DESINGULARIZATIONS OF QUIVER GRASSMANNIANS VIA GRADED QUIVER VARIETIES

BERNHARD KELLER AND SARAH SCHEROTZKE

Abstract. Inspired by recent work of Cerulli–Feigin–Reineke on desingularizations of quiver Grassmannians of representations of Dynkin quivers, we obtain desingularizations in considerably more general situations and in particular for Grassmannians of modules over iterated tilted algebras of Dynkin type. Our desingularization map is constructed from Nakajima's desingularization map for graded quiver varieties.

Contents

1. Introduction and main results

A quiver Grassmannian is the variety of subrepresentations with given dimension vector of a fixed quiver representation. To the best of the authors' knowledge, quiver Grassmannians first appeared in Schofield's work [37]. They are projective varieties and Reineke shows in [30] that every projective variety can be realized as a quiver Grassmannian (we refer to the final example of Hille's [18] for a similar, and in fact closely related result, and to Ringel's [35] for an analogous 'universality theorem' in the setting of Auslander algebras). Caldero-Chapoton discovered [2] that the canonical generators of Fomin-Zelevinsky's cluster algebras [13] can be interpreted as generating polynomials of Euler characteristics of quiver Grassmannians. Since then, quiver Grassmannians have played an important role in the additive categorification of (quantum) cluster algebras, cf. for example [2] [3] [8] [27] [28] [9].

In [7], [4], Cerulli–Feigin–Reineke initiated a systematic study of (singular) quiver Grassmannians of Dynkin quivers, starting from the surprising

2010 *Mathematics Subject Classification.* 13F60, 14M15, 16G20.
Key words and phrases. Quiver Grassmannians, Nakajima quiver varieties.

observation that the type A degenerate flag varieties studied in [10] [11] [12] are of this form. An important aspect of their work is the construction of desingularizations, which they achieve in their recent paper [5] generalizing [12]. In [6], they link these desingularizations to a construction by Hernandez–Leclerc [17], which has been generalized by Leclerc–Plamondon [23] and further generalized by the present authors in [20].

In this article, we build on [20] to construct desingularizations of quiver Grassmannians in much more general situations and in particular for all modules over the repetitive algebra of an arbitrary iterated tilted algebra B of Dynkin type, like the algebra B given by the square quiver

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 3 & \longrightarrow & 4 \end{array}$$

with the commutativity relation (B is tilted of type D_4). The main ingredient of our construction is the desingularization map for graded quiver varieties introduced by Nakajima [26] [27] and generalized from bipartite to acylic quivers by Qin [28] [29].

More precisely, we consider a module M over the singular Nakajima category $\mathcal{S}$ associated with an acyclic quiver Q, cf. section 3.3. By [23], such a module corresponds to a point in the graded affine quiver variety $\mathcal{M}_0(d)$ associated with Q and the dimension vector $d = \underline{\dim}\, M$ of M. Nakajima has constructed a pre-desingularization (i.e. a proper, surjective morphism with smooth domain)

$$\pi : \mathcal{M}(d) \to \mathcal{M}_0(d)$$

of $\mathcal{M}_0(d)$. Here the points of $\mathcal{M}(d)$ can be interpreted as (orbits of stable) representations of the regular Nakajima category $\mathcal{R}$, which contains $\mathcal{S}$ as a full subcategory, and the map π takes a representation L of $\mathcal{R}$ to its restriction $\operatorname{res}(L)$ to $\mathcal{S} \subset \mathcal{R}$. We will show that for suitable modules M, each quiver Grassmannian of M admits a desingularization by a disjoint union of connected components of quiver Grassmannians of a distinguished point in the fiber of π over M, namely the so-called intermediate Kan extension $K_{LR}(M)$ of section 2.10 of [20].

Let us describe our main results more precisely. Let M be a finite-dimensional $\mathcal{S}$-module of dimension vector d. Let w be a dimension vector less or equal to d. Using Nakajima's stratification of $\mathcal{M}_0(d)$, we assign a dimension vector (v_C, w) of $\mathcal{R}$ with each irreducible component C of the quiver Grassmannian $\mathrm{Gr}_w(M)$, cf. Lemma 3.9. Let $\mathcal{V}_w(M)$ be the set of the vectors v_C. Recall that a module is *rigid* if its space of selfextensions vanishes. The following result is modeled on Theorem 7.4 of [5] with the intermediate Kan extension $K_{LR}(M)$ playing the role of the module $\widehat{M}$ of [loc. cit.].

Theorem 1.1 (Theorem 3.11)**.** *Suppose that $K_{LR}(M)$ is rigid. Then the map*

$$\pi_{\mathrm{Gr}} : \coprod_{v \in \mathcal{V}_w(M)} \mathrm{Gr}_{(v,w)}(K_{LR}(M)) \to \mathrm{Gr}_w(M)$$

taking $U \subset K_{LR}(M)$ to $\mathrm{res}(U) \subset M$ is a pre-desingularization (a proper, surjective morphism with smooth domain).

We determine the fibres of the map π_{Gr} in Theorem 3.13. To make sure that the generic fibre is reduced to a point, we need to shrink the domain of π_{Gr}. We do this as follows: An $\mathcal{R}$-module is *bistable* if it is isomorphic to the intermediate extension of some $\mathcal{S}$-module. For a dimension vector (v, w) of $\mathcal{R}$, denote by $\mathrm{Gr}^{bs}_{(v,w)}(K_{LR}(M))$ the *bistable Grassmannian*, i.e. the closure of the set of points corresponding to bistable submodules. In analogy with Remark 7.8 of [loc. cit.], we conjecture that the bistable Grassmannian actually equals the whole Grassmannian. The following result is modeled on Corollary 7.7 of [5] with the bistable Grassmannians playing the role of the sets $\overline{S_{[N]}}$ in [loc. cit.].

Theorem 1.2 (Theorem 3.11)**.** *Suppose that $K_{LR}(M)$ is rigid. The map*

$$\pi^{bs} : \coprod_{v \in \mathcal{V}_w(M)} \mathrm{Gr}^{bs}_{(v,w)}(K_{LR}(M)) \to \mathrm{Gr}_w(M)$$

taking $U \subset K_{LR}(M)$ to $\mathrm{res}(U) \subset M$ is a desingularization (a proper, surjective morphism with smooth domain which induces an isomorphism between dense open subsets).

We will give sufficient conditions for $K_{LR}(M)$ to be rigid (Lemmas 2.4 and 2.6) and show by an example that this is not always the case (section 3.14). Nevertheless, as a consequence of the above theorems, we will obtain desingularizations for all modules over the repetitive algebra of an iterated tilted algebra of Dynkin type (Corollary 4.4). We will show that this covers in particular all the cases considered in [5] and yields a natural interpretation for the algebra $\mathcal{H}_Q$ of [loc. cit.] (cf. section 4.5).

The paper is organized as follows. In section 2, we introduce the intermediate extension $F_{\lambda\rho}$ associated to a localization functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ which admits a right and a left adjoint. In Lemma 2.4, we give sufficient conditions for an object in the image of $F_{\lambda\rho}$ to be rigid. In section 2.5, we examine the particular case where F is the restriction

$$\mathrm{mod}(\mathrm{mod}(\mathcal{P})) \to \mathrm{mod}(\mathcal{P})$$

along the Yoneda embedding $\mathcal{P} \to \mathrm{mod}(\mathcal{P})$ from a coherent category $\mathcal{P}$ to its category of finitely presented modules.

In section 3, we recall the definition of the regular and the singular Nakajima categories $\mathcal{R}$ and $\mathcal{S}$ introduced in [23] (cf. also section 2 of [20]) and explain how they relate to Nakajima's graded quiver varieties. In section 3.5, for certain subsets $\sigma^{-1}(C)$ of the set of vertices of $\mathcal{S}$, we consider the quotients $\mathcal{S}_C$ and $\mathcal{R}_C$ obtained by factoring out the ideal generated by the identities of the vertices outside $\sigma^{-1}(C)$. We will later need the extra generality afforded by a suitable choice of C in order to recover the results of Cerulli–Feigin–Reineke [5]. In section 3.7, we state and prove our main Theorems.

In section 4.2, we show that for a suitable choice of C, the category $\mathcal{S}_C$ is equivalent to the category of indecomposable projectives over the repetitive algebra $\widehat{A}$ of an iterated tilted algebra A of Dynkin type and that the category $\mathcal{R}_C$ is equivalent to the category of indecomposable representations of $\widehat{A}$ (Proposition 4.3). By Lemma 2.6, we know that the intermediate extension of a finite-dimensional $\widehat{A}$-module is always rigid. Thus, we obtain a desingularization map for any finite-dimensional module over the repetitive algebra $\widehat{A}$ of A. We then specialize A to the path algebra kQ of a Dynkin quiver Q. The category of finite-dimensional kQ-modules appears naturally as a full subcategory of $\mathrm{mod}(\widehat{kQ})$. In section 4.5, we show that the intermediate extension K_{LR} restricted to the category of finite-dimensional kQ-modules specializes to the functor Λ constructed by Cerulli–Feigin–Reineke [5] and that our desingularization specializes to theirs. In section 5, we illustrate the desingularization theorem using a module over a tilted algebra of type D_4.

Acknowledgments. This article was conceived during the cluster algebra program at the MSRI in fall 2012. The authors are grateful to the MSRI for financial support and ideal working conditions. They thank Giovanni Cerulli Irelli, Bernard Leclerc, Pierre-Guy Plamondon and Markus Reineke for stimulating conversations and Shengyong Pan for pointing out a missing hypothesis in a previous version of Lemma 2.4 c).

