GAEL XIII

Geometric Aspects of Valuation Theory

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Abstract

These notes [1] were taken during the mini-lecture series of the thirteenth edition of GAEL which was held in Luminy, France, from March 21st to March 25th. Professor Teissier made an introduction to the geometry of valuations and presented his latest results concerning the problem of local uniformization ([5]).

Lecture 1

1 Introduction

Valuation theory largely fell in disgrace among algebraic geometers after Hironaka's 1964 proof of resolution of singularities of a field of characteristic zero ([3]). His proof made no use of valuation theory and thus departed from the approach which had been advocated by Zariski. The lack of success of the attempts to adapt Hironaka's techniques to positive characteristic generated a revival of valuation theory about 10 years ago. Before proceeding to modern valuation theory, let us first go back in time to look at its history and evolution.

2 History

2.1 Dedekind and Weber

In 1882, Dedekind and Weber ([7]) introduced *places* which as we shall see are the same thing as valuations in order to construct algebraically the Riemann

[1] For any remark, contact Jimmy Dillies : `jimmyd@math.upenn.edu`

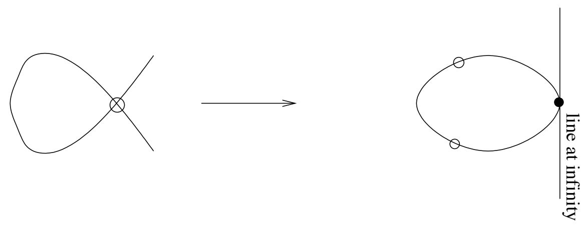


Figure 1: Resolving singularities and adding points at infinity

surface associated to an affine curve over $\mathbb{C}$ (Riemann's construction was topological). The purpose was to construct the Riemann surface from the field of functions of the curve. If C is an irreducible affine curve which is the zero locus of the irreducible polynomial $F(x, y) \in \mathbb{C}[x, y]$, the ring of regular functions on C is defined as

$$A = \mathbb{C}[x, y]/F(x, y);$$

it is an integral $\mathbb{C}$-algebra. The field of rational functions on C is the field $K = \text{Frac}A$ of fractions of A. Note that it is an extension of $\mathbb{C}$; the elements of $\mathbb{C}$ are the constant functions on the curve C. The aim is to construct a one dimensional compact complex manifold (a Riemann surface) with field of fraction K.

Idea: Think of a point $x \in C$ as a map $ev_x : K \longrightarrow \mathbb{C} \cup \{\infty\} : \frac{f}{g} \mapsto \frac{f(x)}{g(x)}$ where $f, g \in A$. Note that here our base field is $\mathbb{C} \hookrightarrow K$.
Define $V_x = \{h \in K | ev_x(h) \text{ is finite}\}$

Definition Given a field extension $k \hookrightarrow K$, a k- place $\mathcal{P}$ of the field K with values in a field L which is also an extension of k is the datum of a subring $V_{\mathcal{P}} \subset K$ together with a homomorphism $\mathcal{P} : V_{\mathcal{P}} \to L$ such that

- If $x \notin V_{\mathcal{P}}$ then $1/x \in V_{\mathcal{P}}$ and $\mathcal{P}(1/x) = 0$.
- There exists $x \in V_{\mathcal{P}}$ such that $\mathcal{P}(x) \neq 0$.
- For all $c \in k$, $\mathcal{P}(c) = c$.

Example Let C be an algebraic curve in $\mathbb{C}^2$

1. Any nonsingular point of C gives a place of $K = \mathbb{C}(C)$

2. A singular point gives rise to several places – one per branch (locally analytically irreducible component at the singular point). This corresponds to the fact that if we take a sequence of non singular points $x_i \in C$ tending to the singular point x, the limit value of $\frac{f(x_i)}{g(x_i)}$ exists if all the x_i are ultimately on the same branch, and then it depends on the branch.

3. Points at infinity also give places (which might be singular or not depending on the behaviour of the curve at ∞.) Consider the limit of ev_x as $x \to \infty$.

If we remember that the normalization $n: \overline{X} \to X$ of an integral algebraic curve X is a finite birational map which has the property that $\overline{X}$ is non singular and for each $x \in X$ there is a bijection between $|n^{-1}(x)|$ and the set of branches of X at the point x, we see that all the ideas encompassing the previous example are actually contained in the following theorem :

Theorem 2.1 *(Dedekind & Weber) Denoting by X the projective closure of an affine algebraic curve C with algebra A, and by $\overline{X}$ its normalization, the map $x \mapsto ev_x(\frac{f}{g})$, where $\frac{f}{g}$ is viewed as a rational function on $\overline{X}$, defines a natural bijection*

$$\{\text{Points of } \overline{X}\} \overset{bij}{\longleftrightarrow} \{\mathbb{C}\text{-places of } K = \text{Frac}(A)\}$$

The point is that considering places simultaneously adds the points at infinity and desingularizes; it gives the set-theoretically the Riemann surface. Of course there remains to define its topology.

