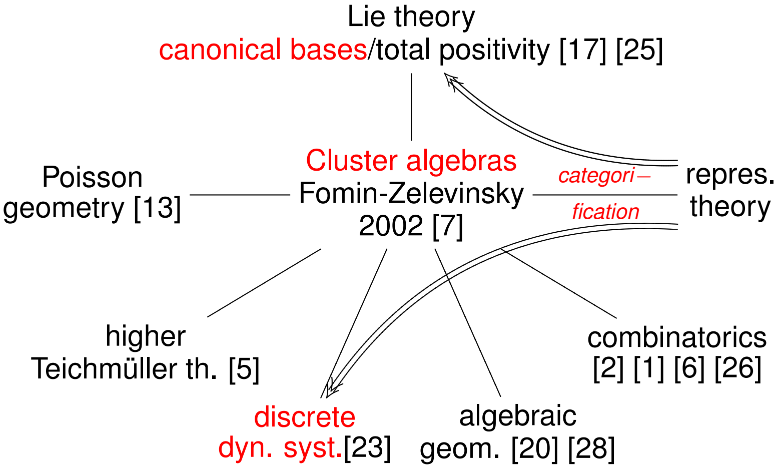
Cluster algebras and applications

Bernhard Keller

Université Paris Diderot – Paris 7

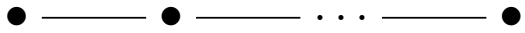
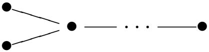
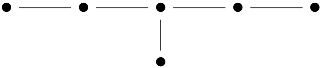
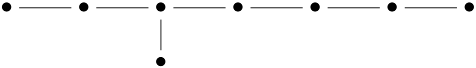
DMV Jahrestagung Köln, 22. September 2011

Context


Lie theory
canonical bases/total positivity [17] [25]
Cluster algebras
Fomin-Zelevinsky
2002 [7]
Poisson
geometry [13]
categori–
fication
repres.
theory
higher
Teichmüller th. [5]
combinatorics
[2] [1] [6] [26]
discrete
dyn. syst.[23]
algebraic
geom. [20] [28]

Plan

The Dynkin diagrams (of type ADE)

Name	Graph	n	Cox. nber
A_n		≥ 1	$n+1$
D_n		≥ 4	$2n-2$
E_6		6	12
E_7		7	18
E_8		8	30

A quiver is an oriented graph

Definition

A *quiver* Q is an oriented graph: It is given by

- a set Q_0 (the set of vertices)
- a set Q_1 (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (taking an arrow to its source)
  - $t : Q_1 \to Q_0$ (taking an arrow to its target).

Remark

A quiver is a ‘category without composition’.

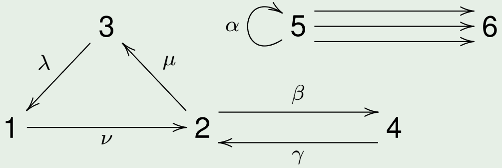
A quiver can have loops, cycles, several components.

Example

The quiver $\vec{A}_3 : \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ is an orientation of the Dynkin diagram $A_3 : \ 1 \longrightarrow 2 \longrightarrow 3$.

Example

Q :



We have $Q_0 = \{1, 2, 3, 4, 5, 6\}$, $Q_1 = \{\alpha, \beta, \dots\}$.
α is a *loop*, (β, γ) is a 2-*cycle*, (λ, μ, ν) is a 3-*cycle*.

Definition of quiver mutation

Let Q be a quiver without loops nor 2-cycles
(from now on always assumed).

Definition (Fomin-Zelevinsky)

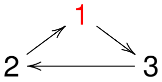
Let $j \in Q_0$. The *mutation* $\mu_j(Q)$ is the quiver obtained from Q as follows

1) for each subquiver $i \xrightarrow{\beta} j \xrightarrow{\alpha} k$, add a new arrow

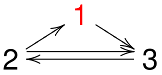
   $i \xrightarrow{[\alpha\beta]} k$;
2) reverse all arrows incident with j;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles (e.g. $\bullet \rightrightarrows\!\!\leftarrow \bullet$ yields $\bullet \longrightarrow \bullet$, '2-reduction').