2. Intermediate Kan extensions

2.1. **Intermediate Kan extensions and rigidity.** We first study the properties of the intermediate extension in the framework of abelian categories: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a localization functor, i.e. F is exact and induces an equivalence

$$\mathcal{A}/\ker(F) \xrightarrow{\sim} \mathcal{B} \ ,$$

where $\mathcal{A}/\ker(F)$ is the localization of $\mathcal{A}$ with respect to the Serre subcategory $\ker(F)$ in the sense of [14]. We assume that F admits both a right adjoint F_ρ and a left adjoint F_λ so that we have three adjoint functors

$$\begin{array}{c} \mathcal{A} \\ F_\lambda \uparrow \;\; \downarrow F \;\; \uparrow F_\rho \\ \mathcal{B}. \end{array}$$

Notice that F_λ and F_ρ are both fully faithful (since F is a localization), that F_λ is right exact and preserves projectivity and that F_ρ is left exact and preserves injectivity. Denote the adjunction morphisms by

$$\phi : F_\lambda F \to \mathbf{1}_\mathcal{A} \ \ , \psi : \mathbf{1}_\mathcal{B} \to FF_\lambda \ \ , \eta : FF_\rho \to \mathbf{1}_\mathcal{B} \ \ , \varepsilon : \mathbf{1}_\mathcal{A} \to F_\rho F.$$

Lemma 2.2. *Let M be an object of $\mathcal{A}$.*

a) *The adjunction morphism $M \to F_\rho F M$ is mono if and only if the group* $\mathrm{Hom}(N, M)$ *vanishes for each object N of* $\ker(F)$*.*
b) *The adjunction morphism $M \to F_\rho F M$ is invertible iff we have* $\mathrm{Hom}(N, M) = 0 = \mathrm{Ext}^1(N, M)$ *for each object N of* $\ker(F)$*.*

We leave the proof of part a) as an exercise for the reader. Part b) is the characterization of the image of the adjoint of a localization functor given in Lemme 1, page 370 of [14]. We call an object M *stable* if it satisfies the conditions part a). Dually, it is *co-stable* if it satisfies the dual conditions: the adjunction morphism $F_\lambda F M \to M$ is epi or, equivalently, we have $\mathrm{Hom}(M, N) = 0$ for each object N of $\ker(F)$.

Lemma 2.3. *The following square is commutative*

$$\begin{array}{ccc} F_\lambda F F_\rho & \xrightarrow[F_\lambda \eta]{\sim} & F_\lambda \\ {\scriptstyle \phi F_\rho}\downarrow & & \downarrow{\scriptstyle \varepsilon F_\lambda} \\ F_\rho & \xrightarrow[F_\rho \psi]{\sim} & F_\rho F F_\lambda \end{array}$$

Proof. Since $F : \mathcal{A} \to \mathcal{B}$ is essentially surjective, it suffices to check the commutativity after pre-composing with F. Consider the diagram

$$\begin{array}{ccccccc} F_\lambda F & \xrightarrow[F_\lambda F \varepsilon]{\sim} & F_\lambda F F_\rho F & \xrightarrow[F_\lambda \eta F]{\sim} & F_\lambda F & \xrightarrow{\phi} & \mathbf{1}_\mathcal{A} \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \phi F_\rho F}\downarrow & & \downarrow{\scriptstyle \varepsilon F_\lambda F} & & \downarrow{\scriptstyle \varepsilon} \\ \mathbf{1}_\mathcal{A} & \xrightarrow[\varepsilon]{} & F_\rho F & \xrightarrow[F_\rho \psi F]{\sim} & F_\rho F F_\lambda F & \xrightarrow[F_\rho F \phi]{\sim} & F_\rho F. \end{array}$$

Here the composition $(F_\lambda \eta F)(F_\lambda F \varepsilon)$ is the identity and so is the composition $(F_\rho F \phi)(F_\rho \psi F)$. Thus, the large rectangle is commutative. The leftmost square is commutative because $\phi : F_\lambda F \to \mathbf{1}_\mathcal{A}$ is a natural transformation and the rightmost square is commutative because so is $\varepsilon : \mathbf{1}_\mathcal{A} \to F_\rho F$. It follows that the central square is commutative as claimed. $\surd$

By Lemma 2.3, we have a canonical morphism

$$\mathrm{can} : F_\lambda \to F_\rho.$$

We define the *intermediate extension* $F_{\lambda\rho}$ to be its image. Notice that, for each object M of $\mathcal{B}$, the object $F_{\lambda\rho}(M)$ is both stable (as a subobject of $F_\rho M$) and co-stable (as a quotient of $F_\lambda M$). We have canonical morphisms

$$F_\lambda \xrightarrow{\pi} F_{\lambda\rho} \xrightarrow{\iota} F_\rho$$

and their images under F are invertible (since $F\iota$ is mono, $F\pi$ is epi and their composition $F(\text{can})$ is invertible). One deduces that $F_{\lambda\rho}$ induces an equivalence from $\mathcal{B}$ onto the full subcategory of $\mathcal{A}$ formed by the objects which are both stable and co-stable.

Lemma 2.4. a) *For all objects L of $\mathcal{A}$ and M of $\mathcal{B}$, we have canonical injections*

$$\mathrm{Ext}^1_{\mathcal{A}}(L, F_\rho M) \to \mathrm{Ext}^1_{\mathcal{B}}(FL, M) \text{ and } \mathrm{Ext}^1_{\mathcal{A}}(F_\lambda M, L) \to \mathrm{Ext}^1_{\mathcal{B}}(M, FL).$$

b) *If L is rigid in $\mathcal{B}$, then $F_\lambda(L)$, $F_\rho(L)$ and $F_{\lambda\rho}(L)$ are rigid in $\mathcal{A}$.*

c) *Conversely, suppose that $F_{\lambda\rho}(L)$ is rigid in $\mathcal{A}$, that $\mathrm{Ext}^1_{\mathcal{A}}(X, N) = 0$ for each stable object X and each object N in $\ker(F)$ and moreover that the group $\mathrm{Ext}^2_{\mathcal{A}}(U, U)$ vanishes, where U is the cokernel of $F_{\lambda\rho}(L) \to F_\rho(L)$. Then L is rigid in $\mathcal{B}$.*

Proof. a) We define the image of the class of an exact sequence

$$0 \longrightarrow F_\rho M \xrightarrow{i} E \xrightarrow{p} L \longrightarrow 0$$

to be the class of the sequence

$$0 \longrightarrow M \xrightarrow{j} FE \xrightarrow{Fp} FL \longrightarrow 0 \quad ,$$

where $j = (Fi)(\eta M)^{-1}$. Clearly, this yields a well-defined map

$$\mathrm{Ext}^1_{\mathcal{A}}(L, F_\rho M) \to \mathrm{Ext}^1_{\mathcal{B}}(L, M).$$

It is not hard to check that if $r : FE \to M$ is a retraction for j, then $(F_\rho r)(\eta E)$ is a retraction for i. Thus, our map is injective. Dually, one obtains the second injection.

b) By part a), we have the injection

$$\mathrm{Ext}^1_{\mathcal{A}}(F_\lambda L, F_\lambda L) \to \mathrm{Ext}^1_{\mathcal{B}}(L, FF_\lambda L) = \mathrm{Ext}^1_{\mathcal{B}}(L, L) = 0$$

and similarly for $F_\rho L$. Now consider the exact sequence

$$0 \longrightarrow F_{\lambda\rho}(L) \longrightarrow F_\rho(L) \longrightarrow U \longrightarrow 0.$$

Here the object U lies in $\ker(F)$. If we apply the functor $\mathrm{Hom}_{\mathcal{A}}(F_{\lambda\rho}(L), ?)$ to the exact sequence, we obtain the exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(F_{\lambda\rho}(L), U) \to \mathrm{Ext}^1_{\mathcal{A}}(F_{\lambda\rho}(L), F_{\lambda\rho}(L)) \to \mathrm{Ext}^1_{\mathcal{A}}(F_{\lambda\rho}(L), F_\rho(L)).$$

Since $F_{\lambda\rho}(L)$ is co-stable and U lies in $\ker(F)$, the left-hand term vanishes. Since we have $FF_{\lambda\rho}(L) = L$ and L is rigid, the right hand term vanishes by part a). Thus, the object $F_{\lambda\rho}(L)$ is rigid.

c) Let

$$0 \longrightarrow L \xrightarrow{i} E \longrightarrow L \longrightarrow 0$$

be a non split exact sequence in $\mathcal{B}$. Then $F_\rho(i)$ is a non split monomorphism of $\mathcal{A}$ (since F_ρ is fully faithful) and so the sequence

$$0 \longrightarrow F_\rho(L) \xrightarrow{F_\rho(i)} F_\rho(E) \longrightarrow \operatorname{cok}(F_\rho(i)) \longrightarrow 0$$

is non split in $\mathcal{A}$. The object $\operatorname{cok}(F_\rho(i))$ is stable since for $N \in \ker(F)$, we have

$$\operatorname{Hom}(N, F_\rho(E)) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(N, F_\rho(L)).$$

Moreover, the image $F \operatorname{cok}(F_\rho(i))$ is isomorphic to L since F is exact. So we have an exact sequence

$$0 \to F_{\lambda\rho}(L) \to \operatorname{cok}(F_\rho(i)) \to V \to 0$$

with V in $\ker(F)$. If we apply $\operatorname{Hom}(?, F_\rho(L))$ to this sequence, we obtain an exact sequence

$$0 \to \operatorname{Ext}^1(\operatorname{cok}(F_\rho(i)), F_\rho(L)) \to \operatorname{Ext}^1(F_{\lambda\rho}(L), F_\rho(L)).$$

Since we have found a non zero element in $\operatorname{Ext}^1(\operatorname{cok}(F_\rho(i)), F_\rho(L))$, we see that the right hand group does not vanish. We claim that $\operatorname{Ext}^1(F_{\lambda\rho}(L), U)$ vanishes. Indeed, this follows from our assumption when we apply $\operatorname{Ext}^1(?, U)$ to the sequence

$$0 \to F_{\lambda\rho}(L) \to F_\rho(L) \to U \to 0.$$

Now we claim that we have an isomorphism

$$\operatorname{Ext}^1(F_{\lambda\rho}(L), F_{\lambda\rho}(L)) \xrightarrow{\sim} \operatorname{Ext}^1(F_{\lambda\rho}(L), F_\rho(L)).$$

Indeed this follows by applying $\operatorname{Ext}^1(F_{\lambda\rho}(L), ?)$ to the sequence

$$0 \to F_{\lambda\rho}(L) \to F_\rho(L) \to U \to 0.$$

We conclude that $\operatorname{Ext}^1(F_{\lambda\rho}(L), F_{\lambda\rho}(L))$ is non zero as claimed. $\surd$

2.5. **The case of the Auslander category.** We consider the special case of the setup of section 2.1 where $\mathcal{B}$ is a module category and $\mathcal{A} = \operatorname{mod}(\mathcal{B})$ the Auslander category.

