreinte des sens*, Ed. du Seuil.
[L] Longo, G., *Incomplétude et incertitude en Mathématiques et en Physique,* to appear. (see www.di.ens.fr/users/longo/download.html)
[LPT] Longo, G., Petitot, J., Teissier, B., 1999, see ``*Motivations générales*", in ``Géométrie et cognition", on http://www.di.ens.fr/users/longo/geocogni.html
[P1] Petitot , J., 1984, "*Applications des mathématiques aux sciences humaines*" in Math. et Sciences humaines n°86. (rapport collectif).
[P2] Petitot, J., 1985, *Morphogénèse du sens*, PUF, Paris.
[T1] Teissier, B., 2004, *Le mur du langage*, in ``Le réel en mathématiques, Mathématiques et psychanalyse", P. Cartier et N. Charraud, éditeurs, Editions Agalma, diffusion Le Seuil.
[T2] Teissier, B., 1994, *Des modèles de la Morphogénèse à la morphogénèse des modèles*, in ``Passion des formes'', Michèle Porte, coordonnateur, ENS Editions Fontenay-Saint-Cloud, diffusion Ophrys.
[Th] Thom, R., 1972, *Stabilité structurelle et Morphogénèse*, W.A. Benjamin, Inc., Reading, Massachusatts, Interédition, Paris.