**Cluster algebras and quantum affine algebras, after B. Leclerc**

Bernhard Keller

This talk, based on [14], is a report on recent work by B. Leclerc on a new type of categorification for cluster algebras.

Cluster algebras were invented by Fomin and Zelevinsky [8] at the beginning of this decade. Since then, a major effort has gone into their categorification (*cf.* for example [15] [1] [2] [3] [10]). Namely, in many cases, it was proved that for a given cluster algebra $\mathcal{A}$, there exists a triangulated (or Frobenius) category $\mathcal{C}$, such that

- the cluster variables $x$ of $\mathcal{A}$ correspond to certain indecomposables $T_x$ of $\mathcal{C}$,
- two cluster variables $x$ and $y$ belong to the same cluster if and only if there are no non split extensions between the corresponding objects $T_x$ and $T_y$,
- the cluster monomial $m = xy \cdots z$ corresponds to the the object $M = T_x \oplus T_y \oplus \cdots T_z$ of $\mathcal{C}$,
- the exchange relations $xx^* = m + m'$ of $\mathcal{A}$ correspond to triangles
$$T_x \to M \to T_{x^*} \to \Sigma T_x \text{ and } T_{x^*} \to M' \to T_x \to \Sigma T_{x^*}$$
of $\mathcal{C}$.

It was shown that in certain cases, the objects $T_x$ are precisely the indecomposable *rigid* objects of $\mathcal{C}$, *i.e.* those without selfextensions. For example, when $\mathcal{A}$ has only a finite number of cluster variables, then all indecomposable objects of $\mathcal{C}$ are rigid and the cluster variables are in bijection with the indecomposables of $\mathcal{C}$. In this case, it was also shown that the cluster algebra $\mathcal{A}$ can be realized as a sort of dual Hall algebra of the triangulated category $\mathcal{C}$ and that its commutativity reflects the fact that $\mathcal{C}$ is 2-Calabi-Yau, *i.e.* the space $\mathsf{Ext}^1_{\mathcal{C}}(L, M)$ is in natural duality with $\mathsf{Ext}^1_{\mathcal{C}}(M, L)$ for all objects $L$ and $M$ of $\mathcal{C}$.

This type of categorification is very useful: it has allowed to prove properties of cluster algebras which appear to be beyond the reach of the purely combinatorial methods, *cf.* for example [4]. However, it is perhaps not the most natural notion of categorification which we could expect for a cluster algebra.

In order to categorify an algebra $\mathcal{A}$ defined over the integers and endowed with a distinguished $\mathbb{Z}$-basis $B$, one would rather look for an abelian category $\mathcal{M}$ which is monoidal (*i.e.* endowed with a tensor product) and whose Grothendieck ring is isomorphic to $\mathcal{A}$ in such a way that the elements of $B$ correspond to the classes of the simple objects of $\mathcal{M}$, *cf.* for example [12]. The definition of a 'canonical basis' for a general cluster algebra is still an open problem (*cf.* for example [18]) but in many cases, this basis is known, for example when there is only a finite number of clusters or when the algebra already admits a canonical basis in the sense of Kashiwara and Lusztig. One then expects [8] that the cluster monomials, and in particular the cluster variables, belong to this canonical basis.

The natural notion of 'tensor-indecomposability' is primality: an object of $\mathcal{M}$ is *prime*, if it does not admit a non trivial tensor factorization. In order to categorify a cluster algebra $\mathcal{A}$, one would therefore look for an abelian monoidal category $\mathcal{M}$ whose Grothendieck ring is $\mathcal{A}$ and such that

- the cluster variables $x$ of $\mathcal{A}$ are the classes of certain prime simple objects $S_x$ of $\mathcal{M}$,
- two cluster variables $x$ and $y$ belong to the same cluster if and only if $S_x \otimes S_y$ is simple,
- the cluster monomial $m = xy \cdots z$ in $\mathcal{A}$ is the class of the simple object $M = S_x \otimes S_y \otimes \ldots \otimes S_z$ of $\mathcal{M}$,
- the exchange relations $xx^* = m + m'$ come from exact sequences

$$0 \to M \to S_x \otimes S_{x*} \to M' \to 0.$$

This last condition lacks in symmetry. But if we remember that the cluster algebra is commutative, and thus the tensor product induces a commutative multiplication in the Grothendieck group, we can save symmetry by also requiring the existence of an exact sequence

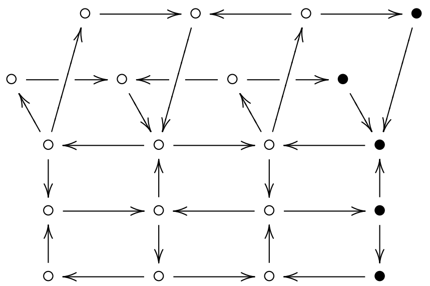
$$0 \to M' \to S_{x^*} \otimes S_x \to M \to 0.$$

The natural notion which replaces rigidity in a monoidal category appears to be 'reality': an object of $\mathcal{M}$ is *real* if its tensor square is simple (*cf.* [13]). The objects $S_x$ should exactly be the real prime simple objects of $\mathcal{M}$. When the cluster algebra $\mathcal{A}$ has only finitely many cluster variables, all the prime simple objects of $\mathcal{M}$ should be real and the cluster variables of $\mathcal{A}$ should be in bijection with the prime simples.