2.2 Hensel

In 1897, Hensel ([2]) defined p-adic valuation [2] on the field of rationals.

$$\nu_p : \mathbb{Q}^* \to \mathbb{Z} : q = p^n \frac{a}{b} \mapsto n$$

where $(a, p) = (b, p) = 1$. It is easy to check that this is well defined. The valuation verifies the following properties :

1. $\nu_p(xy) = \nu_p(x) + \nu_p(y)$

2. $\nu_p(x + y) \geq \min(\nu_p(x), \nu_p(y))$ (in case $\nu_p(x) \neq \nu_p(y)$ we have an equality.)

[2]Stricto sensu, he defined the absolute value $\|q\| = p^{-\nu_p(q)}$

3. $\nu_p(0) = +\infty$

Hensel's motivation was to apply ... Hensel's lemma. The aim was to study diophantine equations by looking at their solutions in the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ and Hensel's lemma is an adapted version of the implicit function theorem which allows one to give fairly efficient criteria for the existence of solutions in $\mathbb{Q}_p$. The approach was so successful that it has changed the course of algebraic number theory and has remained a very active subject up to these days.

2.3 The parallel

In the case of the algebraic curve C we can associate to each non-singular point x a valuation ν_x which assigns to each rational function f/g its order of vanishing at x, $\nu_x(f/g) = \nu_x(f) - \nu_x(g)$.
This is in fact among the first historical parallels between function fields and number fields :

$$\begin{array}{ccc} \{\, p: \text{ primes } \} & \longleftrightarrow & \{\, x: \text{ non singular points } \} \\ \{\nu_p\} & \longleftrightarrow & \{\nu_x\} \end{array}$$

This analogy suggested in particular to take the $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{O}}_{C,x}$ and later the henselization, and opened vast new fields in commutative algebra.

3 Valuation rings

Definition A *valuation ring* [3], is an integral domain V such that if K is its field of fraction and we have $x \in K \setminus V$ then $x^{=1} \in V$.

This is equivalent to saying that given any two nonzero elements $a, b \in V$, either $a|b$ or $b|a$ in V, or also that any finitely generated ideal $I = (a_1, \ldots, a_k)$ of V is principal and generated by one of the a_i.

Example Let $x \in C$ be a non-singular point of an algebraic curve : the local ring $\mathcal{O}_{C,x}$ is a valuation ring. More generally, any one dimensional regular local ring is a valuation ring.

[3] For a general reference, consult [6]

Example Integers localized at a prime p form a valuation ring : $\mathbb{Z}_p = \{a/b \mid (b,p) = 1\}$.

Exercise Show that a valuation ring is a local ring.

Exercise (The parallel between Dedekind-Weber and Hensel)
Given an extension $k \to K$ of fields , show that there is a bijection

$$\{ \text{ Valuation ring in K s.t. } \mathfrak{m}_V \cap k = \{0\} \}$$
$$\updownarrow$$
$$\{ \text{ Places of K with values in extension L of the basefield } k \text{ and constant on } k\}/$$

Here the equivalence means that we identify two places if they differ only by composition by an injection $L \hookrightarrow L'$. Modulo this equivalence we can always assume that L is the residue field $V/\mathfrak{m}_V$.

The map is given by $\mathcal{P} \mapsto V_{\mathcal{P}} \subset K$.

3.1 Pairing valuation rings and valuations

Let V be a valuation ring, we define a pre-order on V :

$$a \leq b \Leftrightarrow (a,b)V = (a)V \Leftrightarrow a|b$$

Remark that it is only a pre-order as $(a \leq b) \wedge (b \leq a)$ does not imply that $a = b$; it merely says that they are related by an invertible element of V. That is, there exists some multiplicative unit $u \in U_V \subset K^*$ such that $a = ub$. Naturally, the preorder on V gives rise to a total order on the group $\Phi = K^*/U_V$. In turn, we get a map

$$K^* \xrightarrow{\nu} \Phi$$

(First extend to V/U_V, then symmetrize and embed in K^*)

Theorem 3.1 *The function ν is a valuation.*

The order on Φ permits us to decompose it into $\Phi_- \cup \{0\} \cup \Phi_+$. Define $R_V = \nu^{-1}(\{0\} \cup \Phi_+)$ it is a valuation ring of maximal ideal $\mathfrak{m}_V =$
$nu^{-1}(\Phi_+)$.