Examples of quiver mutation

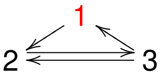
A simple example:



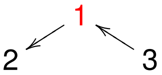
1)



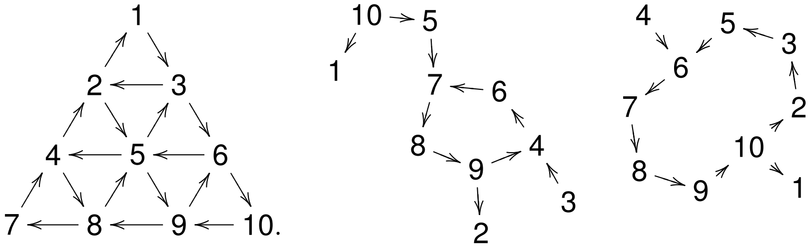
2)



3)



More complicated examples: Google ‘quiver mutation’!



Recall: We wanted to define cluster algebras!

Seeds and their mutations

Definition

A seed is a pair (R, u), where

a) R is a quiver with n vertices;

b) $u = \{u_1, \dots, u_n\}$ is a free generating set of the field $\mathbb{Q}(x_1, \dots, x_n)$.

Example: $(1 \to 2 \to 3, \{x_1, x_2, x_3\}) = (x_1 \to x_2 \to x_3)$.

Definition

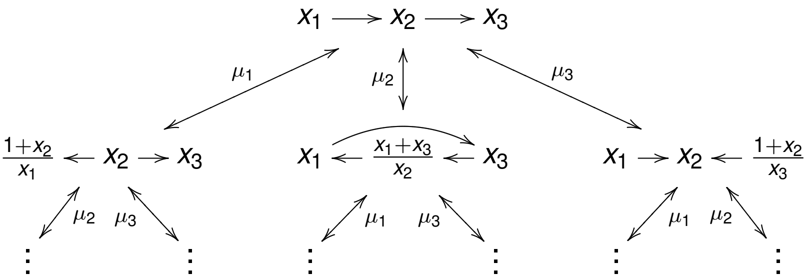
For a vertex j of R, the mutation $\mu_j(R, u)$ is (R', u'), where

a) $R' = \mu_j(R)$;

b) $u' = u \setminus \{u_j\} \cup \{u_j'\}$, with u_j' defined by the exchange relation

$$u_j u_j' = \prod_{\substack{\text{arrows} \\ i \to j}} u_i + \prod_{\substack{\text{arrows} \\ j \to k}} u_k .$$

An example


x1 → x2 → x3
μ1
μ2
μ3
(1+x2)/x1 ← x2 → x3
x1 ← (x1+x3)/x2 ← x3
x1 → x2 ← (1+x2)/x3
μ2 μ3
μ1 μ3
μ1 μ2

Clusters, cluster variables and the cluster algebra

Let Q be a quiver with n vertices.

Definition

a) The initial seed is $(Q, x) = (Q, \{x_1, \ldots, x_n\})$.
b) A cluster is an n-tuple u appearing in a seed (R, u) obtained from (Q, x) by iterated mutation.
c) The cluster variables are the elements of the clusters.
d) The cluster algebra $\mathcal{A}_Q$ is the subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by the cluster variables.
e) A cluster monomial is a product of powers of cluster variables which all belong to the same cluster.

Remark

If Q is mutation equivalent to Q', then $\mathcal{A}_Q \xrightarrow{\sim} \mathcal{A}_{Q'}$.

Fundamental properties

Let Q be a connected quiver.

Theorem (Fomin-Zelevinsky, 2002-03 [7] [8])

a) *All cluster variables are Laurent polynomials.*

b) *There is only a finite number of cluster variables iff Q is mutation-equivalent to an orientation $\vec{\Delta}$ of a Dynkin diagram Δ. Then Δ is unique and called the cluster type of Q.*

Examples for a) and b): A_3, D_4.

Positivity conjecture (Fomin-Zelevinsky)

All cluster variables are Laurent polynomials with non negative coefficients.

Remark

Partial results: [16] [29] [4] [27] [3] [30] [24] . . . . Still wide open in the general case.

