Sketch of a proof that Zariski dimensionality type
can be computed with linear projections, when it is $\leqslant 2$

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B. TEISSIER

Let $0 \in X_o \subset \mathbb{C}^3$ be a representative of a germ of a reduced complex analytic surface, defined by $f(x_1,x_2,x_3) = 0$, $f \in \mathcal{O}_3 \simeq \mathbb{C}\{x_1,x_2,x_3\}$. We consider projections $p : \mathbb{C}^3 \to \mathbb{C}^2$ described by :

$$(*)_N \qquad \left\{ \begin{array}{l} x_1^* = x_1 + \sum\limits_{2 \leqslant |A| \leqslant N} u_{1,A} x^A \\ \\ x_2^* = x_2 + \sum\limits_{2 \leqslant |A| \leqslant N} u_{2,A} x^A \end{array} \right. \qquad \text{for each } N \geqslant 2.$$

and the space with coordinates $(u_{1,A}, u_{2,A})$ is denoted by U_N.

Thus, we obtain a projection $p^* : \mathbb{C}^3 \times U_N \to \mathbb{C}^2 \times U_N$
$(x_1,x_2,x_3,\underline{u}) \mapsto (x_1^*,x_2^*,\underline{u})$, and its restriction π^* to $X_o \times U_N$, $\pi^* : X_o \times U_N \to \mathbb{C}^2 \times U_N$ has a discriminant which is a hypersurface $\Delta^* \subset \mathbb{C}^2 \times U_N$ containing $\{0\} \times U_N$ since $0 \in X_o$ is a singular point.

We now consider the Nash modification $\nu : \tilde{X}_o \to X_o$ of X_o, which in this case is nothing but the blowing up on X_o of the ideal generated in $\mathcal{O}_{X_o,0} \simeq \mathcal{O}_{3/(f)}$ by the three partial derivatives of f. This ideal, and therefore its blowing up, are independent of the choice of coordinates and of the generator of the ideal $f.\mathcal{O}_3$. Considering ν as this blowing-up, we have a diagram

$$\begin{array}{ccc} \tilde{X}_o & \xrightarrow{\ j\ } & \mathbb{C}^3 \times \check{\mathbb{P}}^2 \\ \downarrow \nu & & \downarrow pr_1 \\ X_o & \xrightarrow{\ i\ } & \mathbb{C}^3 \end{array} \qquad pr_1 \circ j = i \circ \nu .$$

where $\check{\mathbb{P}}^2$ denotes the projective space of 2-planes through the origin in $\mathbb{C}^3$, with homogeneous coordinates (a:b:c) such that the hypersurface corresponding to the point (a:b:c) is $ax_1 + bx_2 + cx_3 = 0$.

We have the inequality $\dim \nu^{-1}(0) \leq 1$ since $\nu^{-1}(0)$ is contained in the exceptionnal divisor of ν. Furthermore, we see that $\nu^{-1}(0) \subset \check{\mathbb{P}}^2$.

Let us consider the subset B of $\nu^{-1}(0)$ defined as follows :
A point $\tilde{x} \in \nu^{-1}(0)$ belongs to B if either $\tilde{x}$ is an isolated point of $\nu^{-1}(0)$, i.e., $\dim_{\tilde{x}} \nu^{-1}(0) = 0$, or $\dim_{\tilde{x}} \nu^{-1}(0) = 1$, but X_o is <u>not equisingular</u> along $\nu^{-1}(0)_{red}$ at $\tilde{x}$ in the following sense:

<u>Definition</u> : A reduced complex space Z of pure dimension k is said to be equisingular along a subspace $Y \subset Z$ of dimension k-1 at a point $y \in Y$ if :

1) Y is non-singular at y
2) Z is equisaturated along Y at y, i.e. the saturation $Z^s \xrightarrow{s} Z$ has the property that Z^s is locally an analytic product by Y at the point $s^{-1}(y)$.

By the theory of saturation, this is <u>equivalent</u> to saying that for a generic linear projection $Z \xrightarrow{\pi} \mathbb{C}^{k+1}$, the image of Z is a hypersurface which is Zariski-equisingular along $\pi(Y)$ at the point $\pi(y)$.

Let us denote by $L \subset \check{\mathbb{P}}^2$ the projective line composed of the points of $\check{\mathbb{P}}^2$ which represent planes in $\mathbb{C}^3$ containing the line $x_1 = x_2 = 0$.

As we see, the datum of a linear projection $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$ (in the coordinates x_1, x_2, x_3) is equivalent to the datum of such a line in $\check{\mathbb{P}}^2$. We can now state .