Let k be a field and $\mathcal{P}$ a skeletally small k-category (i.e. its isomorphism classes form a set). Let $\operatorname{mod}(\mathcal{P})$ be the category of finitely presented $\mathcal{P}$-modules, i.e. of k-linear functors $M : \mathcal{P}^{op} \to \operatorname{Mod} k$ admitting an exact sequence

$$P_1 \to P_0 \to M \to 0$$

where P_0 and P_1 are finitely generated projective $\mathcal{P}$-modules, i.e. direct factors of finite direct sums of representable $\mathcal{P}$-modules $P^\wedge = \mathcal{P}(?, P)$, $P \in \mathcal{P}$. Notice that the category $\operatorname{mod}(\mathcal{P})$ is still skeletally small. We assume that $\operatorname{mod}(\mathcal{P})$ is abelian or, equivalently, that $\mathcal{P}$ is coherent, i.e. the kernel of any morphism between finitely generated projective $\mathcal{P}$-modules is finitely generated. We have the Yoneda embedding $\mathcal{P} \to \operatorname{mod}(\mathcal{P})$ taking P to $P^\wedge$.

Let $\mathcal{B} = \mathrm{mod}(\mathcal{P})$, $\mathcal{A} = \mathrm{mod}(\mathcal{B})$ and let $F = \mathrm{res} : \mathcal{A} \to \mathcal{B}$ be the restriction along the Yoneda embedding. Thus, we have functors

$$\begin{array}{ccc} \mathrm{mod}(\mathcal{P}) & \quad & \mathrm{mod}(\mathrm{mod}(\mathcal{P})) = \mathcal{A} \\ \text{Yoneda} \uparrow & & K_L \uparrow \;\; \downarrow \mathrm{res} \;\; \uparrow K_R \\ \mathcal{P} & & \mathrm{mod}(\mathcal{P}) = \mathcal{B}, \end{array}$$

where $F_\lambda = K_L$ and $F_\rho = K_R$ are the left and right Kan extensions adjoint to the restriction functor $F = \mathrm{res}$. As in section 2.1, let $K_{LR} = F_{\lambda\rho}$ be the intermediate extension.

Lemma 2.6. a) *The functor $K_R : \mathcal{A} \to \mathcal{B}$ is isomorphic to the Yoneda embedding*

$$\mathrm{mod}(\mathcal{P}) \to \mathrm{mod}(\mathrm{mod}(\mathcal{P})),\ M \mapsto M^\wedge = \mathrm{Hom}(?, M).$$

In particular, for each injective I of $\mathrm{mod}(\mathcal{P})$, the module $K_R(I)$ is both projective and injective.

b) *The canonical morphism $K_L(P) \to K_R(P)$ is invertible for all finitely generated projective $\mathcal{P}$-modules P.*

c) *Let M be in $\mathrm{mod}(\mathcal{P})$ and*

$$0 \longrightarrow \Omega M \xrightarrow{g} P_0 \xrightarrow{f} M \longrightarrow 0$$

be an exact sequence with finitely generated projective P_0. Then the induced sequence

$$0 \longrightarrow (\Omega M)^\wedge \xrightarrow{g^\wedge} P_0^\wedge \longrightarrow K_{LR}(M) \longrightarrow 0$$

is a projective resolution of $K_{LR}(M)$. If P_0 is also injective, then $K_{LR}(M)$ is rigid.

We refer to section 3.14 for an example where $K_{LR}(M)$ is not rigid.

Proof. a) For L in $\mathrm{mod}(\mathcal{P})$, we have functorial isomorphisms

$$(K_R M)(L) = \mathrm{Hom}(L^\wedge, K_R M) = \mathrm{Hom}(\mathrm{res}(L^\wedge), M) = \mathrm{Hom}(L, M) = M^\wedge(L).$$

b) We have $K_L(P) = P^\wedge$ and by a), we have $P^\wedge = K_R(P)$.

c) We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & K_L(P_0) & \longrightarrow & K_L(M) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \\ 0 \longrightarrow & (\Omega M)^\wedge \longrightarrow & P_0^\wedge & \xrightarrow{f^\wedge} & K_R(M) = M^\wedge & & \end{array}$$

Thus, the image of $f^\wedge$ is $K_{LR}(M)$. Now suppose that P_0 is injective. Let $h : (\Omega M)^\wedge \to K_{LR}(M)$ represent an element of $\mathrm{Ext}^1(K_{LR}(M), K_{LR}(M))$.

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & (\Omega M)^\wedge & \xrightarrow{g^\wedge} & P_0^\wedge & \longrightarrow & K_{LR}(M) \longrightarrow 0 \\ & & & {\scriptstyle l}\swarrow & \downarrow {\scriptstyle h} & & & & \\ P_0^\wedge & \longrightarrow & & & K_{LR}(M) & & & & \end{array}$$

Since $(\Omega M)^\wedge$ is projective, the morphism h lifts along $P_0^\wedge \to K_{LR}(M)$ to a morphism $l : (\Omega M)^\wedge \to P_0^\wedge$. Since $P_0^\wedge$ is injective, the morphism l extends along $g^\wedge : (\Omega M)^\wedge \to P_0^\wedge$. Thus, the morphism h extends along $g^\wedge$ and its class in $\mathrm{Ext}^1(K_{LR}(M), K_{LR}(M))$ vanishes. $\checkmark$

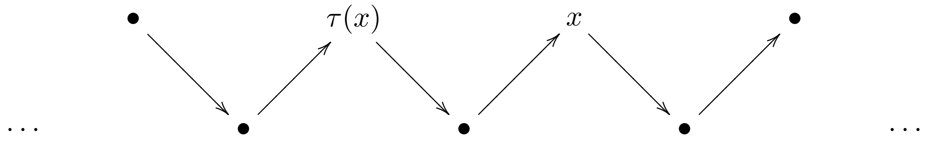
3. Desingularization of quiver Grassmannians

3.1. **Repetition quivers and the derived category.** Let Q be a quiver. We write Q_0 for its set of vertices and Q_1 for its set of arrows. We assume that Q is *finite*, i.e. both Q_0 and Q_1 are finite, and *acyclic*, i.e. it has no oriented cycles.

Let k be a field. The path algebra kQ is a finite-dimensional hereditary k-algebra. Let $\mathrm{mod}\, kQ$ be the category of all k-finite-dimensional right kQ-modules. The projective indecomposable modules are given up to isomorphism by $P_i = e_i kQ$, where e_i denotes the path of length zero at the vertex i. The head of P_i is the simple module S_i concentrated at the vertex i.

We denote by $\mathcal{D}_Q$ the bounded derived category $\mathcal{D}^b(\mathrm{mod}\, kQ)$. Endowed with the shift (=suspension) functor Σ it is a triangulated category. By [16] the derived category $\mathcal{D}_Q$ is a Krull–Schmidt category which admits Auslander-Reiten triangles or, equivalently, a Serre functor, cf. [31]. Let $\tau_{\mathcal{D}_Q}$ be the Auslander-Reiten translation. The Serre functor is then given by $S = \Sigma \circ \tau_{\mathcal{D}_Q}$ and is isomorphic to the derived tensor product with the bimodule $D(kQ) = \mathrm{Hom}_k(kQ, k)$. Let us denote by $\mathrm{ind}(\mathcal{D}_Q)$ a full subcategory of $\mathcal{D}_Q$ whose set of objects contains exactly one representative of each isomorphism class of indecomposable objects of $\mathcal{D}_Q$. If Q is an orientation of an ADE Dynkin diagram, Happel showed that $\mathrm{ind}(\mathcal{D}_Q)$ can be fully described in combinatorial terms using the so-called repetition quiver. The *repetition quiver* $\mathbb{Z}\, Q$, cf. [33], has the set of vertices $Q_0 \times \mathbb{Z}$. We obtain its set of arrows from Q_1 as follows: For each arrow $\alpha : i \to j$ in Q_1 and each integer p, we have an arrow $(\alpha, p) : (i, p) \to (j, p)$ and an arrow $\sigma(\alpha, p) : (j, p-1) \to (i, p)$. We define the automorphism τ of $\mathbb{Z}\, Q$ to be the shift by one unit to the left, so that we have in particular $\tau(i, p) = (i, p-1)$ for all vertices $(i, p) \in Q_0 \times \mathbb{Z}$.

Following [15] [32], we define the *mesh category* $k(\mathbb{Z}\, Q)$ to be the k-category whose objects are the vertices of $\mathbb{Z}\, Q$ and whose morphism space from a to b is the space of all k-linear combinations of paths from a to b modulo the subspace spanned by all elements $u r_x v$, where u and v are paths and r_x is the sum of all paths from $\tau(x)$ to x. For example if $Q = \vec{A}_2 : 1 \to 2$, the repetition quiver is



In the mesh category $k(\mathbb{Z}\vec{A}_2)$ associated with the quiver $Q = \vec{A}_2 : 1 \to 2$, the composition of any two consecutive arrows vanishes. The mesh category $k(\mathbb{Z}Q)$ and $\mathrm{ind}(\mathcal{D}_Q)$ are related as follows:

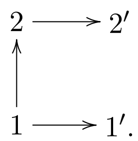
Theorem 3.2 (Prop. 4.6 of [16]). *There is a canonical fully faithful functor*

$$H : k(\mathbb{Z}Q) \to \mathrm{ind}(\mathcal{D}_Q)$$

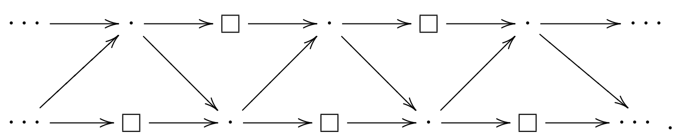
taking each vertex $(i, 0)$ *to the indecomposable projective module* P_i, $i \in Q_0$. *It is an equivalence iff* Q *is an orientation of an ADE Dynkin diagram.*

Note that the map τ induces naturally an autoequivalence on $k(\mathbb{Z}Q)$. Happel showed that $H \circ \tau$ is isomorphic to $\tau_{\mathcal{D}_Q} \circ H$. We will therefore denote $\tau_{\mathcal{D}_Q}$ by τ.

