Approximating rational valuations by Abhyankar valuations

BERNARD TEISSIER

Let k be an algebraically closed field and let $X \subset \mathbf{A}^n(k)$ be an affine algebraic variety. According to the "viewpoint on resolution of singularities" of [2], one hopes to prove embedded resolution of singularities by proving the existence of re-embedding $X \subset \mathbf{A}^N(k)$ such that there exist coordinate systems on $\mathbf{A}^N(k)$ such that the intersection of X and the torus (complement of the coordinate hyperplanes) is dense in X and there exist proper birational toric maps $Z \to \mathbf{A}^N(k)$ of non singular toric varieties such that the strict transform of $X \subset \mathbf{A}^N(k)$ is non singular and transversal to the toric boundary of Z.

The same problem makes sense for projective varieties and Tevelev (see [5]) has proved that given an embedded resolution of an irreducible projective variety $X \subset \mathbf{P}^n(k)$, one can find re-embeddings $\mathbf{P}^n(k) \subset \mathbf{P}^N(k)$ (built from the given embedded resolution) and projective coordinates on $\mathbf{P}^N(k)$ such that the given embedded resolution is obtained by strict transforms from a proper birational toric map $Z \to \mathbf{P}^N(k)$ of non singular toric varieties. This means that toric embedded resolutions are in a sense "universal" among embedded resolutions.

In the absence of a given embedded resolution, how can one try to build suitable re-embeddings, say for $X \subset \mathbf{A}^n(k)$?

The idea is to first find "embedded local uniformizations" for valuations centered in X. This means to find re-embeddings $X \subset \mathbf{A}^N(k)$ and toric maps $Z \to \mathbf{A}^N(k)$ which will at least make the center of the valuation non singular on the strict transform $X' \subset Z$ of X, and X' transversal to the toric boundary of Z at that point. This strategy is suggested by the case of branches, corresponding to analytically irreducible one dimensional excellent local domains. In this case local embedded local uniformization of the unique valuation ν given by the normalization coincides with embedded resolution and the appropriate re-embedding is given by elements of the local ring R of the branch whose valuations (in the normalization) generate the semigroup of values $\Gamma = \nu(R \setminus \{0\}) \subset \mathbf{N}$.

Therefore we study valuations of the local ring R of X at a given closed singular point and we may assume that R is a domain. In [1, Proposition 3.20] it is shown that it suffices to uniformize *rational* valuations, which are those valuations centered in R for which the residual extension $R/m \subset R_\nu/m_\nu$ is trivial. These valuations correspond to rational points of the Zariski-Riemann manifold of the fraction field of R.

A valuation ν determines a filtration on each subring R' of R_ν by the ideals $\mathcal{P}_\phi(R') = \{x \in R'/\nu(x) \geq \phi\}$, and $\mathcal{P}^+_\phi(R') = \{x \in R'/\nu(x) > \phi\}$.

The graded ring $\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi_{\geq 0}} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$ associated to the ν-filtration on R is the graded k-subalgebra of $\mathrm{gr}_\nu R_\nu$ whose homogeneous elements have degree in the semigroup $\Gamma = \nu(R \setminus \{0\})$. Since the valuation is rational, each homogeneous component of $\mathrm{gr}_\nu R$ is a one dimensional k-vector space and in fact $\mathrm{gr}_\nu R$ is isomorphic to the semigroup algebra $k[t^\Gamma]$. Since R is noetherian the semigroup Γ, which is not finitely generated in general, is well ordered and so has a minimal system of generators $\Gamma = \langle \gamma_1, \ldots, \gamma_i, \ldots \rangle$. We emphasize here that the γ_i are indexed by a countable ordinal $I \leq \omega^h$, where h is the rank (or height) of the valuation, which is less than its rational rank. In [1, Proposition 4.2], it is shown that the graded k-algebra $\mathrm{gr}_\nu R$ is then generated by homogeneous elements $(\overline{\xi}_i)_{i\in I}$ with $\deg \overline{\xi}_i = \gamma_i$ and we have a surjective map of graded k-algebras

$$k[(U_i)_{i\in I}] \longrightarrow \mathrm{gr}_\nu R, \quad U_i \mapsto \overline{\xi}_i,$$

where $k[(U_i)_{i\in I}]$ is graded by giving U_i the degree γ_i. Its kernel is generated by binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell \in L}$, $\lambda_\ell \in k^*$, where U^m represents a monomial in the U_i's. These binomials correspond to a generating system of relations between the generators γ_i of the semigroup.

By a result of Piltant (see [1, Proposition 3.1]), for rational valuations, the Krull dimension of the k-algebra $\mathrm{gr}_\nu R$ is the rational rank of the group Φ of the valuation ν, so that Abhyankar's inequality reduces to $\dim \mathrm{gr}_\nu R \leq \dim R$. The valuations for which equality holds are called *Abhyankar valuations* and it was shown in [3] that embedded local uniformization holds for them.