Example Take $\Phi = \mathbb{Z}^2_{\text{lex}}$ the set of bi-integers lexicographically ordered. The set $\Phi_+ = (a,b) \in \mathbb{Z}^2 | a = 0$ and $b > 0$ or $a > 0$ is displayed on the picture.

One can check that Φ_+ is neither finitely generated nor well ordered.

Consider now the tower of subrings of $\mathbb{C}(x, y)$:

$$\mathbb{C}[x,y]_{\mathfrak{m}} \subset \ldots \subset \underbrace{\mathbb{C}[x, \frac{y}{x^n}]_{\mathfrak{m}}}_{R_n} \subset \ldots$$

Each of those rings is isomorphic to a polynomial ring in 2 variables, localized at the origin.

Claim: The $\bigcup_{n\geq 1} R_n$ is a valuation ring with value group $\mathbb{Z}^2_{\text{lex}}$. Regardless the fact that this example is not Noetherian, it is an important valuation.

In general, a Noetherian valuation ring is a one dimensional regular local ring because if it is noetherian, its maximal ideal is principal.

4 Zariski-Riemann "Manifold"

We work with an algebraic variety X/k. Consider the extension $K = k(X)$ of k, where $k(X)$ is the field of rational functions on X. We want to understand the valuations of K, that is the valuation rings V where $k \subset V \subset Tot(V) = K = k(X)$ which are such that the valuation is trivial on k (in other words $k \cap \mathfrak{m}_V = \{0\}$).

Lemma 4.0.1 *Given a valuation ring V, the set of points of X such that $\mathcal{O}_{X,x} \subset V$ and $\mathfrak{m}_V \cap \mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ is a closed irreducible subvariety of X. It is called the center of the valuation on X.*

Note that the center may be empty; this is the case if X is an affine algebraic curve and the valuation corresponds to a place at infinity. If the center is not empty, it corresponds to a prime ideal p is some affine chart Spec$A$ where $A \subset k(X)$ and if R_ν is the ring of ν we have that $A \subset R_\nu$ and $\mathfrak{m}_V \cap A = p$. The following result generalizes what we saw about curves and places:

Theorem 4.1 *(Valuative criterion of properness, first version) X is proper over k (that is compact over the complex field) if and only if every valuation of $k(X)$ has a non-empty center on X.*

Proof See [1]

Now assume X is proper over k; we look at all the proper birational maps $X' \longrightarrow X$ defined over k. (think in terms of Schemes; a point of X is a closed subvariety of X, irreducible over k,)

$$\begin{array}{rcl} Define a map : \{ \text{ Valuations } k(X)/k \, \} & \rightsquigarrow & \varprojlim_{X' \to X} X' \\ \nu & \mapsto & \{ \text{ center of} \nu \text{ on } X' \}_{X'} \end{array}$$

The right-hand side is clearly a projective system of (schema-theoretic) points indexed by birational proper maps $X' \to X$.

Theorem 4.2 *(Zariski) This map is a bijection*

When X is a curve we have $\bar{X} \to X' \to X$ (the birational maps above X are finite as they are proper) and $\bar{X}$ is the normalization of X!

Lecture 2

In last lecture I forgot to mention the

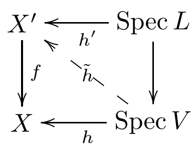
Theorem 4.3 *(Ostrowski) The only valuations on $\mathbb{Q}$ are the trivial one, the absolute value and the p-adic valuations, for all positive primes p.*

Further on we have the analogy, for $x \in \mathbb{Q}$ and $f \in k(X)$ with k algebraically closed

$$\prod_{\text{Places of } \mathbb{Q}} |x|_{\mathcal{P}} = 1 \longleftrightarrow \sum_{\text{Places of } k(X)} \nu_{\mathcal{P}}(f) = 0$$

where the right hand side can be read as the fundamental theorem of algebra if one considers the degree of a polynomial as the order of its pole at infinity.

Behind Zariski's theorem lies the valuative criterion for properness, which, although essentially known in Zariski's era, was only christened by Grothendieck. Let V be a k-valuation ring and L be its field of fractions



Theorem 4.4 (Valuative criterion of properness, second version)

The map f is proper if and only if for any such diagram there exists a lifting $\tilde{h} : \operatorname{Spec} V \to X'$.

Moreover, if the map is separated then the lifting is unique. [The idea is that if the map is not proper, some points are missing in the fibre and there is no center for a valuation "upstairs"]. As a consequence, since we consider birational maps and valuations of the field $k(X)$ of rational functions on X, the map from the Zariski-Riemann manifold $\mathcal{X}$ to its manifold X factors uniquely through any X' dominating X by a proper and birational map. In other words, the valuations of X' are the same as those of X.

Example One should consult the notes of Favre and Jonsson ([4]) regarding the fiber over the origin of Spec$R$ of its Zariski-Riemann manifold, in the case where $R = \mathbb{C}[[x, y]]$.