3.3. **The regular and the singular Nakajima category.** Let Q be a finite acyclic quiver as in section 3.1. The *framed quiver* $\widetilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$. For example, if Q is the quiver $1 \to 2$, the framed quiver is



Let $\mathbb{Z}\widetilde{Q}$ be the repetition quiver of $\widetilde{Q}$. We refer to the vertices (i', p), $i \in Q_0$, $p \in \mathbb{Z}$, as the *frozen vertices* of $\mathbb{Z}\widetilde{Q}$ and mark them by squares . For example, if the underlying quiver of Q is the Dynkin diagram A_2, the repetition $\mathbb{Z}\widetilde{Q}$ is the quiver



For a vertex $x = (i, p)$, we put $\sigma(x) = (i', p - 1)$ and for a vertex (i', p), we put $\sigma(i', p) = (i, p)$.

The *regular Nakajima category* $\mathcal{R}$ is the mesh category $k(\mathbb{Z}\widetilde{Q})$, where we only impose mesh relations associated with the *non frozen vertices.* The *singular Nakajima category* $\mathcal{S}$ is the full subcategory of $\mathcal{R}$ whose objects are the frozen vertices.

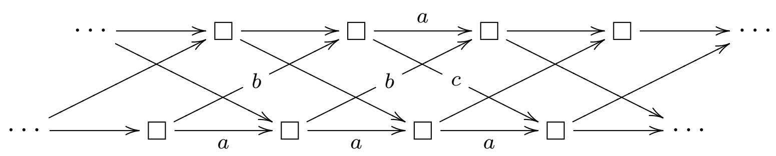
Notice that while $\mathcal{R}$ is given by a quiver with relations, it is not clear how to describe the subcategory $\mathcal{S} \subset \mathcal{R}$ in this way. In Theorem 2.4 of [20], we have shown that $\mathcal{S}$ is given by a quiver $Q_{\mathcal{S}}$ with relations such that the vertices of $Q_{\mathcal{S}}$ are the frozen vertices of $\mathbb{Z}\widetilde{Q}$, the number of arrows in $Q_{\mathcal{S}}$ from $\sigma(x)$ to $\sigma(y)$ equals the dimension of

$$\mathrm{Ext}^1_{\mathcal{D}_Q}(H(y), H(x))$$

and the minimal number of relations in the space of paths from $\sigma(x)$ to $\sigma(y)$ is given by the dimension of

$$\mathrm{Ext}^2_{\mathcal{D}_Q}(H(y), H(x)).$$

For the quiver $Q : 1 \to 2$, we find that $Q_{\mathcal{S}}$ is the quiver



and that $\mathcal{S}$ is isomorphic to the path category of $Q_{\mathcal{S}}$ modulo the ideal generated by all relations of the form $ab - ba$, $ac - ca$, $a^3 - cb$ and $bc - a^3$ (we denote all horizontal arrows by a, all rising arrows by b and all descending arrows by c).

3.4. **Graded quiver varieties.** Although, from a strictly logical point of view, we do not need graded quiver varieties in this article, we include this section for the convenience of the reader. Let us fix a dimension vector $w : \mathcal{S}_0 \to \mathbb{N}$, i.e. a function with finite support. The *affine graded quiver variety* $\mathcal{M}_0(w)$ is the affine variety $\mathrm{rep}_w(\mathcal{S})$ of $\mathcal{S}$-modules M such that $M(u) = k^{w(u)}$ for each vertex $u \in \mathcal{S}_0$. This definition is equivalent to Nakajima's original definition in [26] [27] (Q bipartite) and to the definition in [28] (Q acyclic), cf. the proof of Theorem 2.4 of [23], based on [24] [22].

Now in addition to the dimension vector $w : \mathcal{S}_0 \to \mathbb{N}$, let us fix a dimension vector $v : \mathcal{R}_0 \setminus \mathcal{S}_0 \to \mathbb{N}$ of $\mathcal{R}$. Let $\mathrm{rep}_{v,w}(\mathcal{R})$ denote the variety of $\mathcal{R}$-modules of dimension vector (v, w). Let G_v be the product of the groups $\mathrm{Gl}(k^{v(x)})$, where x runs through the non frozen vertices. The group G_v acts on the variety $\mathrm{rep}_{v,w}(\mathcal{R})$ by base change in the spaces $k^{v(x)}$. To define the smooth graded quiver variety $\mathcal{M}(v, w)$, we consider the set $\widetilde{\mathcal{M}}(v, w) \subset \mathrm{rep}_{v,w}(\mathcal{R})$ formed by the $\mathcal{R}$-modules M with dimension vector (v, w) which are *stable*, i.e. do not have non zero $\mathcal{R}$-submodules which restrict to the zero module of $\mathcal{S}$. The *graded quiver variety* is the quotient

$$\mathcal{M}(v, w) = \widetilde{\mathcal{M}}(v, w)/G_v.$$

For this definition and the following facts, we refer to Nakajima's work [26] [27] for the case where Q is Dynkin or bipartite and to Qin [28] and Kimura-Qin [21] for the extension to the case of an arbitrary acyclic quiver Q. The set $\mathcal{M}(v,w)$ canonically becomes a smooth quasi-projective variety and the projection map

$$\pi : \mathcal{M}(v,w) \to \mathcal{M}_0(w)$$

taking an $\mathcal{R}$-module M to its restriction $\operatorname{res} M$ is a proper map (notice that res is constant on the G_v-orbits). We denote by $\mathcal{M}(w) = \coprod_v \mathcal{M}(v,w)$ the disjoint union over all dimension vectors v.

In [26] Nakajima shows that the affine quiver variety admits a finite stratification

$$\mathcal{M}_0(w) = \coprod_v \mathcal{M}_0^{bs}(v,w)$$

into the locally closed smooth subsets $\mathcal{M}_0^{bs}(v,w)$ formed by the orbits of *bistable* (i.e. stable and co-stable) representations (these are called 'regular' in [26]). He also shows that we have

$$\overline{\mathcal{M}_0^{bs}(v,w)} = \coprod_{v' \leq v} \mathcal{M}_0^{bs}(v',w),$$

where the order on the dimension vectors is given by $v' \leq v$ if and only if $v'(i) \leq v(i)$ for all $i \in \mathcal{R}_0 \setminus \mathcal{S}_0$. As shown in [20], the strata $\mathcal{M}_0^{bs}(v,w)$ can be described by

$$\mathcal{M}_0^{bs}(v,w) = \{L \in \mathcal{M}_0(w) | \underline{\dim}\, K_{LR} L = (v,w)\},$$

where $K_{LR} : \operatorname{Mod} \mathcal{S} \to \operatorname{Mod} \mathcal{R}$ is the intermediate Kan extension in the sense of Section 2 associated with the restriction functor $\operatorname{res} : \operatorname{Mod} \mathcal{R} \to \operatorname{Mod} \mathcal{S}$.

3.5. **Configurations.** Let C be a subset of the set of vertices of the repetition quiver $\mathbb{Z}Q$. Let $\mathcal{R}_C$ be the quotient of $\mathcal{R}$ by the ideal generated by the identities of the frozen vertices not belonging to $\sigma^{-1}(C)$ and let $\mathcal{S}_C$ be the full subcategory of $\mathcal{R}_C$ formed by the vertices in $\sigma^{-1}(C)$. Notice that $\operatorname{Mod}(\mathcal{R}_C)$ is a subcategory of $\operatorname{Mod}(\mathcal{R})$ and similarly $\operatorname{Mod}(\mathcal{S}_C)$ a subcategory of $\operatorname{Mod}(\mathcal{S})$. Let $\operatorname{res}^C : \operatorname{Mod}(\mathcal{R}_C) \to \operatorname{Mod}(\mathcal{S}_C)$ be the restriction functor. Clearly, it is just the restriction of the functor $\operatorname{res} : \operatorname{Mod}(\mathcal{R}) \to \operatorname{Mod}(\mathcal{S})$ to the subcategories under consideration. The left and right adjoints K_L and K_R of res take the subcategory $\operatorname{Mod}(\mathcal{S}_C)$ of $\operatorname{Mod}(\mathcal{S})$ to $\operatorname{Mod}(\mathcal{R}_C)$ and thus induce left and right adjoints K_L^C and K_R^C of res^C so that we have

$$\begin{array}{c} \operatorname{Mod}(\mathcal{R}_C) \\ K_L^C \uparrow \quad \downarrow \operatorname{res}^C \quad \uparrow K_R^C \\ \operatorname{Mod}(\mathcal{S}_C). \end{array}$$

The functor res^C is a localization of abelian categories in the sense of [14], or equivalently, its adjoints are fully faithful. *In the sequel, we will omit the exponents C in the notation for the functors K_L^C and K_R^C and simply write K_L and K_R.*

To ensure that the category of finite-dimensional $\mathcal{R}_C$-modules has global dimension at most two, we make the following assumption on C.

Assumption 3.6. *For each non frozen vertex x of $\mathbb{Z}\widetilde{Q}$, the sequences*
(3.6.1)
$$0 \to \mathcal{R}_C(?, x) \to \bigoplus_{x \to y} \mathcal{R}_C(?, y) \quad \textit{and} \quad 0 \to \mathcal{R}_C(x, ?) \to \bigoplus_{y \to x} \mathcal{R}_C(y, ?)$$
are exact, where the sums range over all arrows of $\mathbb{Z}\widetilde{Q}$ whose source (respectively, target) is x.

Note that the assumption holds if C is the set of all vertices of $\mathbb{Z}Q$. The following situation provides further examples of sets C satisfying the assumption: Assume that $\mathcal{E}$ is a Hom-finite exact Krull–Schmidt category which is standard (in the sense of section 2.3, page 63 of [36]) and whose stable Auslander–Reiten quiver is $\mathbb{Z}Q$. Let us define C as the set of vertices c such that $\sigma^{-1}(c)$ corresponds to a projective indecomposable object of $\mathcal{E}$. Then the sequences (3.6.1) are associated with Auslander–Reiten conflations of $\mathcal{E}$ and hence are exact. In section 4.2, we show how iterated tilted algebras of Dynkin type give rise to such configurations C.