The main purpose of the lecture was to explain how to approximate a rational valuation ν of rational rank r on a complete equicharacteristic noetherian local domain R of dimension d by Abhyankar semivaluations ν_B, that is, Abhyankar valuations ν_B on r-dimensional quotients R/K_B of R, indexed by certain finite subsets of the minimal set of generators of the semigroup Γ of ν on R, which are the generators of the value semigroups of the valuations ν_B and fill up the set of generators of Γ as B grows. The idea is that for large enough B an embedded local uniformization for ν_B will also uniformize ν. The reduction to the case of complete local domains is a separate issue which will not be discussed here.

Theorem 1. *Let R be a complete equicharacteristic noetherian local domain and let ν be a rational valuation centered in R, of rational rank $r = 1$. Let $\Gamma = \langle (\gamma_i)_{i\in L} \rangle$ be the minimal set of generators of Γ. There exist a collection $\mathcal{B}$ of finite subsets $B \subset I$ such that $I = \bigcup_{B\in\mathcal{B}} B$ and prime ideals K_B of R such that each quotient R/K_B is one dimensional and carries a rational Abhyankar valuation ν_B whose value semigroup is the semigroup $\langle (\gamma_i)_{i\in B} \rangle$, and for each $x \in R \setminus \{0\}$ there are $B \in \mathcal{B}$ such that $x \notin K_B$ and $\nu(x) = \nu_B(x \bmod . K_B)$. One may choose the sets B to be nested, but there are no inclusions between the ideals K_B in general.*

Example 2. *If R is a power series ring in two variables over k, we recover the description of "infinitely singular" valuations as limits of "curve valuations", limits which is this case are understood in terms of infinite sequences of point blowing-ups. See* [1, Example 4.20].

We believe that the result is true for arbitrary rational rank.

The idea of the proof is to first present R as a quotient of a power series ring $S = k[[x_1, \ldots, x_n]]$ by a prime ideal $P = (p_1, \ldots, p_s)$ and apply the valuative Cohen theorem of [3, §4] to the valuation μ on S which is composed of the PS_P-adic valuation μ_1 on S and the valuation ν on S/P. The value group of μ is

$\tilde{\Phi} = \mathbf{Z} \oplus \Phi$ with the *lex* order. One shows that one can choose a minimal system of generators $(p_a)_{a=1,\dots,s}$ of the ideal P such that their initial forms $\mathrm{in}_\mu p_a$ are part of a minimal system of generators of the graded k-algebra $\mathrm{gr}_\mu S$, as well as the $\overline{\xi}_i$ which generate the subalgebra $\mathrm{gr}_\nu R \subset \mathrm{gr}_\mu S$.

The valuative Cohen theorem gives us the existence of representatives in S

$$(\tilde{\xi}_i)_{i\in I}, p_1, \dots, p_s, (h_j)_{j\in J}$$

of the elements of a minimal system of generators of the graded k-algebra $\mathrm{gr}_\mu S$ such that there exists a surjective continuous map of k-algebras

$$\Pi\colon k[(u_i)_{i\in I}, \widehat{v_1, \dots,} v_s, (z_j)_{j\in J}] \to S, \qquad u_i \mapsto \tilde{\xi}_i, v_a \mapsto p_a, z_j \mapsto h_j,$$

where the first algebra, whose definition is part of the theorem, is a generalized power series ring topologized by giving each variable the weight (e.g., $w(u_i) = \gamma_i$) of the corresponding element of the semigroup $\tilde{\Gamma}$ of μ and S is topologized by the filtration determined by μ.

The kernel of Π is generated up to closure by deformations of a set of binomials generating the kernel of the surjective map of graded k-algebras

$$k[(U_i)_{i\in I}, V_1, \dots, V_s, (Z_j)_{j\in J}] \to \mathrm{gr}_\mu S$$

determined by the minimal set of generators of $\mathrm{gr}_\mu S$. The topological generators of the kernel of Π can be chosen in such a way that each involves only finitely many variables. For each v_a, among these generators must be series of the form

$$u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(p) > w(u^{m^\ell})} c_p u^p + \sum_q c_q v^q = v_a.$$

In characteristic zero, the sum $\sum_q c_q v^q$ does not appear. These equations and the variables they contain are used to define the ideals K_B.

References

[1] B. Teissier, Valuations, deformations, and toric geometry. In *Valuation Theory and its applications, Vol. II*, Fields Inst. Commun. 33, AMS., Providence, RI., 2003, 361-459. Available at http://webusers.imj-prg.fr/ bernard.teissier/

[2] B. Teissier, A viewpoint on local resolution of singularities. Oberwolfach Workshop on Singularities, September 2009. *Oberwolfach Reports*, Vol. 6, No. 3, 2009. European Math. Soc. Publications. Available at http://webusers.imj-prg.fr/ bernard.teissier/

[3] B. Teissier, Overweight deformations of affine toric varieties and local uniformization, in *Valuation theory in interaction*, Proceedings of the second international conference on valuation theory, Segovia–El Escorial, 2011. Edited by A. Campillo, F-V. Kuhlmann and B. Teissier. European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, 474-565.

[4] B. Teissier, Overweight deformations of affine toric varieties and local uniformization, II. in preparation.

[5] J. Tevelev, On a question of B. Teissier. *Collectanea Math.* **65** (1) (2014), 61-66. (Published on line February 2013. DOI 10.1007/s13348-013-0080-9)