Toric degenerations of complete local domains equipped with a rational valuation

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In this note we work with algebraic varieties over an algebraically closed field k. An approach to embedded resolution of singularities of an affine variety $X \subset \mathbf{A}^N(k)$ by a single toric map after a suitable re-embedding $\mathbf{A}^N(k) \subset \mathbf{A}^M(k)$ was proposed in [5]. It received serious encouragement after a 2009 Oberwolfach workshop when Jenia Tevelev proved in [7] a theorem for projective embeddings. It states that *any* embedded resolution of an irreducible $X \subset \mathbf{P}^N(k)$ is obtained by base change from an embedded resolution of $X \subset \mathbf{P}^M(k)$ by a single toric birational map $Z(\Sigma) \to \mathbf{P}^M(k)$ of non singular toric varieties through a suitable embedding $\mathbf{P}^N(k) \subset \mathbf{P}^M(k)$. Here suitable means in particular that the toric structure on $\mathbf{P}^M(k)$ is such that X meets the torus, and of course the embedding $\mathbf{P}^N(k) \subset \mathbf{P}^M(k)$ depends on the given embedded resolution.

Tevelev's result means in particular that embedded resolution by a suitable toric birational map is not only possible whenever embedded resolution is, and in particular in characteristic zero, but that it is also in some sense "universal".

How can one find suitable re-embeddings when no embedded resolution is known to exist? One possibility, going back to the local case $X \subset \mathbf{A}^N(k)$, is to try to find local re-embeddings $\mathbf{A}^N(k) \subset \mathbf{A}^M(k)$ and toric maps which will uniformize a given valuation centered at a point x of X. This means that we can find, after re-embedding, a birational toric map such that the strict transform $X' \subset Z(\Sigma)$ of X will be non singular and transversal to the toric boundary but only at the point $x' \in X'$ picked by the valuation. According to [4] the valuations which are *rational* concentrate the difficulty from this viewpoint. Rational valuations are those which are such that the inclusion $\mathcal{O}_{X,x} = R \subset R_\nu$ which determines the valuation, where R_ν is the valuation ring, satisfies $m_\nu \cap R = m$ and $R/m \simeq R_\nu/m_\nu$. Here we assume that $\mathcal{O}_{X,x} = R$ is an integral domain, which for the problems at hand is permissible.

What suggests to look at valuations is that if X is a germ of analytically irreducible curve X in $\mathbf{A}^N(k)$ and x its singular point, its local ring R has a unique valuation with value group $\mathbf{Z}$ and the semigroup of values $\Gamma = \nu(R \setminus \{0\})$ is therefore finitely generated, say that it is minimally generated by $\gamma_1, \ldots, \gamma_s$. If we choose a system of generators $\xi_1, \ldots, \xi_s$ of the maximal ideal of R having those valuations, they determine a re-embedding $X \subset \mathbf{A}^s(k)$ which has an embedded resolution by a single toric map. The reason is that in $\mathbf{A}^s(k)$ the curve X can degenerate (or specialize) in an *overweight* manner (see [6]) to the monomial curve $\mathrm{Spec}k[t^\Gamma] \subset \mathbf{A}^s(k)$ which is an affine toric variety and as such has toric embedded resolutions *in any characteristic*, some of which also resolve X (see [2]). The ring $k[t^\Gamma]$ is the associated graded ring of R with respect to the valuation.

When one tries to generalize this to higher dimension things begin well: given a rational valuation ν on R one defines a filtration of R by the valuation ideals

$$\mathcal{P}_\phi(R) = \{x \in R \setminus \{0\} | \nu(x) \geq \phi\} \cup \{0\}, \quad \mathcal{P}^+_\phi(R) = \{x \in R \setminus \{0\} | \nu(x) > \phi\} \cup \{0\},$$

and the associated graded ring $\mathrm{gr}_\nu R = \sum_{\phi\in\Gamma} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$.
Because of the properties of rational valuations, all components of this graded algebra are 1-dimensional vector spaces over k. This implies that if we take any system of homogeneous generators $(\overline{\xi}_j)_{j\in J}$ of the k-algebra $\mathrm{gr}_\nu R$, the surjective map of k-algebras

$$k[(U_j)_{j\in J}] \to \mathrm{gr}_\nu R, \quad U_j \mapsto \overline{\xi}_j$$

has a kernel generated by binomials; in fact it is isomorphic to the semigroup algebra $k[t^\Gamma]$. So we have our toric variety to which we want to degenerate, although it may be of infinite embedding dimension as we are going to see.