Sergey Fomin
University of Michigan

Andrei Zelevinsky
Northeastern University

Construction of a large part of the dual semi-canonical basis

Let $\mathfrak{g}$ be a simple complex Lie algebra of type *ADE* and $U_q^+(\mathfrak{g})$ the positive part of the Drinfeld-Jimbo quantum group.

Theorem (Geiss-Leclerc-Schröer)

a) *(April 2011 [11]): $U_q^+(\mathfrak{g})$ admits a canonical structure of quantum cluster algebra.*

b) *(2006 [12]): All cluster monomials belong to Lusztig's dual semi-canonical basis of the specialization of $U_q^+(\mathfrak{g})$ at $q = 1$.*

Remarks

1) This agrees with Fomin–Zelevinsky's original hopes.
2) Main tool: add. categorification using preproj. algebras.

The periodicity conjecture, I

- Origin: Alexey Zamolodchikov's study of the thermodyn. Bethe ansatz (1991).
- Applications in number theory: identities for the Rogers dilogarithm.

Notation

- Δ and Δ' two Dynkin diagrams with vertex sets I, I',
- h, h' their Coxeter numbers, A, A' their adjacency matrices,
- $Y_{i,i',t}$ variables where $i \in I$, $i' \in I'$, $t \in \mathbb{Z}$.

Y-system associated with (Δ, Δ')

(i, j')
|
$(i, i') - (j, i')$

$$Y_{i,i',t-1} Y_{i,i',t+1} = \frac{\prod_{j \in I} (1 + Y_{j,i',t})^{a_{ij}}}{\prod_{j' \in I'} (1 + Y_{i,j',t}^{-1})^{a'_{i'j'}}}.$$

The periodicity conjecture, II

Periodicity conjecture (Al. Zamolodchikov 1991, [31] [21] [22])

All solutions to this system are periodic of period dividing $2(h+h')$.

Case	Authors
(A_n, A_1)	Frenkel-Szenes (1995) and Gliozzi-Tateo (1996)
(Δ, A_1)	Fomin-Zelevinsky (2003)
(A_n, A_m)	Volkov (2007), Szenes (2006), Henriques (2007)

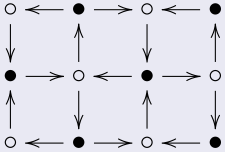
Theorem (K, 2010)

The conjecture holds for (Δ, Δ').

Tools: Link to cluster algebras, additive categorification using the cluster category constructed from quiver representations.

Link to cluster algebras: Square product quivers

Example: $A_4 \square A_3$



$\mu_+ =$ sequence of mutations at all $\circ$.
$\mu_- =$ sequence of mutations at all $\bullet$.

Key Lemma

The conjecture holds iff the sequence of seeds

$$\cdots \xrightarrow[\mu_-]{} S_0 \xrightarrow[\mu_+]{} S_1 \xrightarrow[\mu_-]{} S_2 \xrightarrow[\mu_+]{} \cdots$$

of the (principal extension of the) square product $Q = \Delta \square \Delta'$ is periodic of period dividing $2(h + h')$.

Combinatorial periodicity

Theorem

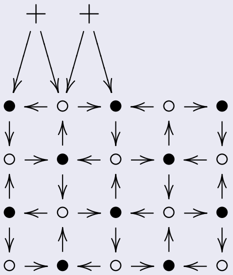
Let D be any quiver containing $\Delta \square \Delta'$ as a full subquiver. Then

$$(\mu_+\mu_-)^{h+h'}(D) = D.$$

A local and an exotic example

local: (A_5, A_4) : $h + h' = 6 + 5 = 11$

exotic: (E_6, E_7) : $h + h' = 12 + 18 = 30$



Summary

- Cluster algebras are commutative algebras with a rich combinatorial structure.
- They have important applications in Lie theory, in discrete dynamical systems and in many other subjects.
- Do not forget to google ‘quiver mutation’!

References

[1] Frédéric Chapoton, *Enumerative properties of generalized associahedra*, Sém. Lothar. Combin. **51** (2004/05), Art. B51b, 16 pp. (electronic).

[2] Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2002), no. 4, 537–566, Dedicated to Robert V. Moody.