In fact, we have shown in Theorem 5.23 of [20] that when the assumption holds and Q is a Dynkin quiver, then the set C always comes from the choice of a Hom-finite exact Krull–Schmidt category which is standard and whose stable Auslander–Reiten quiver is $\mathbb{Z}Q$.

3.7. **The desingularization theorem.** Let M be a finite-dimensional $\mathcal{S}_C$-module such that $K_{LR}(M)$ is rigid. Recall that a variety is *equidimensional* if all its irreducible components have the same dimension.

Lemma 3.8. *Each quiver Grassmannian $\mathrm{Gr}_e(K_{LR}(M))$ is smooth and equidimensional.*

Proof. Indeed, by Proposition 7.1 of [5], we only need to check the following: The module $K_{LR}(M)$ is finite-dimensional, the space $\mathrm{Ext}^i(K_{LR}M, K_{LR}M)$ vanishes for all $i \geq 1$ and the category of finite-dimensional $\mathcal{R}_C$-modules is of global dimension at most 2. The first condition is satisfied by section 4.8 of [20] and the last one is shown in Lemma 3.5 of [20]. Finally, the module $K_{LR}(M)$ is both of projective and of injective dimension at most one by Lemma 4.15 of [20] and $K_{LR}(M)$ is rigid by our assumption. Therefore $\mathrm{Ext}^i(K_{LR}(M), K_{LR}(M))$ vanishes for all $i \geq 1$. $\surd$

We introduce a stratification with finitely many strata on $\mathrm{Gr}_w(M)$ using Nakajima's stratification of the representation spaces $\mathcal{M}_0(w)$. Following [7, 2.3], we write

$$\mathrm{Hom}^0(w, M) = \{(N, f) | N \in \mathrm{mod}_w(\mathcal{S}_C) \text{ and } f : N \to M \text{ is injective}\}.$$

Then $\mathrm{Gr}_w(M)$ is isomorphic to the quotient $\mathrm{Hom}^0(w, M)/G_w$ where G_w is the product of the groups $\mathrm{GL}(k^{w(x)})$ for all $x \in \mathcal{S}_0$. We have a canonical map

$$p : \mathrm{Hom}^0(w, M) \to \mathrm{mod}_w(\mathcal{S}_C)$$

given by the projection. For a function with finite support $v : \mathcal{R}_0 \setminus \mathcal{S}_0 \to \mathbb{N}$, we define the locally closed subset $\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$ of $\mathrm{Gr}_w(M)$ by

$$\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}} = p^{-1}(\mathcal{M}_0^{bs}(v, w))/G_w.$$

For fixed w, the subset $\mathcal{M}_0^{bs}(v, w)$ is non empty only for finitely many functions v. Thus, the variety $\mathrm{Gr}_w(M)$ decomposes into the disjoint union of finitely many strata $\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$. We have

$$\overline{\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}} \subset p^{-1}(\overline{\mathcal{M}_0^{bs}(v, w)})/G_w = \coprod_{v' \leq v} \mathcal{M}_0^{bs}(v', w)^{\mathrm{Gr}}. \tag{3.8.1}$$

Thus, if v is minimal with the property that $\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$ is not empty, then it is closed.

Lemma 3.9. *Let C be an irreducible component of $\mathrm{Gr}_w(M)$. Then there is a unique dimension vector v_C such that*

$$C \cap \mathcal{M}_0^{bs}(v_C, w)^{\mathrm{Gr}}$$

is an open dense subset in C. The vector v_C is the unique maximal element in the set of vectors v such that

$$C \cap \mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$$

is non empty.

Proof. Let V be the set of functions v such that $C \cap \mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$ is not empty. Then it follows from (3.8.1) that for a subset $V' \subset V$ stable under taking predecessors for the componentwise order, the union of the subsets $C \cap \mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$, where v ranges over V', is closed in C. In particular, this happens if V' is the complement of a maximal element v of V. Thus, if v_C is maximal in V, the set $C \cap \mathcal{M}_0^{bs}(v_C, w)^{\mathrm{Gr}}$ is open and dense in C. Since the strata $\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$ are pairwise disjoint, the maximal element v_C is uniquely determined by the irreducible component C. $\surd$

As $K_{LR}(M)$ is stable, each submodule of $K_{LR}(M)$ is stable and in particular, the points of $\mathrm{Gr}_{(v,w)}(K_{LR}(M))$ yield points of $\mathcal{M}(v, w)$. We define the subset $\mathcal{M}^{bs}(v, w)^{\mathrm{Gr}}$ of $\mathrm{Gr}_{(v,w)}(K_{LR}(M))$ in analogy with $\mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}$, so that a point of $\mathrm{Gr}_{(v,w)}(K_{LR}(M))$ lies in $\mathcal{M}^{bs}(v, w)^{\mathrm{Gr}}$ if and only if the corresponding submodule is bistable.

Lemma 3.10. a) *The restriction functor induces an isomorphism*

$$\mathcal{M}^{bs}(v, w)^{\mathrm{Gr}} \xrightarrow{\sim} \mathcal{M}_0^{bs}(v, w)^{\mathrm{Gr}}.$$

b) *The varieties* $\mathcal{M}^{bs}(v,w)^{\mathrm{Gr}}$, $\overline{\mathcal{M}^{bs}(v,w)^{\mathrm{Gr}}}$ *and* $\mathcal{M}_0^{bs}(v,w)^{\mathrm{Gr}}$ *are smooth and equidimensional.*

Proof. a) Let us first check that this map is bijective: Indeed, it is surjective, since a submodule $L \subset M$ is obtained by restricting the bistable submodule $K_{LR}(L) \subset K_{LR}(M)$. It is injective because if a submodule $N \subset K_{LR}(M)$ is costable, it is generated by the spaces $N(x)$, $x \in \mathcal{S}_0$, and is thus determined by its restriction to $\mathcal{S}$. This also shows how to construct an inverse of the map: a submodule $L \subset M$ is sent to the submodule of $K_{LR}(M)$ generated by the spaces $L(x)$, $x \in \mathcal{S}_0$.

b) By Lemma 3.8, the variety $\mathrm{Gr}_{(v,w)}(K_{LR}(M))$ is smooth and equidimensional. Thus, the same holds for its open subset $\mathcal{M}^{bs}(v,w)^{\mathrm{Gr}}$. The closure of this subset is the union of the connected (=irreducible) components of $\mathrm{Gr}_{(v,w)}(K_{LR}(M))$ which meet $\mathcal{M}^{bs}(v,w)^{\mathrm{Gr}}$. Thus, the closure is also smooth and equidimensional. By a), we obtain the same assertion for $\mathcal{M}_0^{bs}(v,w)^{\mathrm{Gr}}$. $\surd$

As in section 1, we say that a morphism of algebraic varieties $\pi : X \to Y$ is a *desingularization* if X is smooth, π is proper and surjective and induces an isomorphism from an open dense subset of X onto an open dense subset of Y. Recall that the *bistable quiver Grassmannian* is defined as

$$\mathrm{Gr}^{bs}_{(v,w)}(K_{LR}(M)) = \overline{\mathcal{M}^{bs}(v,w)^{\mathrm{Gr}}}.$$

Theorem 3.11. *As above, we assume that M is an $\mathcal{S}_C$-module such that $K_{LR}(M)$ is rigid. Let w be a dimension vector less or equal to the dimension vector of M. Let $\mathcal{V}_w(M)$ be the set of the vectors v_C, where C ranges over the irreducible components of* $\mathrm{Gr}_w(M)$ *(cf. Lemma 3.9). Let*

$$\pi^{bs} : \coprod_{v \in \mathcal{V}_w(M)} \mathrm{Gr}^{bs}_{(v,w)}(K_{LR}(M)) \to \mathrm{Gr}_w(M),$$

be the map taking a submodule L to its restriction to $\mathcal{S}_C$.

a) *The map π^{bs} is a desingularization. It induces an isomorphism between the dense open subsets*

$$\coprod_{v \in \mathcal{V}_w(M)} \mathcal{M}^{bs}(v,w)^{\mathrm{Gr}} \to \coprod_{v \in \mathcal{V}_w(M)} \mathcal{M}_0^{bs}(v,w)^{\mathrm{Gr}} \subset \mathrm{Gr}_w(M).$$

b) *Let C be an irreducible component of* $\mathrm{Gr}_w(M)$ *and v_C the unique vector such that $C \cap \mathcal{M}_0^{bs}(v_C,w)^{\mathrm{Gr}}$ is a dense open subset of C (Lemma 3.9). Then the map*

$$\pi_C : \pi^{-1}(C) \cap \mathrm{Gr}^{bs}_{(v,w)}(K_{LR}(M)) \to C$$

taking a submodule L to its restriction $\mathrm{res}(L)$ *to $\mathcal{S}_C$ is a desingularization. It induces an isomorphism between the dense open subsets*

$$\pi^{-1}(C) \cap \mathcal{M}^{bs}(v_C,w)^{\mathrm{Gr}} \xrightarrow{\sim} C \cap \mathcal{M}_0^{bs}(v_C,w)^{\mathrm{Gr}}.$$

Remark 3.12. *Theorems 1.1 and 1.2 of Section 1 are immediate consequences of part a).*

Proof. Part a) is an immediate consequence of part b). To prove b), we note that the domain of π_C is smooth by part b) of Lemma 3.10. The map π_C is proper since its domain is projective. It induces the isomorphism between dense open sets by part a) of Lemma 3.10. $\surd$

Generalizing remark 7.8 of [5] we conjecture that $\overline{\mathcal{M}^{bs}(v,w)^{\mathrm{Gr}}}$ equals the whole Grassmannian $\mathrm{Gr}_{(v,w)}(K_{LR}M)$. If this is true, we have an easy description of the fibres using the next theorem.

Theorem 3.13. *The fibre of $\pi_{v,w} : \mathrm{Gr}_{(v,w)}(K_{LR}M) \to \mathrm{Gr}_w(M)$ over a submodule $U \subset M$ is isomorphic to the quiver Grassmannian of submodules of dimension $(v,w) - \underline{\dim}\, K_{LR}(U)$ of the module $K_{LR}(M)/K_{LR}(U)$.*

Proof. The claim is equivalent to the statement that the fibre is isomorphic to the variety of submodules $V \subset K_{LR}(M)$ containing $K_{LR}(U)$ and such that $\underline{\dim}\, V = (v,w)$.