The Zariski topology was introduced by Zariski precisely to endow the Zariski-Riemann manifold with a topology for which it is quasi-compact. If X is an algebraic variety over a field k, the Zariski (k-)topology has as a basis of open sets the complements of algebraic subvarieties defined over k and X is quasi-compact for that topology. Then the Zariski-Riemann manifold viewed as a limit of birational (proper) maps can be given the projective limit topology, the coarsest for which all the projection maps are continuous. It is then also quasi-compact. In general, if when we have the Zariski-Riemann manifold dominating $\mathcal{X} \xrightarrow{\pi} X$, the points of $\mathcal{X}$ are valuations and $\mathcal{X}$ can be given the structure of a ringed space with local rings which are the associated valuation rings :

$$\mathcal{O}_{\mathcal{X},\nu} = R_\nu$$

Given an algebraic variety X, the map $\mathcal{X} \to X$ from the Zariski-Riemann manifold to X is in some sense "the" resolution of the singularities of X but, $\mathcal{X}$ is, alas, very large; as soon as X is of dimension > 1, it is not an algebraic variety .

Therefore it is a fundamental problem to get a grip on rings R' :

$$R = \mathcal{O}_{X,\pi(\nu)} \subset R' \subset R_\nu$$

which are local, regular and essentially of finite type over R (i.e. $R' = R[a_1, \ldots, a_n]_p$); this is the problem of *local uniformization* which was solved by Zariski for characteristic 0 in 1944 ([8]).

We recall the following definitions :

Definition The *rational rank* of an abelian group Φ is the dimension of $\Phi \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space. We will write $\mathrm{r.rk}(\Phi)$. If the valuation ν has group of values Φ we shall also say that ν has rational rank $\mathrm{r.rk}(\Phi)$ and write $\mathrm{r.rk}\nu$.

Definition A subgroup $\Psi \subset \Phi$ is *convex* if and only if Φ/Ψ can be ordered in such a way that the canonical surjection from Φ is monotonous. This is equivalent to saying that if we have $x, y \in \Phi$ and $0 \leq y \leq x$ and $x \in \Psi$, then $y \in \Psi$.

Example In the example with $\mathbb{Z}^2_{\text{lex}}$ we can take $\Psi = \{(0, b)\}$. Note that this is the only possibility.

Any totally ordered abelian group with no nontrivial convex subgroup embeds in $\mathbb{R}$ as an ordered group. It is not difficult to see that that the convex subgroups of a totally ordered abelian group form a totally ordered sequence for inclusion. Moreover if R_ν is a valuation ring with value group Φ, the correspondance between convex subgroups of Φ and ideals $m \subset R_\nu$ given by:

$$\Psi \mapsto \{x \in R_\nu/\nu(x) \notin \Psi\}$$

is a bijection between the set of convex subgroups of Φ and the set of prime ideals or R_ν.

Definition The *rank* (or *height*) of Φ is the length (the ordinal, if it is infinite) of the sequence (with respect to inclusion) of all convex subgroups of Φ; it is also the length of the sequence (with respect to inclusion) of the prime ideals $\mathfrak{m}_{\nu_i}$ of R_ν that is, the Krull dimension of R_ν. It is also the length of the maximal sequence of valuation rings containing R_ν : $R_\nu \subset R_{\nu_1} \subset \cdots \subset R_{\nu_n}$, or $\mathfrak{m}_{\nu_n} \subset \cdots \subset \mathfrak{m}_{\nu_1} \subset \mathfrak{m}_\nu$. (we write R_{ν_i} for $R_{\mathfrak{m}_{\nu_i}}$) where $\mathfrak{m}_{\nu_i}$ is the i-th prime ideal of R_ν. We will write $\mathrm{rk}(\Phi)$ or $\mathrm{rk}(\nu)$.

In the example with $\mathbb{Z}^2_{\text{lex}}$, we have a surjective monotonous map $\mathbb{Z}^2_{\text{lex}} \to \mathbb{Z} : (a, b) \mapsto a$. We have $R_\nu = \bigcup k[x, \frac{y}{x^n}]_{\mathfrak{m}} \subset k(x)[y]_{\mathfrak{m}} = R_{\nu_1}$. Note that $x \notin yR_{\nu_1} \cap R_\nu$ (we use the same notation as in the definition of the example) and that therefore there is no domination between the two valuation rings. This is expected since any valuation ring is a maximal local ring in its field of fractions with respect to the domination relation.

Theorem 4.5 *A valuation of rank 1 takes values is a subgroup $\Phi \subset \mathbb{R}$ (up to ordered isomorphism of the value group).*

Theorem 4.6 *(Abhyankar's inequality) If R be a Noetherian ring and $R \subset R_\nu \subset K$, with K the field of fractions of R, then*

$$\mathrm{r.rk}(\Phi) + (\mathrm{rk}\Phi) \leq \dim R$$

In the case of equality we call the valuation an Abhyankar valuation.