Using classical results on representations of quantum affine algebras [5] [6] [9] [16] [17] B. Leclerc has shown [14] that the cluster algebras of types $A_n$, $n \in \mathbb{N}$, and $D_4$ (with suitable coefficients) do admit monoidal categorifications given by tensor abelian subcategories of categories of finite-dimensional representations of quantum affine algebras. He conjectures that this holds in many more cases. More precisely, the main conjecture of [14] is the following.

**Conjecture** (Leclerc)**.** *Let $\Delta$ be a Dynkin diagram and $l \geq 1$ an integer. Let $\mathfrak{g}$ be the complex simple Lie algebra of type $\Delta$, $q$ a non zero complex number which is not a root of unity and $U_q(\widehat{\mathfrak{g}})$ the corresponding quantum affine algebra. Then the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ admits a monoidal abelian subcategory $\mathcal{M}_{\Delta,l}$ which is a monoidal categorification of the cluster algebra associated with a quiver $Q_{\Delta,l}$.*

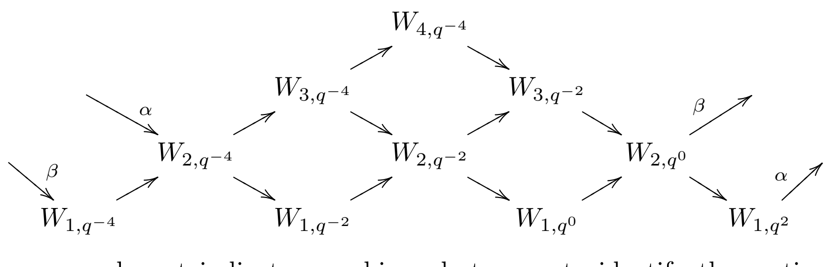
In [14], Leclerc explicitly describes the subcategory $\mathcal{M}_{\Delta,l}$ and the quiver $Q_{\Delta,l}$. For example, if $\Delta = D_5$ and $l = 3$, then the quiver $Q_{\Delta,l}$ is as follows



The vertices marked by $\bullet$ correspond to 'frozen variables' of the initial cluster. For $\Delta = A_1$ and $l = 3$, the quiver $Q_{\Delta,l}$ is

$$\circ \longleftarrow \circ \longrightarrow \circ \longleftarrow \bullet$$

In this last case, the subcategory $\mathcal{M}_{\Delta,l}$ is the full subcategory on the finite-dimensional $U_q(\widehat{sl_2})$-modules all of whose simple subfactors have Drinfeld polynomials with roots in $q^4, q^2, q^0, q^{-2}$. The isomorphism between the cluster algebra $\mathcal{A}(Q_{A_1,3})$ and the Grothendieck group $K_0(\mathcal{M}_{A_1,3}) \otimes_{\mathbb{Z}} \mathbb{Q}$ sends the variables $x_1$, $x_2$, $x_3$, $x_4$ of the initial cluster to the classes of the Kirillov-Reshetikhin modules $W_{1,q^0}$, $W_{2,q^{-2}}$, $W_{3,q^{-2}}$ and $W_{4,q^{-4}}$. The complete list of the prime simples (up to isomorphism) is




The arrows do not indicate morphisms but serve to identify the vertices other than $W_{4,q^{-4}}$ with those of the Auslander-Reiten quiver of the cluster category of type $A_3$ (the arrows on the left and on the right of the diagram are identified as indicated by their labels). Every simple module in $\mathcal{M}_{\Delta,l}$ is a tensor product of modules in this list. A given tensor product of modules in the list other than $W_{4,q^{-4}}$ is simple iff the corresponding direct sum of indecomposables of the cluster category is rigid.

Thus, at least in certain examples, one obtains two rather different categorifications of a given cluster algebra. Table 1 sums up the correspondences. The category $\mathcal{C}$ is much 'smaller' than $\mathcal{M}$ and $\mathcal{M}$ is much less well understood than $\mathcal{C}$. It does not seem to be known whether it has enough projectives, for example. The table suggests that $\mathcal{M}$ should be an 'exponential' of $\mathcal{C}$ or $\mathcal{C}$ a 'linearisation' of $\mathcal{M}$ ...

| cluster algebra $\mathcal{A}$ | additive categorification $\mathcal{C}$ | monoidal categorification $\mathcal{M}$ |
|---|---|---|
| $+$ | ? | $\oplus$ |
| $\times$ | $\oplus$ | $\otimes$ |
| cluster monomial | rigid object | real simple object |
| cluster variable | rigid indecomposable | real prime simple |

Table 1. Correspondences between categorifications

Finally, let us point out [11] [7] for a very different link between cluster algebras and quantum affine algebras, which does not seem to be related to categorification.

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