By the definition of $\pi_{v,w}$, this is equivalent to the following statement: Suppose that $V \subset K_{LR}(M)$ is a submodule of dimension vector (v,w). Then the restriction $\mathrm{res}(V)$ equals U if and only if we have $K_{LR}(U) \subset V$.

Indeed, if we have $K_{LR}(U) \subset V$, then we have

$$U = \mathrm{res}\, K_{LR}(U) \subset \mathrm{res}(V)$$

and

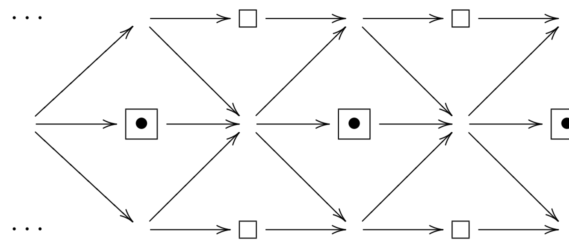
$$\underline{\dim}\, U = \underline{\dim}\, \mathrm{res}(V).$$

Hence we have $U = \mathrm{res}(V)$ as claimed.

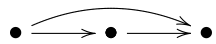
Conversely, suppose that we have $\mathrm{res}(V) = U$. By assumption, we have $V \subset K_{LR}(M)$. Since $K_{LR}(M)$ is stable, so is V. Thus, we have $K_{LR}(\mathrm{res}(V)) \subset V$. But we also have $K_{LR}(V) = K_{LR}(U)$. Thus, we have $K_{LR}(U) \subset V$ as claimed. $\surd$

3.14. **Example of a non rigid intermediate extension.** In section 3.7, to prove that π_{Gr} is indeed a desingularization, we made the assumption that the intermediate extension $K_{LR}(M)$ is rigid. We gave some sufficient conditions for this to hold in Lemma 2.6. Let us show by an example that $K_{LR}(M)$ is not always rigid.

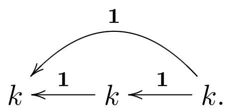
Let Q be some orientation of A_3 and let w be a dimension vector which takes the value 1 in the marked boxes and zero everywhere else in the quiver of $\mathcal{R}$ associated with Q:



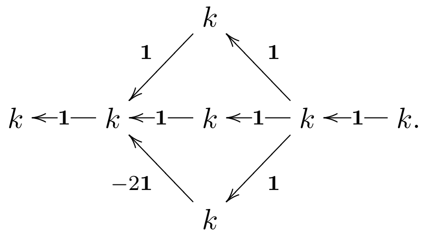
By Theorem 2.4 of [20], the modules in $\mathcal{S}$ with dimension vector w are given by the representations with dimension vector $(1, 1, 1)$ of the quiver



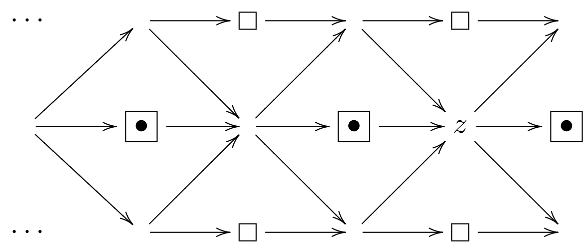
The following module $M \in \operatorname{mod} \mathcal{S}$ is not rigid and has dimension vector w:



Its intermediate extension $K_{LR}M$ is the $\mathcal{R}_C$-module given by



Using Corollary 3.6 of [20] it is easy to see that the space $\operatorname{Ext}^1(S_x, K_{LR}(M))$ vanishes for all non-frozen vertices $x \in \mathcal{R}_0 - \mathcal{S}_0$ except for the vertex z:



Hence, by Lemma 4.13 of [20], the cokernel of $K_{LR}M \hookrightarrow K_R M$ is represented as a module of $\mathcal{D}_Q \cong \mathcal{R}/\langle \mathcal{S} \rangle$ by z, i.e. it is the injective module $z^{\vee}_{\mathcal{D}}$ defined by

$$z^{\vee}_{\mathcal{D}} = D \operatorname{Hom}_{\mathcal{D}_Q}(z, -),$$

where we identify the vertex z with its image in $\mathcal{D}_Q$ under the Happel functor. Now using the projective resolution of $z^{\vee}_{\mathcal{D}}$ from Theorem 3.7 of [20], we have

$$\operatorname{Ext}^2_{\mathcal{R}}(z^{\vee}_{\mathcal{D}}, z^{\vee}_{\mathcal{D}}) \cong \operatorname{Hom}((\Sigma^{-1} z)^{\vee}, z^{\vee}_{\mathcal{D}}) \cong D \operatorname{Hom}_{\mathcal{D}_Q}(z, \Sigma^{-1} z)$$

which vanishes, since z is an indecomposable object of $\mathcal{D}_Q$. So using part c) of Lemma 2.4 (the vanishing of extensions from stables to objects in the kernel follows from part e) of Corollary 3.6 of [20]), we deduce that $K_{LR}(M)$ is not rigid, since M is not rigid.

4. Quiver Grassmannians over repetitive algebras

4.1. **Repetitive algebras.** Let A be a finite-dimensional k-algebra and $DA = \mathrm{Hom}_k(A, k)$ the bimodule dual to A. Let $T(A)$ be the *trivial extension*, i.e. the algebra $A \oplus DA$ with the multiplication

$$(a, f)(b, g) = (ab, ag + fb), a, b \in A, \quad f, g \in DA.$$

We endow $T(A)$ with the $\mathbb{N}$-grading such that $T(A)^0 = A$, $T(A)^1 = DA$ and $T(A)^p = 0$ for all $p \geq 2$. A $\mathbb{Z}$-graded module M over $T(A)$ is given by a sequence M^p, $p \in \mathbb{Z}$, of A-modules and A-linear maps

$$m^p : \nu(M^p) \to M^{p+1} \quad , p \in \mathbb{Z} \quad ,$$

where $\nu(L) = L \otimes_A DA$, such that $m^{p+1} \circ \nu(m^p) = 0$ for all $p \in \mathbb{Z}$. Equivalently, such a module may be interpreted as a module over the *repetitive algebra*, which is a suitably defined (locally unital) infinite matrix algebra, cf. section 10 of [16]. Let $\mathrm{grm}(T(A))$ be the category of $\mathbb{Z}$-graded $T(A)$-modules of finite dimension over A with morphisms the homogeneous $T(A)$-linear maps of degree 0. Let $\mathcal{P} \subset \mathrm{grm}(T(A))$ be the full subcategory of the projective graded modules. We have a canonical equivalence

$$\mathrm{grm}(T(A)) \xrightarrow{\sim} \mathrm{mod}(\mathcal{P})$$

taking a module M to the restriction of $\mathrm{Hom}(?, M)$ to $\mathcal{P}$ and the category $\mathcal{P}$ fits into the setup of section 2.5: it is coherent since its projectives are of finite dimension and the finitely presented $\mathcal{P}$-modules coincide with the finite-dimensional $\mathcal{P}$-modules.

For a graded $T(A)$-module M, let $M\langle 1\rangle$ denote the shifted module defined by

$$M\langle 1\rangle^p = M^{p+1}, m^p_{M\langle 1\rangle} = m^{p+1}_M.$$

For a graded left $T(A)$-module M, let DM denote the k-dual right module with $(DM)^p = D(M^{-p})$, $p \in \mathbb{Z}$. We have a canonical isomorphism of A-modules

$$DT(A) \xrightarrow{\sim} (T(A))\langle 1\rangle.$$

This shows that the projectives coincide with the injectives in $\mathrm{mod}(\mathcal{P})$, which is therefore a Frobenius category. Thus, the associated stable category $\underline{\mathrm{mod}}(\mathcal{P})$ (the quotient of $\mathrm{grm}\, T(A)$ by the ideal of morphisms factoring through a projective) is canonically triangulated, cf. [16]. By a theorem of Happel [16], the canonical embedding

$$\mathrm{mod}\, A \to \underline{\mathrm{mod}}(\mathcal{P})$$

taking an A-module to the corresponding graded $T(A)$-module concentrated in degree 0 extends to a fully faithful triangle functor

$$\mathcal{D}^b(\mathrm{mod}\, A) \to \underline{\mathrm{mod}}(\mathcal{P}) \tag{4.1.1}$$

and this functor is an equivalence if and only if A is of finite global dimension.

4.2. **The case of iterated tilted algebras of Dynkin type.** From now on, let us suppose that A is derived equivalent to the path algebra kQ for a Dynkin quiver Q with underlying graph Δ (for example, we can take $A = kQ$ or A a tilted algebra of type Q, cf. [16] [19]). For a Krull–Schmidt category $\mathcal{C}$, let us denote by $\mathrm{ind}(\mathcal{C})$ the full subcategory whose objects form a set of representatives for the isomorphism classes of the indecomposable objects of $\mathcal{C}$. By combining Happel's theorem 3.2 with the equivalence (4.1.1), we obtain an isomorphism

$$k(\mathbb{Z}\,Q) \xrightarrow{\sim} \mathrm{ind}(\underline{\mathrm{mod}}(\mathcal{P})). \tag{4.2.1}$$

Let s_i, $i \in Q_0$, denote the vertices of $\mathbb{Z}\,Q$ corresponding to the simple A-modules S_i (considered as $T(A)$-modules concentrated in degree 0). Let h be the Coxeter number of Δ. Let C be the set of the following vertices of $\mathbb{Z}\,Q$:

$$\tau^{p(h-1)}\tau^{-1}\Sigma^{-1}s_i \ \ , p \in \mathbb{Z} \ \ , i \in Q_0. \tag{4.2.2}$$