5 The Strategy for local uniformization

The main idea is to make the (compplicated) valuations on a noetherian ring appear as deformations of simple valuations on a non-noetherian ring. This is best explained by examples:

Example Let $\Gamma = \langle \gamma_1, \cdots, \gamma_{g+1} \rangle \subset \mathbb{N}$ where $\langle A \rangle$ means the semi-group of all non negative integral linear combinations of the elements of A, and where the γ_i are coprime, ordered in accordance to their indices. Assume that no γ_i belongs to the semigroup generated by the previous ones; this means that the γ_i form a minimal system of generators of $\langle \gamma_1, \cdots, \gamma_{g+1} \rangle$.

Example Let $(s_i)_{i \geq 1}$ be a sequence of positive integers such that $s_i \geq 2$ for $i \geq 2$. We can define the family of rational numbers : $\gamma_1 = s_1^{-1}$, $\gamma_{i+1} = s_i \gamma_i + \frac{1}{s_1 \cdots s_{i+1}}$. Take $\Gamma = \langle \gamma_1, \cdots, \gamma_i, \cdots \rangle \subset \mathbb{Q}_+$. If $s_i := i$ then the group generated by Γ is $\mathbb{Q}$.

Lecture 3

To such semigroups we can associate geometric objects, the spectra of their semigroup algeras over an algebraically closed field k. The Krull dimension of the semigroup algebra is equal to the rational rank $\dim_{\mathbb{Q}}(\Phi \otimes_{\mathbb{Z}} \mathbb{Q})$ of the group Φ generated by Γ.
In our case this rank is equal to one, so that our semigroup algebras correspond to curves. In the first example, we have a mononial curve in $\mathbb{A}^{g+1}(k)$:

$$u_i = t^{\gamma_i},\ 1 \leq i \leq g+1,\ \gamma_i \in \mathbb{N}$$

In the second example, we can also consider we are working with a monomial curve $u_i = t^{\gamma_i}$, but embedded in an infinite dimensional space.
We are now going to deform these rings, and for that we need equations for them.

These equations correspond to relations with integral coefficients between the generators γ_i. The equations defining the monomial curve of example 1, that is, the relations between the γ_i, may be fairly complicated. We shall make the following simplifying assumptions :

1. If e_i is the greatest common divisor of $(\gamma_1, \cdots, \gamma_i)$ and if we write $e_i = s_{i+1}e_{i+1}$, then for $1 \leq i \leq g$

$$s_{i+1}\gamma_{i+1} \in <\gamma_1, \cdots, \gamma_i>$$

2. $s_i\gamma_i < \gamma_{i+1}$ for $2 \leq i \leq g$.

Then the relations are generated by the following g expressions of the first condition :
$s_{i+1}\gamma_{i+1} = \sum_{k=1}^{i} \ell_k^{i+1}\gamma_k$, with $\ell_k^{i+1} \in \mathbb{N}$. These relations are not uniquely determined but in view of the first condition there is a unique way of writing each relation satisfying the condition that $\ell_k^{i+1} < s_k$ for $2 \leq k \leq i$. The first condition implies that in the special case considered, the monomial curve is a complete intersection with equations

$$u_{i+1}^{s_{i+1}} - \prod_{k=1}^{i} u_k^{\ell_k^{i+1}} = 0,\ 1 \leq i \leq g.$$

In the second example, it is not difficult to see, using the fact that $(\gamma_1, \cdots, \gamma_i)$ are in the subgroup of $\mathbb{Q}$ consisting of rational numbers which can be written with denominator $s_1, \cdots, s_i$, that all relations are generated by the :

$$s_{i+1}\gamma_{i+1} = \sum_{k=1}^{i} \ell_k^{i+1}\gamma_k, \text{ with } \ell_k^{i+1} \in \mathbb{N}, i \geq 1,$$

so that the equations of our monomial curve are

$$u_{i+1}^{s_{i+1}} - \prod_{k=1}^{i} u_k^{\ell_k^{i+1}} = 0,\ 1 \leq i.$$

All these equations are binomial equations defining irreducible varieties in a possibly infinite dimensional affine space. They are (non normal) toric varieties.