<u>Theorem</u> : If the line $L \subset \check{\mathbb{P}}^2$ corresponding to the projection (x_1, x_2) does not contain any point of the set $B \subset \nu^{-1}(0)$ defined above, and in addition is transversal in $\check{\mathbb{P}}^2$ to $\nu^{-1}(0)_{red}$, then, for every integer $N \geq 2$, the hypersurface $\Delta_N^* \subset \mathbb{C}^2 \times U_N$ is equisingular along $\{0\} \times U_N$ at the point $\{0\} \times \{0\}$.

Remark that B contains all the singular points of $\nu^{-1}(0)_{red}$, and is a finite set of points. The condition on L is therefore realized for a Zariski open dense subset of the set $\mathbb{P}^2$ of lines in $\check{\mathbb{P}}^2$.

<u>Proof</u> : Let us consider the Nash modification of $X_o \times U_N$; by its definition, it is naturally isomorphic to $\nu_N : \tilde{X}_o \times U_N \xrightarrow{\nu \times id_{U_N}} X_o \times U_N$ and therefore we have a diagram :

$$\begin{array}{ccc} \tilde{X}_o \times U_N & \hookleftarrow & \mathbb{C}^3 \times \check{\mathbb{P}}^2 \times U_N \\ \Big\downarrow \nu_N & & \Big\downarrow \\ X_o \times U_N & \hookleftarrow & \mathbb{C}^3 \times U_N \end{array}$$

Let us consider the hypersurface P in $\mathbb{C}^3 \times \mathbb{P}^2 \times U_N$ defined by :

$$\det \begin{vmatrix} a & b & c \\ \dfrac{\partial x_1^*}{\partial x_1} & \dfrac{\partial x_1^*}{\partial x_2} & \dfrac{\partial x_1^*}{\partial x_3} \\ \dfrac{\partial x_2^*}{\partial x_1} & \dfrac{\partial x_2^*}{\partial x_2} & \dfrac{\partial x_2^*}{\partial x_3} \end{vmatrix} = 0$$

We remark that the induced projection $P \longrightarrow U_N$ is flat and that furthermore in view of the form of the equations $(*)_N$ the fibre over 0 of this induced projection is $L \times \mathbb{C}^3 \subset \check{\mathbb{P}}^2 \times \mathbb{C}^3$. Therefore the map $P \longrightarrow U_N$ is smooth (= flat and with non-singular fibres) over a neighbourhood of 0 in U_N.

If L satisfies the conditions of the theorem, since B contains the intersection of $\nu^{-1}(0)$ with the strict transforms by ν of the components of the singular locus of X_o, we have that :

The closure of the part of the critical locus C_{π^*} of the projection $\pi^* : X_o \times U_N \longrightarrow \mathbb{C}^2 \times U_N$ which is not contained in the singular locus of $X_o \times U_N$ is the image by ν of $(\tilde{X}_o \times U_N) \cap P$. In fact if we set $C^o_{\pi^*} = \overline{C_{\pi^*} - \operatorname{sing}(X_o \times U_N)}$ we have :

$\big|$ $(\tilde{X}_o \times U_N) \cap P$ is the strict transform by ν of $C^o_{\pi^*}$.

Now I claim that $(\tilde{X}_o \times U_N) \cap P$ is equisingular along a finite number of subspaces $W_{\tilde{x}}$, each of which is locally isomorphic to U_N.

Let $\tilde{x} \in \nu^{-1}(0) \cap L$. By the assumption on L, $\tilde{X}_o$ is equisingular along $\nu^{-1}(0)_{red}$ at $\tilde{x}$. Therefore, $\tilde{X}_o \times U_N$ is equisingular along $\nu^{-1}(0)_{red} \times U_N$ at $\tilde{x} \times \{0\}$ and since we know that $L \times \mathbb{C}^3$ is transversal in $\check{\mathbb{P}}^2 \times \mathbb{C}^3$ to $\nu^{-1}(0)_{red}$ at $\tilde{x}$, we have that the hypersurface P is transversal to $\nu^{-1}(0)_{red} \times U_N$ in $\mathbb{C}^3 \times \check{\mathbb{P}}^2 \times U_N$.

In other words, if we set $W = (\nu^{-1}(0)_{red} \times U_N) \cap P$, we have that in a neighbourhood of each of the points $\tilde{x} \in L \cap \nu^{-1}(0)$, W induces a non-singular subspace $W_{\tilde{x}}$, transversal intersection of $\nu^{-1}(0)_{red} \times U_N$ with P, and furthermore the induced map $W_{\tilde{x}} \longrightarrow U_N$ is a local isomorphism at $\tilde{x}$.