Proposition 4.3. *The isomorphism (4.2.1) lifts to an isomorphism*

$$\mathcal{R}_C \xrightarrow{\sim} \mathrm{ind}(\mathrm{mod}(\mathcal{P})) \tag{4.3.1}$$

taking s_i to the simple module S_i, $i \in Q_0$. It induces an isomorphism

$$\mathcal{S}_C \xrightarrow{\sim} \mathrm{ind}(\mathcal{P}). \tag{4.3.2}$$

Proof. The category $\mathcal{P}$ is locally bounded and locally representation-finite. Moreover, it is directed. It follows from [1] that $\mathcal{P}$ is standard, i.e. the category $\mathrm{ind}(\mathrm{mod}(\mathcal{P}))$ is isomorphic to the mesh category of the Auslander–Reiten quiver $\Gamma_{\mathrm{mod}(\mathcal{P})}$ (with the mesh relations associated with the non projective vertices), cf. [34]. It remains to be checked that the Auslander–Reiten quiver $\Gamma_{\mathrm{mod}(\mathcal{P})}$ is indeed obtained from $\mathbb{Z}\,Q$ by adding a new vertex $\sigma(x)$ and new arrows $\tau(x) \to \sigma(x) \to x$ to $\mathbb{Z}\,Q$ for each vertex x in C. Indeed, the quiver $\mathbb{Z}\,Q$ is isomorphic to the stable Auslander–Reiten quiver $\Gamma_{\underline{\mathrm{mod}}(\mathcal{P})}$ via the isomorphism induced by (4.2.1), and we know that we have to insert the vertex $v(P)$ corresponding to an indecomposable projective P in the mesh starting at $v(\mathrm{rad}(P))$. If P_M is the projective cover of a simple module M, we have the exact sequence

$$0 \to \mathrm{rad}(P_M) \to P_M \to M \to 0 \ \ ,$$

which shows that $\mathrm{rad}(P_M) = \Sigma^{-1}M$ in the triangulated category $\underline{\mathrm{mod}}(\mathcal{P})$. Thus, we have to insert $v(P_M)$ in the mesh *ending* at $\tau^{-1}\Sigma^{-1}v(M)$. Now the simple $\mathcal{P}$-modules are of the form $S_i\langle p\rangle$, $i \in Q_0$, $p \in \mathbb{Z}$. It is well-known and not hard to check that the shift $\langle 1\rangle$ induces the composition $S\Sigma$ in the stable category $\underline{\mathrm{mod}}(\mathcal{P})$ (equivalent to $\mathcal{D}_Q$), where S is the Serre functor. Now we have

$$S\Sigma = \tau\Sigma\Sigma = \tau\Sigma^2 = \tau\tau^{-h} = \tau^{-(h-1)}.$$

Here, the isomorphism $\Sigma^2 = \tau^{-h}$ follows from Happel's theorem 3.2 and from Gabriel's description of the Serre functor (alias Nakayama functor) in Proposition 6.5 of [15]. A detailed proof of a more precise statement is given by Miyachi-Yekutieli in Theorem 4.1 of [25]. Thus, we get $\tau^{-1}\Sigma^{-1}S_i\langle -p\rangle = \tau^{p(h-1)}\tau^{-1}\Sigma^{-1}S_i$, which proves the claim. This construction of C also makes the second assertion clear. $\surd$

Let $C \subset \mathbb{Z}Q_0$ be as in the Proposition. It satisfies Assumption 3.6 by the remark following the statement of the assumption. Let $M \in \mathrm{mod}(\mathcal{P})$ be a finite-dimensional module (i.e. a finite-dimensional module over the repetitive algebra). The proposition shows that we may consider M as a module over the singular Nakajima category $\mathcal{S}_C$, and that we can consider its intermediate extension

$$K_{LR}(M) \in \mathrm{mod}(\mathrm{mod}(\mathcal{P}))$$

as a module over the regular Nakajima category $\mathcal{R}_C$. By part c) of Lemma 2.6, the intermediate extension $K_{LR}(M)$ is rigid, since each projective in $\mathrm{mod}(\mathcal{P})$ is also injective. Thus, from Theorem 3.11, we obtain the following Corollary.

Corollary 4.4. *The map π^{bs} of Theorem 3.11 provides a desingularization of the quiver Grassmannian of M.*

4.5. **Link to Cerulli–Feigin–Reineke's desingularization.** In their article [5], Cerulli–Feigin–Reineke have constructed desingularizations of quiver Grassmannians of representations of Dynkin quivers. We will show how their construction fits into the framework of desingularizations of quiver Grassmannians of modules over repetitive algebras of section 4.2.

Let Q be a connected Dynkin quiver and A the path algebra of Q. Following section 4 of [5], we define $\mathcal{H}_Q$ to be the full subcategory of the category of morphisms of $\mathrm{mod}(A)$ whose objects are the injective morphisms

$$f : P_1 \to P_0$$

such that

1) P_0 and P_1 are finitely generated projective A-modules and
2) f does not admit a non zero direct factor of the form $0 \to P$, where P is a finitely generated projective A-module.

We define a commutative square of functors

$$\begin{array}{ccc} \mathcal{H}_Q & \longrightarrow & \mathrm{mod}(\mathcal{P}) = \mathrm{grm}(T(A)) \\ \uparrow & & \uparrow \\ \mathrm{proj}(A) & \longrightarrow & \mathcal{P} \end{array}$$

as follows:

- the functor $\mathrm{proj}(A) \to \mathcal{H}_Q$ takes a module P to the identity $P \to P$,
- the functor $\mathcal{P} \to \mathrm{mod}(\mathcal{P})$ is the Yoneda embedding,
- the functor $\mathcal{H}_Q \to \mathrm{grm}(T(A))$ takes a morphism $f : P_1 \to P_0$ to the graded module $P_1 \rightsquigarrow \nu(P_0)$ which has P_1 in degree 0, $\nu(P_0)$ in degree 1, $\nu(f) : \nu(P_1) \to \nu(P_0)$ as the structural morphism and all other components equal to zero,
- the functor $\mathrm{proj}(A) \to \mathcal{P}$ takes a module P to the graded $T(A)$-module $(P \rightsquigarrow \nu(P))$, where P in degree 0 is linked to $\nu(P)$ in degree 1 by the identity map $\nu(P) \to \nu(P)$ and all other components vanish.

Notice that all four functors are fully faithful. Using the horizontal functors, we can restrict a $\mathcal{P}$-module to $\mathrm{proj}(A)$ and a $\mathrm{mod}(\mathcal{P})$-module to $\mathcal{H}_Q$. In this way, the category $\mathrm{mod}(A)$ identifies with the subcategory of $\mathrm{mod}(\mathcal{P})$ formed by the modules supported on $\mathrm{proj}(A)$ and the category $\mathrm{mod}(\mathcal{H}_Q)$ with the full subcategory of $\mathrm{mod}(\mathrm{mod}(\mathcal{P}))$ formed by the modules supported on $\mathcal{H}_Q$.

Following section 5 of [5], for an A-module M, we define the $\mathcal{H}_Q$-module $\widehat{M}$ by

$$\widehat{M}(P_1 \to P_0) = \mathrm{im}(\mathrm{Hom}(P_0, M) \to \mathrm{Hom}(P_1, M)).$$

The following proposition shows that the functor $\Lambda : M \mapsto \widehat{M}$ of [loc. cit.] is a particular case of the intermediate extension $M \mapsto K_{LR}(M)$. Therefore, Corollary 4.4 generalizes Corollary 7.7 of [5].

Proposition 4.6. *Let M be an A-module identified with a $\mathcal{P}$-module supported on $\mathrm{proj}(A) \subset \mathcal{P}$. Then the $\mathrm{mod}(\mathcal{P})$-module $K_{LR}(M)$ is supported on $\mathcal{H}_Q$ and its restriction to $\mathcal{H}_Q$ is canonically isomorphic to $\widehat{M}$.*

Proof. Let $M^\wedge : \mathrm{mod}(\mathcal{P}) \to \mathrm{mod}(k)$ be the functor represented by M. We have $\mathrm{res}(M^\wedge) = M$. Thus, by part c) of Lemma 5.4 of [20], the module $K_{LR}(M) = K_{LR}(\mathrm{res}(M^\wedge))$ is the submodule of $M^\wedge$ generated by the images of all morphisms $P^\wedge \to M^\wedge$, where P belongs to $\mathcal{P}$. Let us check that the restriction of $K_{LR}(M)$ to $\mathcal{H}_Q$ is isomorphic to $\widehat{M}$. Let

$$0 \longrightarrow P_1^M \longrightarrow P_0^M \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of M in $\mathrm{mod}(A)$. We deduce that we have a minimal projective resolution of $(M \rightsquigarrow 0)$ in $\mathrm{mod}(\mathcal{P})$ given by

$$0 \longrightarrow (P_1^M \rightsquigarrow \nu(P_0^M)) \longrightarrow (P_0^M \rightsquigarrow \nu(P_0^M)) \longrightarrow (M \rightsquigarrow 0) \longrightarrow 0.$$

For an arbitrary object L of $\mathrm{mod}(\mathcal{P})$, a morphism $L \to (M \rightsquigarrow 0)$ factors through a projective if and only if it factors through $(P_0^M \rightsquigarrow \nu(P_0^M))$. Using this we see that for an object $P_1 \to P_0$ of $\mathcal{H}_Q$, the module

$$K_{LR}(P_1 \rightsquigarrow \nu(P_0))$$

is the image

$$\mathrm{im}(\mathrm{Hom}((P_1 \rightsquigarrow \nu(P_0)), (P_0^M \rightsquigarrow \nu(P_0^M))) \to \mathrm{Hom}((P_1 \rightsquigarrow \nu(P_0)), (M \rightsquigarrow 0)).$$

Clearly this image identifies with

$$\mathrm{im}(\mathrm{Hom}(P_0, M) \to \mathrm{Hom}(P_1, M)) = \widehat{M}(P_1 \to P_0).$$

It remains to be shown that $K_{LR}(M)$ vanishes at all indecomposables L not belonging to the image of $\mathcal{H}_Q$ in $\mathrm{mod}(\mathcal{P})$. Indeed, for an object L of $\mathrm{mod}(\mathcal{P})$, let $\Omega(L)$ denote the kernel of a projective cover and $\Omega^{-1}(L)$ the cokernel of an injective hull. Since Q is a Dynkin quiver and the stable category of $\mathrm{mod}(\mathcal{P})$ is equivalent to the derived category of $\mathrm{mod}(A)$, the indecomposable objects of $\mathrm{mod}(\mathcal{P})$ are exactly the projective-injective indecomposables and the objects $\Omega^p(L)$, where $p \in \mathbb{Z}$ and L is an indecomposable A-module. Clearly, the only indecomposable projective objects possibly admitting a non zero morphism to $M \rightsquigarrow 0$ are the $P \rightsquigarrow \nu(P)$, where P is an indecomposable projective A-module. Now let L be an indecomposable A-module and

$$0 \longrightarrow P_1^L \longrightarrow P_0^L \longrightarrow L \longrightarrow 0$$

a minimal projective resolution. We have $K_{LR}(M)(L \rightsquigarrow 0) = 0$ since

$$\mathrm{Hom}((L \rightsquigarrow 0), (P_0^M \rightsquigarrow \nu(P_0^M))) = 0.$$