Les us now remark that in both examples we have $\gamma_{i+2} > s_{i+1}\gamma_{i+1}$ and let us deform the equations in the following manner: in the first example we consider a variable v and the equations

$$u_{i+1}^{s_{i+1}}-\prod_{k=1}^{i}u_k^{\ell_k^{i+1}}-vu_{i+2} = 0,\ 1\leq i\leq g-1 \quad (1)$$

$$u_{g+1}^{s_{g+1}}-\prod_{k=1}^{g}u_k^{\ell_k^{g+1}} = 0 \quad (2)$$

In the second example, we introduce a variable v_i, for each index $i\geq 2$ and consider for $i\geq 1$ the equations

$$u_{i+1}^{s_{i+1}}-\prod_{k=1}^{i}u_k^{\ell_k^{i+1}}-v_{i+1}u_{i+2}=0$$

In both cases we have an obvious elimination process in the polynomial ring $k[(v_j^{\pm 1};(u_i)]$. In the first example, setting $v=1$, the result is an isomorphism $R=k[u_1,u_2]/(F)=k[u_1,\cdots,u_{g+1}]/(u_2^{s_2}-u_1^{\ell_1^(2)}-u_3,u_3^{s_3}-u_1^{\ell_1^3}u_2^{\ell_2^3}-u_4,\cdots)$, where $F(u_1,u_2)$ is the result of the elimination; for example if $\Gamma=<4,6,13>$, the equations of the monomial curve are

$$u_2^2-u_1^3 = 0 \quad (3)$$

$$u_3^2-u_1^5u_2 = 0 \quad (4)$$

and since the deformation affects only the first equation and is $u_2^2-u_1^3-vu_3=0$, we find

$$F(u_1,u_2)=(u_2^2-u_1^3)^2-u_1^5u_2=0.$$

and for the second example, stting all $v_j=1$:

$$R=k[u_1,u_2]=k[u_1,\cdots,u_i,\cdots]/(u_2^{s_2}-u_1^{\ell_1^(2)}-u_3,u_3^{s_3}-u_1^{\ell_1^3}u_2^{\ell_2^3}-u_4,\cdots)$$

In both cases, giving to the variable u_i the weight γ_i determines a valuation on the ring and the isomorphism gives a way to compute it: in the right hand side the value of a polynomial $P(u_1,u_2)$ rewritten replacing systematically each $u_i^{s_i}$ by $\prod_{k<i}u_k^{\ell_k^i}+u_{k+1}$ is the minimum of the values (i.e.,weights) of its monomials, since now there can be no more cancellation because no $u_i^{s_i}$ appears, and this determines a valuation of R with semigroup Γ.
Giving the variable u_i the weight γ_i determines a monomial order on the polynomial ring $k[(u_i)]$, and therefore a filtration by the minimal order of the monomials in a polynomial. Each of the equations which we have created by deformation

has an initial form with respect to this filtration which is precisely the binomial equation which we have deformed.

In fact, we have a faithfully flat family parametrized by $k[(v_j)]$ specializing the ring R to the ring of the monomial curve defined by the initial binomial equations. The equations of R play the role of a standard, or Gröbner, basis with respect to the monomial order.

Since we have seen that the Krull dimension of that ring is equal to one, the second example contradicts the semicontinuity of fiber dimensions in a family.

This is due to what I call the *abyssal phenomenon.* Let us write out the system of equations appearing in the second example:

$$\begin{aligned} u_2^{s_2} - u_1^{l_1^2} - v_2u_3 &= 0 & (5)\\ u_3^{s_3} - u_1^{l_1^3}u_2^{l_2^3} - v_3u_4 &= 0 & (6)\\ \vdots\quad &\quad \vdots & (7)\\ u_{i+1}^{s_{i+1}} - \prod_{k=1}^{i} u_k^{l_k^{i+1}} - v_{i+1}u_{i+2} &= 0 & (8)\\ \vdots\quad &\quad \vdots & (9) \end{aligned}$$

When all $v_i \neq 0$, say equal to one, it amounts actually to an endless sequence of substitutions and therfore it cannot decrease the dimension, while when we specialize to the monomial curve, making all $v_{i+1} = 0$, we obtain equations which also express all $u_i, i \geq 3$ algebraically in terms of (u_1, u_2), but now u_1 and u_2 are algebraically dependent so that the dimension drops to 1.
However if k is of characteristic zero we can also view these equations as defining a very transcendental curve in $\mathbb{A}^2(k)$ whose Zariski closure, which is all our equations see, is the entire affine plane.