From the theory of saturation, it follows that $(\tilde{X}_o \times U_N) \cap P$ is equisingular along W at each of the points $\tilde{x} \in L \cap \nu^{-1}(0)$. (Remark : if $L \cap \nu^{-1}(0) = \emptyset$, which happens if $\dim \nu^{-1}(0) = 0$, this means that $(X_o \times U_N) \cap P = \emptyset$).

In particular, $(\tilde{X}_o \times U_N) \cap P$ has a strong simultaneous resolution along W, hence $C^o_{\pi^*}$ has at least a weak simultaneous resolution. Since Δ^*_N is the union of the images of $C^o_{\pi^*}$ and $\mathrm{Sing}(X_o \times U_N)$ by π^*, we deduce that Δ^*_N has at least a weak simultaneous resolution along $\{0\} \times U_N$. In my algebraic proof of " μ constant $\Rightarrow$ equisingularité" I proved in particular that for hypersurfaces, weak simultaneous resolution <u>in codimension 1</u> implies equisingularity (and hence also strong simultaneous resolution). This completes the proof of the theorem.

To complete the proof of the fact that dimensionality type can be computed by generic linear projections, one then proceeds as follows : take $X^r \subset \mathbb{C}^{r+1}$, and $Y \subset X^r$, of codimension $\leqslant 2$. Take a linear projection $X \longrightarrow \mathbb{C}^r$ which is "generic" in the sense of the theorem above with respect to all the slices of X by some foliation of $\mathbb{C}^{r+1}$ by 3-planes which are transversal to Y, and move it as above to obtain a truly generic projection. By the Theorem, we obtain a family of discriminants (in which the image of Y is of codim 1) the slices of which by 2-planes on $\mathbb{C}^r$ transversal to the image of Y are equisingular. The result now follows from the fact that in codimension 1, Zariski equisingularity can be tested by looking at slices (= considering the total space as a family of curves).

Some steps towards the end of the proof in arbitrary codimension : Let $X^r \subset \mathbb{C}^N$; if we no longer assume $N = r+1$ we can still define the Zariski dimensionality type of X at $x \in X$ as follows : Consider $r+1$ series $x_i^* = \sum_{0<|A|} v_{iA} X^A$ where $X = (X_1, \ldots, X_N)$ and $1 \leqslant i \leqslant r+1$; Set $k^* = \mathbb{C}(v_{i,A})$. Define $dt_{\mathbb{C}}(X,x) = d.t_{k^*}(X^*, x^*)$ where $X^* \subset A^{r+1}_{k^*}$ is the image of X by the map $\mathbb{A}^N_{k^*} \longrightarrow A^{r+1}_{k^*}$ defined by the x_i.

Then hopefully one can prove the existence of a stratification of X by dimensionality type, along the lines of Zariski.

Now we consider $X \subset \mathbb{C}^{r+1}$ and the Nash modification $\nu : \tilde{X} \dashrightarrow X$. We look at the decomposition $\nu^{-1}(0) = \cup A_i$ where $A_i = (\tilde{X})_i \cap \nu^{-1}(0)$ and $\cup (\tilde{X})_i$ is the stratification of $\tilde{X}$ by dimensionality type.

Given $Y \subset X$, we can also consider the partition $\nu^{-1}(Y) = \cup S_i$ defined in the same manner. It is natural to <u>conjecture</u> that Zariski equisingularity implies the following :

1) $\tilde{X}$ is equisingular along each of the S_i
2) The natural morphism $S_i \dashrightarrow Y$ is a submersion.

Now if this is true in any dimensions we can proceed as follows : Let again $X_o^r \subset \mathbb{C}^{r+1}$ and consider projections described by

$$x_i^* = x_i + \sum_{2 \leq |A| \leq N} u_{i,A} x^A \qquad (1 < i \leq r).$$

Let U_N be the affine space with coordinates $(u_{i,A})$ and

$$\begin{array}{ccc} \tilde{X}_o \times U_N & \hookrightarrow & \mathbb{C}^{r+1} \times \check{\mathbb{P}}^r \times U_N \\ \downarrow \nu & & \\ X_o \times U_N & & \end{array}$$

where $\check{\mathbb{P}}^r$ is the projective space with coordinates $(a_1 : \ldots : a_{r+1})$ such that the corresponding hyperplanes $\Sigma a_i x_i = 0$.