Here, one meets the difficulty that the semigroup $\Gamma = \nu(R \setminus \{0\})$ is not finitely generated in general. Since R is noetherian, Γ has nevertheless some important properties:
1) It is well ordered, which implies that it has a minimal system of generators $\Gamma = \langle \gamma_1, \gamma_2, \dots, \gamma_i, \dots \rangle$ where the generators $(\gamma_i)_{i\in I}$ are indexed by an ordinal I which is $\leq \omega^{\mathrm{dim} R}$ by results of Krull and Zariski.
2) By a result of Campillo-Galindo (see [1]), it is combinatorially finite, which means that the number of dinstinct ways of writing an element of Γ as a sum of other elements is finite.
3) By a result of Piltant (see [4]), even though $\mathrm{gr}_\nu R$ may not be noetherian, its Krull dimension is the rational rank of the value group Φ of the valuation.
4) In view of this, Abhyankar's inequality reads, for rational valuations: $\mathrm{dimgr}_\nu R \leq \mathrm{dim} R$.
The lack of noetherianity makes it difficult to use the valuation algebra of [4]:

$$\mathcal{A}_\nu(R) = \sum_{\phi\in\Phi} \mathcal{P}_\phi(R) v^{-\phi} \subset R[v^\Phi]$$

which describes the "natural" specialization of R to its associated graded ring; in fact in the absence of noetherianity we cannot get equations to describe this specialization.

We are going to describe this specialization in another way, assuming that the ring R is complete. The reduction to the complete case is a separate issue which is treated separately and involves an assumption of excellence on the ring R, which is satisfied in our case; see [3] and [6].

Assume now that the equicharacteristic noetherian local domain R is complete and endowed with a rational valuation with semigroup of values $\Gamma = \langle (\gamma_i)_{i\in I} \rangle$. Choose a field of representatives $k \subset R$.
Let $(u_i)_{i\in I}$ be variables indexed by the elements of the minimal system of generators $(\gamma_i)_{i\in I}$ of the semigroup Γ. Give each u_i the weight $w(u_i) = \gamma_i$ and let us consider the k-vector space of power series $\sum_{e\in E} d_e u^e$ where $(u^e)_{e\in E}$ is any set of monomials in the variables u_i and $d_e \in k$. Since Γ is combinatorially finite, for any given series the map $w\colon E \to \Gamma,\ e \mapsto w(u^e)$ has finite fibers. Each of these fibers is a finite set of monomials in variables indexed by a totally ordered set, and so can be given the lexicographical order and order-embedded into an interval $1 \leq i \leq n$ of $\mathbf{N}$. This defines an injection of the set E into $\Gamma \times \mathbf{N}$ equipped with

the lexicographical order and thus induces a total order on E, for which it is well ordered. When E is the set of all monomials, this gives a total monomial order.

The combinatorial finiteness also implies that this vector space of series is a k-algebra, which we denote by $k[\widehat{(u_i)_{i\in I}}]$.

The weight of a series is defined to be the lowest weight of its terms. The filtration by weight determines a topology on our ring. It has many nice properties, in particular of completeness with respect to this topology. We can think of it as a generalized power series ring with weights on the variables.

Theorem (The valuative Cohen theorem) 1) *There exist choices of representatives* $\xi_i \in R$ *of the* $\overline{\xi}_i$ *minimally generating the* k*-algebra* $\mathrm{gr}_\nu R$ *such that the application* $u_i \mapsto \xi_i$ *determines a surjective map of* k*-algebras*

$$\pi\colon k[\widehat{(u_i)_{i\in I}}] \longrightarrow R$$

which is continuous with respect to the topologies associated to the filtrations by weight and by valuation respectively. The associated graded map with respect to these filtrations is the surjective map

$$\mathrm{gr}_w\pi\colon k[(U_i)_{i\in I}] \longrightarrow \mathrm{gr}_\nu R \simeq k[t^\Gamma], \quad U_i \mapsto \overline{\xi}_i = \mathrm{in}_\nu \xi_i$$

whose kernel is a prime ideal generated by binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L},\ \lambda_\ell \in k^*$.
2) *There exist elements* $F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(p)>w(m^\ell)} c_p^{(\ell)} u^p$ *which, as the binomials run through a set of generators of the kernel of* $\mathrm{gr}_w\pi$*, topologically generate the kernel* F *of* π.

Now we have equations for our degeneration! If the valuation is of rank one or if Γ is finitely generated, any choice of the representatives ξ_i is allowed.

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