To see this, use the order on the γ_j to order the equations $u_j^{s_j} - \cdots$ as we did above, and for a given $n \in \mathbb{N}$ truncate the system at order n in the folowing sense: keep all equations of index $< n$, replace the equation of index n by its initial form, which involves only variables of index $\leq n$, and forget all the other equations. We are now reduced to the case of example 1 except that we have

to multiply $\gamma_1, \cdots, \gamma_n$ by their common denominator, obtaining a sequence of coprime integers.
If k is of characteristic zero we can solve the corresponding equation $F_n(u_1, u_2) = 0$ by a Puiseux expansion

$$u_2 = \sum_{j=1}^{\infty} a_j^{(n)} u_1^{\frac{j}{s_2 \cdots s_n}}$$

Using the Smith-Zariski formula for the intervsection number of parmetrized curves, one can show that as n increases these Puiseux expansions converge in the ring $k[[t^{\mathbb{Q}_+}]]$ of series with well ordered sets of exponents to a series $w(u_1)$ of fractional powers of u_1 whose exponents have unbounded denominators. Because of Newton-Puiseux, setting $u_2 = w(u_1)$ can not cause the vanishing of a polynomial (or even a power series) in (u_1, u_2). It parametrizes a transcendental curve whose Zariski closure is $\mathbb{A}^2(k)$.
We have produced it as a deformation of an algebraic, indeed toric, curve of infinite embedding dimension.

This curve defines a valuation on $k[u_1, u_2]$ by taking, for any polynomial $P(u_1, u_2)$, the order in u_1 of $P(u_1, w(u_1))$, which is finite as we just saw. It is of the same valuation as that which is obtained as explained above, as one verifies with a little work.
One of the morals of this story is that although algebraic equations do not "see" this very transcendental curve, it is visible in algebraic geometry *as a valuation.* This is a very general fact.

Now, what is the use of this for local uniformization?

Remember that we are interested in building a regular local ring R' between R and R_ν, which is essentially of finite type over R (i.e. the problem of local uniformization). The basic idea is that over an algebraically closed field of any characteristic it is not difficult to resolve by toric maps the singularities of an irreducible variety (of finite embedding dimension) defined by binomials. In the case of the first example, that toric resolution will also resolve the singularities of the plane curve obtained by elimination, viewed as embedded in affine $g+1$-space; this is due to the fact that the deformation adds only terms of higher weight than the initial binomial equation.

The second example is not so convincing as far as local uniformazaiton is concerned since the ring is regular, and also because there is no resolution in the usual sense of a space defined by infinitely many binomial equations. But we can make it more complicated as follows: in some of the equations instead of adding a linear term u_{j+2}, we can add with v_{i+2} a series in the variables of index $\leq j+2$ of weight greater that the weight of the initial binomial and were no two terms have the same weight, and thus manufacture a singular ring, which may be of dimension > 2. We can also add u_{j+1} instead of u_3, u_{j+2} instead of u_4 and so on, to deform our curve into a polynomial ring $k[u_1, \cdots, u_j]$ in j variables. When we grow tired, we go back to adding linear terms as above, which we must do anyway if we want the result to be noetherian. The resulting ring still specializes to the ring of the monomial curve which admits as equations the initial forms, and therefore a toric resolution of the binomial variety corresponding the finitely many binomials which we have deformed in this new manner will provide a local uniformization of the valuation defined on R.

In short, if we have manufactured a complicated singular ring R by adding non linear terms of higher weight to a finite number of the binomial equations of our curve we may be glad to exchange its noetherianity for the simplicity of dealing with a toric variety, provided we can show that it suffices resolve finitely many of the binomials to uniformize the valuation on R. One can show that this is indeed the case.

Now that the claim is that this is essentially the general situation, at least when R is a complete equicharacteristic local ring with an algebraically closed residue field.

Let $R \subset R_\nu$ be the inclusion of a ring in a valuation ring. The only really important case for local uniformization is when R is an excellent equicharacteristic local ring and R_ν dominates R, i.e., $\mathfrak{m}_\nu \cap R = \mathfrak{m}$, and the residue field extension $R/\mathfrak{m} \subset R_\nu/\mathfrak{m}_\nu$ is trivial; we say then that ν is a rational valuation of R. We will assume that we are in this situation.

We may consider the filtration of R by the ideals $\mathcal{P}_\phi(R) = \{x \in R/\nu(x) \geq \phi\}$ and $\mathcal{P}^+_\phi(R) = \{x \in R/\nu(x) > \phi\}$, and this gives us an associated graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\phi\in\Gamma} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}^+_\phi(R)},$$

where $\Gamma \subset \Phi_+ \cup \{0\}$ is the semigroup of the values taken on elements of $R \setminus \{0\}$ by the valuation ν with value group Φ.

The first basic fact is

Proposition 5.1 *If ν is a rational valuation of the noetherian ring local R, the associated graded ring can be presented as a quotient of a polynomial ring in countably many variables by a binomial ideal:*

$$gr_\nu R = k[(U_i)_{i\in I}]/(U^m - \lambda_{mn} U^n)_{m,n\in E}.$$

Now one can associate to (R, ν) a *valuation algebra*

$$\mathcal{A}_\nu(R) = \bigoplus_{\phi\in\Phi} \mathcal{P}_\phi(R) v^{-\phi} \subset R[v^\Phi],$$

where $R[v^\Phi]$ is the group algebra of Φ with coefficients in R.