We have

$$\Omega(L) = (P_1^L \rightsquigarrow \nu(P_0^L))$$

and so this object belongs to $\mathcal{H}_Q$. It is easy to check that for $p \geq 2$, the object $\Omega^p(L)$ has vanishing component in degree 0 and so does not admit non zero morphisms to M. Now let

$$0 \longrightarrow L \longrightarrow I_L^0 \longrightarrow I_L^1 \longrightarrow 0$$

be a minimal injective coresolution. We have

$$\Omega^{-1}(L) = (\nu^{-1}(I_L^0) \rightsquigarrow I_L^1)$$

and so $\mathrm{Hom}(\Omega^{-1}(L), (0 \rightsquigarrow M)) = 0$, where now the two components are concentrated in degrees -1 and 0. For $p \geq 2$, the object $\Omega^{-p}(L)$ has vanishing component in degree 0 and so we have $\mathrm{Hom}(\Omega^{-p}(L), (0 \rightsquigarrow M)) = 0$ for $p \geq 2$ as well. $\surd$

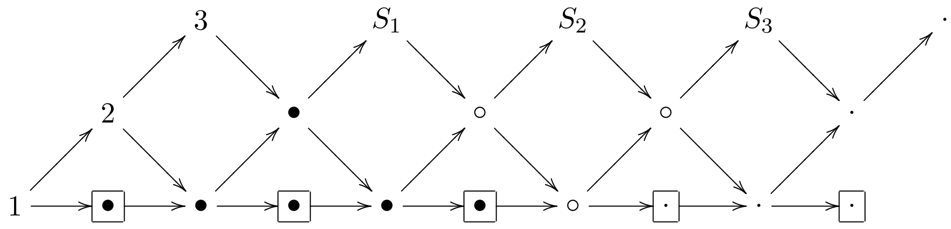


FIGURE 1. Example A_3

4.7. **The example A_3.** We illustrate section 4.5 by an example. Let Q be the linearly oriented quiver

$$1 \longrightarrow 2 \longrightarrow 3$$

of type A_3. Figure 1 shows the quiver of $\mathcal{R}_C$, where C is the configuration obtained from Proposition 4.3. Thus, the quiver is also the Auslander-Reiten quiver of the repetitive algebra of Q. It contains the Auslander-Reiten quiver of the path algebra kQ as the triangle whose base is formed by the simples S_1, S_2, S_3. The vertices marked by $\bullet$ correspond to the indecomposables in the image of the functor $\mathcal{H}_Q \to \mathrm{mod}(\mathcal{P})$.

5. An example in tilted type D_4

We illustrate Corollary 4.4 with an example of tilted type D_4. We consider the algebra B given by the square

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 3 & \longrightarrow & 4 \end{array}$$

with the commutativity relation (B is tilted of type D_4). Let M be the B-module given as the direct sum of the three modules $I_1 = P_4$, P_2 and I_3:

$$\begin{array}{ccc} k & \xleftarrow{\mathbf{1}} & k \\ \mathbf{1}\uparrow & & \uparrow\mathbf{1} \\ k & \xleftarrow[\mathbf{1}]{} & k \end{array} \qquad \begin{array}{ccc} k & \xleftarrow{\mathbf{1}} & k \\ \uparrow & & \uparrow \\ 0 & \longleftarrow & 0 \end{array} \qquad \begin{array}{ccc} 0 & \longleftarrow & 0 \\ \uparrow & & \uparrow \\ k & \xleftarrow{\mathbf{1}} & k. \end{array}$$

All submodules of M with dimension vector $(1,1,1,1)$ are isomorphic to one of the modules I_1 and $P_2 \oplus I_3$. The space $\mathrm{Hom}(I_1, P_2 \oplus I_3)$ is of dimension one. Let us denote by L the submodule isomorphic to $P_2 \oplus I_3$, where we embed the factor P_2 into P_4. We obtain $\mathrm{Hom}(L, M/L) \cong \mathrm{Hom}(P_2 \oplus I_3, P_2 \oplus I_3)$, which is two-dimensional. Let N denote the submodule isomorphic to $P_2 \oplus I_3$, where we embed N into $P_2 \oplus I_3$. In this case, we have $\mathrm{Hom}(N, M/N) \cong \mathrm{Hom}(P_2 \oplus I_3, I_1)$, which is one-dimensional. Thus, the tangent spaces of the quiver Grassmannian $\mathrm{Gr}_{(1,1,1,1)^t}(M)$ at the points L and N do not have the same dimension. In fact the quiver Grassmannian consists of two irreducible

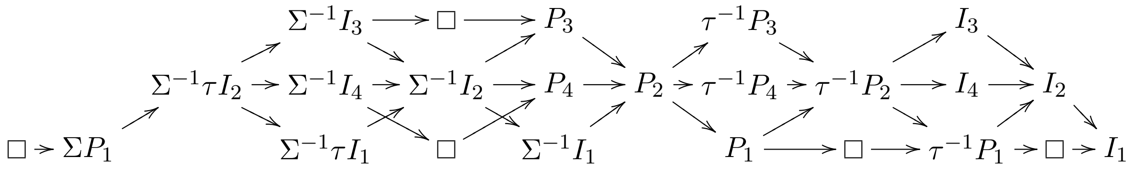
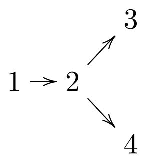


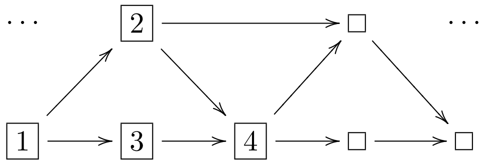
FIGURE 2. The AR-quiver of the repetitive algebra of D_4

components, both isomorphic to the projective line, and which intersect at the singular point L. The two components are given by the closures of $S_{[I_1]}$ and $S_{[P_2 \oplus I_3]}$, where $S_{[N]}$ denotes the irreducible and locally closed subset in $\mathrm{Gr}_{(1,1,1,1)^t}(M)$ of submodules isomorphic to N. Hence $\{v_1, v_2\} = \mathcal{V}_w(M)$ is given by $\underline{\dim}\, K_{LR}(I_1) = (v_1, w)$ and $\underline{\dim}\, K_{LR}(P_2 \oplus I_3) = (v_2, w)$.

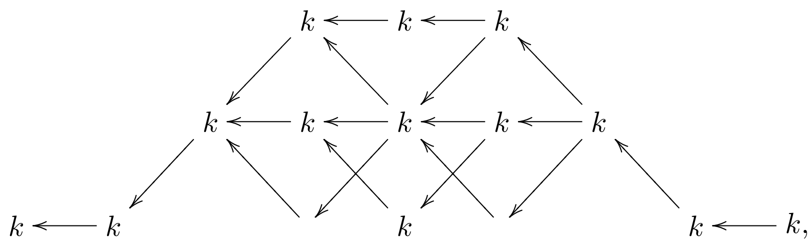
Let us choose the orientation

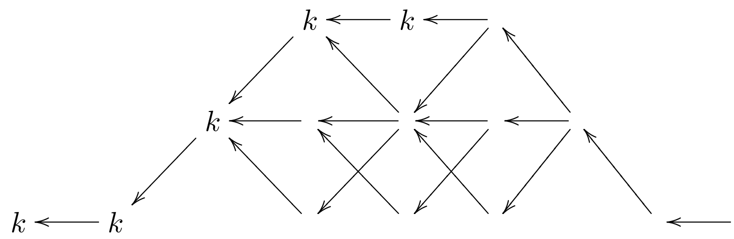


of the Dynkin diagram D_4. Removing the boxes from the quiver in Figure 2 gives the Auslander-Reiten quiver of the derived category $\mathcal{D}_Q$. In this category, the suspension functor Σ is isomorphic to τ^{-3} and the Coxeter number of D_4 equals $h = 6$. Thus, the configuration C of section 4.2 is given by the vertices corresponding to the objects $\tau^{5p+2} S_i$ of $\mathcal{D}_Q$ and the quiver of $\mathcal{R}_C$ is the one displayed in Figure 2. Using Theorem 2.4 of [20] one verifies that the quiver with relations of $\mathcal{S}_C$ is the quiver

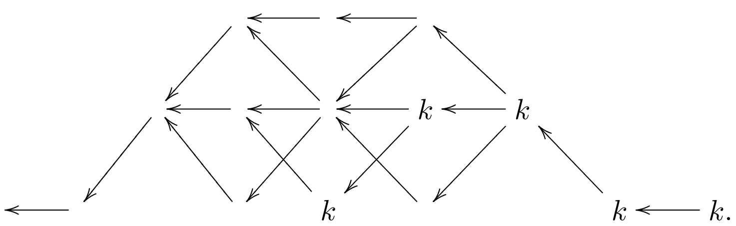


with commuting relations in the squares and no relations in the triangles. The intermediate extension of M is the direct sum of the intermediate extensions of its three summands, namely the modules





and



Here all non-labelled vertices are represented by zero spaces and all possibly non zero maps are the identity. The intermediate extension of the generic subrepresentations is given by the first summand $K_{LR}(I_1)$ and the sum of the last two summands $K_{LR}(P_2) \oplus K_{LR}(I_3)$ of $K_{LR}(M)$. We conclude that the desingularization map of Corollary 4.4 is the map

$$\mathrm{Gr}_{\underline{\dim}\, K_{LR}(I_1)}(K_{LR}(M)) \coprod \mathrm{Gr}_{\underline{\dim}\, K_{LR}(I_3 \oplus P_2)}(K_{LR}(M)) \to \mathrm{Gr}_{(1,1,1,1)^t}(M)$$

taking U to $\mathrm{res}(U)$. Finally, it is easy to see that both Grassmannians on the left hand side are isomorphic to projective lines.

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