[3] Philippe Di Francesco and Rinat Kedem, *Q-systems as cluster algebras. II. Cartan matrix of finite type and the polynomial property*, Lett. Math. Phys. **89** (2009), no. 3, 183–216.

[4] G. Dupont, *Positivity in coefficient-free rank two cluster algebras*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 98, 11.

[5] Vladimir V. Fock and Alexander B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Annales scientifiques de l'ENS **42** (2009), no. 6, 865–930.

[6] Sergey Fomin, Michael Shapiro, and Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Math. **201** (2008), no. 1, 83–146.

[7] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic).

[8] ———, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121.

[9] ———, *Y-systems and generalized associahedra*, Ann. of Math. (2) **158** (2003), no. 3, 977–1018.

[10] Edward Frenkel and András Szenes, *Thermodynamic Bethe ansatz and dilogarithm identities. I*, Math. Res. Lett. **2** (1995), no. 6, 677–693.

[11] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Cluster structures on quantum coordinate rings*, arXive: 1104.0531 [math.RT].

[12] ———, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), no. 3, 589–632.

[13] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical Surveys and Monographs, vol. 167, American Mathematical Society, Providence, RI, 2010.

[14] F. Gliozzi and R. Tateo, *Thermodynamic Bethe ansatz and three-fold triangulations*, Internat. J. Modern Phys. A **11** (1996), no. 22, 4051–4064.

[15] André Henriques, *A periodicity theorem for the octahedron recurrence*, J. Algebraic Combin. **26** (2007), no. 1, 1–26.

[16] David Hernandez and Bernard Leclerc, *Cluster algebras and quantum affine algebras*, arXiv:0903.1452v1 [math.QA].

[17] Masaki Kashiwara, *Bases cristallines*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 6, 277–280.

[18] Bernhard Keller, *Cluster algebras, quiver representations and triangulated categories*, arXive:0807.1960 [math.RT].

[19] ———, *The periodicity conjecture for pairs of Dynkin diagrams*, arXiv:1001.1531.

[20] Maxim Kontsevich and Yan Soibelman, *Stability structures, Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[21] A. Kuniba and T. Nakanishi, *Spectra in conformal field theories from the Rogers dilogarithm*, Modern Phys. Lett. A **7** (1992), no. 37, 3487–3494.

[22] Atsuo Kuniba, Tomoki Nakanishi, and Junji Suzuki, *Functional relations in solvable lattice models. I. Functional relations and representation theory*, Internat. J. Modern Phys. A **9** (1994), no. 30, 5215–5266.

[23] Atsuo Kuniba, Tomoki Nakanishi, and Junji Suzuki, *T-systems and y-systems in integrable systems*, Journal of Physics A: Mathematical and Theoretical **44** (2011), no. 10, 103001.

[24] Kyungyong Lee and Ralf Schiffler, *A combinatorial formula for rank* 2 *cluster variables*, arXive:1106.0952 [math.CO].

[25] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[26] Gregg Musiker, *A graph theoretic expansion formula for cluster algebras of type B_n and D_n*, arXiv:0710.3574v1 [math.CO], to appear in the Annals of Combinatorics.

[27] Gregg Musiker, Ralf Schiffler, and Lauren Williams, *Positivity for cluster algebras from surfaces*, arXiv:0906.0748.

[28] Kentaro Nagao, *Donaldson-Thomas theory and cluster algebras*, arXive:1002.4884 [math.AG].

[29] Hiraku Nakajima, *Quiver varieties and cluster algebras*, arXiv:0905.0002v3 [math.QA].

[30] Fan Qin, *Quantum cluster variables via Serre polynomials*, math.RT/1004.4171.

[31] F. Ravanini, A. Valleriani, and R. Tateo, *Dynkin TBAs*, Internat. J. Modern Phys. A **8** (1993), no. 10, 1707–1727.

[32] András Szenes, *Periodicity of Y-systems and flat connections*, Lett. Math. Phys. **89** (2009), no. 3, 217–230.

[33] Alexandre Yu. Volkov, *On the periodicity conjecture for Y-systems*, Comm. Math. Phys. **276** (2007), no. 2, 509–517.

[34] Al. B. Zamolodchikov, *On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*, Phys. Lett. B **253** (1991), no. 3-4, 391–394.