If R contains a field k such that the valuation takes the value 0 on k^*, we have a natural composed map

$$k[v^{\Phi_+}] \to R[v^{\Phi_+}] \to \mathcal{A}_\nu(R),$$

corresponding to a map of schemes

$$\mathrm{Spec}\, \mathcal{A}_\nu(R) \to \mathrm{Spec}\, k[v^{\Phi_+}].$$

And the second basic fact is:

Proposition 5.2 *If in addition the ring R contains a field of representatives, the $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$ is faithfully flat, the general fiber of the corresponding map of schemes is isomorphic to* $\mathrm{Spec}\, R$ *and its special fiber is* $\mathrm{Spec}\, gr_\nu R$.

By a result of Piltant, the Krull dimension of $\mathrm{Spec}\,\mathrm{gr}_\nu R$ is the rational rank of the group Φ of the valuation, and by Abhyankar's inequality, we have for a rational valuation

$$\mathrm{dim}\mathrm{gr}_\nu R \leq \mathrm{dim} R.$$

Strict inequality can occur, as we saw in example 2. This example displays a faithfully flat family of schemes in which the dimension of the special fiber is *less* than the dimension of the general fiber. There is no contradiction with the semicontinuity in the other direction of fiber dimensions which is usual in algebraic geometry, because the finiteness assumptions under which that result is proved are not satisfied here.

Using the properties of flatness, one can deduce from this a valuative version of Cohen's theorem :

Theorem 5.3 *If the noetherian local ring R is complete and equicharacteristic, given a field of representatives $k \subset R$ elements $\xi_i \in R$ whose initial forms $\bar{xi}_i$ generate the k-algebra gr_ν e surjective map of k-algebras*

$$k[(U_i)_{i\in I}] \to gr_\nu R$$

of the first proposition, mapping U_i to $\bar{\xi}_i$, extends to a continous surjective map of $k - algebras$

$$k[\widehat{(u_i)_{i\in I}}] \to R$$

mapping u_i to $\xi_i \in R$, and such that the associated graded map with respect to the natural filtrations coincides with the first map.

Here the hat means a scalewise completion taking into account the structure of the group Φ of the valuation. The natural filtration on the first ring is the filtration by the weight, with weight(u_i)=γ_i.

Moreover the kernel of the second map is generated, up to closure, by equations which are deformations of the binomial equations $U^m - \lambda_{mn}U^n$ generating the kernel of the first map. They are of the form

$$F_{mn} = u^m - \lambda_{mn}u^n + \sum_p c_p u^p,$$

with $c_p \in k^*$, $w(u^p) > w(u^m) = w(u^n)$, and w is the monomial weight giving to u_i the weight $\gamma_i = \nu(\xi_i) \in \Gamma$.

More precisely, taking the trivial character $\chi\colon \Phi \to k^*$ mapping ϕ to 1, a suitable scalewise completion of the valuation algebra is isomorphic to a quotient

$$k[v^{\Phi_+}][\widehat{(u_i)_{i\in I}}]/\overline{((\tilde{F}_{mn})_{m,n})},$$

where the bar denotes the topological closure, $\tilde{F}_{mn} = u^m - \lambda_{mn} u^n + \sum_p \tilde{c_p}(v^\phi) u^p$ and the $\tilde{c_p}(v^\phi)$ are in $k[v^{\Phi_+}]$, with $\tilde{c_p}(0) = 0$, and finally the series obtained by replacing all v^ϕ by $1 \in k^*$ is F_{mn}. Now, because R is noetherian, its maximal ideal is generated by finitely many of the ξ_i and any variable u_j not in that finite set must appear linearly in one of the equations F_{mn}. In fact, modulo an implicit function theorem whose proof is not yet written in full, one can prove that all the equations F_{mn} except finitely many must be of the form

$$F_i = u^{n(i)} - \lambda_i u^{m(i)} + c_i u_{i+1} + \sum_p c_p^{(i)} u^p,$$

where This begins to look a lot like our second example. It suffices to resolve the toric variety defined by finitely many binomials which do not appear in the F_i, this will resolve, and all the F_i add nothing: it is just a graph.

This reduces us to the case where the graded k-algebra $\mathrm{gr}_\nu R$ is finitely generated: one proves that an irreducible binomial variety over an algebraically closed field of any characteristic has embedded resolutions by toric maps (joint work with P.González Pérez), and then that the same toric map also resolves the space defined by the deformed equations at the point picked by the valuation provided one deforms by adding terms of higher weight.

Then there is the difficulty of reducing the excellent equicharacteristic case to the complete case. This is not entirely settled yet, although there is a precise program to deal with it.

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