Let $P \subset \mathbb{C}^{r+1} \times \mathbb{P}^r \times U_N$ be the hypersurface defined by

$$\det \begin{vmatrix} a_1, \ldots\ldots\ldots\ldots, & a_{r+1} \\ \dfrac{\partial x_1^*}{\partial x_1}, \ldots\ldots\ldots\ldots, & \dfrac{\partial x_1^*}{\partial x_{r+1}} \\ \vdots & \vdots \\ \dfrac{\partial x_r^*}{\partial x_1} & \dfrac{\partial x_r^*}{\partial x_{r+1}} \end{vmatrix} = 0$$

and choose the direction of projection $\mathbb{C}^{r+1} \longrightarrow \mathbb{C}^r$ $(x_1,\ldots,x_{r+1}) \longmapsto (x_1,\ldots,x_r)$ in such a way that the corresponding hypersurface $L \subset \check{\mathbb{P}}^r$ is transversal to all the strata $A_i \subset \nu^{-1}(0)$. Such a linear projection we call strongly transversal.

Then if X is Zariski equisingular along Y, L is also transversal to the S_i and therefore the induced map $P \cap (S_i \times U_N) \longrightarrow U_N \times Y$ is a submersion, having fibre over 0 equal to $L \cap (A_i)$.

Since $\check{X}_0 \times U_N$ is "equisingular" along $S_i \times U_N$, we have that $(X_0 \times U_N) \cap P$ is "equisingular" along $(S_i \times U_N) \cap P$ and therefore the discriminant $\Delta^* \subset \mathbb{C}^r \times U_N$ is equisingular along $Y \times U_N$ at least in the sense of weak simultaneous resolution.

The difficulty which remains here is to prove that this discriminant has in fact a strong simultaneous resolution. Essentially this amounts to showing that the map $C^* \xrightarrow{\pi^*} \Delta^*$ from the critical locus to the discriminant $\Delta^* \subset \mathbb{C}^r \times U_N$ is a "generic projection" for C^* in the sense that the image is equisingular (in the sense of hypersurfaces, i.e. Zariski's sense) with the image of the generic projection of C^*. More precisely if we consider a projection $\pi^* : \mathbb{C}^{r+1} \times U_N \times \mathbb{C} \longrightarrow \mathbb{C}^r \times U_N \times \mathbb{C}$ such that $\pi^* \,|\mathbb{C}^{r+1} \times U_N \times \{0\} = \pi^*$, then we want the image by π^* of $C^* \times \mathbb{C}$ to be equisingular along $0 \times 0 \times \mathbb{C}$, at $0 \times 0 \times 0$.

If we can prove that the equisingularity of X along Y at 0 implies the equisingularity of Δ^* along $Y \times U_N$, and also that the equisingularity of X along Y in the sense of linear projections implies the equisingularity of Δ^* along $Y \times U_N$, then we have proved that equisingularity in the sense of Zariski and equisingularity in the sense of linear projections coincide.

There is also a sketch of proof by similar methods of the following fact. Let $X \subset \mathbb{C}^{r+1}$ and $Y \subset X$, and assume that Y is non singular at $0 \in Y$. Let us choose a local retraction $r : \mathbb{C}^{r+1} \longrightarrow Y$ at 0, and a system of coordinates $z_1,\ldots,z_{r+1-d}$ on $r^{-1}(0)$ at 0. Then we can identify $\mathbb{C}^{r+1}$ with $Y \times \mathbb{C}^{r+1-d}$, 0 and we may restrict our choice of projections to those which are of the form : $Y \times \mathbb{C}^{r+1-d} \xrightarrow{id_Y \times \pi_0} Y \times \mathbb{C}^{r-d}$ (and iterate this).

That is, we consider only those projections which are compatible with a product decomposition $\mathbb{C}^{r+1} \simeq Y \times \mathbb{C}^{r+1-d}$.

Now assume that X is equisingular along Y at 0 in the sense of linear projections. Then, supposing that we can prove that this implies that the stratification of $\nu^{-1}(Y)$ is smooth over Y, we prove by the same method as above that the discriminant of $id_Y \times \pi_0$ is equisingular along $Y \times \{0\}$ at 0, that is, X is also equisingular in the sense of restricted linear projections. The converse should be proved by essentially the same method again ; the idea is that if there are many linear projections with respect to which the discriminant is equisingular, there are many hyperplanes transversal to the equisingularity strata of $\nu^{-1}\{y\}$ and that should suffice to show that these equisingularity strata specialize to equisingularity strata in $\nu^{-1}(0)$. Same idea for Zariski equisingularity, using the fact that to compute the dimensionality type along $\nu^{-1}(0)$ of $\tilde{X}_0$ we may extend the base field to $k^* = k(u_{i,s})$ and then our projection is linear in the coordinates x_i^*. (Here we have to use the fact that $\nu^{-1}(0)$ is independent of the choice of coordinates. There must be a short and elegant argument).